Les Houches lectures on laser cooling and trapping
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Lecture 1

Light forces

The research on cold atoms, molecules and ions was triggered by the development of lasers [1]. In 1987, the first magneto-optical trap (MOT) was demonstrated [2]. Since then, the achievement of laser cooling [3, 4] and trapping enabled the fast development of ultra high resolution spectroscopy on atoms and ions, dramatic improvement of time and frequency standards, the application of atom interferometry to inertial sensors [5] and the observation of Bose-Einstein condensation (BEC) [6–8], opening a new area of interdisciplinary physics. The importance of laser cooling techniques in this growing field was recognised shortly after the achievement of BEC in 1995, by the 1997 Nobel prize attributed to Steven Chu, Claude Cohen-Tannoudji and William D. Phillips [9–11].

Light forces are at the basis of atomic position and momentum manipulation with light. The present lecture is devoted to this subject, and is inspired by Jean Dalibard's lectures at École normale supérieure [12]. The proceedings of the Les Houches school on quantum metrology 2007 gives a very brief overview of the topic [13]. For a deeper insight, the following general references on laser cooling and trapping may be useful:

1. Cohen-Tannoudji's and Phillips' lectures in Les Houches, 1990 [14, 15]
2. Cohen-Tannoudji's lectures at Collège de France [16] (in french)
3. the book of Metcalf and van der Straten [17]
4. on atom-photon interactions: the book of Cohen-Tannoudji *et al.* [18]
5. on dipole forces: the review paper by Grimm, Weidemüller and Ovchinnikov [19]
6. from optical pumping to quantum degenerate gases: a very complete book by Cohen-Tannoudji and Guéry-Odelin inspired by the Collège de France lectures [20].

The first deflection of neutral particles by near resonant light was observed in 1933 by Otto Frisch, on a beam of sodium atoms irradiated by an emission lamp [21]. However, the real development of atom manipulation with laser light started when lasers became available.

To understand how light can be used to manipulate the external degrees of freedom of atoms, let us estimate the acceleration undergone by an atom irradiated by a laser. Each time a photon of wavelength λ_L is absorbed or emitted, due to momentum conservation the atomic velocity changes by the *recoil velocity* $v_{\rm rec}$, defined by

$$v_{\rm rec} = \frac{\hbar k_L}{M} \quad \text{with} \quad k_L = \frac{2\pi}{\lambda_L} \tag{1}$$

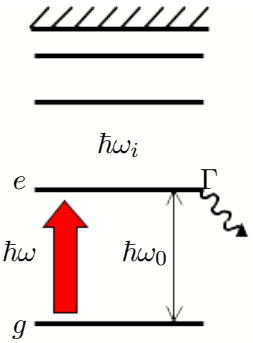
where M is the atomic mass. For atoms in near resonant light, photons are scattered at a rate close to Γ, the inverse lifetime of the excited state. For alkali, one has $\Gamma^{-1} \sim 30$ ns

typically, and $v_{\rm rec}$ ranges between 3 and 30 mm·s^{-1}. The acceleration corresponding to an irradiation by a resonant laser is then of order $a \simeq \Gamma v_{\rm rec} \simeq 10^5$ m·s^{-2}, four orders of magnitude larger than the Earth acceleration! This strong acceleration enables to stop an atom moving at 300 m·s^{-1} within 3 ms, over 50 cm. Note that the Doppler shift $-\mathbf{k}\cdot\mathbf{v}$ makes the process a little more subtle than that.

1 Atom-light interaction

1.1 A two-level model

In lecture 1, we will consider the interaction of an atom with a near resonant laser light, of frequency ω_L.



The atom has many electronic transitions of frequencies ω_i. The two-level approximation is valid if the detuning $\delta = \omega_L - \omega_0$ to a particular transition of frequency ω_0 is such that $|\delta| \ll \omega_0, \omega_L, |\omega_i - \omega_L|$ for all $i \neq 0$. We then restrict the discussion to these two levels.

N.B.1: Later on, the case of a degenerate state will be examined, with a $J \neq 0$ internal structure.
N.B.2: If the two-level condition is not fulfilled, the interaction of the ground state with all the other levels have to be considered. It may be the case for the calculation of light shifts induced by a far off resonant laser [19].

The two-level atom is described by its transition frequency ω_0, or wavelength $\lambda_0 = 2\pi c/\omega_0$, and the lifetime of the excited state Γ^{-1} due to the coupling with the electromagnetic vacuum. It is irradiated with a laser of frequency ω_L and wavelength λ_L. For most atoms which are laser cooled, λ_L is in the visible or near infra-red region. The coupling between atom and laser is ensured by the (electric) dipolar interaction.

1.2 Dipolar interaction

An atom has no permanent dipole which could interact with the light field. However, the laser field itself induces an atomic dipole $\mathbf{D}$ which in turn interacts with the light field. The dipolar interaction energy reads $-\mathbf{D}\cdot\mathbf{E}$. The dipolar operator makes the atom change its internal state. It can be written

$$\hat{\mathbf{D}} = \mathbf{d}|e\rangle\langle g| + \mathbf{d}^\dagger|g\rangle\langle e|, \tag{2}$$

where $\mathbf{d}$ is the reduced dipole:

$$\mathbf{d} = \langle e|\hat{\mathbf{D}}|g\rangle \qquad \mathbf{d}^\dagger = \langle g|\hat{\mathbf{D}}.|e\rangle \tag{3}$$

In the rest of the lecture, we remain in the dipolar approximation and will not consider other coupling processes between atom and light.

1.3 Laser electric field

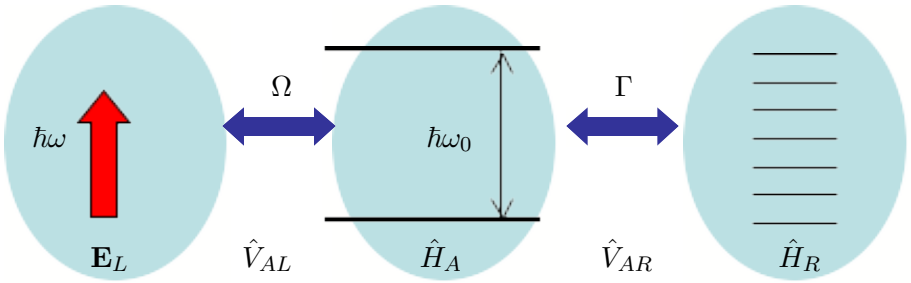
A laser field contains a huge number of photons in the same mode. The Poissonian photon number fluctuations around the mean value $\bar{n}$ are of order $\Delta n = \sqrt{\bar{n}}$, very small as compared to $\bar{n}$. It is thus relevant to describe the laser field by a classical time-dependent field. A quantum description of the laser field will nevertheless be given in section 3.

$$\mathbf{E}_L(\mathbf{r},t) = \frac{1}{2}\mathcal{E}_L(\mathbf{r})\left\{\boldsymbol{\epsilon}_L(\mathbf{r})\, e^{-i\omega_L t}\, e^{-i\phi(\mathbf{r})} + c.c.\right\} \tag{4}$$

The laser amplitude $\mathcal{E}_L$, polarisation $\boldsymbol{\epsilon}_L$ and phase ϕ may depend on position $\mathbf{r}$. The coupling of the atom to this classical laser describes efficiently the absorption and stimulated emission processes. The quantum fluctuations will be included in an additional term describing quantum vacuum in the total Hamiltonian. This term is responsible for spontaneous emission.

1.4 Hamiltonian of the three coupled systems

We finally deal with three coupled systems: laser, atom and quantum field.



The total Hamiltonian reads

$$\hat{H} = \hat{H}_A + \hat{H}_R + \hat{V}_{AL} + \hat{V}_{AR}, \tag{5}$$

the four terms being discussed in the following.

1.4.1 Hamiltonian of the isolated atom

The operators of position and momentum are labelled $\hat{\mathbf{R}}$ and $\hat{\mathbf{P}}$. The atomic Hamiltonian is the sum of the internal and the kinetic terms.

$$\hat{H}_A = \hbar\omega_0 |e\rangle\langle e| + \frac{\hat{\mathbf{P}}}{2M} \tag{6}$$

where the energy of the ground state $|g\rangle$ is taken as the origin of energies.

1.4.2 Hamiltonian of the quantum field

If the quantum modes of the field are labelled by $\ell = (\mathbf{k}, \boldsymbol{\epsilon})$, the energy of the quantum modes is given by

$$\hat{H}_R = \sum_\ell \hbar\omega_\ell \, \hat{a}_\ell^\dagger \hat{a}_\ell \, . \tag{7}$$

1.4.3 Atom to quantum field coupling

The coupling between the atom and the quantum field through the induced dipole is denoted as $\hat{V}_{AR}$. It is responsible for spontaneous emission. It is not necessary to give an explicit form of this term here.

1.4.4 Atom – laser coupling

The coupling to the classical laser field given in Eq.(4) is due to the dipole operator (2). The scalar product of (2) and (4) yields four terms, which can be expressed as a sum of two terms, a resonant term $\hat{V}_{AL}^{\rm res}$ and a non resonant term $\hat{V}_{AL}^{\rm non\,res}$:

$$\begin{aligned}
\hat{V}_{AL} &= -\hat{\mathbf{D}} \cdot \mathbf{E}_L(\hat{\mathbf{R}}, t) = \hat{V}_{AL}^{\rm res} + \hat{V}_{AL}^{\rm non\,res} \\
\hat{V}_{AL}^{\rm res} &= -\frac{1}{2} \left(\mathbf{d} \cdot \boldsymbol{\epsilon}(\hat{\mathbf{R}}) \right) \, \mathcal{E}_L(\hat{\mathbf{R}}) \, |e\rangle\langle g| \, e^{-i\omega_L t} \, e^{-i\phi(\hat{\mathbf{R}})} + h.c. \\
\hat{V}_{AL}^{\rm non\,res} &= -\frac{1}{2} \left(\mathbf{d} \cdot \boldsymbol{\epsilon}^*(\hat{\mathbf{R}}) \right) \, \mathcal{E}_L^*(\hat{\mathbf{R}}) \, |e\rangle\langle g| \, e^{i\omega_L t} \, e^{i\phi(\hat{\mathbf{R}})} + h.c.
\end{aligned} \tag{8}$$

At this stage, we introduce the *Rabi frequency* $\Omega_1(\mathbf{r})$ defined by

$$\hbar\Omega_1(\mathbf{r}) = -\left(\mathbf{d} \cdot \boldsymbol{\epsilon}(\mathbf{r})\right) \mathcal{E}_L(\mathbf{r}) \, . \tag{9}$$

The time origin is chosen such that Ω_1 is real. The atom – laser resonant coupling can then be written as

$$\hat{V}_{AL}^{\rm res} = \frac{\hbar\Omega_1(\hat{\mathbf{R}})}{2} \left\{ |e\rangle\langle g| \, e^{-i\omega_L t} \, e^{-i\phi(\hat{\mathbf{R}})} + h.c. \right\} \tag{10}$$

The Rabi frequency is the oscillation frequency between $|g\rangle$ and $|e\rangle$ at resonance in the strong coupling regime.

2 Light forces

2.1 Orders of magnitude. Approximations

2.1.1 Rotating wave approximation

Let us first consider the two contributions to the atom – laser coupling. In the interaction picture, that is in the frame rotating at frequency ω_0 due to the internal energy of the state $|e\rangle$, the term $\hat{V}_{AL}^{\rm res}$ oscillates at the frequency $\delta = \omega_L - \omega_0$. It is slowly evolving as compared to ω_0 or ω_L.

On the contrary, $\hat{V}_{AL}^{\rm non\,res}$ oscillates at frequency $\omega_0 + \omega_L$, much larger than δ. It has then an amplitude smaller by a factor $|\omega_0 - \omega_L|/(\omega_0 + \omega_L)$, and is negligible as compared to the

resonant process. In the following $\hat{V}_{AL}^{\text{non res}}$ is ignored: $\hat{V}_{AL} = \hat{V}_{AL}^{\text{res}}$. This approximation is known as the rotating wave approximation (RWA). It holds provided that $|\delta|, \Omega_1 \ll \omega_0, \omega_L$. N.B.: RWA is wrong in the case of a very far detuned laser, like a CO_2 laser at 10 μm, and both terms then contribute almost equally to the light shift [19]. For $\lambda_L = 532$ nm and rubidium atoms with $\lambda_0 = 780$ nm, the correction due to the non-resonant term amounts to 20%. The non-resonant term enhances the effect of the resonant term if $\delta < 0$, whereas it decreases its effect if $\delta > 0$.
N.B.: If the laser field is described by a quantum field with the operators $\hat{a}_L^\dagger$ and $\hat{a}_L$, $\hat{V}_{AL}^{\text{res}}$ is proportional to $|g\rangle\langle e|\hat{a}_L^\dagger + |e\rangle\langle g|\hat{a}_L$ and corresponds to resonant processes where either a photon is emitted and the atom changes its internal state from $|e\rangle$ to $|g\rangle$, or a photon is absorbed and the atom state changes from $|g\rangle$ to $|e\rangle$. On the other hand, $\hat{V}_{AL}^{\text{non res}} \propto |e\rangle\langle g|\hat{a}_L^\dagger + |g\rangle\langle e|\hat{a}_L$: simultaneous emission of a photon of frequency ω_L and change in the internal atomic state from $|g\rangle$ to $|e\rangle$, or absorption of a photon and change from $|e\rangle$ to $|g\rangle$.

2.1.2 Time scales

The atomic external and internal variables evolve at different time scales, t_{ext} and t_{int}.

t_{int} is the time necessary for reaching a steady state of the internal matrix density operator $\hat{\sigma}$ (population and coherences). It is related to the lifetime of the excited state, involved in the optical Bloch equations (OBE) describing the internal states dynamics. Hence, its order of magnitude is Γ^{-1}, that is 10 to 100 ns.

t_{ext} is the time necessary to change in a measurable way the external atomic variables. It can be defined for example as the time after which an atom undergoing an acceleration $a = \Gamma v_{\text{rec}}$ due to a resonant light pressure becomes non resonant, due to the Doppler effect. With this definition, it is the time after which the velocity v satisfies $k_L v = \Gamma$, with $v = \Gamma v_{\text{rec}} t_{\text{ext}}$.

$$t_{\text{ext}} = \frac{1}{k_L v_{\text{rec}}} = \frac{\hbar}{M v_{\text{rec}}^2} = \frac{\hbar}{2E_{\text{rec}}} = \frac{1}{2\omega_{\text{rec}}}$$

where $E_{\text{rec}} = M v_{\text{rec}}^2/2$ is the recoil energy and $\omega_{\text{rec}} = E_{\text{rec}}/\hbar$ the recoil frequency. For alkali, the recoil frequency $\omega_{\text{rec}}/(2\pi)$ is a few kHz, which makes t_{ext} of order a few tens of microseconds.

For alkali, as well as for most laser cooled species, one has $t_{\text{ext}} \gg t_{\text{int}}$. For rubidium, $\lambda_0 = 780$ nm, $\Gamma = 2\pi \times 6$ MHz and $M = 1.44 \times 10^{-25}$ kg which yields $v_{\text{rec}} = 5.89$ mm·s^{-1} and $t_{\text{ext}}/t_{\text{int}} \simeq 800$. The condition $t_{\text{ext}} \gg t_{\text{int}}$, or

$$\hbar\Gamma \gg \omega_{\text{rec}} \tag{11}$$

is called the *broad band condition*. When it is satisfied, the two time scale clearly separate. For describing the dynamics of the external state, the internal state can therefore be considered as being in its steady state. In the following, the steady state value of the dipole will thus be used for the calculation of the light force.
N.B.: There are cases for which the broad band condition is not fulfilled. Laser cooling can be applied to narrow lines for which $\Gamma < \omega_{\text{rec}}$, allowing a sub-recoil Doppler cooling [22]. This case is beyond the objective of this short course.

2.1.3 Semi-classical approximation

The external motion of the atom will be treated as classical, which means that the force $\mathbf{F}$ is calculated at position $\mathbf{r}$ for a velocity $\mathbf{v}$. This description is correct if the laser field at the position of the atom is well defined, that is if the atomic position is known better than the wavelength:

$$\Delta R \ll \lambda_L \quad \text{or} \quad \Delta R \ll k_L^{-1} \,.$$

In the same way, the velocity should be defined to better than Γ/k_L for the frequency seen by a single atom to be well defined:

$$k_L \Delta v \ll \Gamma \quad \text{or} \quad \Delta P \ll M\frac{\Gamma}{k_L} \,.$$

As $\Delta R \Delta P > \hbar/2$, these two conditions imply

$$\frac{\hbar}{2} \ll M\frac{\Gamma}{k_L^2} \quad \text{or} \quad \Gamma \gg \frac{\hbar k_L^2}{2M} \,.$$

We recover the broad band condition $\Gamma \gg \omega_{\text{rec}}$. The semi-classical approximation is valid down to very small velocities. From now on, we assume that the broad band condition is satisfied and the semi-classical treatment of the external motion will be used.

2.2 The mean light force

To find the expression of the force exerted by the laser light onto the atom, let us write the force operator in the Heisenberg representation. The only term in the Hamiltonian which does not commute with $\hat{\mathbf{R}}$ is $\hat{\mathbf{P}}^2$:

$$\frac{d\hat{\mathbf{R}}}{dt} = \frac{1}{i\hbar}\left[\hat{\mathbf{R}}, \hat{H}\right] = \frac{1}{i\hbar}\left[\hat{\mathbf{R}}, \hat{H}_A\right] = \frac{\hat{\mathbf{P}}}{M} \,. \tag{12}$$

We recover the expression of the atomic velocity.

On the other hand, $\hat{\mathbf{P}}$ commutes with $\hat{H}_A$ and $\hat{H}_R$ but not with $\hat{V}_{AL}$ and $\hat{V}_{AR}$:

$$\hat{\mathbf{F}} = \frac{d\hat{\mathbf{P}}}{dt} = \frac{1}{i\hbar}\left[\hat{\mathbf{P}}, \hat{H}\right] = -\boldsymbol{\nabla}\hat{V}_{AL} - \boldsymbol{\nabla}\hat{V}_{AR} \,. \tag{13}$$

The mean force is then $\mathbf{F} = \langle\hat{\mathbf{F}}\rangle = -\langle\boldsymbol{\nabla}\hat{V}_{AL}\rangle - \langle\boldsymbol{\nabla}\hat{V}_{AR}\rangle$. The second term is zero, see [14] p. 14-15. It is related to the fact that spontaneous emission occurs in random directions, giving to the atom random momentum kicks with equal probabilities in the directions $\mathbf{k}$ and $-\mathbf{k}$. In average, the corresponding force is zero.
N.B.: The average is zero, while $\boldsymbol{\nabla}\hat{V}_{AR} \neq 0$. The fluctuations of this random force induce a Brownian motion in momentum space and contribute to the final finite temperature that can be reached with laser cooling.

With the notations $\mathbf{r} = \langle\hat{\mathbf{R}}\rangle$ and $\mathbf{p} = \langle\hat{\mathbf{P}}\rangle$, the mean force is, in the semi-classical approximation:

$$\begin{aligned}
\mathbf{F} &= -\langle\boldsymbol{\nabla}\hat{V}_{AL}\rangle = \langle\boldsymbol{\nabla}\left(\hat{\mathbf{D}}\cdot\mathbf{E}_L(\mathbf{r},t)\right)\rangle = \boldsymbol{\nabla}\left[\sum_{i=x,y,z}\langle\hat{D}_i E_{Li}(\mathbf{r},t)\rangle\right] \\
\mathbf{F} &= \sum_{i=x,y,z}\langle\hat{D}_i\rangle(t)\boldsymbol{\nabla}E_{Li}(\mathbf{r},t) \,.
\end{aligned} \tag{14}$$

Taking advantage of the different internal and external time scales, the dipole average value $\langle \hat{D}_i \rangle(t)$ can be replaced by its steady state:

$$\mathbf{F} = \sum_{i=x,y,z} \langle \hat{D}_i \rangle_{st} \boldsymbol{\nabla} E_{Li}(\mathbf{r}, t)\,.$$

The mean stationary dipole $\langle \hat{D}_i \rangle_{st}$ is deduced from the optical Bloch equations on the internal state density matrix operator $\hat{\sigma}$:

$$i\hbar \frac{d\hat{\sigma}}{dt} = [\hat{H}_A + \hat{V}_{AL}, \hat{\sigma}] - i\hbar \hat{\Gamma} \hat{\sigma}, \tag{15}$$

where the relaxation from $|e\rangle$ to $|g\rangle$ is taken into account through the operator $\hat{\Gamma}$ [18]. Taking into account that the elements of $\hat{\sigma}$ are linked through $\sigma_{gg} + \sigma_{ee} = 1$ and $\sigma_{ge} = \sigma_{eg}^*$, there are only two coupled equations:

$$\begin{aligned}
\dot{\sigma}_{ee} &= -\Gamma \sigma_{ee} + i \frac{\Omega_1(\mathbf{r})}{2} \left(\sigma_{eg}\, e^{i\omega_L t}\, e^{i\phi(\mathbf{r})} - \sigma_{eg}^*\, e^{-i\omega_L t}\, e^{-i\phi(\mathbf{r})} \right) \\
\dot{\sigma}_{eg} &= -\left(i\omega_0 + \frac{\Gamma}{2} \right) \sigma_{eg} - i \frac{\Omega_1(\mathbf{r})}{2} \left(1 - 2\sigma_{ee}\right)\, e^{-i\omega_L t}\, e^{-i\phi(\mathbf{r})}\,.
\end{aligned}$$

These equations imply three real variables. The stationary solution is conveniently found by introducing three new real variables u, v and w defined by

$$\left\{ \begin{array}{rcl}
u &=& \frac{1}{2} \left(\sigma_{eg}^*\, e^{-i\omega_L t}\, e^{-i\phi(\mathbf{r})} + \sigma_{eg}\, e^{i\omega_L t}\, e^{i\phi(\mathbf{r})} \right) \\
v &=& \frac{1}{2i} \left(\sigma_{eg}^*\, e^{-i\omega_L t}\, e^{-i\phi(\mathbf{r})} - \sigma_{eg}\, e^{i\omega_L t}\, e^{i\phi(\mathbf{r})} \right) \\
w &=& \frac{1}{2}(\sigma_{ee} - \sigma_{gg}) = \sigma_{ee} - \frac{1}{2}
\end{array} \right.$$

and satisfying the coupled equations

$$\left\{ \begin{array}{rcl}
\dot{u} &=& -\frac{\Gamma}{2} u + \delta v\,, \\
\dot{v} &=& -\frac{\Gamma}{2} v - \delta u - \Omega_1 w\,, \\
\dot{w} &=& -\Gamma \left(w + \frac{1}{2} \right) + \Omega_1 v\,.
\end{array} \right.$$

The stationary solution is

$$\left\{ \begin{array}{rcl}
u_{st} &=& \dfrac{\Omega_1 \delta / 2}{\frac{\Omega_1^2}{2} + \delta^2 + \frac{\Gamma^2}{4}}\,, \\
v_{st} &=& \dfrac{\Omega_1 \Gamma / 4}{\frac{\Omega_1^2}{2} + \delta^2 + \frac{\Gamma^2}{4}}\,, \\
w_{st} + \dfrac{1}{2} &=& \sigma_{ee,st} = \dfrac{\Omega_1^2 / 4}{\frac{\Omega_1^2}{2} + \delta^2 + \frac{\Gamma^2}{4}}\,.
\end{array} \right.$$

The population in the excited state and the atomic dipole can be written in terms of the *saturation parameter* $s(\mathbf{r})$, defined by

$$s(\mathbf{r}) = \frac{\Omega_1^2(\mathbf{r})/2}{\delta^2 + \dfrac{\Gamma^2}{4}} = \frac{I/I_s}{1 + \dfrac{4\delta^2}{\Gamma^2}}\,. \tag{16}$$

I_s is the *saturation intensity*[1] $I_s = 2\pi^2\hbar c\Gamma/(3\lambda_0^3)$, typically a few mW·cm^{-2}. With this definition, we can write

$$\sigma_{ee} = \frac{1}{2}\frac{s(\mathbf{r})}{1+s(\mathbf{r})}, \tag{17}$$

$$\langle\hat{\mathbf{D}}\rangle\cdot\boldsymbol{\epsilon}(\mathbf{r}) = 2\mathbf{d}.\boldsymbol{\epsilon}(\mathbf{r})\frac{s(\mathbf{r})}{1+s(\mathbf{r})}\left\{\frac{\delta}{\Omega_1(\mathbf{r})}\cos\left[\omega_L t+\phi(\mathbf{r})\right]-\frac{\Gamma}{2\Omega_1(\mathbf{r})}\sin\left[\omega_L t+\phi(\mathbf{r})\right]\right\}. \tag{18}$$

The dipole has two components: a term in δ/Ω_1 oscillating in phase with the electric field, related to the real part of the polarisability which leads to a *conservative force*. On resonance, this term is zero. The second term is in quadrature with the electric field, proportional to $\Gamma/(2\Omega_1)$. It is maximum on resonance and related to the imaginary part of the polarisability, that is with absorption. It leads to a *dissipative force*.

The gradient applied on E_{Li} gives a term in phase proportional to $\boldsymbol{\nabla}\Omega_1$ and a term in quadrature proportional to $\boldsymbol{\nabla}\phi$. After averaging over one period of the laser field, the final expression of the total force is

$$\mathbf{F}(\mathbf{r}) = -\frac{s(\mathbf{r})}{1+s(\mathbf{r})}\left(\hbar\delta\frac{\boldsymbol{\nabla}\Omega_1}{\Omega_1(\mathbf{r})}+\frac{\hbar\Gamma}{2}\boldsymbol{\nabla}\phi\right). \tag{19}$$

2.3 Interpretation of the mean force

The force is the sum of two contributions, corresponding to the dispersive and the absorptive part of the atomic polarisability. In this section, we discuss these two parts separately.

2.3.1 Radiation pressure

Let us label as $\mathbf{F}_{\rm pr}$ the term in the expression of the force that is proportional to the gradient of the phase:

$$\mathbf{F}_{\rm pr} = -\frac{\hbar\Gamma}{2}\frac{s(\mathbf{r})}{1+s(\mathbf{r})}\boldsymbol{\nabla}\phi. \tag{20}$$

To understand the origin of this force, let us consider the case of a plane wave, for which Ω_1 is uniform, and so is s. The phase $\phi(\mathbf{r})$ is equal to $-\mathbf{k}_L\cdot\mathbf{r}$ for a plane wave of wave vector $\mathbf{k}_L$, such that $\boldsymbol{\nabla}\phi = -\mathbf{k}_L$. It yields

$$\mathbf{F}_{\rm pr} = \frac{\Gamma}{2}\frac{s}{1+s}\hbar\mathbf{k}_L. \tag{21}$$

As already seen, $\sigma_{ee} = \frac{1}{2}\frac{s}{1+s}$ is the population of the excited state, such that

$$\Gamma_{\rm sp} = \frac{\Gamma}{2}\frac{s}{1+s} \tag{22}$$

is the spontaneous scattering rate. The mean force is then simply $\mathbf{F}_{\rm pr} = \Gamma_{\rm sp}\,\hbar\mathbf{k}_L$ and is due to the momentum transfer of one recoil $\hbar\mathbf{k}_L$ each time a photon is absorbed from the laser, which occurs at a rate $\Gamma_{\rm sp}$. The spontaneously emitted photons are randomly distributed in direction and do not contribute to the mean force. The force pushes the atoms in the direction of the light wave vector, and for this reason is it called the *radiation pressure*.

[1] We can also write this relation as $\frac{3\lambda_0^2}{2\pi}I_s = \hbar\omega_0\frac{\Gamma}{2}$: the saturation intensity is the maximum scattered power divided by the resonant absorption cross section.

Dependence on the intensity — On resonance, the saturation parameter is equal to $s = I/I_s$. The saturation intensity I_s is characteristic of the transition, and measures how much intensity is needed to reach the maximum scattering rate of $\Gamma/2$. For $I \gg I_s$, the radiation pressure saturates to its maximum value, $\mathbf{F}_{\rm pr} = \frac{\Gamma}{2}\hbar\mathbf{k}_L$.

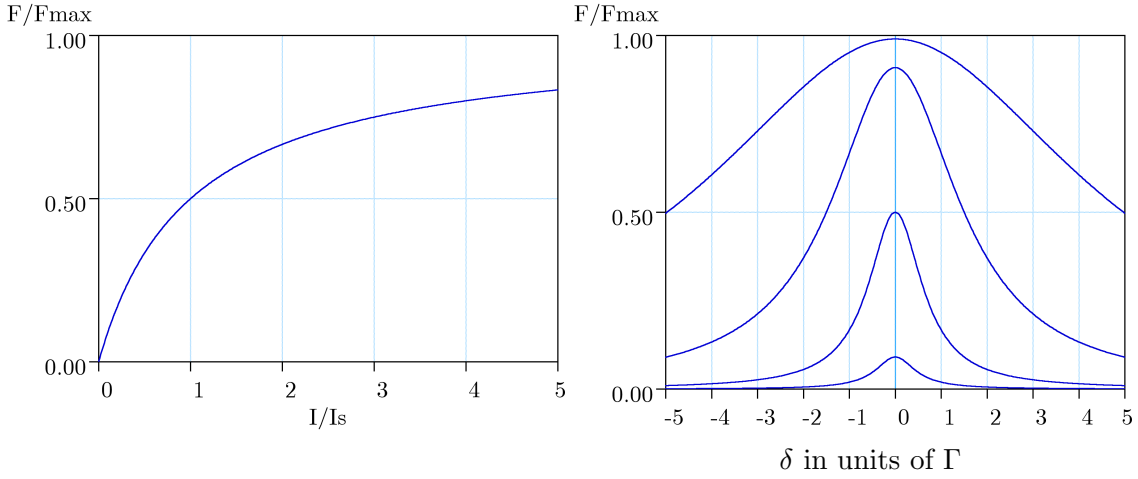


Figure 1: Left: Dependence of the radiation pressure $F/F_{\rm max}$ exerted by a plane wave on the light intensity, on resonance $\delta = 0$. For $I \gg I_s$, it saturates to its maximum value, $F_{\rm max} = \frac{\Gamma}{2}\hbar k_L$. Right: Dependence on the detuning, for different values of the intensity. From bottom to top, $I/I_s = 0.1$, 1, 10 and 100. The line broadening due to saturation is clear on this graph. The lower Lorentzian has a full width at half maximum of about Γ.

Dependence on the detuning — The radiation pressure depends on the detuning just as the scattering rate does: with a Lorentzian shape of full width at half maximum (FWHM) $\sqrt{\Gamma^2 + 2\Omega_1^2} = \Gamma\sqrt{1 + I/I_s}$. In the wings, that is for $\delta \gg \Gamma$, the force scales as $1/\delta^2$.

N.B.: If the line is shifted, for example by the Zeeman effect in the presence of a magnetic field B, such that $\omega_0' = \omega_0 + g\mu_B B/\hbar$, and (or) if the laser frequency is shifted by the Doppler effect, the detuning δ must be replaced by $\delta' = \omega_L - \mathbf{k}_L \cdot \mathbf{v} - \omega_0'$, that is

$$\delta' = \delta - \mathbf{k}_L \cdot \mathbf{v} - \frac{g\mu_B}{\hbar} B\,.$$

Application: the Zeeman slower — An important application of this large force is the Zeeman slower, first demonstrated by W.D. Phillips, H. Metcalf and their colleagues [23]. A laser propagating against an atomic beam can slow it down to $v = 0$ due to radiation pressure. To maintain a strong force during all the deceleration time, the resonance condition should be maintained. A inhomogeneous magnetic field is tailored to compensate for the reduction of the Doppler shift with the velocity.

Consider atoms propagating along the z axis with initial velocity v_0. Clearly, we must chose $\mathbf{k}_L \cdot \mathbf{v} < 0$ to slow down the atoms. With a counter-propagating beam,

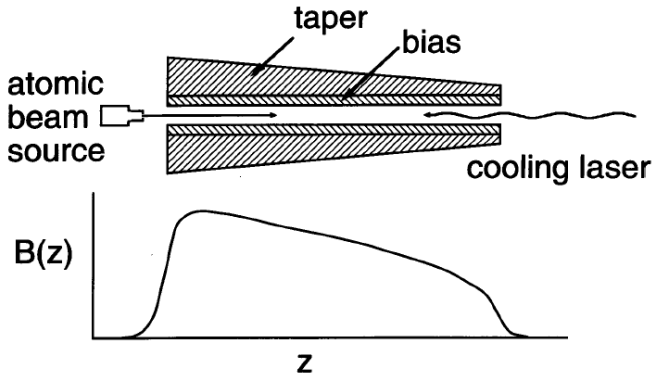


Figure 2: Principle of a Zeeman slower. Top: scheme of the experimental setup. Bottom: typical shape of the bias magnetic field as a function of z. Figure from Ref. [11].

$\mathbf{k}_L \cdot \mathbf{v} = -k_L v$. Writing $\delta' = 0$ gives $B(v) = \hbar k_L v/(g\mu_B)$. If the acceleration is constant $a = a_{\max} = \Gamma v_{\rm rec}/2$, the velocity depends on z as $v(z) = \sqrt{v_0^2 - 2az}$. The magnetic field should then be designed to vary with z like

$$B(z) = B_0 \sqrt{1 - \frac{2az}{v_0^2}} = B_0 \sqrt{1 - \frac{\Gamma v_{\rm rec}}{v_0^2} z} \quad \text{where} \quad B_0 = \frac{\hbar k_L v_0}{g\mu_B}.$$

2.3.2 Dipole force

The second term appearing in the expression of the force (19) is

$$\mathbf{F}_{\rm dip} = -\hbar\delta \frac{s(\mathbf{r})}{1+s(\mathbf{r})} \frac{\boldsymbol{\nabla}\Omega_1}{\Omega_1(\mathbf{r})} = -\frac{\hbar\delta}{2} \frac{\boldsymbol{\nabla} s(\mathbf{r})}{1+s(\mathbf{r})}. \tag{23}$$

This force is equal to zero on resonance ($\delta = 0$) and is also zero in the case of a plane wave, for which Ω_1 or s does not depend on position. From its expression, it is clear that it derives from the following *dipole potential*:

$$U_{\rm dip}(\mathbf{r}) = \frac{\hbar\delta}{2} \ln(1 + s(\mathbf{r}))\,. \tag{24}$$

Hence, the dipole force is a *conservative force*. Its value is zero at resonance. In the case where $|\delta| \gg \Gamma, \Omega_1$, the saturation parameter s is very small and one can expand the logarithm. As a result, the dipole potential becomes proportional to the local intensity:

$$U_{\rm dip}(\mathbf{r}) = \frac{\hbar\delta}{2} s(\mathbf{r}) = \frac{\hbar\Omega_1^2(\mathbf{r})}{4\delta} = \hbar\Gamma \frac{\Gamma}{\delta} \frac{I(\mathbf{r})}{8I_s} \quad \text{for} \quad |\delta| \gg \Gamma, \Omega_1\,. \tag{25}$$

The dependence of the force on the detuning has a dispersive shape, as it is related to the real part of the atomic polarizability. In particular, the force — and the potential — is opposite for opposite detunings. It expels the atoms from a high intensity region when $\delta > 0$, whereas it attracts the atoms to high intensity regions for $\delta < 0$.

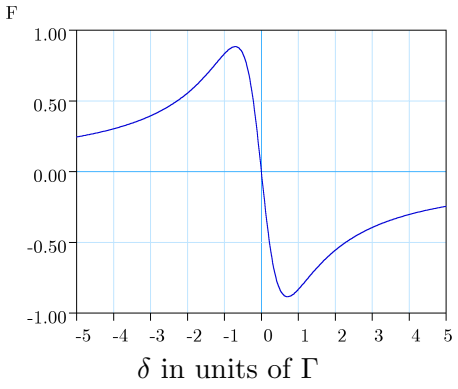


Figure 3: Dependence of the dipole force $F_{\rm dip}$ (arbitrary units) on the detuning, as a function of δ/Γ, for $I = I_s$.

For large values of the detuning δ, the force decreases as $1/\delta$. Recalling that the radiation pressure decreases like $1/\delta^2$, for large values of δ the dipole force dominates over the radiation pressure:

$$\frac{F_{\rm dip}}{F_{\rm pr}} \simeq \frac{|\delta|}{\Gamma}\frac{1}{k_L\ell},$$

where ℓ is the typical scale for the variation of intensity in space. $k_L\ell$ varies from a few units to 10^{-4} typically depending on the intensity profile (from an evanescent wave to a focused beam), such that $F_{\rm dip}$ dominates for far off-resonant beams detuned by more than $10^4\Gamma$, that is a few tens of GHz. In this case, the dipole potential can be used to realise conservative traps.

Example of conservative dipole potentials Far off-resonant lasers are used to tailor conservative dipole potentials. As δ must be large to avoid photon scattering, the laser intensity should also be large to obtain a significant value of $U_{\rm dip}$ as compared to the temperature, or to the external energy.

Blue detuned potentials, that is with $\delta > 0$, repel the atoms from high intensity regions. They can be used for realising an atomic mirror, with an evanescent field at the surface of a dielectric material [24, 25]. It has been shown [26] that the atoms can bounce several times above such a mirror when it is orientated upwards, provided the mirror surface is curved to stabilise the trajectories, see Fig. 4, left. Another example is the guiding of atoms in the centre of blue detuned hollow laser beams. Using a combination of a hollow beam and two sheets of light, a three-dimensional box has recently been realised to confine a Bose-Einstein condensate [27].

With red-detuned light, one can realise conservative atom traps. At the focus point of a laser beam, the intensity is maximum and the dipole potential is minimum. In this way, Steven Chu and his colleagues demonstrated the first dipole trap in 1986 [28]. To increase the oscillation frequency in the direction of the beam, two laser can be used, and a crossed dipole trap is obtained [29], see Fig 4, right.

Finally, optical lattices where atoms are placed in a periodic light potential are obtained by interfering several light beams in a standing wave configuration. In this way, atoms play

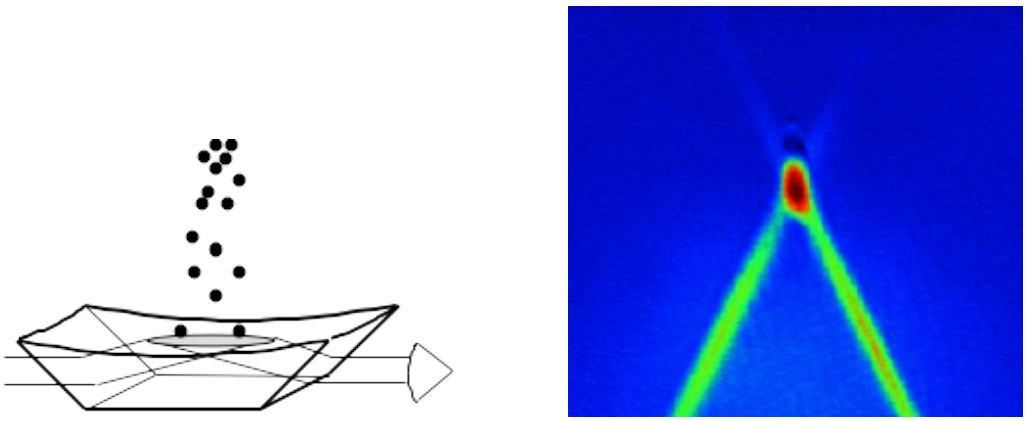


Figure 4: Left: Atoms bouncing off a blue detuned evanescent wave. Courtesy of Jean Dalibard. Right: A conservative trap is realised by crossing two, far off resonant, red detuned laser beams. The trap is loaded from a magneto-optical trap. Caesium atoms that were not initially at the crossing fall due to gravity, preferentially along the axes of the laser beams.

the role of electrons in the periodic potential of a crystal in condensed matter physics. The physics of atoms in optical lattices is very rich, both in near resonant lattices [30, 31] and in far detuned lattices [32], and the analogy with condensed matter has led to important results, like the observation of the Mott insulator state [33], which paved the way for the study of many-body physics with quantum gases [34].
N.B.: The calculation of light shifts is not straightforward in the case of a multi-level atom. Maxim Olshanii made a short but useful document with this calculation in the case of the D lines of alkali [35].

3 The dressed state picture

Another path may be taken to derive the dipole force [14, 20]. In a first step, we will neglect the effect of spontaneous emission. The laser field can be described by a quantum field, with creation and annihilation operators $\hat{a}_L^\dagger$ and $\hat{a}_L$ of photons in the mode of the laser, and a Hamiltonian $\hat{H}_L = (\hat{a}_L^\dagger \hat{a}_L + 1/2)\,\hbar\omega_L$. The idea is now to describe together the internal atomic state and the state of the light field, and to diagonalize the total Hamiltonian $\hat{H}_L + \hat{H}_A + \hat{V}_{AL}$ in a coupled basis. The resulting eigenstates are called the *dressed states*, the atomic states being dressed by the photons.

3.1 System under consideration

The external variables $\mathbf{r}$ and $\mathbf{p}$ are again considered classical. The atomic internal states are $|e\rangle$ and $|g\rangle$, and the eigenstates of the Hamiltonian $\hat{H}_L$ for the light field alone are the photon number states $|n\rangle$. The number states are eigenstates of the number operator $\hat{n} = \hat{a}_L^\dagger \hat{a}_L$, such that $\hat{n}|n\rangle = n|n\rangle$. This corresponds to the photon number in a given volume V, and must be understood as $\langle n\rangle \to \infty$, $V \to \infty$, $\langle n\rangle/V$ being related directly to

the laser intensity. Then, the fluctuations $\Delta n \sim \sqrt{\langle n \rangle} \ll \langle n \rangle$ are small.

The uncoupled atomic + field states are denoted as $|e, n\rangle$ and $|g, n\rangle$. They are eigenstates of $\hat{H}_L + \hat{H}_A$ with energies

$$E_{e,n} = \left(n + \frac{1}{2}\right)\hbar\omega_L + \hbar\omega_0 \qquad\qquad E_{g,n} = \left(n + \frac{1}{2}\right)\hbar\omega_L.$$

For near resonant light, such that $|\delta| \ll \omega_0, \omega_L$, the unperturbed eigenstates are organised in manifolds of two eigenstates $M_n = \{|e, n\rangle, |g, n-1\rangle\}$ with energies separated by only $\hbar|\delta|$ around $E_n = n\hbar\omega_L + \frac{1}{2}\hbar\omega_0$, see Fig. 5. Each manifold M_n is separated from the next one M_{n+1} by a large energy $\hbar\omega_L$:

$$\left\{ \begin{array}{rcl} E_{e,n-1} & = & n\hbar\omega_L + \frac{1}{2}\hbar\omega_0 + \frac{1}{2}\hbar(\omega_0 - \hbar\omega_L) = E_n - \frac{\hbar\delta}{2} \\ E_{g,n} & = & n\hbar\omega_L + \frac{1}{2}\hbar\omega_0 + \frac{1}{2}\hbar(\omega_L - \hbar\omega_0) = E_n + \frac{\hbar\delta}{2}. \end{array} \right. \tag{26}$$

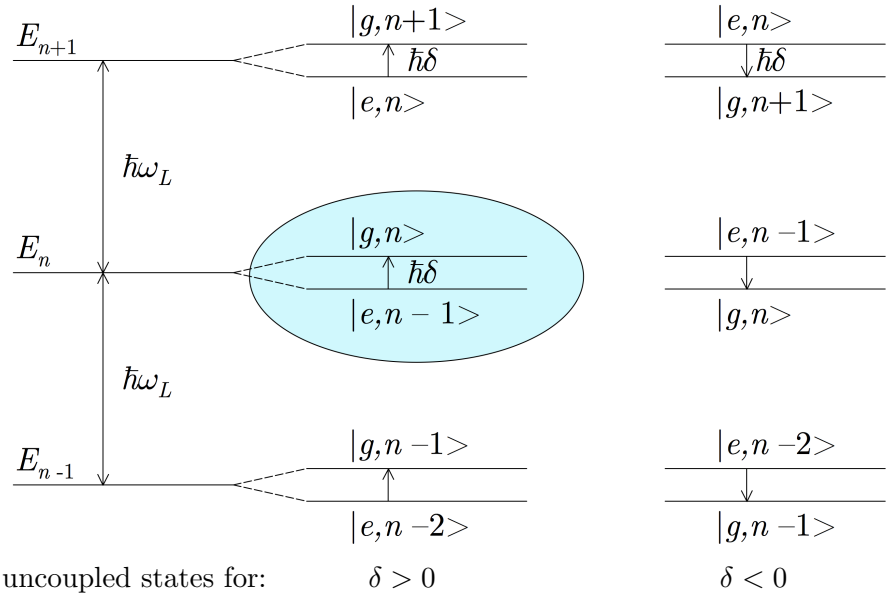


Figure 5: The unperturbed atom + field states can be grouped into manifolds of two states with a small energy difference $\hbar\delta$ compared to the energy spacing between manifolds $\hbar\omega_L$. Depending on the sign of δ, either the state connected to $|g\rangle$ or to $|e\rangle$ has a larger energy. On resonance ($\delta = 0$), the two states are degenerate. The M_n manifold with mean energy E_n and $\delta > 0$ is enlightened.

3.2 Eigenstates for the coupled system: the dressed states

Let us now add the coupling $\hat{V}_{AL} = \frac{\hbar\Omega_0(\mathbf{r})}{2}\left(\hat{a}_L|e\rangle\langle g| + \hat{a}_L^\dagger|g\rangle\langle e|\right)$, in the rotating wave approximation, where non resonant terms $\hat{a}_L|g\rangle\langle e|$ and $\hat{a}_L^\dagger|e\rangle\langle g|$ have been neglected. It

acts inside a given manifold M_n, but doesn't couple different manifolds together. $\Omega_0(\mathbf{r})$ is the Rabi frequency for the 1 photon coupling in the manifold M_1 between $|g,1\rangle$ and $|e,0\rangle$. The matrix element between $|g,n\rangle$ and $|e,n-1\rangle$ in manifold M_n is:

$$\langle g,n|\hat{V}_{AL}|e,n-1\rangle = \langle e,n-1|\hat{V}_{AL}|g,n\rangle = \frac{\hbar\Omega_0(\mathbf{r})}{2}\sqrt{n} \tag{27}$$

As the laser mode is populated by a very large amount of photons with mean number $\langle n\rangle$ and a very small dispersion Δn, the matrix element in manifold M_n and M_{n+1} are almost the same. We thus define the mean Rabi frequency as $\Omega_1(\mathbf{r}) = \sqrt{\langle n\rangle}\Omega_0(\mathbf{r})$ and write

$$\langle g,n|\hat{V}_{AL}|e,n-1\rangle = \langle e,n-1|\hat{V}_{AL}|g,n\rangle \simeq \frac{\hbar\Omega_1(\mathbf{r})}{2}. \tag{28}$$

The Hamiltonian inside the manifold M_n then reads:

$$\hat{H}_n = E_n + \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_1 \\ \Omega_1 & \delta \end{pmatrix}. \tag{29}$$

Its eigenstates $|\pm,n\rangle$ are superpositions of $|e,n\rangle$ and $|g,n-1\rangle$, where atom and field cannot be separated any more, as if the atom were *dressed* by the light. This is the reason why they are called the *dressed states*. The eigenenergies read

$$E_\pm = E_n \pm \frac{\hbar}{2}\sqrt{\delta^2+\Omega_1^2} = E_n \pm \frac{\hbar}{2}\Omega, \tag{30}$$

where $\Omega = \sqrt{\Omega_1^2+\delta^2}$ is the generalised Rabi frequency. Due to the interaction, the states repel each other are are now separated by $\hbar\Omega \geq \hbar|\delta|$. In particular, the degeneracy is lifted on resonance, the two dressed states being separated by $\hbar\Omega_1$.

In the case $|\delta| \gg \Omega_1$ discussed in the previous section, the eigenstates are very close to the unperturbed states. $|g,n\rangle$ is close to $|+\rangle$ for $\delta > 0$ and to $|-\rangle$ if $\delta < 0$. The expression of the energy simplifies and we recover the light shift $\hbar\Omega_1^2/4\delta$:

$$E_{g,n} \simeq E_{sgn(\delta)} \simeq E_n \pm \frac{\hbar|\delta|}{2} \pm \frac{\hbar\Omega_1^2}{4|\delta|} = E_n + \frac{\hbar\delta}{2} + \frac{\hbar\Omega_1^2}{4\delta} = (n+\frac{1}{2})\hbar\omega_L + \frac{\hbar\Omega_1^2}{4\delta}.$$

The dressed levels are decomposed on the unperturbed basis as follows:

$$|+,n\rangle = \sin\frac{\theta}{2}|g,n\rangle + \cos\frac{\theta}{2}|e,n-1\rangle \tag{31}$$

$$|-,n\rangle = -\cos\frac{\theta}{2}|g,n\rangle + \sin\frac{\theta}{2}|e,n-1\rangle \tag{32}$$

where the dressing angle $\theta(\mathbf{r})$ is defined by

$$\cos\theta(\mathbf{r}) = -\frac{\delta}{\Omega(\mathbf{r})}, \quad \sin\theta(\mathbf{r}) = \frac{\Omega_1(\mathbf{r})}{\Omega(\mathbf{r})}.$$

With this notation, the energy of state $|\pm,n\rangle$ is simply $\pm\hbar\Omega(\mathbf{r})/2$. $\theta = 0$ corresponds to large and negative detuning, where $|+,n\rangle \simeq |e,n-1\rangle$, $\theta = \pi$ corresponds to the opposite case of large and positive detuning where $|+,n\rangle \simeq |g,n\rangle$ and $\theta = \pi/2$ to the resonance $\delta = 0$ where $|+,n\rangle$ has equal weight on the two unperturbed states.

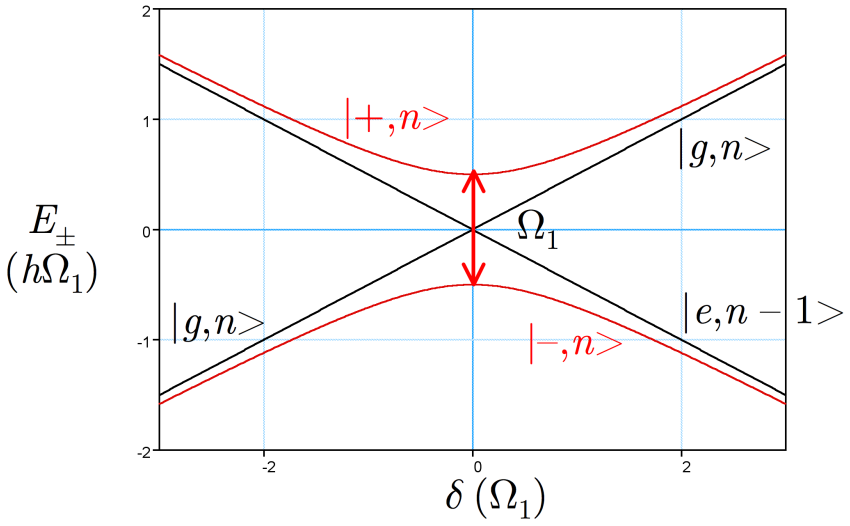


Figure 6: Energy, in units of $\hbar\Omega_1$, of the dressed states (red) and of the unperturbed states (black), as a function of the detuning δ (in units of Ω_1). At resonance, the level spacing is $\hbar\Omega_1$. Far from resonance, the dressed levels essentially coincide with the unperturbed levels.

3.3 Spontaneous emission

The new eigenstates $|\pm, n\rangle$, as linear superpositions of states $|g, n\rangle$ and $|e, n-1\rangle$, have in fact a finite lifetime, due to the coupling of the excited state $|e\rangle$ with the empty modes of the quantum field. As a consequence, transitions between states of the multiplicity M_n to states of the multiplicity M_{n-1} can occur: a photon of the laser mode is scattered into an empty mode of the quantum field. The emitted frequency can be ω_L, $\omega_L + \Omega$ or $\omega_L - \Omega$ depending on the initial and final states.

Let us estimate the reduced dipole element $d_{s's}$ between the initial state $|s, n\rangle$ and the final state $|s', n-1\rangle$, where $s, s' = \pm$.

$$d_{s's} = \langle s', n-1| \left(|e\rangle\langle g| + |g\rangle\langle e|\right) |s, n\rangle$$

Here, $|e\rangle\langle g|$ stands in facts for $|e\rangle\langle g| \otimes \mathbb{1}$, meaning that the laser photon states remains unchanged. The dipole operator acts on $|e\rangle$ or $|g\rangle$, but not on $|n\rangle$. It can then only couple states $|e, n-1\rangle$ (from $|s, n\rangle$) and $|g, n-1\rangle$ (from $|s', n-1\rangle$), which have the same photon number. As a consequence, only the second term $|g\rangle\langle e|$ contributes and we have:

$$d_{s's} = \langle s', n-1|g\rangle\langle e|s, n\rangle \Longrightarrow \begin{cases} d_{++} = d_{--} = -\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} \\ d_{+-} = \sin^2(\theta/2) \\ d_{-+} = \cos^2(\theta/2). \end{cases}$$

The transition rate from $|s, n\rangle$ to $|s', n-1\rangle$ is proportional to the square dipole $d^2_{s's}$. For $\theta = 0$, the linewidth must be equal to Γ for the transition $+ \to -$, corresponding in fact

to the $e \to g$ transition. The linewidth are thus

$$\Gamma_{++} = \Gamma_{--} = \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}\,\Gamma$$

$$\Gamma_{+-} = \sin^4\frac{\theta}{2}\,\Gamma$$

$$\Gamma_{-+} = \cos^4\frac{\theta}{2}\,\Gamma.$$

N.B. The total decay rate from state $|+,n\rangle$ is $\Gamma_+ = \Gamma_{++} + \Gamma_{-+} = \cos^2\frac{\theta}{2}\,\Gamma$, proportional to the weight of state $|+,n\rangle$ on the excited state $|e,n-1\rangle$.

3.4 Dipole force

If the Rabi frequency Ω_1 depends on the position of the atom, both the eigenenergies and the decomposition of the eigenstates on the uncoupled states vary with $\mathbf{r}$. The instant force acting on the atom is $\mathbf{F}_+ = -\boldsymbol{\nabla}E_+ = -\hbar\boldsymbol{\nabla}\Omega/2 = -\hbar\left[\boldsymbol{\nabla}\Omega^2\right]/4\Omega = -\frac{\hbar}{4\Omega}\left[\boldsymbol{\nabla}\Omega_1^2\right]$ if the system is in state $|+,n\rangle$, or $\mathbf{F}_- = -\mathbf{F}_+$ in state $|-,n\rangle$. As the number of photons in the laser field is very large, we neglect the small difference in the coupling for different values of n and consider that the force F_+ is the same for all $|+,n\rangle$ states.

If spontaneous emission is not negligible, the system jumps from one type of state, say $|+\rangle$, to the other $|-\rangle$. To deduce the resulting force, we must evaluate the total population π_+ in all the $|+,n\rangle$ states:

$$\pi_\pm = \Sigma_n\langle\pm,n|\hat{\rho}|\pm,n\rangle$$

where $\hat{\rho}$ is the density matrix of the system. The steady state populations $\pi_\pm$ in states $|\pm\rangle$ are position dependent, and the mean force is

$$\mathbf{F} = \pi_+\mathbf{F}_+ + \pi_-\mathbf{F}_- = (\pi_+ - \pi_-)\mathbf{F}_+ = -(\pi_+ - \pi_-)\frac{\hbar}{4\Omega}\boldsymbol{\nabla}\Omega_1^2 \propto -(\pi_+ - \pi_-)\boldsymbol{\nabla}\Omega_1^2. \quad (33)$$

As we will see below, this is nothing but the dipole force discussed at paragraph 2.3.2. From this simplified expression, we can already identify three cases:

1. if $\delta > 0$, the state $|+\rangle$ has a larger component on $|g,n\rangle$, and is thus more populated. $\pi_+ > \pi_-$ and the force expels the atom from the high intensity regions.

2. $\delta < 0$: in this case, $\pi_+ < \pi_-$ and the force attracts the atom to the high intensity regions.

3. $\delta = 0$: both dressed states have the same weight on $|e\rangle$ and $|g\rangle$. $\pi_+ = \pi_-$ and the mean force is zero.

The full calculation of the populations $\pi_\pm$ allows to recover the expression (23) of the dipole force [36]. The steady state population can be deduced from rate equations:

$$\frac{d\pi_+}{dt} = -\Gamma_{-+}\pi_+ + \Gamma_{+-}\pi_-$$

$$\frac{d\pi_-}{dt} = -\Gamma_{+-}\pi_- + \Gamma_{-+}\pi_+.$$

The solution is

$$\pi_+ = \frac{\Gamma_{+-}}{\Gamma_{+-}+\Gamma_{-+}} = \frac{\sin^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}+\cos^4\frac{\theta}{2}}$$
$$\pi_- = \frac{\Gamma_{-+}}{\Gamma_{+-}+\Gamma_{-+}} = \frac{\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}+\cos^4\frac{\theta}{2}}$$

and for the population difference:

$$\pi_+ - \pi_- = \frac{\sin^4\frac{\theta}{2}-\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}+\cos^4\frac{\theta}{2}} = \frac{\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}}{1-2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} = \frac{-\cos\theta}{1-\frac{1}{2}\sin^2\theta} = \frac{\delta\Omega}{\Omega^2-\frac{\Omega_1^2}{2}},$$

$$\pi_+ - \pi_- = \frac{\delta\Omega}{\delta^2+\frac{\Omega_1^2}{2}}. \tag{34}$$

The force is deduced from Eq. (33) and (34):

$$\mathbf{F} = -\frac{\delta\Omega}{\delta^2+\frac{\Omega_1^2}{2}}\frac{\hbar}{4\Omega}\boldsymbol{\nabla}\Omega_1^2 = -\frac{\hbar\delta}{2}\frac{\boldsymbol{\nabla}\Omega_1^2/2}{\delta^2+\frac{\Omega_1^2}{2}},$$

$$\mathbf{F} = -\boldsymbol{\nabla}\left[\frac{\hbar\delta}{2}\ln\left(1+\frac{\Omega_1^2}{2\delta^2}\right)\right] = -\boldsymbol{\nabla}U_{\rm dip}. \tag{35}$$

We recover the expression of the dipolar force $-\boldsymbol{\nabla}U_{\rm dip}$ of Eq. (23) and (24) in the limit $|\delta| \gg \Gamma/2$ where $s \simeq \Omega_1^2/2\delta^2$.

Appendix

3.5 Rotating wave approximation

Let us come back to the rotating wave approximation and give explicitely the value of $\hat{V}_{AL}^{\rm res}$ and $\hat{V}_{AL}^{\rm non\,res}$ in the interaction picture. We will stick to the case where the broad band condition is fulfilled, such that we apply the semi-classical approximation. External variables then become c-numbers, and we concentrate on the internal dynamics only. The hamiltonian reads:

$$\hat{H} = \hat{H}_0 + \frac{\hbar\Omega_1}{2}\left(|e\rangle\langle g|\,e^{-i\omega_L t} + |g\rangle\langle e|\,e^{i\omega_L t}\right) + \frac{\hbar\Omega_2}{2}|e\rangle\langle g|\,e^{i\omega_L t} + \frac{\hbar\Omega_2^*}{2}|g\rangle\langle e|\,e^{-i\omega_L t}.$$

Here, $\hat{H}_0 = \hbar\omega_0|e\rangle\langle e|$. The first coupling term is the resonant term, the other one is the non resonant term. We have introduced a complex amplitude $\Omega_2 = -\left(\mathbf{d}\cdot\boldsymbol{\epsilon}^*\right)\mathcal{E}_L^*$ for this term in a similar way than we defined the Rabi frequency at Eq.(9). We now remark that $\hat{H}_0$ is the hamiltonian of a 1/2-spin:

$$\hat{H}_0 = \frac{1}{2}\hbar\omega_0\hat{\mathbb{I}} + \frac{1}{2}\hbar\omega_0\hat{\sigma}_z$$

where $\hat{\mathbb{I}}$ is the identity matrix and $\hat{\sigma}_z$ is the z Pauli matrix. These matrices are the generators of the rotations. We recall:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the same spirit, we can write the coupling term with Pauli matrices:

$$\begin{cases} \hat{V}_{AL}^{\text{res}} = \dfrac{\hbar\Omega_1}{2}\left(|e\rangle\langle g|\, e^{-i\omega_L t} + |g\rangle\langle e|\, e^{i\omega_L t}\right) = \dfrac{\hbar\Omega_1}{4}\left(\hat{\sigma}_+\, e^{-i\omega_L t} + \hat{\sigma}_-\, e^{i\omega_L t}\right) \\ \hat{V}_{AL}^{\text{non res}} = \dfrac{\hbar\Omega_2}{2}|e\rangle\langle g|\, e^{i\omega_L t} + \dfrac{\hbar\Omega_2^*}{2}|g\rangle\langle e|\, e^{-i\omega_L t} = \dfrac{\hbar\Omega_2}{4}\hat{\sigma}_+\, e^{i\omega_L t} + \dfrac{\hbar\Omega_2^*}{4}\hat{\sigma}_-\, e^{-i\omega_L t} \end{cases}$$

where $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$, *i.e.* $\hat{\sigma}_+ = 2|e\rangle\langle g|$ and $\hat{\sigma}_- = 2|g\rangle\langle e|$.

The idea now is to remove the part of the internal state oscillating at a frequency close to the laser frequency ω_L. We will thus write the Schrödinger equation in a basis rotating at frequency ω_L around the z axis. The atomic state $|\psi'\rangle$ in the new basis is related to the atomic state $|\psi\rangle$ in the old basis through:

$$|\psi\rangle = e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}|\psi'\rangle. \tag{36}$$

Its time derivative reads

$$i\hbar\partial_t|\psi\rangle = \frac{1}{2}\hbar\omega_L\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}\,\hat{\sigma}_z|\psi'\rangle + i\hbar\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}\partial_t|\psi'\rangle\,.$$

Inserting this expression into the time-dependent Schrödinger equation

$$i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle = \hat{H}_0|\psi\rangle + \hat{V}_{AL}^{\text{res}}|\psi\rangle + \hat{V}_{AL}^{\text{non res}}|\psi\rangle\,,$$

we get:

$$\frac{1}{2}\hbar\omega_L\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}\,\hat{\sigma}_z|\psi'\rangle + i\hbar\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}\partial_t|\psi'\rangle = \hat{H}e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z}|\psi'\rangle.$$

Multiplying on the left by $e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}$, we obtain

$$i\hbar\partial_t|\psi'\rangle = \left(e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{H}e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} - \frac{1}{2}\hbar\omega_L\,\hat{\sigma}_z\right)|\psi'\rangle = \hat{H}_{\text{eff}}|\psi'\rangle.$$

The first term in the effective hamiltonian $\hat{H}_{\text{eff}}$ is the rotated $\hat{H}$ with an angle $\omega_L t$ around the z axis. Let us explicit the effect of the rotation on the three terms of $\hat{H}$. First, $\hat{H}_0$ contains only $\hat{\sigma}_z$ and the indentity. Hence, it commutes with the rotation operator and we get:

$$e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{H}_0\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} = \hat{H}_0 = \frac{1}{2}\hbar\omega_0\hat{\mathbb{I}} + \frac{1}{2}\hbar\omega_0\hat{\sigma}_z.$$

Then, we use the fact that the Pauli matrices $\hat{\sigma}_\pm$ transform under the action of the rotation operator like

$$e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{\sigma}_+e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} = e^{i\omega_L t}\hat{\sigma}_+\,,$$
$$e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{\sigma}_-e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} = e^{-i\omega_L t}\hat{\sigma}_-\,.$$

The transformed resonant and non resonant coupling thus reads:

$$\hat{V}_1 = e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{V}_{AL}^{\text{res}}\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} = \frac{\hbar\Omega_1}{4}\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) = \frac{\hbar\Omega_1}{2}\hat{\sigma}_x\,,$$

$$\hat{V}_2(t) = e^{\frac{i}{2}\omega_L t\hat{\sigma}_z}\hat{V}_{AL}^{\text{nonres}}\, e^{-\frac{i}{2}\omega_L t\hat{\sigma}_z} = \frac{\hbar\Omega_2}{4}\hat{\sigma}_+\, e^{2i\omega_L t} + \frac{\hbar\Omega_2^*}{4}\hat{\sigma}_-\, e^{-2i\omega_L t}\,.$$

The resonant $\hat{V}_{AL}^{\text{res}}$ term is static in the rotated basis, whereas the non resonant term evolves at frequency $2\omega_L$. The effective hamiltonian is the sum of a static term plus a rapidly oscillating term:

$$\hat{H}_{\text{eff}} = \frac{1}{2}\hbar\omega_0\hat{\mathbb{I}} - \frac{1}{2}\hbar\delta\hat{\sigma}_z + \frac{\hbar\Omega_1}{2}\hat{\sigma}_x + \hat{V}_2(t).$$

We recognize in the static term the hamiltonian (29) in the dressed state picture. Its eigenstates are $|\pm\rangle$, with energies

$$E_\pm = \frac{1}{2}\hbar\omega_0 \pm \frac{\hbar}{2}\sqrt{\delta^2 + \Omega_1^2}.$$

The effect of $\hat{V}_1$ is important in the sense that is can significantly mix the states $|e\rangle$ and $|g\rangle$, and even inverse their population (see the eigenstate in section 3.2). This is due to the fact that δ is small as compared to ω_0, and can be made comparable with Ω_1. If, on the contrary, we try to apply the RWA by rotating in the opposite direction, it would make $\hat{V}_2$ static, but we would have a term $(2\omega_L - \delta)$ instead of δ for the term proportional to $\hat{\sigma}_z$. In this case, the eigenstate for the static part would be essentially the bare eigenstates $|e\rangle$ and $|g\rangle$. The effect of $\hat{V}_2$ is hence negligible, as soon as both δ and Ω_1 are very small as compared to ω_0.

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