ANOTHER LOOK AT THE CONTROL PROPERTIES OF THE KORTEWEG-DE VRIES EQUATION

ROBERTO DE A. CAPISTRANO-FILHO* AND FERNANDO GALLEGO

ABSTRACT. This paper represents a new perspective in understanding the controllability of the Korteweg-de Vries (KdV) equation on unbounded domains. By studying the equation on both the right and left half-line with a single control input, we show that a class of solutions exists for which the KdV equation is exactly controllable. This is achieved by introducing a new concept known as *operational controllability*, which provides key insights for establishing exact controllability results for the KdV equation. This approach allows explicitly characterizing both the control input and the controllable solutions. Furthermore, this concept holds significant potential for application to a wide range of nonlinear dispersive equations on the half-line and in bounded intervals.

CONTENTS

Date: September 23, 2024.

2020 *Mathematics Subject Classification.* Primary, 35Q53; Secondary, 37K10, 93B05, 93D15.

Key words and phrases. Korteweg–de Vries equation, exact boundary controllability, operational controllability, half-line, critical set.

*Corresponding author: roberto.capistranofilho@ufpe.br.

Capistrano–Filho was partially supported by CAPES/COFECUB grant number 88887.879175/2023-00, CNPq grant numbers 421573/2023-6 and 307808/2021-1, and Propesqi (UFPE). Gallego was partially supported by Hermes UNAL project no. 61029.

1. Introduction

1.1. **Problem framework.** In recent years, the study of the KdV equation[1] as an initial boundary value problem on the half-line has given the attention of many researchers. This is a challenging problem, mainly when studied with low Sobolev regularity.

The theory of local well-posedness (LWP), i.e., existence, uniqueness, and continuity of the data-to-solution map, of the initial boundary value problem (IBVP) associated with the KdV equation is well studied. On the right half-line $(0,+\infty)$ is considered the following IBVP

$$
\begin{cases} \partial_t u+\partial_x u+\partial_x^3 u+u\partial_x u=0, & \text{for } (x,t)\in(0,+\infty)\times(0,T),\\ u(0,t)=f(t), & \text{for } t\in(0,T),\\ u(x,0)=\phi(x), & \text{for } x\in(0,+\infty), \end{cases} \tag{1.1}
$$

and on the left half-line $(-\infty,0)$ the following one

$$
\begin{cases} \partial_t u+\partial_x u+\partial_x^3 u+u\partial_x u=0, & \text{for } (x,t)\in(-\infty,0)\times(0,T),\\ u(0,t)=g_1(t), \quad \partial_x u(0,t)=g_2(t), & \text{for } t\in(0,T),\\ u(x,0)=\phi(x), & \text{for } x\in(-\infty,0). \end{cases} \tag{1.2}
$$

The rationale behind the presence of one boundary condition in the right half-line problem (1.1) as opposed to two boundary conditions in the left half-line problem (1.2) can be elucidated by integral identities concerning smooth solutions to the linear KdV equation. For such solutions, denoted by v, and any arbitrary time t, where $0<t<T$, the following holds:

$$
\int_0^{+\infty} v^2(x,t)dx=\int_0^{+\infty} v^2(x,0)dx+\int_0^t\left[2v\left(0,t'\right)v_{xx}\left(0,t'\right)-v_x^2\left(0,t'\right)\right]dt' \tag{1.3}
$$

and

$$
\int_{-\infty}^0 v^2(x,t)dx=\int_{-\infty}^0 v^2(x,0)dx-\int_0^t\left[2v\left(0,t'\right)v_{xx}\left(0,t'\right)-v_x^2\left(0,t'\right)\right]dt'. \tag{1.4}
$$

Thus, assuming $v(x,0)=0=v\left(0,t'\right)$ for all $x>0$ and $0<t'<t$, we deduce from (1.3) that $v(x,t)=0$ for all $x>0$. However, the existence of $v(x,t)\neq 0$ for $x<0$, satisfying $v(x,0)=0=v\left(0,t'\right)$ for all $x<0$ and $0<t'<t$, is not precluded by (1.4). Indeed, such nonzero solutions do exist, as demonstrated in [29]. Nevertheless, (1.4) does reveal that homogeneous conditions $v(x,0)=v\left(0,t'\right)=v_x\left(0,t'\right)=0$ for all $x<0$ and $0<t'<t$ imply that $v(x,t)=0$ for all $x<0$.

Remark 1. *Acknowledging the significance of the natural regularity assumptions for the boundary function, it is pertinent to note that these are inspired by the Kato smoothing effects elucidated by Kenig, Ponce, and Vega* [32].

$$
\left\|\psi(t)e^{t\partial_x^3}\phi\right\|_{L_x^\infty H_t^{(k+1)/3}}\leq c_{\psi,s}\|\phi\|_{H^k(\mathbb{R})}
$$

and

$$
\left\|\psi(t)\partial_x e^{t\partial_x^3}\phi\right\|_{L_x^\infty H_t^{k/3}}\leq c_{\psi,s}\|\phi\|_{H^k(\mathbb{R})},
$$

[1] This equation was first introduced by Boussinesq [7], and Korteweg and de Vries rediscovered it twenty years later [33]. Moreover, the linearized system $\partial_t u+\partial_x u+\partial_x^3 u=0$ is also called the first Stokes equation. The small amplitude, and long wave limit of the equations governing inviscid, irrotational water waves yield the KdV equation. Typically, when studying this equation over the entire line, the $\partial_x u$ term is omitted, as it can be eliminated via a Galilean transformation. Yet, on the half-line, applying this transformation would alter the domain to a wedge.

where $\psi(t)$ is a smooth cutoff function and the operator $e^{-t\partial_x^3}$ denotes the free propagator on $\mathbb{R}$. This motivates the following setting for the IBVP (2.1),

$$u_0 \in H^k\left(\mathbb{R}^+\right) \text{ and } f \in H^{\frac{k+1}{3}}\left(\mathbb{R}^+\right).$$

While, for the IBVP (2.2),

$$u_0 \in H^k\left(\mathbb{R}^+\right), f \in H^{\frac{k+1}{3}}\left(\mathbb{R}^+\right) \text{ and } g \in H^{\frac{k}{3}}\left(\mathbb{R}^+\right).$$

1.2. **Why study control problems on unbounded domains?** The well-posedness of the system (1.1), as an initial-boundary value problem, is well established in the literature (see, for example, [1, 2, 3]). In particular, a result by Bona *et al.* [2] guarantees the following:

Theorem A. *Let $\nu > 0$ and $s > -1$ be given with $s \neq 3m + \frac{1}{2}$, $m = 0, 1, 2, \cdots$. Set*

$$H_\nu^s\left(\mathbb{R}^+\right) := \left\{f \in H^s\left(\mathbb{R}^+\right); \mathrm{e}^{\nu x} f \in H^s\left(\mathbb{R}^+\right)\right\}.$$

For any given compatible pair[2] $(\phi, h) \in H_\nu^s(\mathbb{R}^+) \times H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+)$, there exists $T > 0$ such that the IBVP (1.1) admits a unique mild solution $u \in C([0, T]; H_\nu^s(\mathbb{R}^+))$.

As a direct consequence of Theorem A (see, for instance, [41, Section 4]), they deduced that the system (1.1) is not exactly controllable for a certain class of final states. Specifically, this class is given by

$$\mathcal{C} := \left\{\phi_T \in H^s\left(\mathbb{R}^+\right) : \phi_T \notin H_\nu^{s'}(\mathbb{R}^+) \text{ for any } \nu \text{ and } s' > -1\right\}.$$

This result highlights that the exact controllability is obstructed for states that fail to belong to the more regular weighted Sobolev space $H_\nu^{s'}(\mathbb{R}^+)$, regardless of the weight parameter ν and for any $s' > -1$. This naturally raises the question:

Question $\mathcal{A}$: *Is there a class of states for which the system (1.1) (or (1.2)) is either null controllable or exactly controllable?*

In this direction, Rosier [39, Theorem 1.2] shows that for a certain class of solutions, the linear system associated with (1.1) is not null controllable for some states. Specifically, the following result was established:

Theorem B. *Let $T > 0$ be given. Then, there exists an initial data $\phi \in L^2(\mathbb{R}^+)$ such that if $u \in L^\infty\left(0, T; L^2\left(\mathbb{R}^+\right)\right)$ solves*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{in } \mathcal{D}'((0, +\infty) \times (0, T)), \\ u\big|_{t=0} = \phi, \end{cases}$$

then $u\big|_{t=T} \neq 0$.

It means that the bad behavior of the trajectories as $x \to \infty$ is the price to be paid for getting the exact controllability in $(0, +\infty)$. Indeed, for a certain function u_0 in $L^2(0, +\infty)$ and $u_T = 0$ a trajectory u as above cannot be found in $L^\infty\left(0, T, L^2(0, +\infty)\right)$. However, when the bounded energy condition $\left(u \in L^\infty\left(0, T; L^2\left(\mathbb{R}^+\right)\right)\right)$ is relaxed, the exact boundary controllability of the linear KdV equation holds, as demonstrated also by Rosier in [39, Theorem 1.3]:

Theorem C. *Let $T, \epsilon,$ and b be positive numbers, with $\epsilon < \frac{T}{2}$. Let $\phi \in H^0\left(\mathbb{R}^+\right) = L^2\left(\mathbb{R}^+\right)$ and $\psi \in H_{-b}^0(\mathbb{R}^+)$. Then, there exists a solution*

$$u \in L_{\text{loc}}^2([0, \infty) \times [0, T]) \cap C\left([0, \epsilon]; L^2\left(\mathbb{R}^+\right)\right) \cap C\left([T-\epsilon, T]; H_{-b}^0\left(\mathbb{R}^+\right)\right)$$

which satisfies

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{in } \mathcal{D}'((0, +\infty) \times (0, T)) \\ u\big|_{t=0} = \phi, \quad u\big|_{t=T} = \psi. \end{cases}$$

[2]Refer to [1] for the precise definition of s-compatibility in this context.

Additionally, Rosier [40, Corollary 2] provides a fundamental solution with compact support in time for the linear KdV equation. This enables the construction of an explicit trajectory, although with undesirable behavior at infinity, that drives any function u_0 to zero. To conclude, and summarize all results related to the controllability of the KdV equation on the half-line, it is important to highlight the following result, which asserts that the linear KdV equation is indeed exactly boundary controllable in $(0,+\infty)$, as established in [40, Theorem 1].

Theorem D. *Let $T>0$ and let $u_0, u_T \in L^2(0,+\infty)$. Then there exists a function $u \in L^2_{loc}([0,T]\times[0,+\infty))$ fulfilling*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & t \in (0,T), \quad x \in \mathbb{R}^+, \\ u_{|t=0} = u_0, & \\ u\,|_{t=T} = u_T. & \end{cases}$$

Note that, in the context of the unbounded domain, the control theory for the KdV equation still needs to be developed if compared with the theory in bounded domains (see below the discussion in this case). So, given these previous results, the study of exact controllability for systems (1.1) and (1.2) in the half-line framework becomes an interesting problem. Specifically, the following question arises:

Question $\mathcal{A}'$: *Given $T>0$ and $\phi, \phi_T \in L^2(\mathbb{R}^+)$ (or $L^2(\mathbb{R}^-)$). Can one find an appropriate control input $f(t) \in H^{\frac{1}{3}}(0,T)$ (or $g_1(t) \in H^{\frac{1}{3}}(0,T)$ or $g_2(t) \in L^2(0,T)$) such that there exists a class of corresponding solutions $u(x,t)$ of system* (1.1) *(or* (1.2)*) satisfying*

$$u(x,0) = \phi(x) \quad \textit{and} \quad u(x,T) = \phi_T(x)?$$

In the framework of unbounded domains, this work provides new insights into the mathematical theory of control for the KdV equation, applicable on the half-line (and also in bounded domains). Specifically, by characterizing the solutions and controls associated with these problems, we can identify a class of solutions where the exact controllability is satisfied, answering the Questions $\mathcal{A}$ and $\mathcal{A}'$.

1.3. **State of the arts.** Before delving into our main findings, let us provide an overview of the existing literature regarding the well-posedness of the KdV equation in half-line, along with the principal results concerning the control problem for the KdV equation, and consequently, the critical set phenomena associated with it.

1.3.1. *Well-posdness theory.* Research on (IBVP) (1.1) commenced with Ton's investigation [45]. Ton established the existence and uniqueness of solutions, considering smooth initial data $\phi(x)$ and zero boundary data f. Subsequently, Bona and Winther [4] demonstrated the global existence and uniqueness of solutions in $L^\infty_{\text{loc}}(\mathbb{R}^+; H^4(\mathbb{R}^+))$, provided $\phi(x) \in H^4(\mathbb{R}^+)$ and $f \in H^2_{\text{loc}}(\mathbb{R}^+)$. Six years later, they further investigated the system in [5], establishing its continuous dependence. Concurrently, Faminskii [20] explored a generalized version of the IBVP (1.1), establishing well-posedness in weighted $H^1(\mathbb{R}^+)$ Sobolev space.

Nearly fifteen years later, Bona *et al.* [1] established conditional local well-posedness, whereby solutions are deemed unique only if they adhere to certain s-compatibility conditions. This achievement was attained by considering $\psi \in H^s(\mathbb{R}^+)$ and $f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ with $s > \frac{3}{4}$. Additionally, they provided global well-posedness for $\psi \in H^s(\mathbb{R}^+)$ and $f \in H^{\frac{3s+7}{12}}(\mathbb{R}^+)$, where $1 \le s \le 3$.

Colliander and Kenig [14] concurrently explored the initial boundary value problem (IBVP) depicted in equation (1.1) alongside their research. They proposed a more inclusive method to address the generalized Korteweg-de Vries (gKdV) equation situated on $\mathbb{R}^+$. Their approach involved expressing the original system (1.1) as a superposition of three initial value problems on $\mathbb{R}\times\mathbb{R}$. Specifically, for the system examined in our study, their findings provided conditional local well-posedness in $L^2(\mathbb{R}^+) \times H^{\frac{1}{3}}(\mathbb{R}^+)$, wherein solutions are deemed unique only if they adhere to supplementary auxiliary conditions. The authors also established a global a priori estimate for the boundary condition $f \in H^{\frac{7}{12}}(\mathbb{R}^+)$, and conditional global well-posedness was achieved for the case $s = 0$.

Two years later, Faminskii [21] enhanced the global outcomes reported in [14] by considering more natural boundary conditions. The local well-posedness of the IBVP (1.1) above $s = -\frac{3}{4}$, representing the critical Sobolev exponent for the KdV initial value problem, was established by Holmer [29] and Bona et al. [3]. Overviews detailing these findings and others can be found in [1] and [25].

In the case of the left half-line, Holmer [29] established local well-posedness in $H^s(\mathbb{R}^+)$ for $s > -\frac{3}{4}$. Subsequently, Faminskii [22] demonstrated global well-posedness in $H^s(\mathbb{R}^+)$ for $s \geq 0$, assuming natural boundary conditions.

An alternative perspective on (1.1) is offered by employing Inverse Scattering techniques. Fokas [23] introduced a novel approach called the unified transform method (UTM), which extends the Inverse Scattering Transform (IST) method for solving IBVPs. For instance, it was noted in [23] that, subject to appropriate decay and smoothness assumptions, akin to the infinite-line scenario, the solution on the right half-line is expected to depict (for large times) an assembly of (standard KdV) solitons traveling at constant speeds. These techniques were refined further in [25], where the UTM method establishes well-posedness in Sobolev spaces.

Finally, two significant points warrant mention. Firstly, the representation and well-posedness approaches introduced in [1, 14, 23, 29] apply to the left-half line and, consequently, to the IBVP (1.2). Secondly, it is noteworthy that Cavalcante introduced an adaptation of Colliander and Kenig's approach for the context of star graphs in [10].

1.3.2. *Control theory.* The exploration of control strategies for the KdV equation was initiated by the works of Russell and Zhang [42, 43, 44, 46, 47]. As for the control issue, Rosier [38] examined boundary control of the KdV equation on the finite domain $(0, L)$ with Dirichlet boundary conditions

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = u(L,t) = 0,\ \partial_x u(L,t) = g(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{1.5}$$

where the boundary value function $g(t)$ is considered as a control input, has important phenomena that directly affect the control problem related to them, so-called *critical length phenomenon*. The control problem (1.5) was presented in 97' in a pioneering work of Rosier [38]. The author answered the following control problem for the system (1.5), giving the origin of the critical length phenomenon for the KdV equation.

Initially, the linear system associated with (1.5) unveiled the phenomenon known as the "critical length". This phenomenon suggests that the exact controllability of the linear system associated with (1.5) hinges on the length L of the spatial domain $(0, L)$. In other words, the linear system is controllable if and only if

$$L \notin \mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \ : k,\, l\ \in \mathbb{N}^* \right\}. \tag{1.6}$$

In the case of $L \in \mathcal{N}$, Rosier demonstrated in [38] that the linear system linked with (1.5) is not controllable. There exists a finite-dimensional subspace $\mathcal{M} = \mathcal{M}(L)$ within $L^2(0, L)$ that remains unreachable from the origin of this linear system. Specifically, for any non-zero state $\psi \in \mathcal{M}$, $g \in L^2(0, T)$, and $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ fulfilling the conditions of the linear system associated with (1.5) and $u(\cdot, 0) = 0$, it follows that $u(\cdot, T) \neq \psi$. A spatial domain $(0, L)$ is termed "critical" for the linear system related to (1.5) if its length $L \in \mathcal{N}$.

When the spatial domain $(0, L)$ is critical, one would not typically anticipate the corresponding nonlinear system (1.5) to be precisely controllable, given the lack of controllability in the associated linear system. Therefore, it was surprising when Coron and Crépeau demonstrated in [16] that the nonlinear system (1.5) remains locally exactly controllable, even when its spatial domain is critical with a length of $L = 2k\pi$, where $k \in \mathbb{N}^*$ satisfies the condition:

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n.$$

Subsequently, Cerpa [11], and Cerpa and Crépeau in [13], demonstrated that the nonlinear system (1.5) is locally exactly controllable for large time considering the critical lengths.

It is important to point out, that if we change the control of position in the boundary condition of (1.5), for example

$$u(0,t) = h(t),\ u(L,t) = 0,\ \partial_x u(L,t) = 0 \quad \text{in } (0,T) \tag{1.7}$$

or

$$u(0,t) = 0,\ u(L,t) = f(t),\ \partial_x u(L,t) = 0 \quad \text{in } (0,T), \tag{1.8}$$

we can not characterize explicitly the critical sets for the KdV equation with the boundary conditions (1.7) and (1.8). For details, we infer [12, 27].

After $97'$, some authors tried to prove the critical set phenomenon for the KdV equation with some boundary condition, we can cite, for example, [27, 15], and the references therein. However, for the sets considered in these works, the authors were not allowed to characterize explicitly the set where the linear controllability fails. Twenty years later, in [8], another set of boundary conditions was considered. The authors introduced the KdV equation with Neumann conditions. Capistrano–Filho *et al.* investigated the KdV equation with the following boundary control

$$\partial_x^2 u(0,t) = 0,\ \partial_x u(L,t) = h(t),\ \partial_x^2 u(L,t) = 0.$$

First, the authors studied the following linearized system

$$\begin{cases} \partial_t u + (1+\beta)\partial_x u + \partial_x^3 u = 0. & \text{in } (0,L)\times(0,T), \\ \partial_x^2 u(0,t) = 0,\ \partial_x u(L,t) = h(t),\ \partial_x^2 u(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{1.9}$$

where β is a given real constant. For any $\beta \neq -1$, considering the following set

$$\mathcal{R}_\beta := \left\{ \frac{2\pi}{\sqrt{3(1+\beta)}}\sqrt{k^2 + kl + l^2} \,:\, k,\ l \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{\beta+1}} : k \in \mathbb{N}^* \right\}.$$

The authors showed that if $\beta \neq -1$, the linear system (1.9) is exactly controllable in the space $L^2(0,L)$ if and only if the length L of the spatial domain $(0,L)$ does not belong to the set $\mathcal{R}_\beta$. Moreover, if $\beta = -1$, then the system (1.9) is not exactly controllable in the space $L^2(0,L)$ for any $L > 0$. The result is also extended to the nonlinear problem using a point-fixed argument.

Note that, as in [38], the set $\mathcal{R}_\beta$ is completely characterized. Moreover, when $\beta = 0$, $\mathcal{N}$ (see (1.6)) is a proper subset of $\mathcal{R}_0$. The linear system (1.9) has more critical length domains than that of the linear system associated with (1.5). In the case of $\beta = -1$, every $L > 0$ is critical for the system (1.9). By contrast, removing the term $\partial_x u$ from the equation in (1.5), every $L > 0$ is not critical for the system (1.5).

In recent work, Capistrano-Filho and da Silva [9] gave a necessary first step to understanding the critical set phenomenon for the KdV equation posed on the interval $[0,L]$ considering the Neumann boundary conditions with only one control input. They showed that the KdV equation is controllable in the critical case, i.e., when the spatial domain L belongs to the set $\mathcal{R}_\beta$. This result is achieved using the *return method* together with a fixed point argument.

Finally, it is important to mention that in [31], the author presented an alternative for the null controllability of the heat equation on the half line since the lack of null controllability for this equation in this framework. This article introduced the relation via the UTM operator (see [23]) to ensure the controllability result (see next subsection for a detailed discussion of the KdV equation in this context).

1.4. **Operational controllability: The method.** In the next sections, we focused on studying the controllability of the KdV equation in the half-line. To achieve our objective we will use the Hilbert uniqueness (HUM) method introduced by Lions [35]. This classical method consists of proving an observability inequality for solutions of the adjoint system associated with the system that we are interested in controlling.

However, to apply the HUM, we need to present a new concept of controllability which we are calling *the operational controllability*. The core concept of this kind of controllability involves deriving a global relation for the initial and boundary conditions from the integral representation of solutions to problems posed on both the half-line and the finite interval. The strategy presented here has the potential to be applied to exact controllability problems governed by various nonlinear evolution partial differential equations. Notably, this form of integral representation has been explored in several works in the last decade, for instance, by Fokas [23], Kenig *et al.* [14], Holmer [29] and Bona *et al.* [1].

Let us consider the exact controllability problem (1.1), the other cases in this work can be treated analogously. An important point is that, without any loss of generality, we shall consider only the case when the initial data $\phi = 0$. This is possible since considering ϕ, ϕ_T in $L^2(\mathbb{R}^+_x)$, and f in $H^{\frac{1}{3}}(\mathbb{R}^+_t)$ is the control which leads the solution $(\widetilde{u}, \widetilde{v})$ of the system KdV equation posed on the right half-line from the zero initial data to the final state $\phi_T - u(T)$, where u is the mild solution corresponding to (1.1) with initial data ϕ and boundary condition $f = 0$, it follows immediately that these controls also lead to the solution $\widetilde{u} + u$ of (1.1) from ϕ to the final state ϕ_T.

1.4.1. *Operational controllability via forcing operator.* We will use the representation introduced by Kenig *et al.* [14], and after explored by Holmer [29]. Suppose that the initial data $\phi = 0$. Then, the integral representation of the solution of (1.1) given by the forcing operator takes the form

$$u(x,t) = -\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t) + \theta(t)\mathscr{L}^{\lambda}_{+}\left(e^{-\pi i\lambda} f\right)(x,t) + \theta(t)\mathscr{L}^{\lambda}_{+}\left(\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right)(x,t). \tag{1.10}$$

Now, if we consider the linearized system with initial data zero, the solution could be written as

$$u(x,t) = \theta(t)\mathscr{L}^{\lambda}_{+}\left(e^{-\pi i\lambda} f\right)(x,t).$$

Thus, we obtain the following definition for the exact controllability problem of the linearized system associated with the problem (1.1).

Definition 1.1. *Let $\phi_T \in L^2(\mathbb{R}^+_x)$, thus the system*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (0,+\infty) \times (0,T), \\ u(0,t) = f(t), & \text{for } t \in (0,T), \\ u(x,0) = 0, & \text{for } x \in (0,+\infty), \end{cases} \tag{1.11}$$

is operational exact controllable at time $t = T$ if and only if there exists $f = f(t) \in H^{\frac{1}{3}}(\mathbb{R}^+_t)$ such that

$$\theta(T)\mathscr{L}^{\lambda}_{+}\left(e^{-\pi i\lambda} f(t)\right)(x,T) = \phi_T(x), \tag{1.12}$$

for some $\lambda \in \mathbb{R}$.

From the identities (B.10) and (B.12), the relation (1.12) is equivalent to

$$\frac{3}{\Gamma(\lambda)}\int_x^{\infty}\int_0^T (y-x)^{\lambda-1} A\left(\frac{y}{(T-t')^{1/3}}\right)\frac{\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}} f(t')}{(T-t')^{1/3}}\,dt'dy = \phi_T(x), \tag{1.13}$$

for some $\lambda \in \mathbb{R}$. It is crucial to emphasize that the presence of the parameter λ in the aforementioned relation must adhere to the conditions ensuring the well-posedness of the system under consideration.

To establish the local exact controllability of the nonlinear system (1.1), with zero initial data, it is necessary to employ the fixed-point argument along with the formula (1.10). Observe that the *operational controllability relation* (1.13) remains effective in achieving exact controllability in $H^s(\mathbb{R}_x^+)$ for $-\frac{3}{4} < s \leq \frac{3}{2}$, excluding the case when $s = \frac{1}{2}$ (see the proof of Theorems 1.1 and 1.2, for details).

Remark 2. *Observe that we can extend the results of this article for two classes of operators, namely: boundary operator and UTM operator. Precisely, we have the following:*

(a) ***Operational controllability via boundary operator:*** *Using the representation giving by Bona et al.* [1], *we can establish the operational controllability of the linear system* (1.11). *Note that* [1, Proposition 2.2] *implies that the solution of the system* (1.11) *is given by*

$$u(x,t) = \left[W_b(t)f\right](x) = \left[U_b(t)f\right](x) + \overline{\left[U_b(t)f\right](x)} \quad \text{in } H^s(\mathbb{R}_x^+), \text{ for } \frac{3}{4} < s \leq 3.$$

Here, for $x \geq, t \geq 0$,

$$\left[U_b(t)f\right](x) = \frac{1}{2\pi}\int_1^\infty e^{i\mu^3 t - i\mu t} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x}\left(3\mu^2-1\right)\int_0^\infty e^{-\left(\mu^3 i - i\mu\right)\xi} f(\xi)d\xi d\mu.$$

Thus, a definition for the exact controllability problem for the linearized system (1.11) *is obtained as follows.*

Definition 1.2. *Let* $\phi_T \in H^s(\mathbb{R}_x^+)$, *thus the system is operational exact controllable at time* $t = T$ *if and only if there exists* $f = f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$ *such that*

$$2Re\left(\int_1^\infty \int_0^\infty e^{i\mu^3 T - i\mu T} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x}\left(3\mu^2-1\right) e^{-\left(\mu^3 i - i\mu\right)\xi} f(\xi)d\xi d\mu\right) = 2\pi\phi_T(x).$$

(b) ***Operational controllability via UTM operator:*** *Following the Fokas approach* [23, 24, 25, 30], *we can consider the case of* $H^s(\mathbb{R}_x^+)$, *for* $-\frac{3}{4} < s$ *with* $\frac{s+1}{3} \notin \mathbb{N} + \frac{1}{2}$, *and we can consider the operational controllability by using the following integral formula*[3]

$$u(x,t) = S\left[0,f,0\right](x,t) = -\frac{1}{2\pi}\int_{\partial D_R^+} \mathrm{e}^{\mathrm{i}kx - \mathrm{i}(k-k^3)t}(3k^2-1)\tilde{f}\left(k-k^3,T\right)dk, \tag{1.14}$$

where $\tilde{f}_0\left(k^3,T\right)$ *are defined by*

$$\tilde{f}\left(k^3,T\right) = \int_0^T \mathrm{e}^{\mathrm{i}(k-k^3)t'} f\left(t'\right)dt', \quad k \in \mathbb{C}, 0 < t < T,$$

and $\mathbb{C}^+$ *and* $\mathbb{C}^-$ *will denote the upper half* $(\operatorname{Im} k > 0)$ *and the lower half* $(\operatorname{Im} k < 0)$ *of the complex* k*-plane. The domain* D *is defined by*

$$D = \{k \in \mathbb{C},\ \operatorname{Re}(ik - ik^3) < 0\} = \{k \in \mathbb{C},\ (\operatorname{Im} k)\left(3(\operatorname{Re} k)^2 - (\operatorname{Im} k)^2 - 1\right) < 0\}.$$

Moreover, D^+ *and* D^- *will denote the part of* D *in* $\mathbb{C}^+$ *and* $\mathbb{C}^-$, *namely*

$$D^+ = D \cap \mathbb{C}^+ \quad \text{and} \quad D^- = D \cap \mathbb{C}^-.$$

The asymptotic form of D, D^+, D^- *as* $k \to \infty$ *will be denoted by* D_R, D_R^+, D_R^-, *respectively, i.e.,*

$$D_R = \{k \in D,\quad |w(k)| > R,\quad R \text{ large }\},\quad D_R^+ = D_R \cap \mathbb{C}^+,\quad D_R^- = D_R \cap \mathbb{C}^-. \tag{1.15}$$

From [25, Theorem 2], *the function* $u(x,t)$ *given by* (1.14) *defines a solution to the system* (1.11) *in the space* $C([0,T];H^s(\mathbb{R}_x^+))$. *Thus, in this case, the definition for the exact controllability problem of the system* (1.11) *is given as follows.*

Definition 1.3. *Pick* $\phi_T \in H^s(\mathbb{R}_x^+)$, *the system is operational exact controllable at time* $t = T$ *if and only if there exists* $f = f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$ *such that*

$$\int_{\partial D_R^+}\int_0^T \mathrm{e}^{\mathrm{i}kx - \mathrm{i}(k-k^3)(T-t')}(3k^2-1)f\left(t'\right)dt'dk = -2\pi\phi_T(x).$$

[3]See for instance [23, Chapter 1, Proposition 1.2 and Example 1.12].

1.5. **Notations and main results.** As stated, there are no results in the literature that show the controllability of the KdV equation in half-line. In this case, we are interested in relating the IBVP with controllability and presenting an approach that serves not only for the KdV equation but also for a series of dispersive equations when posed in half-line.

So, in this spirit, this article deals with a class of distributed parameter control systems described by the KdV equation posed on an unbounded domain $\mathbb{R}^+ = (0,+\infty)$ and the left half-line $\mathbb{R}^- = (-\infty, 0)$. To deal with the well-posedness theory and control problems, consider first the equation (1.1) when $-\frac{3}{4} < s < \frac{3}{2}$ and $s \neq \frac{1}{2}$. In this set of conditions over s, we have that

$$\phi \in H^s\left(\mathbb{R}_x^+\right), \quad f \in H^{\frac{s+1}{3}}\left(\mathbb{R}_t^+\right), \quad \text{and if } \frac{1}{2} < s < \frac{3}{2}, \ \phi(0) = f(0). \tag{1.16}$$

Moreover, considering the system (1.2), if $-\frac{3}{4} < s < \frac{3}{2}$, for $s \neq \frac{1}{2}$, we get

$$\phi \in H^s\left(\mathbb{R}_x^-\right), g_1 \in H^{\frac{s+1}{3}}\left(\mathbb{R}_t^+\right), g_2 \in H^{\frac{s}{3}}\left(\mathbb{R}_t^+\right), \quad \text{and if } \frac{1}{2} < s < \frac{3}{2}, \ \phi(0) = g_1(0). \tag{1.17}$$

With this in hand, we define the solution for the systems (1.1) and (1.2), respectively, as follows.

Definition 1.4. *We will call a solution $u(x,t)$ of (1.1)-(1.16) (resp. (1.2)-(1.17)) on $[0,T]$ if the following holds:*

a) *Well-defined nonlinearity: The function for some (appropriated) space X, if $u \in X$ we have $\partial_x u^2$ is a well-defined, in a distribution sense. Moreover, the function $u(x,t)$ satisfies system (1.1) (resp. (1.2)) in the sense of distributions on the set $(x,t) \in (0,+\infty) \times (0,T)$ (resp. $(x,t) \in (-\infty,0) \times (0,T)$).*

b) *Space traces: The function $u \in C([0,T]; H_x^s)$ and in this sense $u(\cdot,0) = \phi$ in $H^s(\mathbb{R}_x^+)$ (resp. $u(\cdot,0) = \phi$ in $H^s(\mathbb{R}_x^-)$).*

c) *Time traces: Considering $u \in C\left(\mathbb{R}_x; H^{\frac{s+1}{3}}(0,T)\right)$ and in this sense $u(0,\cdot) = f$ in $H^{\frac{s+1}{3}}(0,T)$ (resp. $u(0,\cdot) = g_1$ in $H^{\frac{s+1}{3}}(0,T)$).*

d) *Derivative traces: If $\partial_x u \in C\left(\mathbb{R}_x; H^{\frac{s}{3}}(0,T)\right)$, considering only the system (1.2)-(1.17) we require that, in this sense, $u(0,\cdot) = g_2$ in $H^{\frac{s}{3}}(0,T)$.*

In this case, X shall be the modified Bourgain space $X_{s,b} \cap D_\alpha$ with $b < \frac{1}{2}$ and $\alpha > \frac{1}{2}$, where

$$\|u\|_{X_{s,b}} = \left(\iint_{\xi,\tau} \langle\xi\rangle^{2s} \left\langle \tau - \xi^3 \right\rangle^{2b} |\hat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2},$$

and

$$\|u\|_{D_\alpha} = \left(\iint_{|\xi|\leq 1} \langle\tau\rangle^{2\alpha} |\hat{u}(\xi,\tau)|^2 d\xi d\tau\right)^{1/2}.$$

Let us now introduce two sets. For any $\phi \in H^s(\mathbb{R}^+)$, $-\frac{3}{4} < s \leq \frac{3}{2}$, we define the admissible final state class for the linearized systems (1.1) and (1.2) by the sets

$$\mathcal{A}_r^s(\phi,T) = \Big\{\phi_T \in H^s(\mathbb{R}^+) : \text{the solution given by Definition 1.4 for the linear system associated with (1.1) satisfies } u(x,T) = \phi_T, \text{ for a boundary control } f \in H^{\frac{s+1}{3}}(\mathbb{R}^+)\Big\}$$

and

$$\mathcal{A}_l^s(\phi,T) = \Big\{\phi_T \in H^s(\mathbb{R}^-) : \text{the solution given by Definition 1.4 for the linear system associated with (1.2) satisfies } u(x,T) = \phi_T, \text{ for a boundary controls } g_1 \in H^{\frac{s}{3}}(\mathbb{R}^+),\ g_2 \in H^{\frac{s+1}{3}}(\mathbb{R}^+)\Big\},$$

respectively. Note that from [40, Theorem 1] it follows that the admissible final state class for the linearized system associated with (1.1) is not an empty set, i.e $\mathcal{A}_r^0(\phi,T) \neq \emptyset$.

With this in hand, this article presents two results that give answers for Question $\mathcal{A}'$, and consequently, for Question $\mathcal{A}$. Precisely, in this work, we will prove the exact boundary controllability in the sense of the definition 1.1. The first one is related to the control properties for the system (1.1) and can be read as follows.

Theorem 1.1. *Let $T>0$. Then, there exists $\delta>0$ such that for any $\phi\in L^2(\mathbb{R}_x^+)$ and $\phi_T\in \mathcal{A}_r^0(\phi,T)$, verifying*

$$\|\phi\|_{L^2(\mathbb{R}_x^+)}+\|\phi_T\|_{L^2(\mathbb{R}_x^+)}\leq\delta,$$

the system (1.1) admits a unique solution $u\in X=X_{0,b}\cap D_\alpha$ operational exactly controllable at time T, it means that there exist $f\in H^{\frac{1}{3}}(\mathbb{R}_t^+)$ such that

$$\phi_T(x)=e^{-T(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\mathscr{D}\partial_x u^2(x,T)+\mathscr{L}_+^\lambda h(x,T,\phi),$$

where

$$h(t,\phi)=e^{-\pi i\lambda}\left[f(t)-\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right],$$

with $\mathscr{L}_+^\lambda$ the forcing operator given by (B.11).

The second main result considers the control problem (1.2) when only one control input. The result is the following one.

Theorem 1.2. *Let $T>0$. Then, there exists $\delta>0$ such that for any $\phi\in L^2(\mathbb{R}_x^+)$ and $\phi_T\in \mathcal{A}_l^0(\phi,T)$, verifying*

$$\|\phi\|_{L^2(\mathbb{R}_x^+)}+\|\phi_T\|_{L^2(\mathbb{R}_x^+)}\leq\delta,$$

the system (1.1) admits a unique solution $u\in X=X_{0,b}\cap D_\alpha$ operational exactly controllable at time T. Moreover, we have:

(i) *If $g_1=0$, one can find $g_2\in L^2(\mathbb{R}_t^+)$ such that*

$$\phi_T(x)=e^{-T(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\mathscr{D}\partial_x u^2(x,T)+\mathscr{L}_-^{\lambda_1}h_1(x,T,\phi)+\mathscr{L}_-^{\lambda_2}h_2(x,T,\phi)$$

where

$$\begin{bmatrix} h_1(t,\phi)\\ h_2(t,\phi)\end{bmatrix}=M\begin{bmatrix} -\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\mathscr{D}\partial_x u^2(0,t)\\ \theta(t)\mathcal{I}_{1/3}\left(g_2-\theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\end{bmatrix}.$$

(ii) *If $g_2=0$, one can find $g_1\in H^{\frac{1}{3}}(\mathbb{R}_t^+)$ such that*

$$\phi_T(x)=e^{-T(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\mathscr{D}\partial_x u^2(x,T)+\mathscr{L}_-^{\lambda_1}h_1(x,T,\phi)+\mathscr{L}_-^{\lambda_2}h_2(x,T,\phi),$$

where

$$\begin{bmatrix} h_1(t,\phi)\\ h_2(t,\phi)\end{bmatrix}=M\begin{bmatrix} g_1(t)-\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\mathscr{D}\partial_x u^2(0,t)\\ \theta(t)\mathcal{I}_{1/3}\left(-\theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\end{bmatrix}.$$

Here, M is a matrix given by (B.15).

Both theorems are shown true using the classical tools of control theory, namely, the duality theory of Dolecki and Russell [19] in the set-up of Lions [35], which reduces our problem to provide an observability inequality. Moreover, to ensure the controllability of the full system, we employ a classical fixed-point argument.

Finally, considering the case where the initial data ϕ is zero in the linearized system associated with (1.1) and considering the explicit characterization of the solutions to the systems under consideration in this work, it is possible to characterize precisely the control and the controllable solutions for the control problem. To be precise, the following theorem is achieved using the formulas presented in [1, 23, 29], and in the sense of the definitions 1.1, 1.2 and 1.3:

Theorem 1.3. *Let $\frac{3}{4} < s \leq \frac{3}{2}$ and $f \in H^{\frac{1}{3}}(0,T)$ such that system*

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (0,+\infty) \times (0,T), \\ u(0,t) = f(t), & \text{for } t \in (0,T), \\ u(x,0) = 0, & \text{for } x \in (0,+\infty), \end{cases}$$

is (operational) exactly controllable at time T. Then the control f satisfies the following relations

$$\frac{6\pi}{\Gamma(\lambda)} \int_x^\infty \int_0^T (y-x)^{\lambda-1} A\left(\frac{y}{(T-t')^{1/3}}\right) \frac{\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}} f(t')}{(T-t')^{1/3}} dt' dy \\ + \int_{\partial D_R^+} \int_0^T \mathrm{e}^{\mathrm{i}kx - \mathrm{i}(k-k^3)(T-t')} (3k^2-1) f(t')\, dt' dk = 0$$

or

$$\frac{6\pi}{\Gamma(\lambda)} \int_x^\infty \int_0^T (y-x)^{\lambda-1} A\left(\frac{y}{(T-t')^{1/3}}\right) \frac{\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}} f(t')}{(T-t')^{1/3}} dt' dy \\ - 2Re\left(\int_1^\infty \int_0^\infty e^{i\mu^3 T - i\mu T} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x} \left(3\mu^2-1\right) e^{-\left(\mu^3 i - i\mu\right)\xi} f(\xi) d\xi d\mu\right) = 0$$

or

$$2Re\left(\int_1^\infty \int_0^\infty e^{i\mu^3 T - i\mu T} e^{-\left(\frac{\sqrt{3\mu^2-4}+i\mu}{2}\right)x} \left(3\mu^2-1\right) e^{-\left(\mu^3 i - i\mu\right)\xi} f(\xi) d\xi d\mu\right) \\ + \int_{\partial D_R^+} \int_0^T \mathrm{e}^{\mathrm{i}kx - \mathrm{i}(k-k^3)(T-t')} (3k^2-1) f(t')\, dt' dk = 0.$$

Here, A is the Airy function and $\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}$ is the Caputo fractional derivative given by (B.5) *and* (B.4)*, respectively. Moreover, the region D_R^+ is defined by* (1.15)*.*

1.6. **Comments and novelties.** There are important facts about the theorems presented in this work. We mention some of them below.

i. Theorem 1.3 can be proved for the control system (1.2) with only one control input.
ii. Theorem 1.3 provides how to determine explicitly the control function $f(t)$ (or $g_1(t)$ or $g_2(t)$). This can be done since we have a relationship between the forcing operator, UTM operator, and Boundary operator with the controllability of the KdV equation in the following sense. Suppose, for instance, the control problem (1.1). In this case, the control function $f(t)$ can be given by

$$\left(\{\text{Forcing operator}\}\, f\right)(x,t) + \left(\{\text{UTM operator}\}\, f\right)(x,t) = 0,$$
$$\left(\{\text{Forcing operator}\}\, f\right)(x,t) - \left(\{\text{Boundary operator}\}\, f\right)(x,t) = 0,$$

or

$$\left(\{\text{Boundary operator}\}\, f\right)(x,t) + \left(\{\text{UTM operator}\}\, f\right)(x,t) = 0.$$

Thus, the previous relations ensure that, if the inverse of the operators presented before exists, we have that:

$$f(t) = \left[\left(\{\text{Forcing operator}\} + \{\text{UTM 0perator}\}\right)(x,t)\right]^{-1}(\{0\}),$$
$$f(t) = \left[\left(\{\text{Forcing operator}\} - \{\text{Boundary operator}\}\right)(x,t)\right]^{-1}(\{0\}),$$

or

$$f(t) = \left[\left(\{\text{Bondary operator}\} + \{\text{UTM operator}\}\right)(x,t)\right]^{-1}(\{0\}).$$

iii. Given that operational controllability can be extended to encompass bounded intervals, with all operators effectively operating within these constraints, we can similarly extend the Theorems 1.1, 1.2 and 1.3 to the bounded cases, provided suitable boundary conditions.

iv. Regarding the critical set phenomenon, it is notable that several authors have explored this aspect when considering the KdV equation on a bounded domain. We highlight two specific works. In references [38] and [27], the authors delve into this phenomenon for two sets of boundary conditions: condition $(1.5)_2$ and (1.7). However, in our case, where fewer boundary conditions are imposed and the drift term $(\partial_x u)$ is included in the equation, the critical set phenomenon does not manifest. This represents a novel observation for the KdV equation. Specifically, we can maintain the same positioning of control, either $\partial_x u(L,t) = f(t)$ or $u(0,t) = g_1(t)$.

v. The main contribution of this work can be seen in two directions. The first is to relate the three types of operators that guarantee the existence of solutions to the KdV equation (see [1, 23, 29]) with the control theory for this equation, which guarantees an explicit form for the control that drives the initial data to the final data, considering a class of solution as mentioned before. Furthermore, the second contribution is that, in any configuration of half-line controls, that is, considering $f(t)$, $g_1(t)$ or $g_2(t)$, the phenomenon of "critical length" does not appear.

The scenario presented in this work is completely new in terms of control theory. Employing a comprehensive approach allows us to formulate a broader relationship, thereby achieving precise control, denoted by f, essential for ensuring the exact controllability of the system under consideration, precisely, Theorem 1.3, when only one control input is considered. Our analysis can be seen in the scheme below (see Figure 1 below), which shows the range of s in the control space H^s, in each operator that Theorem 1.3 remains valid.

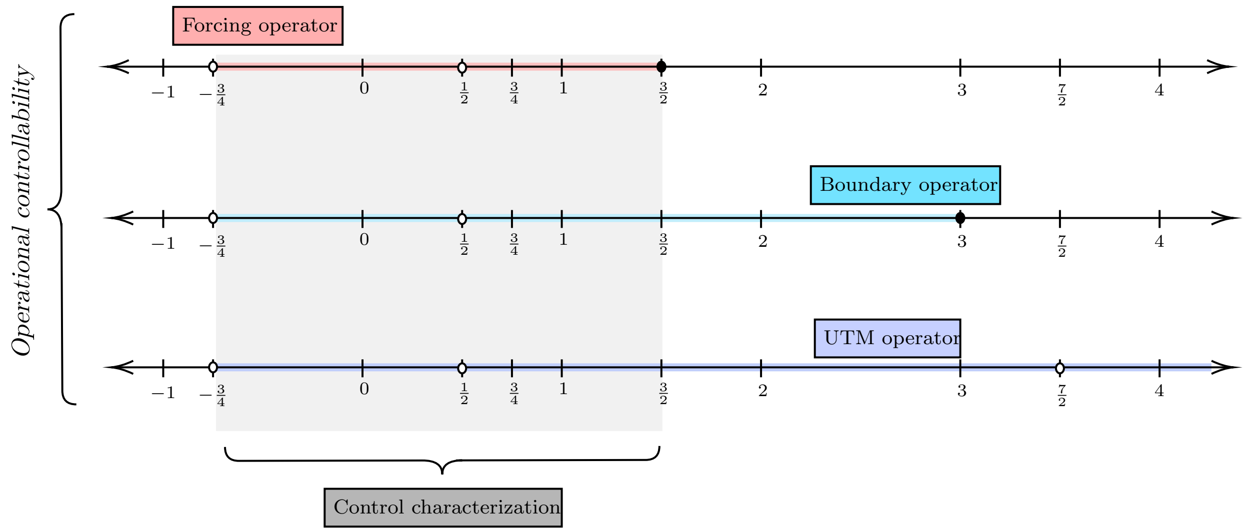


FIGURE 1. Operational controllability and control characterization relations

1.7. **Paper's outline.** We complete our introduction by outlining the structure of the paper. Section 2 provides an overview of the well-posedness theory. In Section 3, we address the exact controllability result on the right half-line, offering the proof of Theorem 1.1. Section 4 contains the detailed proof of Theorem 1.2, demonstrating the exact controllability of the KdV equation on the left half-line. Section 5 gives some perspectives and discusses open issues. Finally, we include two appendices. Appendix A provides supplementary results for the homogeneous systems discussed in this work. Additionally, Appendix B presents the boundary forcing operator formulas for the KdV system, along with the key estimates for this operator.

Acknowledgment. This research was conducted after several visits by the authors to the National University of Colombia (sede Manizales) and the Federal University of Pernambuco, and it was completed during the visit of the authors to the Department of Mathematics at Ewha Womans

University, Korea, in September 2024. The authors sincerely thank the host institutions for their generous hospitality.

2. Well-posedness theory

In this section, we are interested in revisiting the well-posedness theory presented in the classical papers [1, 2, 3, 14, 23, 25, 29]. These preliminary analyses will be paramount for us to present the main novelty of this work.

2.1. **Preliminaries.** Let us consider $s \geq 0$. In this case, we say that $\phi \in H^s(\mathbb{R}^+)$ if exists $\tilde{\phi} \in H^s(\mathbb{R})$ such that $\phi = \tilde{\phi}|_{\mathbb{R}^+}$. Moreover, we set

$$\|\phi\|_{H^s(\mathbb{R}^+)} := \inf_{\tilde{\phi}} \|\tilde{\phi}\|_{H^s(\mathbb{R})}.$$

Here,

$$H_0^s(\mathbb{R}^+) = \Big\{\phi \in H^s(\mathbb{R}^+);\ \operatorname{supp}(\phi) \subset [0,+\infty)\Big\},$$

otherwise, that is, $s<0$, define $H^s(\mathbb{R}^+)$ and $H_0^s(\mathbb{R}^+)$ as the dual space of $H_0^{-s}(\mathbb{R}^+)$ and $H^{-s}(\mathbb{R}^+)$, respectively. The first results summarize useful properties of the Sobolev spaces on the half-line and the proofs can be found in [14].

Lemma 2.1. *For all $f \in H^s(\mathbb{R})$ with $-\frac{1}{2} < s < \frac{1}{2}$ we have*

$$\|\chi_{(0,+\infty)} f\|_{H^s(\mathbb{R})} \lesssim \|f\|_{H^s(\mathbb{R})}.$$

Lemma 2.2. *If $\frac{1}{2} < s < \frac{3}{2}$ the following statements are valid:*

(a) $H_0^s(\mathbb{R}^+) = \big\{f \in H^s(\mathbb{R}^+); f(0) = 0\big\}$,

(b) *If $f \in H^s(\mathbb{R}^+)$ with $f(0)=0$, then $\|\chi_{(0,+\infty)} f\|_{H_0^s(\mathbb{R}^+)} \lesssim \|f\|_{H^s(\mathbb{R}^+)}$.*

Lemma 2.3. *If $f \in H_0^s(\mathbb{R}^+)$ with $s \in \mathbb{R}$, we then have*

$$\|\psi f\|_{H_0^s(\mathbb{R}^+)} \lesssim \|f\|_{H_0^s(\mathbb{R}^+)}.$$

Let us consider the well-known Bourgain theory [6]. We denote by $X^{s,b}$ the Fourier transform space associated with linear KdV equation, precisely, space $X^{s,b}$ is the completion of $S'(\mathbb{R}^2)$ concerning the norm

$$\|w\|_{X^{s,b}(\phi)} = \|\langle\xi\rangle^s \langle \tau - \xi^3\rangle^b \hat{w}(\xi,\tau)\|_{L^2_\tau L^2_\xi}.$$

To obtain our results we also need to define the following auxiliary modified Bougain space. Let $U^{s,b}$ and V^α the completion of $S'(\mathbb{R}^2)$ with respect to the norms

$$\|w\|_{U^{s,b}} = \left(\int\int \langle\tau\rangle^{2s/3}\langle\tau-\xi^3\rangle^{2b}|\widehat{w}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}} \quad \text{and} \quad \|w\|_{V^\alpha} = \left(\int\int \langle\tau\rangle^{2\alpha}|\widehat{w}(\xi,\tau)|^2 d\xi d\tau\right)^{\frac{1}{2}}.$$

Next nonlinear estimates, in the context of the KdV equation, for $b<\frac{1}{2}$, were derived by Holmer in [29].

Lemma 2.4. *The following holds:*

(a) *For $s > -\frac{3}{4}$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$ we have*

$$\big\|\partial_x(v_1v_2)\big\|_{X^{s,-b}} \lesssim \|v_1\|_{X^{s,b}\cap V^\alpha}\|v_2\|_{X^{s,b}\cap V^\alpha}.$$

(b) *Considering $s \in \big(-\frac{3}{4}, 3\big)$, there exists $b = b(s) < \frac{1}{2}$ such that for all $\alpha > \frac{1}{2}$*

$$\big\|\partial_x(v_1v_2)\big\|_{X^{s,-b}} \lesssim \|v_1\|_{X^{s,b}\cap V^\alpha}\|v_2\|_{X^{s,b}\cap V^\alpha},$$

is verified.

2.2. **Overview of well-posedness results.** In our analysis, we will consider the KdV equation posed in the unbounded domains, that is, the positive real line and the negative real line.

Let us start considering the case of the positive real line. We firstly quote the work of Bona *et. al.* [1, 2] that considers the system (1.1) in $\mathbb{R}^+$. Notably, the conventional approach of eliminating the term $\partial_x u$ from the equation by transitioning to traveling coordinates comes with a substantial trade-off in the quarter-plane problem. Introducing a change of variables $v(x,t) = u(x+t,t)$, effectively eliminates the problematic term in the evolution equation. However, this transformation alters the landscape of the boundary condition, now expressed as $v(-t,t) = f(t)$ for $t \geq 0$. Consequently, the boundary condition is enforced at a dynamically shifting spatial point, framing the problem within the unconventional domain $\{(x,t) : t \geq 0, x+t\}$.

Essentially, in [1] the authors establish a Kato smoothing effect in the following form: For $s > \frac{3}{4}$, if $\phi \in H^s(\mathbb{R}^+)$and $f \in H_{loc}^{\frac{s+1}{3}}(\mathbb{R}^+)$ satisfy certain compatibility conditions at $(x,t) = (0,0)$, then the IBVP (1.1) admits a unique solution

$$u \in C\left(0,T; H^s\left(\mathbb{R}_x^+\right)\right) \cap L^2\left(0,T; H_{loc}^{s+1}\left(\mathbb{R}_x^+\right)\right),$$

which satisfies the following additional properties

$$\left(\sup_{0<x<+\infty} \int_0^T \left|\partial_x^{s+1} u(x,t)\right|^2 dt\right)^{\frac{1}{2}} \leq C\left(\|\phi\|_{H^s(\mathbb{R}_x^+)} + \|f\|_{H^{\frac{s+1}{3}}(0,T)}\right).$$

On the other hand, in [3], the authors proved a boundary smoothing property for the corresponding linear problem, where the $u\partial_x u$ term is dropped and $\phi(x) = 0$, which states

$$\|u\|_{L^2\left([0,T];H^{s+\frac{3}{2}}\left(\mathbb{R}_X^+\right)\right)} \leq \mathrm{c}\|f\|_{\mathrm{H}^{\frac{s+1}{3}}\left(\mathbb{R}_t^+\right)},$$

where $c = c(s,T)$. As an application of this property and variants of it, the authors obtain local well-posedness of mild solutions of the nonlinear problem for s $> -\frac{3}{4}$, where mild solutions are defined as ones that can be appropriately approximated by smoother solutions. Next, we present the results that can be found in [1] and [3], respectively.

Theorem 2.5. *The initial-boundary-value problem* (1.1) *is locally well-posed for initial data* $\phi \in H^s\left(\mathbb{R}_x^+\right)$ *and boundary data* $f \in H_{loc}^{(s+1)/3}\left(\mathbb{R}_t^+\right)$ *satisfying certain compatibility conditions for* $s > 3/4$*, whereas global well-posedness holds for* $\phi \in H^s\left(\mathbb{R}_x^+\right), f \in H^{\frac{7+3s}{12}}\left(\mathbb{R}_t^+\right)$ *when* $1 \leq s \leq 3$ *and for* $\phi \in H^s\left(\mathbb{R}_x^+\right)$*,* $f \in H_{loc}^{(s+1)/3}\left(\mathbb{R}_t^+\right)$*when* $s \geq 3$*. Furthermore, the corresponding solution map is an analytic correspondence between the space of initial and boundary data and the solution space.*

Theorem 2.6. *Let* $s \geq -3/2$ *and* $T > 0$ *be given. There exists a constant* C *such that for any* $f \in H_0^{(s+1)/3}\left(\mathbb{R}_t^+\right)$*, the corresponding solution* u *of the linearized system associated to* (1.1) *belongs to the space* $L^2\left(0,T; H_0^{s+3/2}\left(\mathbb{R}_x^+\right)\right)$ *and satisfies*

$$\|u\|_{L^2\left(0,T;H^{s+\frac{3}{2}}\left(\mathbb{R}_x^+\right)\right)} \leq C\|f\|_{H^{\frac{s+1}{3}}\left(\mathbb{R}_t^+\right)},$$

for a constant C *depending only on* s *and* T*.*

To obtain the boundary controllability for the KdV posed in half-line (positive or negative), it is necessary to consider more low regularity in the initial data than $s > \frac{3}{4}$ as in the previous results. In this sense, Holmer [29] proves the existence of a solution of the KdV equation (1.1) and (1.2), with $\beta = 0$, posed either on a left half-line and right half-line. The main accomplishment of his work is to show that initial and boundary data may be given in Sobolev spaces of negative index. Indeed, the author showed the existence of solutions for initial data in the Sobolev space H^s as long as s is greater than $-\frac{3}{4}$, and similar restrictions are enforced for the boundary data. As is shown in the article, the right half-line problem requires one Dirichlet condition, while the left half-line problem requires an additional Neumann condition. On the finite interval, the problem is solved with two Dirichlet conditions and one Neumann condition on the right boundary. Now, to establish the well-posedness of systems (1.1) and (1.2), for $-\frac{3}{4} < s < \frac{3}{2}$ with $s \neq \frac{1}{2}$.

Now on, we will consider the case when $\beta = 0$ and the presence of source function $h \in L^1(0,T,H^s(\Omega))$, where $\Omega = \mathbb{R}^+$ or $\Omega = \mathbb{R}^-$, namely

$$\begin{cases} \partial_t u + \partial_x^3 u = h, & \text{for } (x,t) \in (0,+\infty) \times (0,T), \\ u(0,t) = f(t), & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (0,+\infty), \end{cases} \tag{2.1}$$

and

$$\begin{cases} \partial_t u + \partial_x^3 u = h, & & \text{for } (x,t) \in (-\infty,0) \times (0,T), \\ u(0,t) = g_1(t), & \partial_x u(0,t) = g_2(t), & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & & \text{for } x \in (-\infty,0). \end{cases} \tag{2.2}$$

Thus, the well-posedness result, for the linear systems, in [29] can be read as follows.

Theorem 2.7. *Let $-\frac{3}{4} < s < \frac{3}{2}, s \neq \frac{1}{2}$, and consider $h = 0$ in the system (2.1) and (2.2).*

i. *Given (ϕ, f) satisfying (1.16), exist $T > 0$, depending only on the norms of ϕ, f in (1.16), and $u(x,t)$ that is a mild and distributional solution to (2.1)-(1.16) on $[0,T]$.*
ii. *Given (ϕ, g_1, g_2) satisfying (1.17), exist $T > 0$, depending only on the norms of ϕ, g_1, g_2 in (1.17), and $u(x,t)$ a mild and distributional solution to (2.2)-(1.17) on $[0,T]$.*

Additionally, the results in [29] also ensure the well-posedness theory for the systems (1.1) and (1.2), respectively.

Theorem 2.8. *Let $-\frac{3}{4} < s < \frac{3}{2}, s \neq \frac{1}{2}$, it follows that*

i. *Given (ϕ, f) satisfying (1.16), exist $T > 0$, depending only on the norms of ϕ, f in (1.16), and $u(x,t)$ solution to (1.1)-(1.16) on $[0,T]$.*
ii. *Given (ϕ, g_1, g_2) satisfying (1.17), exist $T > 0$, depending only on the norms of ϕ, g_1, g_2 in (1.17), and $u(x,t)$ solution to (1.2)-(1.17) on $[0,T]$.*

In both cases, the data-to-solution map is analytic as a map from the spaces in (1.16) and (1.17), to the spaces giving in the Definition 1.4, which means that following solutions maps,

$$\begin{array}{rccc} \Gamma_r: & H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) & \longrightarrow & X = X_{s,b} \cap D_\alpha \\ & (\phi, f) & \longrightarrow & \Gamma_r(\phi, f) = u \end{array}$$

and

$$\begin{array}{rccc} \Gamma_l: & H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) \times H^{\frac{s}{3}}(\mathbb{R}_t^+) & \longrightarrow & X = X_{s,b} \cap D_\alpha \\ & (\phi, g_1, g_2) & \longrightarrow & \Gamma_l(\phi, g_1, g_2) = u, \end{array}$$

are analytics, where Γ_r and Γ_l are the map solution of the systems (1.1) and (1.2), respectively.

2.3. **Boundary formulas.** In [29], Holmer introduced the boundary forcing operator, which is the key point to prove the previous results. This operator gives us a chance to express the solution of the system (1.1) and (1.2), as well as, the boundary terms in terms of this operator (for details see Appendix B.1). Thus, the local solution of the systems (1.1) and (1.2) are given by

$$\begin{array}{c}\text{Right}\\ \text{Half-Line}\\ \text{Problem}\end{array} \begin{cases} u(x,t) = \theta(t) e^{-t(\partial_x + \partial_x^3)} \phi(x) - \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t) + \theta(t)\mathscr{L}_+^\lambda h(x,t,\phi), \\ \\ h(t,\phi) = e^{-\pi i \lambda} \left[f(t) - \theta(t) e^{-t(\partial_x+\partial_x^3)}\phi \Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t) \right]. \end{cases} \tag{2.3}$$

and

(2.4)

$$\text{Left Half-Line Problem}\quad\begin{cases} u(x,t) = \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x) - \dfrac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t) \\ \qquad\qquad + \theta(t)\mathscr{L}_-^{\lambda_1}h_1(x,t,\phi) + \theta(t)\mathscr{L}_-^{\lambda_2}h_2(x,t,\phi), \\ \begin{bmatrix} h_1(t,\phi) \\ h_2(t,\phi) \end{bmatrix} = M \begin{bmatrix} g_1(t) - \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t) \\ \theta(t)\mathcal{I}_{1/3}\left(g_2 - \theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t) \end{bmatrix}, \end{cases}$$

respectively, where M is a matrix given by (B.15).

Finally, it is important to say that there is another approach for solving initial-boundary value problems for integrable nonlinear evolution equations proposed by Fokas [24, 23, 25], the so-called unified transform method (UTM). Fokas *et. al*, in [25], studied the validity of the UTM formula for the KdV equation with data in Sobolev spaces. There, the authors studied the KdV equation without drift term, i.e., $\beta = 0$. For more details, see the Appendix B and [23, Chapter 1, examples 1.1 and 1.12].

3. Exact controllability: The right half-line

In this section, our primary focus is to attain the exact controllability of the system described by (1.1) and establish the proof for Theorem 1.1.

3.1. Backward system.

Initially, consider the following homogeneous linearized system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (0,+\infty) \times (0,T), \\ u(0,t) = 0, & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (0,+\infty). \end{cases}$$

Note that this system could be rewritten as

$$\begin{cases} \partial_t u = Au, \\ u(0) = u_0, \end{cases} \quad \text{where} \quad \begin{cases} Au = -\partial_x u - \partial_x^3 u, \\ D(A) := \left\{u \in H^3(\mathbb{R}_x^+) : u(0) = 0\right\} \subset L^2(\mathbb{R}_x^+). \end{cases}$$

Using the Semigroup theory (see, for instance, [37, Cor. 4.4 chapter 1]) is not difficult to see the following result.

Proposition 3.1. *The operator A generates a C_0-semigroup of contraction $(S(t))_{t\geqslant 0}$ in $L^2(\mathbb{R}_x^+)$.*

So on, we will consider the backward adjoint system given by

$$\begin{cases} -\varphi_t = A^*\varphi, \\ \varphi(T) = \varphi_T, \end{cases}$$

which implies that

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_x^3 \varphi = 0, & \text{for } (x,t) \in (0,+\infty) \times (0,T), \\ \varphi(0,t) = \partial_x\varphi(0,t) = 0, & \text{for } t \in (0,T), \\ \varphi(x,T) = \varphi_T(x), & \text{for } x \in (0,+\infty). \end{cases} \tag{3.1}$$

As a direct consequence of Proposition 3.1 and the general theory of evolution equation, the existence and uniqueness of this system holds.

Proposition 3.2. *Let $\varphi_T \in L^2(\mathbb{R}_x^+)$, then there exists a unique mild solution $\varphi(t) = S(T-t)\varphi_T$ of (3.1) such that $\varphi \in C\left([0,T]; L^2(\mathbb{R}_x^+)\right)$. Moreover, if $\varphi_T \in D(A)$, then (3.1) has a unique (classical) solution φ such that*

$$\varphi \in C([0,T]; D(A)) \cap C^1\left(0,T; L^2(\mathbb{R}_x^+)\right).$$

To establish some trace estimates for the backward system, remark that the change of variable $x = -x$ and $t = T - t$ reduces system (3.1) in

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_x^3 \varphi = 0, & \text{for } (x,t) \in (-\infty, 0) \times (0,T), \\ \varphi(0,t) = \partial_x \varphi(0,t) = 0, & \text{for } t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), & \text{for } x \in (-\infty, 0). \end{cases} \tag{3.2}$$

Also the well-posedness of system (3.2) follows from Theorem 2.8, with $-\frac{3}{4} < s < \frac{3}{2}$ and $s \neq \frac{1}{2}$. By using the boundary forcing operator, we have that the solution φ of (3.2) is given by:

$$\varphi(x,t) = \theta(t) e^{-t(\partial_x + \partial_x^3)} \varphi_0(x) + \theta(t) \mathscr{L}_-^{\lambda_1} h_1(x,t) + \theta(t) \mathscr{L}_-^{\lambda_2} h_2(x,t),$$

where

$$\begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} = M \begin{bmatrix} -\theta(t) e^{-t(\partial_x + \partial_x^3)} \varphi_0 \Big|_{x=0} \\ \theta(t) \mathcal{I}_{1/3} \left(-\theta \partial_x e^{-t(\partial_x + \partial_x^3)} \phi \Big|_{x=0} \right)(t) \end{bmatrix}$$

with A a matrix given by (B.15). From Lemmas B.4 and B.6, the estimations of the group and the Duhamel boundary forcing operator, respectively, ensure the following space and time trace estimations

$$\text{(Space traces)} \quad \|\varphi\|_{C(\mathbb{R}_t, H_x^s)} \leq C \|\varphi_0\|_{H^s(\mathbb{R}_x^+)}, \tag{3.3}$$

$$\text{(Time traces)} \quad \|\varphi\|_{C\left(\mathbb{R}_x, H_t^{\frac{s+1}{3}}\right)} \leq C \|\varphi_0\|_{H^s(\mathbb{R}_x^+)}, \tag{3.4}$$

and

$$\text{(Derivative time traces)} \quad \|\partial_x \varphi\|_{C\left(\mathbb{R}_x, H_t^{\frac{s}{3}}\right)} \leq C \|\varphi_0\|_{H^s(\mathbb{R}_x^+)}. \tag{3.5}$$

In particular, for $\varphi_0 \in L^2(\mathbb{R}_x^-)$, we have the solution φ of system (3.2) belonging of $C([0,T]; L^2(\mathbb{R}_x^-))$ with $\varphi(0,\cdot) \in H^{\frac{1}{3}}(0,T)$ and $\varphi_x(0,\cdot) \in L^2(0,T)$, and the following results are verified.

Proposition 3.3. *Any solution φ of adjoint system (3.1) satisfies*

$$T \|\varphi_T\|_{L^2(\mathbb{R}_x^+)}^2 = \int_0^T \|\varphi(t)\|_{L^2(\mathbb{R}_x^+)}^2 dt.$$

Proof. Multiplying the first equation of (3.2) by $t\varphi$ and integrating by parts in $(0,T) \times (0,\infty)$, the results hold using the boundary conditions. □

The following result reveals a notable improvement in the regularity of the solution to the linear system (3.1) and will be proved in Appendix A.

Theorem 3.4. *Let u be the solution of problem (3.2). In addition, if $\chi^\alpha u_0 \in L^2(\mathbb{R}_x^-)$ for $\alpha = 2,3$, then $\|xu\|_{L^2(0,T;H^1(\mathbb{R}_x^-))} \leqslant c$, where $c\left(T, \|u_0\|_{L^2(\mathbb{R}_x^-)}, \|\chi^\alpha u_0\|_{L^2(\mathbb{R}_x^-)}\right)$. Moreover,*

$$\int_0^T \int_{x_0}^{x_0+1} (\partial_x u)^2 dx dt \leqslant c\left(T, \|u_0\|_{L^2(\mathbb{R}_x^-)}\right),$$

for any $x_0 \in (-\infty, 0]$.

3.2. **Controllability: Linear system.** The first lemma gives an optimality condition that will be paramount for our analysis.

Lemma 3.5. *Consider the initial data $\phi \in L^2(\mathbb{R}_x^+)$. Then, the linear system associated to (1.1) is exactly controllable if and only if there exists $f \in H^{\frac{1}{3}}(0,T)$ such that*

$$\left\langle f(\cdot), \partial_x^2 \varphi(0,\cdot) \right\rangle_{H^{1/3}(0,T), H^{-1/3}(0,T)} = \int_0^\infty \phi \varphi(0) dx - \int_0^\infty u(T) \varphi_T dx, \tag{3.6}$$

for all $\varphi_T \in L^2(0,\infty)$ and φ is the solution of the backward system (3.1).

Proof. To prove this result multiply the system (1.1) by φ, solution of the backward system (3.1), and integrate by parts in $\mathbb{R}_x^+ \times (0,T)$. $\square$

The optimality condition (3.6) guarantee that the critical points of the functional

$$\mathcal{J} : L^2(0,\infty) \to \mathbb{R},$$

defined by

$$\mathcal{J}(\varphi_T) = \frac{1}{2}\left\|\partial_x^2\varphi(0,\cdot)\right\|^2_{H^{-1/3}(0,T)} + \int_0^\infty u(T)\varphi_T dx - \int_0^\infty \phi\varphi(0)dx, \tag{3.7}$$

where φ is the solution of (3.2) with final data $\varphi_T \in L^2(\mathbb{R}_x^+)$, is the control that drives my initial data to my final data, precisely, we have the following classical result.

Proposition 3.6. *Let $\phi \in L^2(\mathbb{R}_x^+)$ and suppose that $\widehat{\varphi}_T \in L^2(\mathbb{R}_x^+)$ is a minimizer of $\mathcal{J}$. If $\widehat{\varphi}$ is the corresponding solution of (3.1) with final data $\widehat{\varphi}_T$ then $f(t) = \widehat{\partial_x^2\varphi}(0,t)$ is a desired control.*

Let us now give a general condition that ensures the existence of a minimizer for $\mathcal{J}$. To prove the previous result, we first establish an observability inequality associated with the solutions of the system (3.1).

Proposition 3.7. *For any $T > 0$, there exists a constant $C(T,L) > 0$, such that*

$$\|\varphi_T\|^2_{L^2(\mathbb{R}_x^+)} \leq C\left\|\partial_x^2\varphi(0,\cdot)\right\|^2_{H^{-\frac{1}{3}}(0,T)}, \tag{3.8}$$

for any $\varphi_T \in L^2(\mathbb{R}_x^+)$, where φ is the solution of the backward system (3.1).

Proof. We proceed in a standard way (see, e.g., [38, Lemma 3.5]). Let us suppose that (3.8) does not hold. In this case, it follows that there exists a sequence $\{\varphi_{n,T}\}_{n\in\mathbb{N}}$ of final data, such that

$$1 = \|\varphi_{n,T}\|^2_{L^2(\mathbb{R}_x^+)} \geq n\ \left\|\partial_x^2\varphi_n(0,\cdot)\right\|^2_{H^{-\frac{1}{3}}(0,T)}, \tag{3.9}$$

where, for each $n \in \mathbb{N}$, $\{\varphi_n\}_{n\in\mathbb{N}}$ is the solution of (3.1). Inequality (3.9) imply that

$$\partial_x^2\varphi_n(0,\cdot) \to 0 \quad \text{in} \quad H^{-\frac{1}{3}}(0,T).$$

Moreover, from (3.3), Theorem 3.4 and (3.9), we obtain that the sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1_{loc}(\mathbb{R}_x^+))$. On the other hand, the adjoint system implies that $\{\partial_t\varphi_n\}_{n\in\mathbb{N}}$, is bounded in $L^2(0,T;H^{-2}_{loc}(\mathbb{R}_x^+))$, and the compact embedding

$$H^1_{loc}(\mathbb{R}_x^+) \hookrightarrow_{cc} L^2_{loc}(\mathbb{R}_x^+) \hookrightarrow H^{-2}_{loc}(\mathbb{R}_x^+),$$

allows us to conclude that $\{\varphi_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2_{loc}(\mathbb{R}_x^+))$ and consequently, we obtain a subsequence, still denoted by the same index n, satisfying

$$\varphi_n \to \varphi \text{ in } L^2(0,T;L^2_{loc}(\mathbb{R}_x^+)), \text{ as } n \to \infty.$$

Furthermore, the hidden regularity given by (3.4) implies that $\{\varphi_n(0,\cdot)\}_{n\in\mathbb{N}}$ is bounded in $H^{\frac{1}{3}}(0,T)$. Then, the embedding $H^{\frac{1}{3}}(0,T) \hookrightarrow_{cc} L^2(0,T)$ ensures that the above sequences are relatively compact in $L^2(0,T)$. Thus, we obtain a subsequence, still denoted by the same index n, satisfying

$$\varphi_n(0,\cdot) \to \varphi(0,\cdot), \quad \text{in} \quad L^2(0,T).$$

From the boundary condition of the adjoint system, we deduce that $\varphi(0,\cdot) = 0$. In addition, according to Proposition 3.3, we have

$$T\|\varphi_{n,T}\|^2_{L^2(\mathbb{R}_x^+)} = \int_0^T \|\varphi_n(t)\|^2_{L^2(\mathbb{R}_x^+)}dt.$$

Then, if follows that $\{\varphi_{n,T}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}_x^+)$. Thus,

$$\varphi_{n,T} \to \varphi_T \text{ in } L^2(\mathbb{R}_x^+), \text{ as } n \to \infty, \tag{3.10}$$

which implies that

$$\|\varphi_T\|_{L^2(\mathbb{R}_x^+)} = 1. \tag{3.11}$$

On the other hand, by using (3.5), (3.10) and (3.9), we see that

$$\partial_x\varphi_n(0,\cdot)\to\partial_x\varphi(0,\cdot) \quad \text{in } L^2(0,T), \text{ as } n\to\infty,$$

and

$$\partial_x^2\varphi_n(0,\cdot)\to\partial_x^2\varphi(0,\cdot) \quad \text{in } H^{-\frac{1}{3}}(0,T), \text{ as } n\to\infty.$$

Finally, taking $n\to\infty$, from above converges, we obtain that φ is solution of the system

$$\begin{cases}\partial_t\varphi+\partial_x\varphi+\partial_x^3\varphi=0, & \text{for } (x,t)\in(0,\infty)\times(0,T),\\ \varphi(0,t)=\partial_x\varphi(0,t)=\partial_x^2\varphi(0,t)=0, & \text{for } t\in(0,T).\end{cases}$$

Note that in this situation, we have $\varphi\equiv 0$ because of the unique continuation property ($\varphi(0,t)=\partial_x\varphi(0,t)=\partial_x^2\varphi(0,t)=0,\ t\in(0,T)$), which is a contradiction with (3.11), showing the result. □

We are in a position to present the controllability result for the linearized system associated with (1.1). Indeed, the linear functional (3.7) is continuous and convex. It is evident from Proposition 3.7 that the functional is coercive. Consequently, a minimizer for $\mathcal{J}$ exists. Therefore, based on Lemma 3.5, Proposition 3.6, Lemmas B.4 and B.6, the following theorem is verified.

Theorem 3.8. *Let $T>0$, $\phi,\phi_T\in L^2(\mathbb{R}_x^+)$. Then, there exist $f\in H^{\frac{1}{3}}(\mathbb{R}_t^+)$ such that the distributional solution u of the linear system (1.11) satisfies $u(x,T)=\phi_T(x)$ for $x\in(0,\infty)$. Moreover, the following estimates hold*

$$\|u\|_{C(\mathbb{R}_t^+,L^2(\mathbb{R}_x^+))}\le C\left(\|\phi\|_{L^2(\mathbb{R}_x^+)}+\|f\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\right).$$

3.3. **Controllability: Nonlinear system.** Let $T>0$, thanks to the Theorem 3.8, we can define the bounded linear operator

$$\begin{array}{rccc}\Lambda_l: & L^2(\mathbb{R}_x^+)\times L^2(\mathbb{R}_x^+) & \longrightarrow & H^{\frac{1}{3}}(\mathbb{R}_t^+)\\ & (\phi,\phi_T) & \longrightarrow & \Lambda_l(\phi,\phi_T)=f\end{array}$$

where f is the control defined in Proposition 3.6. Now, we are in a position to prove one of the main results of the work.

3.3.1. **Proof of Theorem 1.1.** We treat the nonlinear problem using a classical fixed-point argument. According to (2.3) the solution of (1.1) can be written as

$$u(x,t)=\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\theta(t)\mathscr{D}\partial_xu^2(x,t)+\theta(t)\mathscr{L}_+^\lambda h(x,t),$$

with

$$h(t)=e^{-\pi i\lambda}\left[f(t)-\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta(t)\mathscr{D}\partial_xu^2(0,t)\right].$$

Consider the map

$$\Gamma(u):=\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\theta(t)\mathscr{D}\partial_xu^2(x,t)+\theta(t)\mathscr{L}_+^\lambda\widehat{h}_T(x,t),$$

where

$$\widehat{h}_T(t)=e^{-\pi i\lambda}\left[\Lambda_l\left(\phi,\phi_T+\frac{1}{2}\theta(T)\mathscr{D}\partial_xu^2(x,T)\right)(t)-\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta(t)\mathscr{D}\partial_xu^2(0,t)\right].$$

If we choose

$$f(t)=\Lambda_l\left(\phi,\phi_T+\frac{1}{2}\theta(T)\mathscr{D}\partial_xu^2(x,T)\right)(t) \tag{3.12}$$

from Theorem 3.8, we get

$$\Gamma(u)|_{t=0}=\phi \quad \text{and} \quad \Gamma(u)|_{t=T}=\phi_T.$$

The next steps are devoted to proving that the map Γ is a contraction in an appropriate metric space, then its fixed point u is the solution of (1.1), with f defined by (3.12). To prove the existence of the fixed point, we apply the Banach fixed-point theorem to the restriction of Γ on the closed ball

$$B_r = \left\{ u \in X_{0,b} \cap D_\alpha : \|u\|_{X_{0,b}\cap D_\alpha} \leq r \right\},$$

for some $r > 0$. In the sequel, C denotes a generic positive constant; C_0, C_1, etc, and others positive (specific) constants.

(i) **Γ maps B_r into itself.**

Indeed,using Lemmas B.4, B.5 and B.6, there exists a constant $C_>0$, such that

$$\begin{aligned}
\|\Gamma(u)\|_{X_{0,b}\cap D_\alpha} \leq& \left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)\right\|_{X_{0,b}\cap D_\alpha} + \left\|\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t)\right\|_{X_{0,b}\cap D_\alpha} \\
&+ \left\|\theta(t)\mathscr{L}_+^\lambda \widehat{h}_T(x,t)\right\|_{X_{0,b}\cap D_\alpha} \\
\leq& C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)} \|\phi\|_{L^2(\mathbb{R}_x^+)} + \left\|\partial_x u^2\right\|_{X_{0,-b}} + \|\widehat{h}_T\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \right).
\end{aligned}$$

From [29, Lemma 5.10], it follows that

$$\left\|\partial_x u^2\right\|_{X_{0,-b}} \leq C\,\|u\|^2_{X_{0,b}\cap D_\alpha} \leq Cr^2.$$

Thus,

$$\begin{aligned}
\|\widehat{h}_T\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \leq& \left\|\Lambda_l\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \\
&+ \left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} + \left\|\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \\
\leq& \|\Lambda_l\|\left(\|\phi\|_{L^2(\mathbb{R}_x^+)} + \left\|\phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right\|_{L^2(\mathbb{R}_x^+)} \right) \\
&+ C\,\|\phi\|_{L^2(\mathbb{R}_x^+)} + 2C\,\|u\|^2_{X_{0,b}\cap D_\alpha} \\
\leq& \|\Lambda_l\|\left(\|\phi\|_{L^2(\mathbb{R}_x^+)} + \|\phi_T\|_{L^2(\mathbb{R}_x^+)}\right) + C\,\|\phi\|_{L^2(\mathbb{R}_x^+)} + 2C\left(1+\|\Lambda_l\|\right)\|u\|^2_{X_{0,b}\cap D_\alpha} \\
\leq& \|\Lambda_l\|\,\delta + C\delta + 2C\left(1+\|\Lambda_l\|\right)r^2.
\end{aligned}$$

Finally, we have that

$$\|\Gamma(u)\|_{X_{s,b}\cap D_\alpha} \leq C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)} + \|\Lambda_l\| + C\right)\delta + C^2\left(3+\|\Lambda_l\|\right)r^2.$$

To obtain (i), we take δ with

$$\delta = \min\left\{ \frac{r}{2C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)} + \|\Lambda_l\| + C\right)}, \frac{1}{4C^3\left(\|\theta\|_{H^1(\mathbb{R}_t^+)} + \|\Lambda_l\| + C\right)\left(3+\|\Lambda_l\|\right)} \right\}.$$

Hence, it follows that

$$C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)} + \|\Lambda_l\| + C\right)\delta + C^2\left(3+\|\Lambda_l\|\right)r^2 \leq r.$$

(ii) **Γ is a contraction.**

Let $u, v \in B_r$, and consider

$$\Gamma(u) - \Gamma(v) = -\frac{1}{2}\theta(t)\mathscr{D}\partial_x\left(u^2(x,t) - v^2(x,t)\right) + \theta(t)\mathscr{L}_+^\lambda\left(\widehat{h}_T^u(x,t) - \widehat{h}_T^v(x,t)\right).$$

Here,

$$\widehat{h}_T^u(t) = e^{-\pi i\lambda}\left[\Lambda_l\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t) - \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right]$$

and

$$\widehat{h}_T^v(t) = e^{-\pi i\lambda}\left[\Lambda_l\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x v^2(x,T)\right)(t) - \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x v^2(0,t)\right].$$

Note that

$$\widehat{h}_T^u(t) - \widehat{h}_T^v(t) = e^{-\pi i\lambda}\left[\Lambda_l\left(0, \frac{1}{2}\theta(T)\mathscr{D}\partial_x\left(u^2(x,T) - v^2(x,T)\right)\right)(t) + \frac{1}{2}\theta(t)\mathscr{D}\partial_x\left(u^2(0,t) - v^2(0,t)\right)\right].$$

Thus, we get

$$\begin{aligned}
\|\Gamma(u) - \Gamma(v)\|_{X_{0,b}\cap D_\alpha} \leq & \left\|\frac{1}{2}\theta(t)\mathscr{D}\partial_x\left(u^2(x,t) - v^2(x,t)\right)\right\|_{X_{0,b}\cap D_\alpha} \\
& + \left\|\theta(t)\mathscr{L}_+^\lambda\left(\widehat{h}_T^u(x,t) - \widehat{h}_T^v(x,t)\right)\right\|_{X_{0,b}\cap D_\alpha} \\
\leq & C\left\|\partial_x(u+v)(u-v)\right\|_{X_{s,-b}} + C\left\|\widehat{h}_T^u(t) - \widehat{h}_T^v(t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \\
\leq & 2C^2\|u+v\|_{X_{0,b}\cap D_\alpha}\|u-v\|_{X_{0,b}\cap D_\alpha} \\
& + C^2(2 + \|\Lambda_l\|)\|u+v\|_{X_{0,b}\cap D_\alpha}\|u-v\|_{X_{0,b}\cap D_\alpha} \\
\leq & 2C^2(4 + \|\Lambda_l\|)\, r\, \|u-v\|_{X_{0,b}\cap D_\alpha},
\end{aligned}$$

and taking r such that $2C^2(4 + \|\Lambda_l\|)\, r < 1$, (ii) follows.

Therefore, the map Γ is a contraction. Thus, from (i), (ii), and the Banach fixed-point theorem, Γ has a fixed point in B_r, and its fixed point is the desired solution. The proof of Theorem 1.1 is, thus, complete. □

4. Exact Controllability: The left half-line

In this section, our primary focus is on achieving exact controllability for the system (1.2) and establishing the validity of Theorem 1.2. Firstly, consider the homogeneous linear system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (-\infty, 0)\times(0,T), \\ u(0,t) = \partial_x u(0,t) = 0, & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (-\infty, 0), \end{cases}$$

whose adjoint associated system is given by

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_x^3 \varphi = 0, & \text{for } (x,t) \in (-\infty, 0)\times(0,T), \\ \varphi(0,t) = 0, & \text{for } t \in (0,T), \\ \varphi(x,T) = \varphi_T(x), & \text{for } x \in (-\infty, 0). \end{cases} \tag{4.1}$$

Remark that the change of variable $x = -x$ and $t = T - t$ reduces system in

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_x^3 \varphi = 0, & \text{for } (x,t) \in (0,\infty)\times(0,T), \\ \varphi(0,t) = 0, & \text{for } t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), & \text{for } x \in (0,\infty). \end{cases} \tag{4.2}$$

The well-posedness of system (4.2) follows from Theorem 2.8, when $-\frac{3}{4} < s < \frac{3}{2}$ and $s \neq \frac{1}{2}$. Note that this system takes the form

$$\begin{cases} \varphi_t = B\varphi, \\ \varphi(T) = \varphi_T, \end{cases}$$

where the differential operator B is given by

$$\begin{cases} B\varphi = -\partial_x\varphi - \partial_x^3\varphi, \\ D(B) := \left\{\varphi \in H^3(\mathbb{R}_x^-) : \varphi(0) = 0\right\} = H^3(\mathbb{R}_x^-) \cap H_0^1(\mathbb{R}_x^-). \end{cases}$$

Similarly as Propositions 3.2 and 3.3, we have the following result.

Proposition 4.1. *Consider the initial data $\varphi_T \in L^2(\mathbb{R}_x^-)$. Then, there exists a unique mild solution $\varphi(t) = S(T-t)\varphi_T$ of (4.2) such that $\varphi \in C\left([0,T]; L^2(\mathbb{R}_x^-)\right)$. Moreover, if $\varphi_T \in D(B)$, the system (4.2) has a unique (classical) solution φ in the class*

$$\varphi \in C([0,T]; D(B)) \cap C^1\left(0,T; L^2(\mathbb{R}_x^-)\right)$$

and satisfies

$$\|\varphi_x(0,\cdot)\|_{L^2(0,T)} \leq \|\varphi_T\|_{L^2(\mathbb{R}_x^-)}.$$

4.1. **Linear control results: Neumann and Dirichlet cases.** We consider the linearized system associated with (1.2), with the presence of one control acting in the Neumann boundary condition

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (-\infty,0)\times(0,T), \\ u(0,t) = 0, \quad \partial_x u(0,t) = g_2(t), & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (-\infty,0), \end{cases} \tag{4.3}$$

and with one control acting in the Dirichlet boundary condition

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (-\infty,0)\times(0,T), \\ u(0,t) = g_1(t), \quad \partial_x u(0,t) = 0, & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (-\infty,0). \end{cases} \tag{4.4}$$

Now, we will first prove that the system (4.3) is exactly controllable. Here, we will follow the same steps as done in the previous section, the control result is achieved if an observability inequality is shown. The observability inequality is given in the next proposition.

Proposition 4.2. *For any $T>0$, there exists a constant $C(T,L)>0$, such that,*

$$\|\varphi_T\|^2_{L^2(\mathbb{R}_x^-)} \leq C\,\|\partial_x\varphi(0,\cdot)\|^2_{L^2(0,T)}, \tag{4.5}$$

for any $\varphi_T \in L^2(\mathbb{R}_x^-)$, where $\varphi(x,t)$ is the solution of the backward system (4.1).

Proof. Let us argue by contradiction. Supposing that the observability inequality (4.5) does not hold, it follows that there exists a sequence $\{\varphi_{n,T}\}_{n\in\mathbb{N}}$, such that

$$1 = \|\varphi_{n,T}\|^2_{L^2(\mathbb{R}_x^-)} \geq n\ \|\partial_x\varphi_n(0,\cdot)\|^2_{L^2(0,T)} \tag{4.6}$$

where, for each $n\in\mathbb{N}$, $\{\varphi_n\}_{n\in\mathbb{N}}$ is the solution of (4.1). Inequality (4.6) imply that

$$\partial_x\varphi_n(0,\cdot) \to 0 \quad \text{in} \quad L^2(0,T).$$

Moreover, it is important to note that from Lemmas B.4 and B.6 together with the formula solution (2.3), we obtain the estimates (3.3), (3.4), and (3.5) for the adjoint system (4.1). Furthermore, from Theorem A.1 and (4.6), we obtain a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ bounded in $L^2(0,T;H^1_{loc}(\mathbb{R}_x^-))$. On the other hand, the adjoint system implies that $\{\partial_t\varphi_n\}_{n\in\mathbb{N}}$, is bounded in $L^2(0,T;H^{-2}_{loc}(\mathbb{R}_x^-))$, and the compact embedding

$$H^1_{loc}(\mathbb{R}_x^-) \hookrightarrow_{cc} L^2_{loc}(\mathbb{R}_x^-) \hookrightarrow H^{-2}_{loc}(\mathbb{R}_x^-),$$

allows us to conclude that $\{\varphi_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2_{loc}(\mathbb{R}_x^-))$ and consequently, we obtain a subsequence, still denoted by the same index n, satisfying

$$\varphi_n \to \varphi \text{ in } L^2(0,T;L^2_{loc}(0,\infty)), \text{ as } n\to\infty.$$

Furthermore, the trace estimate of the system (4.1) implies that $\{\varphi_n(0,\cdot)\}_{n\in\mathbb{N}}$ is bounded in $H^{\frac{1}{3}}(0,T)$. Then, the embedding $H^{\frac{1}{3}}(0,T) \hookrightarrow_{cc} L^2(0,T)$, guarantees that the above sequences are relatively compact in $L^2(0,T)$. Thus, we obtain a subsequence, still denoted by the same index n, satisfying

$$\varphi_n(0,\cdot) \to \varphi(0,\cdot), \quad \text{in} \quad L^2(0,T).$$

From the boundary condition of the adjoint system, we deduce that $\varphi(0,\cdot)=0$. Additionally, multiplying the first equation of (4.1) by $t\varphi$ and integrating by parts in $(0,T)\times(-\infty,0)$, we deduce that

$$T\|\varphi_{n,T}\|^2_{L^2(\mathbb{R}^-_x)}=\int_0^T\|\varphi_n(t)\|^2_{L^2(\mathbb{R}^-_x)}dt+\int_0^T\frac{t}{2}|\varphi_{x,n}(0,t)|^2dt.$$

So, if follows that $\{\varphi_{n,T}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,\infty)$. Thus,

$$\varphi_{n,T}\to\varphi_T \text{ in } L^2(\mathbb{R}^-_x),\ \text{ as } n\to\infty, \tag{4.7}$$

which implies that

$$\|\varphi_T\|_{L^2(\mathbb{R}^-_x)}=1. \tag{4.8}$$

On the other hand, by using the derivative traces estimation of the system (4.1), (4.7) and (4.6), we see that

$$\partial_x\varphi_n(0,\cdot)\to\partial_x\varphi(0,\cdot)\quad \text{ in } L^2(0,T),\ \text{ as } n\to\infty,$$

and

$$\partial_x^2\varphi_n(0,\cdot)\to\partial_x^2\varphi(0,\cdot)\quad \text{ in } H^{-\frac{1}{3}}(0,T),\ \text{ as } n\to\infty.$$

Finally, taking $n\to\infty$, from above converges, we obtain that φ is solution of the system

$$\begin{cases}\partial_t\varphi+\partial_x\varphi+\partial_x^3\varphi=0, & \text{for } (x,t)\in(-\infty,0)\times(0,T),\\ \varphi(0,t)=\partial_x\varphi(0,t)=0, & \text{for } t\in(0,T),\\ \varphi(x,T)=\varphi_T(x), & \text{for } x\in(-\infty,0),\end{cases}$$

or equivalently,

$$\begin{cases}\partial_t\varphi+\partial_x\varphi+\partial_x^3\varphi=0, & \text{for } (x,t)\in(0,\infty)\times(0,T),\\ \varphi(0,t)=\partial_x\varphi(0,t)=0, & \text{for } t\in(0,T),\\ \varphi(x,0)=\varphi_0(x), & \text{for } x\in(0,\infty).\end{cases}$$

Notice that (4.8) implies that the solutions can not be identically zero. However, from the following Lemma, one can conclude that $\varphi=0$, which drives us to contradict (4.8). □

Lemma 4.3. *For any $T>0$, let N_T denote the space of the initial states $\varphi_0\in L^2(\mathbb{R}^-_x)$, such that the solution of (4.2) satisfies $\partial_x\varphi(0,\cdot)=0$. Then, $N_T=\{0\}$.*

Proof. The proof uses the same arguments as those given in [38], that is if $N_T\neq\{0\}$, the map $\varphi_T\in N_T\to B(\varphi_T)\in\mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue.Hence, there exist $\lambda\in\mathbb{C}$ and $\varphi_0\in H^3(\mathbb{R}^-_x)\backslash\{0\}$, such that

$$\begin{cases}\lambda\varphi_T+\varphi_T'+\varphi_T'''=0, & \text{in } \mathbb{R}^-_x,\\ \varphi_T(0)=\varphi_T'(0)=0.\end{cases} \tag{4.9}$$

To conclude the proof of the Lemma 4.3, we prove that this does not hold. To simplify the notation, henceforth we denote $\varphi_T:=\varphi$. Consider

$$\begin{cases}\lambda\varphi+\varphi'+\varphi'''=0, & \text{in } (-\infty,0),\\ \varphi(0)=\varphi'(0)=0,\end{cases}$$

with $\varphi\neq0$. Note that if $\varphi''(0)=0$, then $\varphi\equiv0$. Otherwise, we use an argument similar to the one used in [38, Lemma 3.5]. Let us introduce the notation $\hat{\varphi}(\xi)=\int_{-\infty}^0 \mathrm{e}^{-ix\xi}\varphi(x)\mathrm{d}x$. Note that, the above representation has the following properties:

$$\widehat{\varphi'}(\xi)=\int_{-\infty}^0\mathrm{e}^{-ix\xi}\varphi'(x)\mathrm{d}x=i\xi\widehat{\psi}+[e^{-ix\xi}\varphi]_{x=-\infty}^{x=0}=i\xi\widehat{\varphi}+\varphi(0)=i\xi\widehat{\varphi},$$

and

$$\begin{aligned}\widehat{\varphi'''}(\xi)&=\int_{-\infty}^0\mathrm{e}^{-ix\xi}\varphi'''(x)\mathrm{d}x=i\xi^3\widehat{\varphi}+[-\xi^2e^{-ix\xi}\varphi+i\xi e^{-ix\xi}\varphi'+e^{-ix\xi}\varphi'']_{x=-\infty}^{x=0}\\&=-i\xi^3\widehat{\varphi}-\xi^2\varphi(0)+i\xi\varphi'(0)+\varphi''(0)=-i\xi^3\widehat{\varphi}+\varphi''(0).\end{aligned}$$

Then, multiplying the equation in (4.9) by $e^{-ix\xi}$ and integrating by part in $(-\infty,0)$ yields

$$\widehat{\varphi}(\xi)=\frac{\varphi''(0)}{\lambda-i\xi-i\xi^3}. \tag{4.10}$$

Using Paley-Wiener theorem (see, for instance, [48, Section 4, p. 161]) and the usual characterization of $H^3(-\infty,0)$ functions using their Fourier transforms, we see that nontrivial solution of (4.9) is equivalent to the existence of $\lambda\in\mathbb{C}$, such that

(i) $\widehat{\varphi}$ is a entire functions in $\mathbb{C}$,
(ii) $\int_{\mathbb{R}}|\widehat{\varphi}(\xi)|^2\left(1+|\xi|^2\right)^2\, d\xi<\infty$,
(iii) $\forall\xi\in\mathbb{C}$, we have that $\widehat{\varphi}|(\xi)|\leq c_1(1+|\xi|)^k e^{L|\operatorname{Im}\xi|}$, for some positive constant c_1.

Notice that if (i) holds, then (ii) and (iii) are satisfied, however, since $\widehat{\varphi}$ is given by (4.10), it can not be an entire function. □

As usual in the control theory (see the previous section), with the previous observability inequality in hands, the following controllability result for the linearized system holds.

Theorem 4.4. *Let $T>0$, $\phi,\phi_T\in L^2(\mathbb{R}_x^-)$. Then, there exist $g_2\in L^2(\mathbb{R}_t^+)$ such that the distributional solution u of (4.3) satisfies that $u(x,T)=\phi_T(x)$ for $x\in\mathbb{R}_x^-$. Moreover, the following estimates hold*

$$\|u\|_{C(\mathbb{R}_t^+,L^2(\mathbb{R}_x^-))}\leq C\left(\|\phi\|_{L^2(\mathbb{R}_x^-)}+\|g_2\|_{L^2(\mathbb{R}_t^+)}\right).$$

Remark 3. *Note that similar results can be obtained for the system (4.4). Indeed, we have the following observability inequality for the solution of the adjoint system associated with the system (4.4):*

$$\|\varphi_T\|^2_{L^2(\mathbb{R}_x^-)}\leq C\left\|\partial_x^2\varphi(0,\cdot)\right\|^2_{H^{-\frac{1}{3}}(0,T)},$$

for any $\varphi_T\in L^2(\mathbb{R}_x^-)$, where φ is the solution of the backward system (4.1). So, with this in hand, the following holds:

Theorem 4.5. *Let $T>0$, $\phi,\phi_T\in L^2(\mathbb{R}_x^-)$. Then, there exist $g_1\in H^{\frac{1}{3}}(\mathbb{R}_t^+)$ such that the distributional solution u of (4.4) satisfies that $u(x,T)=\phi_T(x)$ for $x\in\mathbb{R}_x^-$. Moreover, the following estimates hold*

$$\|u\|_{C(\mathbb{R}_t^+,L^2(\mathbb{R}_x^-))}\leq C\left(\|\phi\|_{L^2(\mathbb{R}_x^-)}+\|g_1\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\right).$$

4.2. **Controllability of the nonlinear systems.** Let $T>0$, from Theorems 4.4 and 4.5, we can define the bounded linear operators

$$\begin{array}{rccc}\Lambda_{r,D}: & L^2(\mathbb{R}_x^+)\times L^2(\mathbb{R}_x^+) & \longrightarrow & L^2(\mathbb{R}_t^+)\\ & (\phi,\phi_T) & \longrightarrow & \Lambda_{r,D}(\phi,\phi_T)=g_1\end{array}$$

and

$$\begin{array}{rccc}\Lambda_{r,N}: & L^2(\mathbb{R}_x^+)\times L^2(\mathbb{R}_x^+) & \longrightarrow & L^2(\mathbb{R}_t^+)\\ & (\phi,\phi_T) & \longrightarrow & \Lambda_{r,N}(\phi,\phi_T)=g_2,\end{array}$$

where g_1 and g_2 are the controls of the linear system (4.3) and (4.4), respectively. Now, we are in a position to prove Theorem 1.2.

4.2.1. **Proof of Theorem 1.2.** Since the proof of this theorem is analogous as done before in Theorem 1.1, that is, the nonlinear problem is treated using a classical fixed-point argument, for the sake of completeness we will give the sketch of the proof. According to (2.4) the solution of (1.2) can be written as

$$u(x,t)=\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)-\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t)+\theta(t)\mathscr{L}_-^{\lambda_1}h_1(x,t,\phi)+\theta(t)\mathscr{L}_-^{\lambda_2}h_2(x,t,\phi),$$

with

$$\begin{bmatrix}h_1(t,\phi)\\ h_2(t,\phi)\end{bmatrix}=M\begin{bmatrix}-\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\\ \theta(t)\mathcal{I}_{1/3}\left(g_2-\theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}+\frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\end{bmatrix},$$

where M is a matrix given by (B.15). Consider the map

$$\Gamma(u) := \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x) - \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t) + \theta(t)\mathscr{L}_-^{\lambda}\widehat{h}_{1,T}(x,t) + \theta(t)\mathscr{L}_-^{\lambda}\widehat{h}_{2,T}(x,t).$$

In this case, $\widehat{h}^u_{1,T}(t) = h_1(t,\varphi)$ and $\widehat{h}^u_{2,T}(t) = h_2(t,\varphi)$ with

$$g_2(t) = \Lambda_{r,N}\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t).$$

Thanks to the Theorem 4.4, we get that

$$\Gamma(u)|_{t=0} = \phi \quad \text{and} \quad \Gamma(u)|_{t=T} = \phi_T.$$

Consider the closed ball

$$B_r = \left\{u \in X_{0,b}\cap D_\alpha : \|u\|_{X_{0,b}\cap D_\alpha} \le r\right\},$$

for some $r > 0$.

Note, first, that Γ maps B_r into itself. Indeed, using Lemmas B.4, B.5 and B.6, there exists a constant $C > 0$, such that

$$\|\Gamma(u)\|_{X_{0,b}\cap D_\alpha} \le C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)}\|\phi\|_{L^2(\mathbb{R}_x^-)} + \left\|\partial_x u^2\right\|_{X_{0,-b}} + \|\widehat{h}^u_{1,T}\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} + \|\widehat{h}^u_{2,T}\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\right).$$

From [29, Lemma 5.10], it follows that

$$\left\|\partial_x u^2\right\|_{X_{0,-b}} \le C\,\|u\|^2_{X_{0,b}\cap D_\alpha} \le Cr^2.$$

Note that from definition of Γ, the function $\widehat{h_1}$ and $\widehat{h}_2$ are given by

$$\begin{aligned}\widehat{h}^u_{1,T}(t) =& a_{1,1}\left(\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right)\\ &+ a_{1,2}\left(\theta\mathcal{I}_{1/3}\left(\Lambda_{r,N}\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t)\right.\right.\\ &\left.\left. - \theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\right)\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\widehat{h}^u_{2,T}(t) =& a_{2,1}\left(\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right)\\ &+ a_{2,2}\left(\theta\mathcal{I}_{1/3}\left(\Lambda_{r,N}\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t)\right.\right.\\ &\left.\left. - \theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\right),\end{aligned} \tag{4.12}$$

where

$$\begin{cases} a_{1,1} = \dfrac{\sin\left(\frac{\pi}{3}\lambda_2 - \frac{\pi}{6}\right)}{2\sqrt{3}\sin\left[\frac{\pi}{3}(\lambda_2-\lambda_1)\right]} & a_{1,2} = -\dfrac{\sin\left(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6}\right)}{2\sqrt{3}\sin\left[\frac{\pi}{3}(\lambda_2-\lambda_1)\right]}\\ a_{2,1} = \dfrac{\sin\left(\frac{\pi}{3}\lambda_1 - \frac{\pi}{6}\right)}{2\sqrt{3}\sin\left[\frac{\pi}{3}(\lambda_2-\lambda_1)\right]} & a_{2,2} = -\dfrac{\sin\left(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6}\right)}{2\sqrt{3}\sin\left[\frac{\pi}{3}(\lambda_2-\lambda_1)\right]}.\end{cases}$$

Thus, for $i = 1, 2$, we get

$$\begin{aligned}\|\widehat{h}_{i,T}\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} \le C&\left(\left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)} + \left\|\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\right.\\ &+ \left\|\mathcal{I}_{1/3}\Lambda_{r,N}\left(\phi, \phi_T + \frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\\ &\left.+ \left\|\mathcal{I}_{1/3}\left(-\theta\;\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\right).\end{aligned}$$

Note that

$$\left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0}\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}+\left\|\frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\leq C\delta+2Cr^2$$

and

$$\left\|\mathcal{I}_{1/3}\Lambda_{r,N}\left(\phi,\phi_T+\frac{1}{2}\theta(T)\mathscr{D}\partial_x u^2(x,T)\right)(t)\right\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\leq\|\Lambda_{r,N}\|\,\delta+2C\,\|\Lambda_{r,N}\|\,r^2.$$

Therefore, we have that

$$\|\widehat{h}_{i,T}\|_{H^{\frac{1}{3}}(\mathbb{R}_t^+)}\leq\left(C+\|\Lambda_{r,N}\|\right)\delta+2C\left(1+\|\Lambda_{r,N}\|\right)r^2.$$

Finally, we have that

$$\|\Gamma(u)\|_{X_{s,b}\cap D_\alpha}\leq C\left(\|\theta\|_{H^1(\mathbb{R}_t^+)}+\|\Lambda_{r,N}\|+2C\right)\delta+C^2\left(6+\|\Lambda_{r,N}\|\right)r^2,$$

showing that Γ maps B_r into itself.

Now on, let us show that Γ is a contractive. To do that, let $u,v\in B_r$, and consider

$$\begin{aligned}\Gamma(u)-\Gamma(v)=&-\frac{1}{2}\theta(t)\mathscr{D}\partial_x\left(u^2(x,t)-v^2(x,t)\right)+\theta(t)\mathscr{L}_-^\lambda\left(\widehat{h}^u_{1,T}(x,t)-\widehat{h}^v_{1,T}(x,t)\right)\\&+\theta(t)\mathscr{L}_-^\lambda\left(\widehat{h}^u_{2,T}(x,t)-\widehat{h}^v_{2,T}(x,t)\right).\end{aligned}$$

Here $\widehat{h}^u_{i,T}$ and $\widehat{h}^v_{i,T}$ for $i=1,2$ are given by (4.11) and (4.12), respectively. Observing that

$$\begin{aligned}\widehat{h}^u_{i,T}(t)-\widehat{h}^v_{i,T}(t)=&a_{i,1}\left(\frac{1}{2}\theta(t)\mathscr{D}\partial_x\left(u^2(0,t)-v^2(0,t)\right)\right)\\&+a_{i,2}\left(\theta\mathcal{I}_{1/3}\left(\Lambda_{r,N}\left(0,\frac{1}{2}\theta(T)\mathscr{D}\partial_x\left(u^2(x,T)-v^2(x,T)\right)\right)(t)\right.\right.\\&\left.\left.+\frac{1}{2}\theta\partial_x\mathscr{D}\partial_x\left(u^2(0,t)-v^2(0,t)\right)\right)(t)\right).\end{aligned}$$

we have

$$\|\Gamma(u)-\Gamma(v)\|_{X_{0,b}\cap D_\alpha}\leq 4C^2\left(5+\|\Lambda\|\right)r\,\|u-v\|_{X_{0,b}\cap D_\alpha},$$

and taking r such that $2C^2\left(4+\|\Lambda\|\right)r<1$, Γ is a contractive. Therefore, the results hold by using Banach fixed-point theorem. The proof of the Theorem is complete. □

5. Further comments and perspectives

Let us present some comments and perspectives on our analysis in this work.

5.1. **Noncritical length phenomenon.** Rosier [38] showed that considering $L\notin\mathcal{N}$, where $\mathcal{N}$ is defined by (1.6), that the associated linear system (1.5) posed on the bounded interval

$$\begin{cases}\partial_t u+\partial_x u+\partial_x^3 u=0, & \text{in } (0,L)\times(0,T),\\ u(0,t)=u(L,t)=0,\ \partial_x u(L,t)=g(t), & \text{in } (0,T),\\ u(x,0)=u_0(x), & \text{in } (0,L),\end{cases}\tag{5.1}$$

is controllable; roughly speaking, if $L\in\mathcal{N}$ system (5.1) is not controllable, that is, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M}=\mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi\in\mathcal{M}$, $g\in L^2(0,T)$ and $u\in C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ satisfying (5.1) and $u(\cdot,0)=0$, one has $u(\cdot,T)\neq\psi$.

Definition 5.1. *A spatial domain $(0,L)$ is called critical for the system (5.1) if its domain length L belongs to critical set $\mathcal{N}$.*

Following the work of Rosier [38], the boundary control system of the KdV equation posed on the finite interval $(0, L)$ with various control inputs has been intensively studied (cf. [8, 11, 13, 17, 18, 26, 27, 28] and see [12, 41] for more complete reviews). Essentially, the critical length phenomenon arises from employing HUM, utilizing a compactness-uniqueness argument to establish certain observability inequalities. Certainly, through a contradiction argument, we have determined that the adjoint system is observable, which allows us to explore a complex function. The critical length phenomenon manifests when this complex function is potentially entire.

Thus, we are interested in analyzing if the critical length phenomenon appears for the KdV equation in unbounded domains. Let us consider the three cases treated in this work.

5.1.1. **The case of the left-half line**. We already know that the critical length phenomenon naturally arises in the case of Neumann boundary control within a bounded domain. However, when considering the left half-line, we can further examine the following control problem

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (-\infty, L) \times (0,T), \\ u(L,t) = 0, \quad \partial_x u(L,t) = g_2(t), & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (-\infty, L). \end{cases} \tag{5.2}$$

By using the HUM, let us recall that the backward system associated with (5.2) is

$$\begin{cases} \partial_t \varphi + \partial_x \varphi + \partial_x^3 \varphi = 0, & \text{for } (x,t) \in (-\infty, L) \times (0,T), \\ \varphi(L,t) = 0, & \text{for } t \in (0,T), \\ \varphi(x,T) = \varphi_T(x), & \text{for } x \in (-\infty, L). \end{cases} \tag{5.3}$$

and its observability inequality is given by

$$\|\varphi_T\|^2_{L^2(-\infty,L)} \le C \|\partial_x \varphi(L,\cdot)\|^2_{L^2(0,T)} .$$

Through the application of the compactness-uniqueness argument, the function in the complex plane associated with the observability inequality is

$$\widehat{\varphi}(\xi) = \frac{e^{-iL\xi}\varphi''(L)}{\lambda - i\xi - i\xi^3}.$$

Note that for any $L > 0$, the function $\widehat{\varphi}$ can not be entire. Thus, we do not have any restriction in the length L.

Also, in the case of the control acting in the Dirichlet condition

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (-\infty, L) \times (0,T), \\ u(L,t) = g_1(t), \quad \partial_x u(L,t) = 0, & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (-\infty, L), \end{cases}$$

the backward system and its observability inequality are given by (5.3) and

$$\|\varphi_T\|^2_{L^2(-\infty,L)} \le C \left\|\partial_x^2 \varphi(L,\cdot)\right\|^2_{H^{-\frac{1}{3}}(0,T)} ,$$

respectively. In this case, the function in the plane complex is given by

$$\widehat{\varphi}(\xi) = \frac{\xi e^{-iL\xi}\varphi'(L)}{\xi^3 + \xi + i\lambda},$$

and the function $\widehat{\varphi}$ can not be entire. Again, no restriction in the length L is necessary.

5.1.2. **The case of the right-half line**. Consider the control problem posed on the right half-line

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u = 0, & \text{for } (x,t) \in (L, +\infty) \times (0,T), \\ u(L,t) = f(t), & \text{for } t \in (0,T), \\ u(x,0) = \phi(x), & \text{for } x \in (L, +\infty). \end{cases} . \tag{5.4}$$

Again, using HUM, the backward system associated with (5.4) is

$$\begin{cases} \partial_t\varphi + \partial_x\varphi + \partial_x^3\varphi = 0, & \text{for } (x,t) \in (L,+\infty)\times(0,T),\\ \varphi(L,t) = \partial_x\varphi(L,t) = 0, & \text{for } t\in(0,T),\\ \varphi(x,T) = \varphi_T(x), & \text{for } x \in (L,+\infty), \end{cases}$$

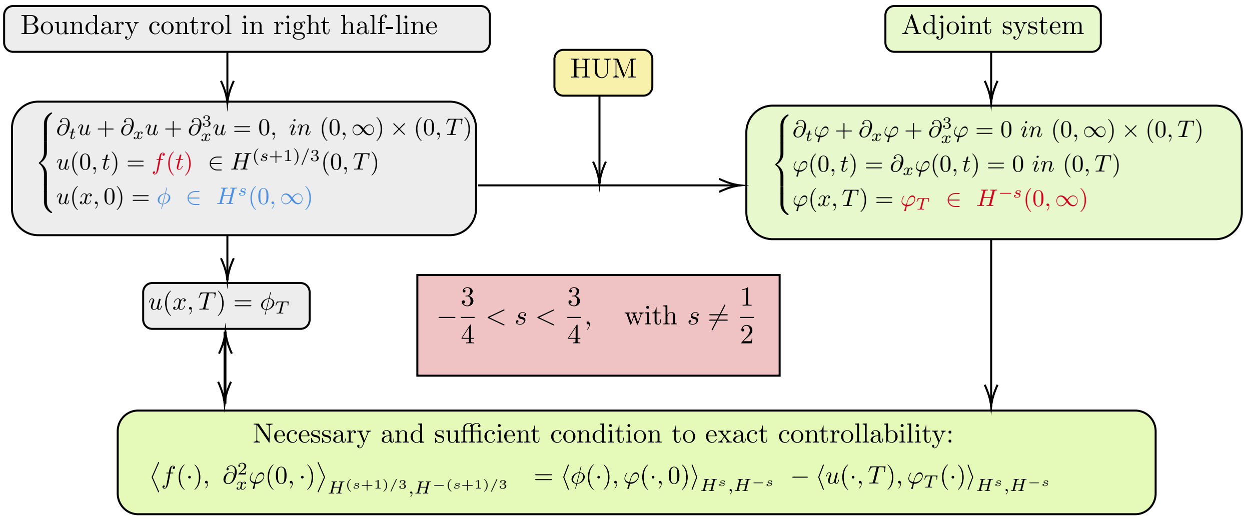
and its observability inequality is given by

$$\|\varphi_T\|^2_{L^2(L,\infty)} \leq C \left\|\partial_x^2\varphi(L,\cdot)\right\|^2_{H^{-\frac{1}{3}}(0,T)}.$$

Through the application of the compactness-uniqueness argument, we do not associate any function in the complex plane, as we directly derive a contradiction from the spectral analysis of the operator.

To summarize, considering all the boundary controllability on the half-line in our work, the critical length phenomenon does not exist, suggesting a behavior completely different compared with the bounded interval, as in the case of the following works [38] and [27].

5.2. **Controllability in H^s.** From the preceding analysis, considering the case $-\frac{3}{4} < s \leq \frac{3}{2}$, with $s \neq \frac{1}{2}$, we can extend our understanding of controllability for the KdV equation for this range of s. By leveraging the well-posedness principles observed in half-line scenarios, we can derive the following scheme



This scheme represents the method of controllability in the right half-line (the left half-line can be done similarly). Note that the necessary and sufficient condition to obtain exact controllability may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : H^{-s}(\mathbb{R}^+_x) \longrightarrow \mathbb{R}$, defined by

$$\mathcal{J}(\varphi_T) = \frac{1}{2}\left\|\partial_x^2\varphi(0,\cdot)\right\|^2_{H^{-(s+1)/3}(0,T)} - \langle\phi(\cdot),\varphi(\cdot,0)\rangle_{H^s,H^{-s}} + \langle u(\cdot,T),\varphi_T(\cdot)\rangle_{H^s,H^{-s}},$$

where φ is the solution of (3.1) with final data $\varphi_T \in H^{-s}(\mathbb{R}^+_x)$. Moreover, we already known that if $\widehat{\varphi}_T \in H^{-s}(\mathbb{R}^+_x)$ is a minimizer of $\mathcal{J}$, with $\widehat{\varphi}$ the corresponding solution of (3.1), with final data $\widehat{\varphi}_T$, then $f(t) = \widehat{\partial_x^2\varphi}(0,t)$ is a desired control. Thus, the observability inequality is given by

$$\|\varphi_T\|^2_{H^{-s}(\mathbb{R}^+_x)} \leq C\left\|\partial_x^2\varphi(0,\cdot)\right\|^2_{H^{-\frac{s+1}{3}}(0,T)},$$

for any $\varphi_T \in H^{-s}(\mathbb{R}^+_x)$, where φ is the solution of the backward system (3.1). To prove the observability inequality, we can utilize several approaches. Our intuition regarding the observability inequality is that it is valid in $-\frac{3}{4} < s \leq \frac{3}{2}$, with $s \neq \frac{1}{2}$, given the favorable trace estimates and smoothing effects exhibited by the solution of the backward system, however, this issue in show the observability inequality in H^s, when $-\frac{3}{4} < s \leq \frac{3}{2}$, with $s \neq \frac{1}{2}$, is still an open problem.

Data Availibility. It does not apply to this article as no new data were created or analyzed in this study.

Conflict of interest. This work does not have any conflicts of interest.

References

[1] J. L. Bona, S.-M. Sun, and B.-Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane.* Transactions of the American Mathematical Society, 354(2), (2002), 427–490.

[2] J. L. Bona, S.-M. Sun, and B.-Y. Zhang, *Non-homogeneous boundary value problems for the Korteweg–de Vries and the Korteweg–de Vries–Burgers equations in a quarter plane.* Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 25(6), (2008), 1145–1185.

[3] J. L Bona, S.-M. Sun, and B.-Y. Zhang, *Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications.* Dyn. Partial Differ. Equ., (3), (2006), 1–69.

[4] J. L. Bona R. Winther, *The Korteweg-de Vries equation, posed in a quarter-plane.* SIAM J. Math. Anal. 14(6), (1983), 1056–1106.

[5] J. L. Bona and R. Winther, *The Korteweg-de Vries equation in a quarter plane, continuous dependence results.* Differ. Integral Equ. 2(2), (1989), 228–250.

[6] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II: The KDV-equation.* Geom. Funct. Anal., 3(3), (1993), 209–262.

[7] J. M. Boussinesq, *Thórie de l'intumescence liquide, applelée onde solitaire ou de, translation, se propageant dans un canal rectangulaire.* C. R. Acad. Sci. Paris. 72, (1871), 755–759.

[8] M. A. Caicedo, R. A. Capistrano–Filho, and B.-Y Zhang, *Neumann Boundary Controllability of the Korteweg–de Vries Equation on a Bounded Domain.* SIAM Journal on Control and Optimization. 55, (2017), 3503–3532.

[9] R. A. Capistrano-Filho and J. S. da Silva, *Boundary controllability of the Korteweg-de Vries equation: The Neumann case*, arXiv:2310.04977 [math.AP].

[10] M. Cavalcante, *The Korteweg-de Vries on a metric star graph.* Z. Angew. Math. Phys. 69:(124), (2018).

[11] E. Cerpa, *Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain.* SIAM J. Control Optim., 46 (2007), 877–899.

[12] E. Cerpa, *Control of a Korteweg-de Vries equation: a tutorial.* Math. Control Relat. Fields, 4 (2014), 45–99.

[13] E. Cerpa and E. Crépeau, *Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain.* Ann. I.H. Poincaré, 26 (2009), 457–475.

[14] J. E. Colliander and C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line.*Comm. Partial Differential Equations, 27 (2002), 2187–2266.

[15] E. Cerpa, I. Rivas, and B.-Y. Zhang, *Boundary controllability of the Korteweg-de Vries equation on a bounded domain.* SIAM J. Control Optim., 51:4 (2013), 2976–3010.

[16] J.-M. Coron and E. Crépeau, *Exact boundary controllability of a nonlinear KdV equation with a critical length.* J. Eur. Math. Soc., 6 (2004), 367–398.

[17] J.-M. Coron and E. Crépeau, *Exact boundary controllability of a nonlinear KdV equation with critical lengths.* Journal of the European Mathematical Society, no. 6 (2004), 367-398.

[18] E. Crépeau, *Exact boundary controllability of the Korteweg–de Vries equation around a non-trivial stationary solution.* International Journal of Control, 74:11 (2001), 1096–1106.

[19] S. Dolecki, D.L. Russell, *A general theory of observation and control.* SIAM J. Control Opt. 15 (1977), 185–220.

[20] A. V. Faminskii, *A mixed problem in a semistrip for the Korteweg-de Vries equation and its generalisations.* Din. Sploshn. Sredy, 258, (1988), 54–94.

[21] A. V. Faminskii, *An initial boundary-value problem in a half-strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces.* Comm. Partial Differential Eq., 29 (2004), 1653–1695.

[22] A. V. Faminskii, *Global well-posedness of two initial-boundary-value problems for the Korteweg-de Vries equation.* Differ. Integral Equ. 20(6), (2007), 601–642.

[23] A. S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs.* Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 453 (1997), 1411–1443.

[24] A.S. Fokas, *A unified approach to boundary value problems.* Society for Industrial and Applied Mathematics, (2008).

[25] A. S. Fokas, A. A. Himonas, and D. Mantzavinos, *The Korteweg–de Vries equation on the half-line.* Nonlinearity, 29(2), (2016).

[26] O. Glass and S. Guerrero, *Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit.* Asymptot. Anal., 60 (2008), 61–100.

[27] O. Glass and S. Guerrero, *Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition.* Systems Control Lett., 59 (2010), 390–395.

[28] J.-P. Guilleron, *Null controllability of a linear KdV equation on an interval with special boundary conditions.* Math. Control Signals Syst., 26 (2014), 375–401.

[29] J. Holmer, *The initial-boundary value problem for the Korteweg-de Vries equation.* Comm. Partial Differential Equations, 31, (2006), 1151–1190.
[30] A.A. Himonas and F. Yan, *A higher dispersion KdV equation on the half-line.* J. Differ. Equ. 333, (2022), 55-102.
[31] K. Kalimeris and T. Özsarı, *An elementary proof of the lack of null controllability for the heat equation on the half line.* Applied Mathematics Letters, 104 (2020), 106241.
[32] C. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations.* Indiana Univ. Math. J. 40, (1991), 33–69.
[33] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. 39 (1895), 422–443.
[34] S. N. Kruzhkov and A.V. Faminskii, *Generalized solutions of the Cauchy problem for the Korteweg–de Vries equation.* Mat. Sb., 120, (1984), 396–425; English transl. in: Sb. Math., 48 (2), (1984), 391–421. .
[35] J.-L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systémes Distribués, Tome 1: Contrôlabilité Exacte.* RMA 8, Masson, Paris (1988).
[36] F. Linares and A. F. Pazoto. *Asymptotic behavior of the Korteweg–de Vries equation posed in a quarter plane.* Journal of Differential Equations, 246 (4), (2009), 1342–1353.
[37] A. Pazy, *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, New York, 1983
[38] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain.* ESAIM Control Optim. Cal. Var. 2 (1997), 33–55.
[39] L. Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line.* SIAM J. Control Optim. 39 (2000), 331–351.
[40] L. Rosier, *A fundamental solution supported in a strip for a dispersive equation.* Computational and Applied Mathematics, 21 (2002), 355–367.
[41] L. Rosier and B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: Recent progress.* J. Syst Sci & Complexity, 22 (2009), 647–682.
[42] D. L. Russell, *Computational study of the Korteweg-de Vries equation with localized control action,* Distributed Parameter Control Systems: New Trends and Applications, G. Chen, E. B. Lee, W. Littman, and L. Markus, eds., Marcel Dekker, New York, 1991.
[43] D. L. Russell and B.-Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain.* SIAM J. Control and Optimization, 31 (1993), 659–676.
[44] D. L. Russell and B.-Y. Zhang, *Exact controllability and stabilizability of the Korteweg-de Vries equation.* Trans. Amer. Math. Soc., 348 (1996), 3643–3672.
[45] B. A. Ton, *Initial boundary-value problems for the Korteweg-de Vries equation.* J. Differ. Equ., 25, (1977) 288–309.
[46] B.-Y. Zhang, *Exact boundary controllability of the Korteweg-de Vries equation.* SIAM J. Cont. Optim., 37 (1999), 543–565.
[47] B.-Y. Zhang, *S*ome results for nonlinear dispersive wave equations with applications to control, Ph. D thesis, University of Wisconsin-Madison, 1990.
[48] K. Yosida, *Functional Analysis.* Springer, Berlin (1978).

Appendix A. Auxiliary results

This first appendix is devoted to obtaining additional properties of the solutions of the homogeneous system associated with (1.1) and (1.2), namely

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0, & \text{for } (x,t) \in (0,+\infty)\times(0,T),\\ u(0,t) = 0 & \text{for } t\in(0,T),\\ u(x,0)=\phi(x) & \text{for } x\in(0,+\infty), \end{cases} \tag{A.1}$$

and

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + u\partial_x u = 0, & \text{for } (x,t) \in (-\infty,0)\times(0,T),\\ u(0,t) = 0, \quad \partial_x u(0,t)=0, & \text{for } t\in(0,T),\\ u(x,0)=\phi(x) & \text{for } x\in(-\infty,0). \end{cases} \tag{A.2}$$

Following the ideas contained in [34, Theorem 2.1] and [36, Theorem 2.2], the first result can be read as follows.

Theorem A.1. *Let u be the solution of problem* (A.1) *given by Theorem 2.8. In addition, if $\chi^\alpha u_0 \in L^2(0,\infty)$ for $\alpha = 2,3$, then $\|xu\|_{L^2(0,T;H^1(0,\infty))} \leqslant c$, where $c\left(T, \|u_0\|_{L^2(0,\infty)}, \|\chi^\alpha u_0\|_{L^2(0,\infty)}\right)$. Moreover,*

$$\int_0^T \int_{x_0}^{x_0+1} u_x^2 dxdt \leqslant c\left(T, \|u_0\|_{L^2(0,\infty)}\right)$$

for any $x_0 \in [0,\infty)$.

Proof. The proof is obtained following closely the arguments developed in [34]. Therefore, we will only present the main steps. Let $\psi_0 \in \mathcal{C}^\infty(0,\infty)$ be a nondecreasing function such that $\psi_0(x) = 0$ for $x \leqslant \frac{1}{2}$ and $\psi_0(x) = 1$ for $x \geqslant 1$. For $\alpha \geqslant 0$ we set $\psi_\alpha(x) = x^\alpha \psi_0(x)$ and note that $\psi_\alpha \in \mathcal{C}^\infty(0,\infty)$ and $\psi_\alpha'(x) \geqslant 0$ for any $x \in (0,\infty)$. Multiplying the equation in (A.1) by $u(x,t)\psi_\alpha(x-x_0)$ and integrating by parts over $(0,\infty)$, we get

$$\int_0^\infty u(x,t)u_t(x,t)\psi_\alpha(x-x_0)\,dx = \frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)\,dx,$$

$$\int_0^\infty u_x(x,t)u(x,t)\psi_\alpha(x-x_0)\,dx = -\frac{1}{2}\int_0^\infty u^2(x,t)\psi_\alpha'(x-x_0)\,dx,$$

$$\int_0^\infty u_{xxx}(x,t)u(x,t)\psi_\alpha(x-x_0)\,dx = \frac{3}{2}\int_0^\infty u_x^2(x,t)\psi_\alpha'(x-x_0)\,dx - \frac{1}{2}\int_0^\infty u^2(x,t)\psi_\alpha'''(x-x_0)\,dx,$$

$$\int_0^\infty u_x(x,t)u^2(x,t)\psi_\alpha(x-x_0)\,dx = -\frac{1}{3}\int_0^\infty u^3(x,t)\psi_\alpha'(x-x_0)\,dx,$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)\,dx - \frac{1}{2}\int_0^\infty u^2(x,t)\psi_\alpha'(x-x_0)\,dx + \frac{3}{2}\int_0^\infty u_x^2(x,t)\psi_\alpha'(x-x_0)\,dx \\ -\frac{1}{2}\int_0^\infty u^2(x,t)\psi_\alpha'''(x-x_0)\,dx - \frac{1}{3}\int_0^\infty u^3(x,t)\psi_\alpha'(x-x_0)\,dx = 0\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)\,dx &+ \frac{3}{2}\int_0^\infty u_x^2(x,t)\psi_\alpha'(x-x_0)\,dx \\ &\leqslant \frac{1}{3}\sup_{x\in(0,\infty)}\left|u(x,t)\sqrt{\psi_\alpha'(x-x_0)}\right|\int_0^\infty u^2(x,t)\sqrt{\psi_\alpha'(x-x_0)}dx \\ &+\frac{1}{2}\int_0^\infty u^2(x,t)\left\{\psi_\alpha'(x-x_0)+\psi_\alpha'''(x-x_0)\right\}dx\end{aligned}$$

Now, since

$$\sup_{x\in(0,\infty)} v^2(x,t) \leqslant \frac{1}{2}\int_0^\infty |v(x)|\left|v'(x)\right|dx, \quad \forall v \in H^1(0,\infty),$$

thus, letting $v(x) = u(x,t)\sqrt{\psi_\alpha'(x-x_0)}$ we have that

$$\begin{aligned}\sup_{x\in(0,\infty)}\left|u\sqrt{\psi_\alpha'}\right| &\leqslant \frac{1}{\sqrt{2}}\left(\int_0^\infty \left|u\sqrt{\psi_\alpha'}\right|\left|u_x\sqrt{\psi_\alpha'}+\frac{u\psi_\alpha''}{2\sqrt{\psi_\alpha'}}\right|\right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{\sqrt{2}}\left(\int_0^\infty u_x^2\psi_\alpha' dx\right)^{\frac{1}{4}}\left(\int_0^\infty u^2\psi_\alpha' dx\right)^{\frac{1}{4}} + \frac{1}{2}\left(\int_0^\infty u^2\psi_\alpha'' dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty u^2(x,t)\psi_\alpha(x-x_0)\,dx+\frac{3}{2}\int_0^\infty u_x^2(x,t)\psi_\alpha'(x-x_0)\,dx$$
$$\leqslant\frac{1}{6}\left(\int_0^\infty u^2(x,t)\left|\psi_\alpha''(x-x_0)\right|dx\right)^{\frac{1}{2}}\int_0^\infty u^2(x,t)\sqrt{\psi_\alpha'(x-x_0)}dx$$
$$+\frac{1}{3\sqrt{2}}\left(\int_0^\infty u_x^2(x,t)\psi_\alpha'(x-x_0)\,dx\right)^{\frac{1}{4}}\left(\int_0^\infty u^2(x,t)\psi_\alpha'(x-x_0)\,dx\right)^{\frac{1}{4}}\int_0^\infty u^2(x,t)\sqrt{\psi_\alpha'(x-x_0)}dx$$
$$+\frac{1}{2}\int_0^\infty u^2(x,t)\left\{\psi_\alpha'(x-x_0)+\left|\psi_\alpha'''(x-x_0)\right|\right\}dx,$$

and taking the above inequality into account, the result is obtained arguing as in [34, Lemma 2.1 and Theorem 2.2]. □

Analogously as done in the previous theorem, we have the following one.

Theorem A.2. *Let u be the solution of problem (A.2), given by Theorem 2.7. If $\chi^\alpha u_0\in L^2(-\infty)$ for $\alpha=2,3$, then $\|xu\|_{L^2(0,T;H^1(-\infty,0))}\leqslant c$, where $c\left(T,\|u_0\|_{L^2(-\infty,0)},\|\chi^\alpha u_0\|_{L^2(-\infty,0)}\right)$. Moreover,*

$$\int_0^T\int_{x_0}^{x_0+1}u_x^2dxdt\leqslant c\left(T,\|u_0\|_{L^2(-\infty,0)}\right)$$

for any $x_0\in(-\infty,0]$.

The last theorem of this first appendix can be read as follows.

Proposition A.3. *Let u be the solution of problem (A.1) given by Theorem 2.7. Then, for any $T>0$,*

$$\|u\|_{L^\infty(0,T;H^1(0,\infty))}\leqslant C,$$

where $C=C\left(T,\|u_0\|_{H^1(0,\infty)}\right)$ is a positive constant.

Proof. Multiplying the equation in (A.1) by u and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(0,\infty)}=-\frac{1}{2}u_x^2(0,t)\leq 0.$$

Consequently, we deduce that

$$\|u(t)\|^2_{L^2(0,\infty)}\leqslant\|u_0\|^2_{L^2(0,\infty)},\quad\forall t>0. \tag{A.3}$$

Now, we multiply the equation in (A.1) by $-2u_{xx}-u^2$ to bound u in $L^2\left(0,T;H_0^1(0,\infty)\right)$. Indeed, integrating by parts over $(0,\infty)\times(0,T)$, we get

$$\int_0^\infty u_x^2(x,t)dx-\int_0^\infty u_x^2(x,0)dx+\int_0^t u_x^2(0,s)ds+\int_0^t u_{xx}^2(0,s)ds+\int_0^t\int_0^\infty u_x^3(x,s)dxds$$
$$-\frac{1}{3}\int_0^\infty u^3(x,t)dx+\frac{1}{3}\int_0^\infty u_0^3(x)dx-\int_0^s\int_0^\infty u_x^3(x,s)dxds=0.$$

So,

$$\int_0^\infty u_x^2(x,t)dx\leq\frac{1}{3}\int_0^\infty u^3(x,t)dx-\frac{1}{3}\int_0^\infty u_0^3(x)dx+\int_0^\infty u_{0,x}^2(x)dx. \tag{A.4}$$

The terms on the right-hand side of the above identity may be estimated as

$$\int_0^\infty u^3(x,t)dx\leqslant\frac{C^2}{2}\|u_0\|^4_{L^2(0,\infty)}+\frac{1}{2}\|u(t)\|^2_{H^1(0,\infty)},$$

where $C>0$ denotes the constant given by the Sobolev embedding theorem. A combination of the (A.3) and (A.4) yields

$$\|u(t)\|^2_{H^1(0,\infty)}\leqslant\|u_0\|^2_{L^2(0,\infty)}+\frac{1}{6}\|u(t)\|^2_{H^1(0,\infty)}+\frac{C^2}{6}\|u_0\|^4_{L^2(0,\infty)}+\frac{1}{3}\|u_0\|^3_{H^1(0,\infty)}+\|u_0\|^2_{H^1(0,\infty)}.$$

Then, we have that

$$\|u(t)\|_{H^1(0,\infty)} \leqslant c,$$

for any $t \in [0,T]$, where $c = c\left(\|u_0\|_{H^1(0,\infty)}\right)$ with $c(s) = \sqrt{\frac{C^2}{5}s^4 + \frac{2}{5}s^3 + \frac{12}{5}s^2}$. □

Appendix B. Initial boundary value problem: Half-line cases

B.1. **Boundary forcing operator.** Following the ideas contained in [29], we can establish a formula for the solution of the initial valued problem (1.1). The local existence and uniqueness of solutions of the systems (3.1) and (3.2) is established through Theorem 2.8. Consider a cut-off function $\theta(t) := \theta$, $\theta \in C_0^\infty(\mathbb{R})$. Denote $\theta(t) = \frac{1}{T}\psi\left(\frac{t}{T}\right)$, for $T > 0$, such that

$$\begin{cases} 0 \leq \psi \leq 1, \quad \theta \equiv 1 & \text{on } [0,1], \\ \psi \equiv 0 & \text{for } |t| \geq 2, \end{cases}$$

Let us now give a summary of the Riemann-Liouville fractional integral operator, the reader can see [14, 29] for more details.

Define the function t_+ as follows

$$t_+ = t \quad \text{if} \quad t > 0, \qquad t_+ = 0 \quad \text{if} \quad t \leq 0.$$

The tempered distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function for Re $\alpha > 0$ by

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, f \right\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t)dt.$$

It follows that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k \left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)} \right), \tag{B.1}$$

for all $k \in \mathbb{N}$. Expression (B.1) can be used to extend the definition of $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ to all $\alpha \in \mathbb{C}$ in the sense of distributions. A change of contour shows the Fourier transform of $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is the following one

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^{\widehat{}}(\tau) = e^{-\frac{1}{2}\pi i \alpha}(\tau - i0)^{-\alpha}, \tag{B.2}$$

where $(\tau - i0)^{-\alpha}$ is the distributional limit. For $\alpha \notin \mathbb{Z}$, let us rewrite (B.2) on the following way

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^{\widehat{}}(\tau) = e^{-\frac{1}{2}\alpha\pi i}|\tau|^{-\alpha}\chi_{(0,\infty)} + e^{\frac{1}{2}\alpha\pi i}|\tau|^{-\alpha}\chi_{(-\infty,0)}. \tag{B.3}$$

Note that from (B.2) and (B.3), we have that

$$(\tau - i0)^{-\alpha} = |\tau|^{-\alpha}\chi_{(0,\infty)} + e^{\alpha\pi i}|\tau|^{-\alpha}\chi_{(-\infty,0)}.$$

For $f \in C_0^\infty(\mathbb{R}^+)$, define $\mathcal{I}_\alpha f$ as

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f,$$

so, for Re $\alpha > 0$, follows that

$$\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \tag{B.4}$$

The following properties easily holds $\mathcal{I}_0 f = f$, $\mathcal{I}_1 f(t) = \int_0^t f(s)\, ds$, $\mathcal{I}_{-1} f = f'$ and $\mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}$. Moreover, the lemmas below can be found in [29], and we will omit their proofs.

Lemma B.1. [29, Lemma 2.1] *If $f \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+)$, for all $\alpha \in \mathbb{C}$.*

Lemma B.2. [29, Lemma 5.3] *If* $0 \leq \text{Re } \alpha < \infty$ *and* $s \in \mathbb{R}$*, then* $\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)}$*, where* $c = c(\alpha)$*.*

Lemma B.3. [29, Lemma 5.4] *If* $0 \leq \text{Re } \alpha < \infty$*,* $s \in \mathbb{R}$ *and* $\mu \in C_0^\infty(\mathbb{R})$*, then* $\|\mu \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}$*, where* $c = c(\mu, \alpha)$*.*

B.1.1. **Oscillatory integral.** In this subsection, we will define the oscillatory integral which is the key to defining, in the next section, the Duhamel boundary forcing operator. The Airy function is

$$A(x) = \frac{1}{2\pi} \int_\xi e^{ix\xi} e^{i\xi^3} d\xi \tag{B.5}$$

From [29], the function Airy function has the following properties:

i. $A(x)$ is a smooth function with the asymptotic properties

$$A(x) \sim c_1 x^{-1/4} e^{-c_2 x^{3/2}} \left(1 + O\left(x^{-3/4}\right)\right) \quad \text{as } x \to +\infty$$

and

$$A(-x) \sim c_2 x^{-1/4} \cos\left(c_2 x^{3/2} - \frac{\pi}{4}\right) \left(1 + O\left(x^{-3/4}\right)\right) \quad \text{as } x \to +\infty,$$

where $c_1, c_2 > 0$.

ii. We can compute:

$$A(0) = \frac{1}{2\pi} \int_\xi e^{i\xi^3} d\xi = \frac{1}{6\pi} \int_\eta \eta^{-2/3} e^{i\eta} d\eta = \frac{\frac{\sqrt{3}}{2} \Gamma\left(\frac{1}{3}\right)}{3\pi} = \frac{1}{3\Gamma\left(\frac{2}{3}\right)},$$

$$A'(0) = \frac{1}{2\pi} \int_\xi i\xi e^{i\xi^3} d\xi = -\frac{1}{3\Gamma\left(\frac{1}{3}\right)}, \quad \text{and}$$

$$\int_0^{+\infty} A(y) dy = \frac{1}{3}.$$

On the other hand, consider the following group as

$$e^{-t(\partial + \partial_x^3)} \phi(x) = \frac{1}{2\pi} \int_\xi e^{ix\xi} e^{it(\xi + \xi^3)} \hat{\phi}(\xi) d\xi, \tag{B.6}$$

so that

$$\begin{cases} \left(\partial_t + \partial_x + \partial_x^3\right) \left[e^{-t\partial_x^3} \phi\right](x,t) = 0, & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \left[e^{-t\partial_x^3} \phi\right](x,0) = \phi(x), & \text{for } x \in \mathbb{R}. \end{cases} \tag{B.7}$$

On the other hand, we define the Duhamel inhomogeneous solution operator $\mathscr{D}$ as

$$\mathscr{D} w(x,t) = \int_0^t e^{-(t-t')(\partial_x + \partial_x^3)} w\left(x, t'\right) dt', \tag{B.8}$$

so that

$$\begin{cases} \left(\partial_t + \partial_x + \partial_x^3\right) \mathscr{D} w(x,t) = w(x,t), & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R} \\ \mathscr{D} w(x,0) = 0, & \text{for } x \in \mathbb{R}. \end{cases} \tag{B.9}$$

We now introduce the Duhamel boundary forcing operator (see [14]). For $f \in C_0^\infty\left(\mathbb{R}^+\right)$, let

$$\begin{aligned} \mathscr{L}^0 f(x,t) &= 3 \int_0^t e^{-(t-t')\partial_x^3} \delta_0(x) \mathcal{I}_{-2/3} f\left(t'\right) dt' \\ &= 3 \int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right) \frac{\mathcal{I}_{-2/3} f\left(t'\right)}{(t-t')^{1/3}} dt' \end{aligned} \tag{B.10}$$

so that

$$\begin{cases} \left(\partial_t + \partial_x^3\right) \mathscr{L}^0 f(x,t) = 3\delta_0(x) \mathcal{I}_{-2/3} f(t), & \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \mathscr{L}^0 f(x,0) = 0, & \text{for } x \in \mathbb{R}. \end{cases}$$

We define *the Duhamel boundary forcing operator class* for $\operatorname{Re}\lambda > -3$, and $f \in C_0^\infty(\mathbb{R}^+)$ as

$$\begin{aligned}\mathscr{L}_-^\lambda f(x,t) &= \left[\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \mathscr{L}^0\left(\mathcal{I}_{-\lambda/3}f\right)(-,t)\right](x)\\ &= \frac{1}{\Gamma(\lambda)}\int_{-\infty}^x (x-y)^{\lambda-1}\mathscr{L}^0\left(\mathcal{I}_{-\lambda/3}f\right)(y,t)dy \\ &= 3\frac{x_+^{(\lambda+3)-1}}{\Gamma(\lambda+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t) - \int_{-\infty}^x \frac{(x-y)^{(\lambda+3)-1}}{\Gamma(\lambda+3)}\mathscr{L}^0\left(\partial_t\mathcal{I}_{-\frac{\lambda}{3}}f\right)(y,t)dy\end{aligned} \tag{B.11}$$

and, with $\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} = e^{i\pi\lambda(-x)_+^{\lambda-1}}\frac{\Gamma(\lambda)}{(\lambda)}$ define

$$\begin{aligned}\mathscr{L}_+^\lambda f(x,t) &= \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \mathscr{L}^0\left(\mathcal{I}_{-\lambda/3}f\right)(-,t)\right](x)\\ &= \frac{1}{\Gamma(\lambda)}\int_x^\infty x(x-y)^{\lambda-1}\mathscr{L}^0\left(\mathcal{I}_{-\lambda/3}f\right)(y,t)dy \\ &= 3\frac{x_-^{(\lambda+3)-1}}{\Gamma(\lambda+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t) + e^{i\pi\lambda}\int_{-\infty}^x \frac{(-x+y)^{(\lambda+3)-1}}{\Gamma(\lambda+3)}\mathscr{L}^0\left(\partial_t\mathcal{I}_{-\frac{\lambda}{3}}f\right)(y,t)dy.\end{aligned} \tag{B.12}$$

It is straightforward from these definitions, in the sense of distributions

$$\left(\partial_t+\partial_x+\partial_x^3\right)\mathscr{L}_-^\lambda f(x,t) = 3\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t) + 3(\lambda+2)\frac{x_+^{\lambda+1}}{\Gamma(\lambda+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t)$$

and

$$\left(\partial_t+\partial_x+\partial_x^3\right)\mathscr{L}_+^\lambda f(x,t) = 3\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t) + 3(\lambda+2)\frac{x_-^{\lambda+1}}{\Gamma(\lambda+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t).$$

B.1.2. **Solution of the systems.** To address the nonlinear problem (1.1) with given data f and ϕ, take $-1<\lambda<1$, we set

$$u(x,t) = \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x) - \frac{1}{2}\theta(t)\mathscr{D}\partial u^2(x,t) + \theta(t)\mathscr{L}_+^\lambda h(x,t),$$

where

$$h(t) = e^{-\pi i\lambda}\left[f(t) - \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t)\right]. \tag{B.13}$$

Then

$$\begin{aligned}\left(\partial_t+\partial_x+\partial_x^3\right)u(x,t) = &-\frac{1}{2}\partial_x u^2(x,t) + 3\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\theta(t)\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t)\\ &+ 3(\lambda+2)\frac{x_-^{\lambda+1}}{\Gamma(\lambda+3)}\theta(t)\mathcal{I}_{-\frac{2}{3}-\frac{\lambda}{3}}f(t).\end{aligned}$$

Due to the support properties of $x_-^{\lambda\pm1}$ and (B.13), we have that u is the solution of (1.1).

Additionally, if we take $-1<\lambda_1,\lambda_2<1, \lambda_1\neq\lambda_2$, and set

$$u(x,t) = \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x) - \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(x,t) + \theta(t)\mathscr{L}_-^{\lambda_1}h_1(x,t) + \theta(t)\mathscr{L}_-^{\lambda_2}h_2(x,t),$$

where

$$\begin{bmatrix} h_1(t) \\ h_2(t)\end{bmatrix} = M\begin{bmatrix} g_1(t) - \theta(t)e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta(t)\mathscr{D}\partial_x u^2(0,t) \\ \theta(t)\mathcal{I}_{1/3}\left(g_2 - \theta\partial_x e^{-t(\partial_x+\partial_x^3)}\phi\Big|_{x=0} + \frac{1}{2}\theta\partial_x\mathscr{D}\partial_x u^2(0,\cdot)\right)(t)\end{bmatrix}. \tag{B.14}$$

with

$$M = \frac{1}{2\sqrt{3}\sin\left[\frac{\pi}{3}\left(\lambda_2-\lambda_1\right)\right]}\begin{bmatrix}\sin\left(\frac{\pi}{3}\lambda_2-\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{3}\lambda_2+\frac{\pi}{6}\right)\\ -\sin\left(\frac{\pi}{3}\lambda_1-\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{3}\lambda_1+\frac{\pi}{6}\right),\end{bmatrix}. \tag{B.15}$$

we have that, in the sense of distribution, u satisfies

$$(\partial_t+\partial_x+\partial_x^3)u(x,t)=-\frac{1}{2}\partial_x u^2(x,t)+3\frac{x_+^{\lambda_1-1}}{\Gamma(\lambda_1)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda_1}{3}}h_1(t)+3(\lambda_1+2)\frac{x_+^{\lambda_1+1}}{\Gamma(\lambda_1+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda_1}{3}}h_1(t)$$
$$+3\frac{x_+^{\lambda_2-1}}{\Gamma(\lambda_2)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda_2}{3}}h_2(t)+3(\lambda_2+2)\frac{x_+^{\lambda_2+1}}{\Gamma(\lambda_2+3)}\mathcal{I}_{-\frac{2}{3}-\frac{\lambda_2}{3}}d_2(t).$$

Due to the support properties of $x_+^{\lambda\pm1}$ and (B.14), we have that u is the solution of (1.2).

B.1.3. **Main estimates.** The operator $e^{-t(\partial_x+\partial_x^3)}$ was defined above in (B.6) satisfying (B.7). Thus, the following lemma holds.

Lemma B.4. [29, Lemma 5.5] *Let $s\in\mathbb{R}$. Then:*

(a) *(Space traces)* $\left\|e^{-t(\partial_x+\partial_x^3)}\phi(x)\right\|_{C(\mathbb{R}_t;H_x^s)}\leq c\|\phi\|_{H^s}$*;*

(b) *(Time traces)* $\left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)\right\|_{C\left(\mathbb{R}_x;H_t^{\frac{s+1}{3}}\right)}\leq c\|\phi\|_{H^s}$*;*

(c) *(Derivative time traces)* $\left\|\theta(t)\partial_x e^{-t(\partial_x+\partial_x^3)}\phi(x)\right\|_{C\left(\mathbb{R}_x;H_t^{\frac{s}{3}}\right)}\leq c\|\phi\|_{H^s}$*;*

(d) *(Bourgain space estimate) If $0<b<1$ and $0<\alpha<1$, then*
$$\left\|\theta(t)e^{-t(\partial_x+\partial_x^3)}\phi(x)\right\|_{X_{s,b}\cap D_\alpha}\leq c\|\theta\|_{H^1}\|\phi\|_{H^s},$$
where c is independent of θ.

In addition to the previous lemma, and taking into account that the operator $\mathscr{D}$ was defined above in (B.8) satisfying (B.9). Consider

$$\|u\|_{Y_{s,b}}=\left(\iint_{\xi,\tau}\langle\tau\rangle^{2s/3}\left\langle\tau-\xi^3\right\rangle^{2b}|\hat{u}(\xi,\tau)|^2d\xi d\tau\right)^{1/2}.$$

Thus, the next result is the following one.

Lemma B.5. [29, Lemma 5.6] *Let $s\in\mathbb{R}$. Then:*

(a) *(Space traces) If $0\leq b<\frac{1}{2}$, then*
$$\|\theta(t)\mathscr{D}w(x,t)\|_{C(\mathbb{R}_t;H_x^s)}\leq c\|w\|_{X_{s,-b}};$$

(b) *(Time traces) If $0<b<\frac{1}{2}$, then*
$$\|\theta(t)\mathscr{D}w(x,t)\|_{C\left(\mathbb{R}_x;H_t^{\frac{s+1}{3}}\right)}\leq\begin{cases}c\|w\|_{X_{s,-b}}, & \text{if } -1\leq s\leq\frac{1}{2},\\ c\left(\|w\|_{X_{s,-b}}+\|w\|_{Y_{s,-b}}\right), & \text{for any } s.\end{cases}$$
If $s<\frac{7}{2}$, then $\|\theta(t)\mathscr{D}w(x,t)\|_{C\left(\mathbb{R}_x;H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)\right)}$ has the same bound.

(c) *(Derivative time traces) If $0<b<\frac{1}{2}$, then*
$$\|\theta(t)\partial_x\mathscr{D}w(x,t)\|_{C\left(\mathbb{R}_x;H_t^{\frac{s}{3}}\right)}\leq\begin{cases}c\|w\|_{X_{s,-b}}, & \text{if } 0\leq s\leq\frac{3}{2},\\ c\left(\|w\|_{X_{s,-b}}+\|w\|_{Y_{s,-b}}\right), & \text{for any } s.\end{cases}$$
If $s<\frac{9}{2}$, then $\|\theta(t)\partial_x\mathscr{D}w(x,t)\|_{C\left(\mathbb{R}_x;H_0^{\frac{s}{3}}(\mathbb{R}_t^+)\right)}$ has the same bound.

(d) *(Bourgain space estimate) If $0\leq b<\frac{1}{2}$ and $\alpha\leq 1-b$, then*
$$\|\theta(t)\mathscr{D}w(x,t)\|_{X_{s,b}\cap D_\alpha}\leq c\|w\|_{X_{s,-b}}.$$

Finally, the operators $\mathscr{L}_\pm^\lambda$ defined in (B.11) and (B.12), have the following properties.

Lemma B.6. [29, Lemma 5.8] *Let $s\in\mathbb{R}$. Then:*

(a) *(Space traces) If* $s - \frac{5}{2} < \lambda < s + \frac{1}{2}, \quad \lambda < \frac{1}{2}$*, and supp* $f \subset [0,1]$*, then*

$$\left\| \mathscr{L}_{\pm}^{\lambda} f(x,t) \right\|_{C(\mathbb{R}_t; H_x^s)} \leq c \|f\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)}.$$

(b) *(Time traces) If* $-2 < \lambda < 1$*, then*

$$\left\| \theta(t) \mathscr{L}_{\pm}^{\lambda} f(x,t) \right\|_{C\left(\mathbb{R}_x; H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)\right)} \leq c \|f\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)}$$

(c) *(Derivative time traces) If* $-1 < \lambda < 2$*, then*

$$\left\| \theta(t) \partial_x \mathscr{L}_{\pm}^{\lambda} f(x,t) \right\|_{C\left(\mathbb{R}_x; H_0^{\frac{s}{3}}(\mathbb{R}_t^+)\right)} \leq c \|f\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)}$$

(d) *(Bourgain space estimate) If* $s - 1 \leq \lambda < s + \frac{1}{2}, \lambda < \frac{1}{2}, \alpha \leq \frac{s-\lambda+2}{3}$*, and* $0 \leq b < \frac{1}{2}$*, then*

$$\left\| \theta(t) \mathscr{L}_{\pm}^{\lambda} f(x,t) \right\|_{X_{s,b} \cap D_\alpha} \leq c \|f\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)}.$$

Departamento de Matemática, Universidade Federal de Pernambuco (UFPE), 50740-545, Recife (PE), Brazil.
Email address: roberto.capistranofilho@ufpe.br

Departamento de Matemática, Universidad Nacional de Colombia (UNAL), Cra 27 No. 64-60, 170003, Manizales, Colombia
Email address: fagallegor@unal.edu.co