COUPLED LINEAR SCHRÖDINGER EQUATIONS: CONTROL AND STABILIZATION RESULTS

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Abstract. This article presents some controllability and stabilization results for a system of two coupled linear Schrödinger equations in the one-dimensional case where the state components are interacting through the Kirchhoff boundary conditions. Considering the system in a bounded domain, the null boundary controllability result is shown. The result is achieved thanks to a new Carleman estimate, which ensures a boundary observation. Additionally, this boundary observation together with some trace estimates, helps us to use the Gramian approach, with a suitable choice of feedback law, to prove that the system under consideration decays exponentially to zero at least as fast as the function $e^{-2\omega t}$ for some $\omega > 0$.

1. Introduction

This work is dedicated to the study of the boundary control problem and stabilization issue of a linear system that appears modeling some problems in the context of nonlinear optics. Precisely, our motivation comes from the following system

$$\begin{cases} iu_t(t,x) + pu_{xx}(t,x) - \theta u(t,x) + \bar{u}(t,x)v(t,x) = 0, \quad t \geq 0,\ x \in \mathbb{R}, \\ i\sigma v_t(t,x) + qv_{xx}(t,x) - \alpha v(t,x) + \frac{a}{2}u^2(t,x) = 0, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \end{cases} \tag{1.1}$$

where u and v are complex-valued functions and $\alpha, \theta,$ and $a := 1/\sigma$ are real numbers representing physical parameters of the system, where $\sigma > 0$ and $p, q = \pm 1$. Notice that the system (1.1) is given by the nonlinear coupling of two dispersive equations of Schrödinger type through the quadratic terms.

There are some physical meanings for the previous system, as mentioned before. For example, the complex functions u and v represent amplitude packets of an optical wave's first and second harmonic, respectively. The values of γ_1 and γ_2 depend on the signals provided between the scattering/diffraction ratios, and the positive constant σ measures the scaling/diffraction magnitude indices. For details about this system, the authors suggested the references [13, 19, 28], and the references therein.

Concerning the mathematical context, most of the work related to the system (1.1) is devoted to proving the well-posedness of the Cauchy problem in $\mathbb{R}^n$, for $n \in [1,6]$ or in a periodic framework $\mathbb{T}$. For example, in [17] the authors showed the local well-posedness for the Cauchy problem (1.1) on the spaces $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for $n \leq 4$ and $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for $n \leq 6$. About qualitative properties of Cauchy problem (1.1), the case where $\gamma_1 = \gamma_2 = 1$ was studied in [4] for initial data u_0, v_0 in the same periodic Sobolev space $H^s(\mathbb{T})$. We also cite that in [5], the author studied the well-posedness of the Cauchy problem associated with the coupled Schrödinger equations with quadratic nonlinearities. He obtained the local well-posedness for data in Sobolev spaces with low regularity. Finally, the authors suggest the reference [24] for the recent progress on nonlinear Schrödinger systems with quadratic interactions.

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In the context of the control theory, no author attempted to show controllability results in bounded domains for the system (1.1). Due to this fact, our motivation is to present the boundary control results for the linear system associated with (1.1) posed in a bounded domain, giving a necessary first step in the direction to prove the nonlinear results for the system (1.1).

1.1. **Problem setting.** As mentioned before, our motivation in this work is to present, as a first step, the control results to the linear Schrödinger system associated with (1.1). More precisely, considering $T>0$ be any finite time and $\Omega=(0,1)$, we define $Q_T:=(0,T)\times\Omega$ and $\Sigma_T:=(0,T)\times\partial\Omega$. So, we will study the boundary controllability of the following linearized system

$$\begin{cases} iu_t+\gamma_1 u_{xx}-\alpha_1 u=0 & \text{in } Q_T,\\ i\sigma v_t+\gamma_2 v_{xx}-\alpha_2 v=0 & \text{in } Q_T,\\ u(0,x)=u_0(x),\ v(0,x)=v_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where the constants $\sigma,\gamma_1,\gamma_2,\alpha_1,\alpha_2,\alpha$ are positive and (u_0,v_0) are given initial data in certain spaces which will be specified later. We will consider the system (1.2) with the so-called *Kirchhoff boundary condition* at the right spatial point $x=1$:

$$\begin{cases} u(t,1)=v(t,1) & \text{in } (0,T),\\ \gamma_1 u_x(t,1)+\frac{\gamma_2}{\sigma}v_x(t,1)+\alpha u(t,1)=0 & \text{in } (0,T). \end{cases} \tag{1.3}$$

Here, the boundary control $h\in L^2(0,T)$ acts either on the component u or v at the left spatial point $x=0$; to be more precise we set

$$\text{either}\quad u(t,0)=h(t),\quad v(t,0)=0 \quad \text{in } (0,T), \tag{1.4a}$$

$$\text{or}\quad u(t,0)=0,\quad v(t,0)=h(t) \quad \text{in } (0,T). \tag{1.4b}$$

So, the first goal of this article is to answer the following null controllability problem:

Problem $\mathcal{A}$: Given $T>0$, (u_0,v_0) in a certain space X, can one find an appropriate control input h such that the corresponding solution (u,v) of (1.2) with boundary conditions (1.3)-(1.4b) (or (1.3)-(1.4a)) satisfies

$$(u(0,x),v(0,x))=(u_0(x),v_0(x)) \quad \text{and} \quad (u(T,x),v(T,x))=(0,0),\quad \forall x\in\Omega\ ? \tag{1.5}$$

If we can positively answer the previous question, an interesting problem is to study the boundary stabilization problem for the system (1.2) with boundary conditions (1.3) and with a single boundary control force exerted on the component v (or u), namely (1.4b) (or (1.4a)). In this context, the second main problem in this work treats the following stabilization issue:

Problem $\mathcal{B}$: Can we construct a stationary feedback law $h(t)$, of the form $F_\omega(u(t,\cdot),v(t,\cdot))$, such that the solution of the closed-loop system (1.2) with boundary condition (1.3)-(1.4b) (or (1.3)-(1.4a)) decays exponentially to zero at any prescribed decay rate?

An important point to answer these questions is that we need to assume the parameter σ, in the second equation of (1.2), be a positive number and the choice of σ will play a crucial role in deducing the required controllability result, which will be discussed further up. Moreover, we mention that now, the second problem will be called the *rapid stabilization problem.*

Before giving details about the main results of the article and some important facts, let us give a brief history of the control problem for the Schrödinger type systems, as well as some references to the rapid stabilization issues of partial differential equations (PDEs).

1.2. **Literature review: Control theory.** We will divide our state-of-the-art into three fronts. The first concerns some (non-exhaustive) control results for a single Schrödinger equation. The second part is dedicated to the control results obtained for systems like (1.2). Finally, we will give a brief overview of the rapid stabilization problem for some systems.

1.2.1. *Control results for Schrödinger equation.* It is well known that control properties for a single Schrödinger equation have received a lot of attention in the last decades (see, e.g., [23, 39] for an excellent review of the contributions up to 2014). There is an ongoing effort to show new control results for this equation, and this effort is giving significant progress for control properties. So, in this spirit, we can cite [21, 30, 31, 32, 34, 35] and the references therein for control issues or [8, 10, 22, 29, 36] and the references therein for Carleman estimates and their applications to inverse problems.

1.2.2. *Control results for coupled system.* We are not aware of any results for systems where the coupling is given at the Kirchhoff boundary condition for the Schrödinger type systems, as is our case. However, concerning the coupled (internal) structure in the equation, there are several results considering the cascade system for Schrödinger equations, which we would like to mention.

We first mention that controllability results for systems of parabolic equations are reviewed [3]. Concerning systems of hyperbolic equations, we cite the works [1, 2, 14]. There and the references therein, the reader can find results about the controllability of two coupled wave equations with only one control, under the hypothesis of the geometric control condition. More recently, in [33] a boundary controllability result is shown for a Schrödinger cascade type system with periodic boundary conditions. This result is obtained as a consequence of the controllability result for a cascade system of two wave equations. We also refer to the work [26], where the authors studied the null controllability of a linear system formed by two Schrödinger, controlling only one of them using Carleman estimates. Finally, considering two uncoupled wave equations with potentials on an interval, in [27], the authors established a Carleman estimate for wave systems with simultaneous boundary control, giving a boundary controllability result for uncoupled wave equations.

1.2.3. *Rapid stabilization of PDEs.* For rapid stabilization issues, in recent years, some abstract methods [20, 37, 38] have been developed to obtain answers considering linear PDEs. The method is based on the Gramian approach and the Riccati equations, and several authors employed during the last few years this approach. For example, we can cite, [11] for the KdV equation in a bounded domain, [9] for the KdV-KdV equation with only one boundary feedback acting, [18] for one-dimensional Schrödinger equation and of the beam and plate equations by moving or oblique observations and, additionally, for vibrating strings and beams, we can refer to [6].

We mention that this is only a small sample of the results concerning the control and stabilization problem. We strongly encourage the reader to see the previous references and the references therein.

1.3. **Main results and structure of the article.** We are now in a position to give comments on our main contribution to this article. Consider, now on, the space

$$\mathcal{H} = \{(u_1, u_2) \in [H^1(\Omega)]^2 \mid u_1(0) = u_2(0) = 0,\ u_1(1) = u_2(1)\}, \tag{1.6}$$

as the natural space for belonging of the initial data associated with (1.2) with the following norm

$$\|(u_1, u_2)\|_{\mathcal{H}} = \left(\int_\Omega \left(|u_1'(x)|^2 + |u_2'(x)|^2\right) dx \right)^{\frac{1}{2}}. \tag{1.7}$$

and the associated inner product defined by

$$\langle (u_1, u_2), (v_1, v_2)\rangle_{\mathcal{H}} = \mathrm{Re} \int_\Omega u_1'(x)\overline{v_1'(x)}dx + \mathrm{Re} \int_\Omega u_2'(x)\overline{v_2'(x)}dx,$$

for any (u_1, u_2), $(v_1, v_2) \in \mathcal{H}$. Finally, we denote $\mathcal{H}'$ as the dual space of $\mathcal{H}$ with respect to the pivot space $[L^2(0, 1)]^2$.

The first result of our work gives the control problem for the system (1.2) with boundary conditions (1.3)-(1.4b), that is when the control is acting on the second component. Precisely, considering these boundary conditions, due to a new Carleman estimate with boundary observation, the following result is verified.

Theorem 1.1. *Consider*

$$\mathfrak{S} := \left\{ \sigma > 0 \mid \sigma = \frac{\kappa\gamma_2}{\gamma_1}, \quad \kappa > 3 \right\}, \tag{1.8}$$

where $\gamma_1, \gamma_2 > 0$ are as appearing in (1.2). Let any $T > 0$, initial data $(u_0, v_0) \in \mathcal{H}'$ and parameters $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha$ be given positive. Then, for any $\sigma \in \mathfrak{S}$, there exists a control $h \in L^2(0, T)$ such that the solution (u, v) to the system (1.2) with boundary conditions (1.3)-(1.4b) satisfies (1.5).

Since we can prove an observability inequality (for the associated adjoint system), as mentioned before, naturally, the Problem $\mathcal{B}$ seems reachable. The next result gives, for the coupled Schrödinger equation (1.2), the following positive answer for the rapid stabilization problem.

Theorem 1.2. *Let any parameters $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha$ be given positive. Then, for the same choices of σ as in Theorem 1.1, there exists a continuous linear map $F_\omega : \mathcal{H} \to \mathbb{C}$ and a positive constants C and*

ω, such that for every $(u_0, v_0) \in \mathcal{H}$, the solution (u, v) of the closed-loop system (1.2) with boundary conditions (1.3)-(1.4b), with $h(t) = F_\omega(u(t,\cdot), v(t,\cdot))$ satisfies

$$\|(u(t), v(t))\|_{\mathcal{H}} \le Ce^{-2\omega t} \|(u_0, v_0)\|_{\mathcal{H}}, \quad \forall t \ge 0. \tag{1.9}$$

Remark 1.3. In what concerns our main results, Theorems 1.1 and 1.2, the following remarks are worth mentioning:

- As usual in the literature the answer for the Problem $\mathcal{A}$, that is, Theorem 1.1, is shown by using the Hilbert Uniqueness Method introduced by Lions [25] and the classical duality theory of Dolecki and Russell [15]. For that, it is essential to prove a suitable observability inequality with boundary observation, and to do so, in the present article, we prove a new Carleman estimate for the associated adjoint system to (1.2)-(1.3)-(1.4b).
- Problem $\mathcal{B}$ is answered using a classical Gramian approach [20, 37, 38], giving the proof of Theorem 1.2.
- We study the control and stabilization problems for the system (1.2) with boundary conditions (1.3)-(1.4b), however, we point out that a similar analysis can be performed if we instead consider (1.3)-(1.4a).
- The choice of such $\sigma \in \mathfrak{S}$ in Theorem 1.1 is important to deduce the required controllability result via Carleman estimate; for more details, we refer to Remark 2.4.

Our work is composed of five parts, including the introduction. In Section 2, the boundary controllability is considered. We obtain a new Carleman estimate with boundary observation, which is the key point to prove the observability inequality. So, with this observability in hand, the Theorem 1.1 is verified. In Section 3, we recall Urquiza's approach and use it to achieve the second main result of the article, i.e., Theorem 1.2. Section 4 is devoted to presenting some comments and open issues. Finally, in Appendix A, we present an overview of the well-posedness results, for the direct and adjoint systems associated with (1.2). Additionally, a key lemma, essential to prove the rapid stabilization result, is proved.

2. Boundary controllability

Let us first study the global null-controllability properties of the system (1.2)-(1.3) when the control acts on the component v, that is precisely (1.4b). The main tool is to establish a suitable Carleman estimate that yields a proper observability inequality and utilizing that, we prove the required controllability result for the concerned model.

Definition 2.1. For a given $T > 0$, the system (1.2) with boundary condition (1.3)-(1.4b) is null controllable at time T if for any given initial data $(u_0, v_0) \in \mathcal{H}'$, there exists a control function $h \in L^2(0,T)$, such that solution (u, v) to (1.2) satisfies (1.5).

Additionally, the solution by transposition of (1.2)-(1.3)-(1.4b) is given below.

Definition 2.2. Let $(u_0, v_0) \in \mathcal{H}'$ and $h \in L^2(0,T)$. We say that $(u, v) \in L^\infty(0,T;\mathcal{H}')$ is solution of

$$\begin{cases} iu_t + \gamma_1 u_{xx} - \alpha_1 u = 0 & \text{in } Q_T, \\ i\sigma v_t + \gamma_2 v_{xx} - \alpha_2 v = 0 & \text{in } Q_T, \\ u(t,0) = 0 & \text{in } (0,T), \\ v(t,0) = h(t) & \text{in } (0,T), \\ u(t,1) = v(t,1) & \text{in } (0,T), \\ \gamma_1 u_x(t,1) + \frac{\gamma_2}{\sigma} v_x(t,1) + \alpha u(t,1) = 0 & \text{in } (0,T), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

in the transposition sense if and only if

$$\operatorname{Re} \int_0^T \langle (u(t), v(t)), (\bar{g}_1, \bar{g}_2) \rangle_{\mathcal{H}',\mathcal{H}} dt = \langle (u_0, v_0), (\overline{\varphi_1(0,\cdot)}, \overline{\varphi_2(0,\cdot)}) \rangle_{\mathcal{H}',\mathcal{H}} - \operatorname{Re} \int_0^T h(t) \overline{\varphi_{2,x}(t,0)}\, dt,$$

for every $(g_1, g_2) \in L^1(0,T;\mathcal{H})$ where (φ_1, φ_2) are the mild solution to the problem

$$\begin{cases} i\varphi_{1,t} + \gamma_1\varphi_{1,xx} - \alpha_1\varphi_1 = g_1, & \text{in } Q_T, \\ i\sigma\varphi_{2,t} + \gamma_2\varphi_{2,xx} - \alpha_2\varphi_2 = g_2, & \text{in } Q_T, \\ \varphi_1(t,1) = \varphi_2(t,1), & \text{in } (0,T), \\ \gamma_1\varphi_{1,x}(t,1) + \frac{\gamma_2}{\sigma}\varphi_{2,x}(t,1) + \alpha\varphi_1(t,1) = 0, & \text{in } (0,T), \\ \varphi_1(t,0) = 0,\ \varphi_2(t,0) = 0, & \text{in } (0,T), \end{cases} \tag{2.2}$$

on the space $C([0,T];\mathcal{H})$, with $(\varphi_1(T,\cdot), \varphi_2(T,\cdot)) = (0,0)$ in Ω.

We remark that the discussion of the system (2.2) is given in Appendix A. Furthermore, the following result is verified.

Proposition 2.3. *Let $(u_0, v_0) \in \mathcal{H}'$ and $h \in L^2(0,T)$. Then the control system (2.1) has a unique solution (u,v) in $C([0,T];\mathcal{H}')$.*

2.1. **Global Carleman estimate.** With the previous definitions, we are in a position to obtain the global Carleman estimate for the adjoint system associated with the system (1.2) with boundary conditions (1.3)-(1.4b), namely,

$$\begin{cases} i\varphi_{1,t} + \gamma_1\varphi_{1,xx} - \alpha_1\varphi_1 = 0, & \text{in } Q_T, \\ i\sigma\varphi_{2,t} + \gamma_2\varphi_{2,xx} - \alpha_2\varphi_2 = 0, & \text{in } Q_T, \\ \varphi_1(t,1) = \varphi_2(t,1), & \text{in } (0,T), \\ \gamma_1\varphi_{1,x}(t,1) + \frac{\gamma_2}{\sigma}\varphi_{2,x}(t,1) + \alpha\varphi_1(t,1) = 0, & \text{in } (0,T), \\ \varphi_1(t,0) = 0,\ \varphi_2(t,0) = 0, & \text{in } (0,T), \\ \varphi_1(T,x) = \zeta_1(x),\ \ \varphi_2(T,x) = \zeta_2(x), & \text{in } \Omega, \end{cases} \tag{2.3}$$

with $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}$.

To this end, we introduce the space

$$\mathcal{Q} := \Big\{ (\varphi_1,\varphi_2) \in [C^2(\overline{Q_T})]^2 \mid \varphi_1(t,0) = \varphi_2(t,0) = 0,\ \varphi_1(t,1) = \varphi_2(t,1), \\ \gamma_1\varphi_{1,x}(t,1) + \frac{\gamma_2}{\sigma}\varphi_{2,x}(t,1) + \alpha\varphi_1(t,1) = 0,\ \forall t \in [0,T] \Big\}.$$

Now recall that $\sigma \in \mathfrak{S}$ (see (1.8)) and thus

$$\sigma = \frac{\kappa\gamma_2}{\gamma_1} \quad \text{for some } \kappa > 3. \tag{2.4}$$

With this in hand, we consider the following auxiliary functions (motivated from [7]):

$$\begin{cases} \beta_j(x) = 2 + c_j(x-1), \quad j = 1,2, \\ c_1 = 1,\ c_2 = -\kappa \ \text{ and thus } |c_2| > 3. \end{cases} \tag{2.5}$$

Therefore, $\beta_j \in C^2([0,1])$ and satisfy

$$\beta_2 \geq \beta_1 > 0, \ \text{ in } [0,1], \ \ \beta_1(1) = \beta_2(1).$$

Next, for any parameter $\lambda > 1$, we introduce the following weight functions:

$$\xi_j(t,x) = \frac{e^{\lambda\beta_j(x)}}{t(T-t)}, \quad \eta_j(t,x) = \frac{e^{2\lambda\|\beta_j\|_\infty} - e^{\lambda\beta_j(x)}}{t(T-t)}, \quad \forall (t,x) \in Q_T, \quad j = 1,2. \tag{2.6}$$

Note that

$$\xi_j,\, \eta_j > 0 \ \text{ for } j = 1,2, \quad \xi_1(t,1) = \xi_2(t,1), \quad \eta_1(t,1) = \eta_2(t,1),$$

since $\beta_1(1) = \beta_2(1)$. Concerning of the function ξ_j and η_j, we have the following behavior in Q_T:

$$\begin{cases} \xi_{j,x} = \lambda\xi_j c_j, \quad \eta_{j,x} = -\lambda\xi_j c_j, \\ \eta_{j,xx} = -\lambda^2 c_j^2\xi_j, \quad \eta_{j,xxx} = -\lambda^3 c_j^3\xi_j, \\ |\xi_{j,t}| \leq CT\xi_j^2, \quad |\eta_{j,t}| \leq CT\xi_j^2, \\ |\eta_{j,xt}| \leq C\lambda T\xi_j^2, \quad |\eta_{j,tt}| \leq CT^2\xi_j^3, \end{cases} \quad \text{for } j = 1,2. \tag{2.7}$$

Remark 2.4. We point out that the choice of c_2 in (2.5) and the value of $\sigma \in \mathfrak{S}$ in (1.8) are crucial to obtain the Carleman estimate (2.8). More precisely, those choices help us deal with some unusual boundary integrals while proving the underlying Carleman estimate.

The main theorem of this subsection can be read as follows:

Theorem 2.5 (Carleman estimate). *Let the weight functions ξ_1, ξ_2, η_1, η_2 be chosen as in* (2.6) *and $\sigma > 0$ is taken as* (2.4)*. Then, there exist constants $\lambda_0 > 0$, $\mu_0 > 0$ and $C > 0$, depending at most on γ_1, γ_2 and c_2, such that the following estimate holds true*

$$\begin{aligned} &s^3\lambda^4 \iint_{Q_T} \left(e^{-2s\eta_1}\xi_1^3|\varphi_1|^2 + e^{-2s\eta_2}\xi_2^3|\varphi_2|^2\right) dxdt \\ &\quad + s\lambda^2 \iint_{Q_T} \left(e^{-2s\eta_1}\xi_1|\varphi_{1,x}|^2 + e^{-2s\eta_2}\xi_2|\varphi_{2,x}|^2\right) dxdt \\ \leq &C \iint_{Q_T} (e^{-2s\eta_1}|L_1\varphi_1|^2 + e^{-2s\eta_2}|L_2\varphi_2|^2) dxdt \\ &\quad + Cs\lambda \int_0^T e^{-2s\eta_2(t,0)}\xi_2(t,0)|\varphi_{2,x}(t,0)|^2 dt, \end{aligned} \tag{2.8}$$

for every $\lambda \geq \lambda_0$, $s \geq s_0 := \mu_0(T + T^2)$ and for all $(\varphi_1, \varphi_2) \in \mathcal{Q}$, where

$$L_1 = i\partial_t + \gamma_1\partial_{xx} \quad \text{and} \quad L_2 = i\partial_t + \frac{\gamma_2}{\sigma}\partial_{xx}.$$

To prove the above theorem, let us now define the following variables:

$$\psi_j(t,x) = e^{-s\eta_j(t,x)}\varphi_j(t,x), \quad \forall (t,x) \in Q_T, \quad \text{for } j = 1,2, \tag{2.9}$$

so that we have the following boundary conditions:

$$\begin{cases} \psi_j(t,0) = 0, \quad \psi_1(t,1) = \psi_2(t,1), \\ \gamma_1\psi_{1,x}(t,1) + \frac{\gamma_2}{\sigma}\psi_{2,x}(t,1) = -\alpha\psi_1(t,1) + s\lambda\xi_1(t,1)\psi_1(t,1)\left(\gamma_1 c_1 + \frac{\gamma_2}{\sigma}c_2\right). \end{cases} \tag{2.10}$$

We further denote

$$F_j := e^{-s\eta_j}L_j(\varphi_j) = e^{-s\eta_j}L_j(e^{s\eta_j}\psi_j),$$

where L_1 and L_2. So, with this, we find the following relations for each $j = 1, 2$,

$$\begin{aligned} &i(\psi_j e^{s\eta_j})_t = i\psi_{j,t}e^{s\eta_j} + is\psi_j e^{s\eta_j}\eta_{j,t}, \\ &(\psi_j e^{s\eta_j})_x = \psi_{j,x}e^{s\eta_j} + s\psi_j e^{s\eta_j}\eta_{j,x}, \\ &(\psi_j e^{s\eta_j})_{xx} = \psi_{j,xx}e^{s\eta_j} + 2se^{s\eta_j}\psi_{j,x}\eta_{j,x} + s^2\psi_j e^{s\eta_j}|\eta_{j,x}|^2 + s\psi_j e^{s\eta_j}\eta_{j,xx}. \end{aligned}$$

Using the previous relation, F_j can be written as follows

$$M_1\psi_j + M_2\psi_j = F_j, \tag{2.11}$$

where

$$\begin{cases} M_1\psi_1 = 2s\gamma_1\psi_{1,x}\eta_{1,x} + s\gamma_1\psi_1\eta_{1,xx} + is\psi_1\eta_{1,t}, \\ M_2\psi_1 = i\psi_{1,t} + \gamma_1\psi_{1,xx} + s^2\gamma_1|\eta_{1,x}|^2\psi_1, \end{cases}$$

and

$$\begin{cases} M_1\psi_2 = 2s\frac{\gamma_2}{\sigma}\psi_{2,x}\eta_{2,x} + s\frac{\gamma_2}{\sigma}\psi_2\eta_{2,xx} + is\psi_2\eta_{2,t}, \\ M_2\psi_2 = i\psi_{2,t} + \frac{\gamma_2}{\sigma}\psi_{2,xx} + s^2\frac{\gamma_2}{\sigma}|\eta_{2,x}|^2\psi_2. \end{cases} \tag{2.12}$$

Thus, we get from (2.11),

$$\begin{aligned} \iint_{Q_T} \left(|M_1\psi_j|^2 + |M_2\psi_j|^2\right) dxdt + 2\text{Re}\iint_{Q_T} M_1\psi_j\overline{M_2\psi_j}dxdt \\ = \iint_{Q_T} |F_j|^2 dxdt, \end{aligned} \tag{2.13}$$

for $j = 1, 2$. Now, we are in a position to prove Theorem 2.5.

Proof of Theorem 2.5. Our goal is to focus on the following inner product

$$\text{Re}\iint_{Q_T} M_1\psi_j\overline{M_2\psi_j}dxdt$$

that contains 9 terms. We will elaborately make the computations for $j = 2$, similarly, the computations can be done for $j = 1$ as well.

Recall that the quantities $M_1\psi_2$ and $M_2\psi_2$ are given by (2.12) and, for $j=2$, we have the relation (2.13). We further denote

$$\mathrm{Re}\iint_{Q_T} M_1\psi_2\overline{M_2\psi_2}dxdt=\sum_{1\le k,l\le 3} I_{kl},$$

where all the terms I_{kl} for $1\le k,l\le 3$ consists of the integral term with the product involving the k-th term of $M_1\psi_2$ with the l-th term of $M_2\psi_2$, and will be computed below. Now, we split the proof into several steps.

• **Step 1.** Computations of the terms I_{11}, I_{21} and I_{32}.

Let us start with I_{11}. Observe that

$$\begin{aligned}I_{11}=&2s\frac{\gamma_2}{\sigma}\mathrm{Re}\iint_{Q_T}\psi_{2,x}\eta_{2,x}\overline{i\psi_{2,t}}dxdt\\=&2s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,x}\eta_{2,x}\overline{\psi_{2,t}}dxdt\\=&-2s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,x}\eta_{2,xt}\overline{\psi_2}dxdt-2s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,xt}\eta_{2,x}\overline{\psi_2}dxdt\\:=&J_1+J_2,\end{aligned}\tag{2.14}$$

where there is no boundary integral since

$$\lim_{t\to 0^+}\psi_2(t,\cdot)=\lim_{t\to T^-}\psi_2(t,\cdot)=0,$$

thanks to the choices of weight functions (2.6).

Next, for the quantity I_{21}, we see that

$$\begin{aligned}I_{21}=&s\frac{\gamma_2}{\sigma}\mathrm{Re}\iint_{Q_T}\psi_2\eta_{2,xx}\overline{i\psi_{2,t}}dxdt\\=&s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_2\eta_{2,xx}\overline{\psi_{2,t}}dxdt\\=&-s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,x}\eta_{2,x}\overline{\psi_{2,t}}dxdt-s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_2\eta_{2,x}\overline{\psi_{2,tx}}dxdt\\&+s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T\psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt\\=&s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,xt}\eta_{2,x}\overline{\psi_2}dxdt+s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,x}\eta_{2,xt}\overline{\psi_2}dxdt\\&-s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_2\eta_{2,x}\overline{\psi_{2,xt}}dxdt+s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T\psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt,\end{aligned}\tag{2.15}$$

where we have applied integration by parts w.r.t. x to the term involving $\eta_{2,xx}$, then w.r.t. t to the term involving $\overline{\psi_{2,t}}$ in Q_T, and using again the decay of η_2 at $t=0$ and $t=T$.

Now, thanks to the fact that $-\mathrm{Im}(z)=\mathrm{Im}(\overline{z})$ (for any $z\in\mathbb{C}$) in the third integral term in the last equality of (2.15), we get

$$\begin{aligned}I_{21}=&2s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,xt}\eta_{2,x}\overline{\psi_2}dxdt+s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T}\psi_{2,x}\eta_{2,xt}\overline{\psi_2}dxdt\\&+s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T\psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt\\=&-J_2-\frac{J_1}{2}+s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T\psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt.\end{aligned}\tag{2.16}$$

Finally, the term I_{32} is computed as follows.

$$\begin{aligned}
I_{32} =& s\frac{\gamma_2}{\sigma}\mathrm{Re}\iint_{Q_T} i\psi_2\eta_{2,t}\overline{\psi_{2,xx}}dxdt\\
=& - s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T} \eta_{2,t}\psi_2\overline{\psi_{2,xx}}dxdt\\
=& s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T} \left(\eta_{2,tx}\psi_2\overline{\psi_{2,x}} + \eta_{2,t}\psi_{2,x}\overline{\psi_{2,x}}\right)dxdt\\
& - s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T \eta_{2,t}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt\\
=& - s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T} \psi_{2,x}\eta_{2,xt}\overline{\psi_2}dxdt\\
& - s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T \eta_{2,t}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt\\
=& \frac{J_1}{2} - s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T \eta_{2,t}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt,
\end{aligned} \tag{2.17}$$

where we have used the fact that

$$\frac{s\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T} \eta_{2,t}|\psi_{2,x}|^2dxdt = 0$$

since $\eta_{2,t}$ is real-valued function. Therefore, by adding (2.14), (2.16) and (2.17), we get

$$\begin{aligned}
I_{11} + I_{21} + I_{32} =& J_1 + s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T \psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt\\
& - s\frac{\gamma_2}{\sigma}\mathrm{Im}\int_0^T \eta_{2,t}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt,
\end{aligned} \tag{2.18}$$

where J_1 satisfies

$$\begin{aligned}
|J_1| =& \left|2s\frac{\gamma_2}{\sigma}\mathrm{Im}\iint_{Q_T} \psi_{2,x}\eta_{2,xt}\overline{\psi_2}dxdt\right|\\
\leq& C\frac{\gamma_2}{\sigma}s\lambda T\iint_{Q_T} \xi_2^2|\psi_2||\psi_{2,x}|dxdt\\
\leq& \frac{C}{\sigma^2}s^2\lambda^2 T\iint_{Q_T} \xi_2^3|\psi_2|^2dxdt + CT\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt,
\end{aligned} \tag{2.19}$$

finishing step 1.

- **Step 2.** Computations of the terms I_{12} and I_{22}.

Let us, for I_{12}, to perform by integration by parts w.r.t. space variable to ensures that

$$\begin{aligned}
I_{12} =& 2s\frac{\gamma_2^2}{\sigma^2}\mathrm{Re}\iint_{Q_T} \eta_{2,x}\psi_{2,x}\overline{\psi_{2,xx}}dxdt\\
=& - 2s\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \eta_{2,xx}|\psi_{2,x}|^2dxdt - 2s\frac{\gamma_2^2}{\sigma^2}\mathrm{Re}\iint_{Q_T} \eta_{2,x}\psi_{2,xx}\overline{\psi_{2,x}}dxdt\\
& + 2s\frac{\gamma_2^2}{\sigma^2}\int_0^T \eta_{2,x}(t,1)|\psi_{2,x}(t,1)|^2dt - 2s\frac{\gamma_2^2}{\sigma^2}\int_0^T \eta_{2,x}(t,0)|\psi_{2,x}(t,0)|^2dt.
\end{aligned}$$

Recalling the expressions of $\eta_{2,x}$ and $\eta_{2,xx}$ from (2.7), we get

$$\begin{aligned}
2I_{12} =& 2s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt\\
& - 2s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,1)|\psi_{2,x}(t,1)|^2dt + 2s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,0)|\psi_{2,x}(t,0)|^2dt,
\end{aligned} \tag{2.20}$$

where the boundary term $\psi_{2,x}(t,1) > 0$ once we have $c_2 < 0$ (see (2.5)).

The term I_{22} can be computed as

$$\begin{aligned}I_{22} =& s\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \eta_{2,xx}\psi_2\overline{\psi_{2,xx}}dxdt\\ =& - s\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \eta_{2,xx}|\psi_{2,x}|^2dxdt - s\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \eta_{2,xxx}\psi_2\overline{\psi_{2,x}}dxdt\\ &+ s\frac{\gamma_2^2}{\sigma^2}\text{Re}\int_0^T \eta_{2,xx}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt.\end{aligned}$$

Again, using the expressions of $\eta_{2,xx}$ and $\eta_{2,xxx}$, thanks to (2.7), we have that

$$\begin{aligned}I_{22} =& s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt + s\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \xi_2\psi_2\overline{\psi_{2,x}}dxdt\\ &- s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\text{Re}\int_0^T \xi_2(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt.\end{aligned} \tag{2.21}$$

Now, by adding the final expressions of I_{12} and I_{22}, i.e, (2.20) and (2.21), yields that

$$\begin{aligned}I_{12} + I_{22} =& 2s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt + s\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \xi_2\psi_2\overline{\psi_{2,x}}dxdt\\ &- s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,1)|\psi_{2,x}(t,1)|^2dt + s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,0)|\psi_{2,x}(t,0)|^2dt\\ &- s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\text{Re}\int_0^T \xi_2(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt,\end{aligned} \tag{2.22}$$

where the second term of the r.h.s. of (2.22), denoted by J_3, satisfies

$$\begin{aligned}|J_3| =& \left|s\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \xi_2\psi_2\overline{\psi_{2,x}}dxdt\right|\\ \leq& \frac{C}{\epsilon}s\lambda^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_2|^2dxdt + \epsilon s\lambda^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt\\ \leq& \frac{C}{\epsilon}T^4s\lambda^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2^3|\psi_2|^2dxdt + \epsilon s\lambda^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2|\psi_{2,x}|^2dxdt,\end{aligned} \tag{2.23}$$

for any $\epsilon > 0$ small enough, where we have used the fact that $\xi_2 \leq CT^4\xi_2^3$.

So, at this point, it is worth mentioning that the boundary integral consisting of $\psi_{2,x}(t,0)$ in (2.22) will lead to the observation term in our final Carleman estimate (since we have exerted a controlled force on the second component), finalizing the analysis in step 2.

- **Step 3.** Analysis of remaining terms I_{13}, I_{23}, I_{31}, and I_{33}.

Let us first deal with the term I_{13}. Notice that the fact that $2\text{Re}(z) = z + \overline{z}$ (for any $z \in \mathbb{C}$), and then integrating by parts w.r.t. x, gives us

$$\begin{aligned}I_{13} =& 2s^3\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T} \eta_{2,x}^3\psi_{2,x}\overline{\psi_2}dxdt\\ =& s^3\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \eta_{2,x}^3\left(\overline{\psi_2}\psi_{2,x} + \psi_2\overline{\psi_{2,x}}\right)dxdt\\ =& s^3\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \eta_{2,x}^3\left(\psi_2\overline{\psi_2}\right)_x dxdt\\ =& - s^3\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} 3\eta_{2,x}^2\eta_{2,xx}|\psi_2|^2dxdt + s^3\frac{\gamma_2^2}{\sigma^2}\int_0^T \eta_{2,x}^3|\psi_2(t,1)|^2dt\\ =& 3s^3\lambda^4c_2^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T} \xi_2^3|\psi_2|^2dxdt - s^3\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2dxdt,\end{aligned}$$

where the boundary term has a positive sign since $c_2 < 0$.

Now, for the analysis of I_{23} we just need to see that

$$I_{23} = s^3\frac{\gamma_2^2}{\sigma^2}\text{Re}\iint_{Q_T}\eta_{2,xx}|\eta_{2,x}|^2|\psi_2|^2dxdt = -s^3\lambda^4c_2^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt.$$

Let us see the term I_{31}. For this integral, using again the decay properties of the weight function and integrating into the time variable, this term can be written as

$$\begin{aligned}I_{31} =& - s\text{Re}\iint_{Q_T}\eta_{2,t}\psi_2\overline{\psi_{2,t}}dxdt\\ =& -\frac{1}{2}s\iint_{Q_T}\eta_{2,t}\left(\psi_2\overline{\psi_{2,t}}+\overline{\psi_2}\psi_{2,t}\right)dxdt\\ =& \frac{1}{2}s\iint_{Q_T}\eta_{2,tt}|\psi_2|^2dxdt,\end{aligned}$$

which satisfies

$$|I_{31}| \leq CsT^2\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt,$$

using the boundness of $\eta_{2,tt}$ from (2.7).

For the last quantity I_{33}, since $\eta_{2,t}$ and $\eta_{2,x}$ are real valued functions, we get that

$$I_{33} = s^3\frac{\gamma_2}{\sigma}\text{Re}\left(i\iint\eta_{2,t}|\eta_{2,x}|^2|\psi_2|^2dxdt\right) = 0.$$

Adding each peach of I_{13}, I_{23}, I_{31} and I_{33}, we have

$$\begin{aligned}I_{13}+I_{23}+I_{31}+I_{33} \geq& 2s^3\lambda^4c_2^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt - s^3\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\int_0^T\xi_2^3(t,1)|\psi_2(t,1)|^2dt\\ &- CsT^2\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt,\end{aligned} \tag{2.24}$$

achieving the goal of step 3.

• **Step 4.** Finding an intermediate estimate.

With the previous steps 1,2 and 3 in hand, adding all the terms I_{kl} for $1 \leq k,l \leq 3$, that is (2.18)–(2.19), (2.22)–(2.23), and (2.24), we can get the following

$$\begin{aligned}\sum_{1\leq k,l\leq 3} I_{kl} \geq& 2s^3\lambda^4c_2^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt + 2s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T}\xi_2|\psi_{2,x}|^2dxdt\\ &- CsT^2\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt - \frac{C}{\sigma^2}s^2\lambda^2T\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt\\ &- CT\iint_{Q_T}\xi_2|\psi_{2,x}|^2dxdt - \frac{C}{\epsilon}T^4s\lambda^4\frac{\gamma_2^2}{\sigma^2}\iint_{Q_T}\xi_2^3|\psi_2|^2dxdt\\ &- \frac{\gamma_2^2}{\sigma^2}\epsilon s\lambda^2\iint_{Q_T}\xi_2|\psi_{2,x}|^2dxdt - s^3\lambda^3c_2^3\frac{\gamma_2^2}{\sigma^2}\int_0^T\xi_2^3(t,1)|\psi_2(t,1)|^2dt\\ &- s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T\xi_2(t,1)|\psi_{2,x}(t,1)|^2dt + s\lambda c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T\xi_2(t,0)|\psi_{2,x}(t,0)|^2dt\\ &- s\lambda^2c_2^2\frac{\gamma_2^2}{\sigma^2}\text{Re}\int_0^T\xi_2(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt\\ &+ s\frac{\gamma_2}{\sigma}\text{Im}\int_0^T\psi_2(t,1)\overline{\psi_{2,t}(t,1)}\eta_{2,x}(t,1)dt\\ &- s\frac{\gamma_2}{\sigma}\text{Im}\int_0^T\eta_{2,t}(t,1)\psi_2(t,1)\overline{\psi_{2,x}(t,1)}dt.\end{aligned} \tag{2.25}$$

Following a similar approach as the case of ψ_2, we can compute the same for ψ_1, for this, let us denote

$$\text{Re}\iint_{Q_T}M_1\psi_1\overline{M_2\psi_1}dxdt = \sum_{1\leq k,l\leq 3}H_{kl}$$

In this case, one can find

$$
\begin{aligned}
\sum_{1\le k,l\le 3} H_{kl} \ge & 2s^3\lambda^4 c_1^4\gamma_1^2 \iint_{Q_T} \xi_1^3|\psi_1|^2dxdt + 2s\lambda^2c_1^2\gamma_1^2 \iint_{Q_T} \xi_1|\psi_{1,x}|^2dxdt \\
& - CsT^2 \iint_{Q_T} \xi_1^3|\psi_1|^2dxdt - Cs^2\lambda^2T \iint_{Q_T} \xi_1^3|\psi_1|^2dxdt \\
& - CT \iint_{Q_T} \xi_1|\psi_{1,x}|^2dxdt - \frac{C}{\epsilon}T^4s\lambda^4\gamma_1^2 \iint_{Q_T} \xi_1^3|\psi_1|^2dxdt \\
& - \epsilon s\lambda^2 \iint_{Q_T} \xi_1|\psi_{1,x}|^2dxdt - s^3\lambda^3c_1^3\gamma_1^2 \int_0^T \xi_1^3(t,1)|\psi_1(t,1)|^2dt \\
& - s\lambda c_1\gamma_1^2 \int_0^T \xi_1(t,1)|\psi_{1,x}(t,1)|^2dt + s\lambda c_1\gamma_1^2 \int_0^T \xi_1(t,0)|\psi_{1,x}(t,0)|^2dt \\
& - s\lambda^2c_1^2\gamma_1^2\mathrm{Re}\int_0^T \xi_1(t,1)\psi_1(t,1)\overline{\psi_{1,x}(t,1)}dt \\
& + s\gamma_1\mathrm{Im}\int_0^T \psi_1(t,1)\overline{\psi_{1,t}(t,1)}\eta_{1,x}(t,1)dt \\
& - s\gamma_1\mathrm{Im}\int_0^T \eta_{1,t}(t,1)\psi_1(t,1)\overline{\psi_{1,x}(t,1)}dt.
\end{aligned} \tag{2.26}
$$

We are now in a position to combine the above estimate. For simplicity, we denote $\nu_1 = \gamma_1$ and $\nu_2 = \frac{\gamma_2}{\sigma}$. Choose $\epsilon > 0$ small enough in (2.25) and (2.26) so that, combining those estimates, we have

$$
\begin{aligned}
& \frac{1}{2}\sum_{j=1}^2 \iint_{Q_T} \left(|M_1\psi_j|^2 + |M_2\psi_j|^2\right) dxdt \\
& + \sum_{j=1}^2 2s^3\lambda^4c_j^4\nu_j^2 \iint_{Q_T} \xi_j^3|\psi_j|^2dxdt + \sum_{j=1}^2 2s\lambda^2\nu_j \iint_{Q_T} \xi_j|\psi_{j,x}|^2dxdt \\
& - C\left(\frac{1}{\sigma^2}s^2\lambda^2T + T^4s\lambda^4\frac{\gamma^2}{\sigma^2}\right) \iint_{Q_T} \xi_2^3|\psi_2|^2dxdt - CT\iint_{Q_T} \xi_2^3|\psi_{2,x}|^2dxdt \\
& - C\left(s^2\lambda^2T + T^4s\lambda^4\gamma_1^2\right) \iint_{Q_T} \xi_1^3|\psi_1|^2dxdt - CT\iint_{Q_T} \xi_1^3|\psi_{1,x}|^2dxdt \\
& - s^3\lambda^3\sum_{j=1}^2 c_j^3\nu_j^2 \int_0^T \xi_j^3(t,1)|\psi_j(t,1)|^2dt - s\lambda\sum_{j=1}^2 c_j\nu_j^2\int_0^T \xi_j(t,1)|\psi_{j,x}(t,1)|^2dt \\
& + s\lambda\sum_{j=1}^2 c_j\nu_j^2\int_0^T \xi_j(t,0)|\psi_{j,x}(t,0)|^2dt - s\lambda^2\sum_{j=1}^2 c_j^2\nu_j^2\mathrm{Re}\int_0^T \xi_j(t,1)\psi_j(t,1)\overline{\psi_{j,x}(t,1)}dt \\
& + s\sum_{j=1}^2 \nu_j\mathrm{Im}\int_0^T \psi_j(t,1)\overline{\psi_{j,t}(t,1)}\eta_{j,x}(t,1)dt \\
& - s\sum_{j=1}^2 \nu_j\mathrm{Im}\int_0^T \eta_{j,t}(t,1)\psi_j(t,1)\overline{\psi_{j,x}(t,1)}dt \le C\sum_{j=1}^2\iint_{Q_T}|F_j|^2dxdt.
\end{aligned} \tag{2.27}
$$

In the above, it is clear that there exist some positive constants λ_0, μ_0 such that if we choose $\lambda \ge \lambda_0$ and $s \ge s_0 := \mu_0(T + T^2)$, then the lower order integrals

$$
\begin{aligned}
& C\left(\frac{1}{\sigma^2}s^2\lambda^2T + T^4s\lambda^4\frac{\gamma^2}{\sigma^2}\right)\iint_{Q_T} \xi_2^3|\psi_2|^2dxdt, \\
& C\left(s^2\lambda^2T + T^4s\lambda^4\gamma_1^2\right)\iint_{Q_T} \xi_1^3|\psi_1|^2dxdt, \\
& CT\iint_{Q_T} \xi_2^3|\psi_{2,x}|^2dxdt,
\end{aligned}
$$

and

$$CT \iint_{Q_T} \xi_1^3 |\psi_{1,x}|^2 dxdt,$$

can be absorbed by the corresponding leading integrals

$$\sum_{j=1}^{2} 2s^3\lambda^4 c_j^4 \nu_j^2 \iint_{Q_T} \xi_j^3 |\psi_j|^2 dxdt \quad \text{and} \quad \sum_{j=1}^{2} 2s\lambda^2 \nu_j \iint_{Q_T} \xi_j |\psi_{j,x}|^2 dxdt.$$

Now, it remains to find the proper estimates for the boundary integrals. We do this in the following step.

• **Step 5.** Computing the boundary integrals.

We split this step into several parts. We hereby recall the fact that

$$\xi_1(t,1) = \xi_2(t,1), \quad \eta_1(t,1) = \eta_2(t,1), \quad \psi_1(t,1) = \psi_2(t,1),$$

and thus, from now onwards, we shall replace all the quantities $\xi_1(t,1)$, $\eta_1(t,1)$ and $\psi_1(t,1)$ by $\xi_2(t,1)$, $\eta_2(t,1)$ and $\psi_2(t,1)$ respectively. Moreover, we will denote the six boundary terms in (2.27) in the order presented above as B_k given by

$$B_k = \sum_{j=1,2} B_{k,j}, \quad 1 \le k \le 6.$$

Let us now give the details of each part.

– **Part A:** Computing $B_1 := B_{11} + B_{12}$.

First notice that, the term

$$B_{12} = -s^3\lambda^3 c_2^3 \frac{\gamma_2^2}{\sigma^2} \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt$$

is positive, since $c_2 < 0$. We need to absorb the negative term

$$B_{11} = -s^3\lambda^3 c_1^3 \gamma_1^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt$$

by the term B_{12}. To do that, since we have $c_1 = 1$ and, by (2.4) that

$$\sigma = -\frac{c_2\gamma_2}{\gamma_1}$$

we have, by adding B_{11} and B_{12}, that

$$\begin{aligned} B_1 &= B_{11} + B_{12} \\ &= s^3\lambda^3 \left(|c_2|^3 \frac{\gamma_2^2}{\sigma^2} - \gamma_1^2 \right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\ &= s^3\lambda^3 \gamma_1^2 \left(|c_2| - 1 \right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt, \end{aligned} \tag{2.28}$$

which has a positive sign since by choice we have $|c_2| > 3$.

– **Part B:** $B_2 := B_{21} + B_{22}$ estimate.

Now, on one hand, the integral

$$B_{22} = -s\lambda c_2 \frac{\gamma_2^2}{\sigma^2} \int_0^T \xi_2(t,1)|\psi_{2,x}(t,1)|^2 dt$$

is positive since $c_2 < 0$ and we shall use this term to absorb the negative integral

$$B_{21} = -s\lambda c_1 \gamma_1^2 \int_0^T \xi_2(t,1)|\psi_{1,x}(t,1)|^2 dt.$$

To see that, thanks to the boundary conditions (2.10), we can ensure that

$$\gamma_1 \psi_{1,x}(t,1) = -\alpha\psi_2(t,1) - \frac{\gamma_2}{\sigma}\psi_{2,x}(t,1) + s\lambda\xi_2(t,1)\psi_2(t,1)\left(\gamma_1 c_1 + \frac{\gamma_2}{\sigma} c_2\right).$$

Then, by using Young's inequality, we obtain

$$
\begin{aligned}
B_{21} =& s\lambda c_1 \int_0^T \xi_2(t,1)|\gamma_1\psi_{1,x}(t,1)|^2 dt \\
\leq& 3s\lambda c_1 \int_0^T \xi_2(t,1) \left|\frac{\gamma_2}{\sigma}\psi_{2,x}(t,1)\right|^2 dt + 3s\lambda c_1\alpha^2 \int_0^T \xi_2(t,1)|\psi_2(t,1)|^2 dt \\
&+ 3s^3\lambda^3 c_1 \left(\gamma_1 c_1 + \frac{\gamma_2}{\sigma}c_2\right)^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
\leq& 3s\lambda c_1 \int_0^T \xi_2(t,1) \left|\frac{\gamma_2}{\sigma}\psi_{2,x}(t,1)\right|^2 dt + CT^4 s\lambda c_1\alpha^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
&+ 3s^3\lambda^3 c_1 \left(\gamma_1 c_1 + \frac{\gamma_2}{\sigma}c_2\right)^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt.
\end{aligned} \tag{2.29}
$$

So, adding B_{21} and B_{22}, the fact that $c_1 = 1$, and the choice of σ given by (2.4), yields that

$$
\begin{aligned}
B_2 =& B_{21} + B_{22} \\
\geq& \frac{s\lambda\gamma_1^2}{c_2^2}(|c_2| - 3)\int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt - CT^4 s\lambda\alpha^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt.
\end{aligned} \tag{2.30}
$$

Note that, here we exactly need the assumption $|c_2| > 3$ (in other words, $\kappa > 3$ in (2.4) since $c_2 = -\kappa$) to make the first integral of (2.30) positive.

– **Part C:** Computing each term of $B_3 := B_{31} + B_{32}$, that is, B_{31} and B_{32}.

Let us look at these terms separately. We have

$$
B_{31} = s\lambda c_1\gamma_1^2 \int_0^T \xi_1(t,0)|\psi_{1,x}(t,0)|^2 dt \geq 0, \tag{2.31}
$$

since $c_1 = 1$ and $\xi_1(t,0) > 0$. Now, using the expression (2.9), we find

$$
\psi_{2,x}(t,x) = e^{-s\eta_2(t,x)}\varphi_{2,x}(t,x) - se^{-s\eta_2(t,x)}\eta_{2,x}(t,x)\varphi_2(t,x).
$$

But $\varphi_2(t,0) = 0$ and so,

$$
\psi_{2,x}(t,0) = e^{-s\eta_2(t,0)}\varphi_{2,x}(t,0).
$$

Therefore, we have

$$
\begin{aligned}
|B_{32}| =& s\lambda \left| c_2\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,0)e^{-2s\eta_2(t,0)}|\varphi_{2,x}(t,0)|^2 dt \right| \\
\leq& \frac{s\lambda\gamma_1^2}{|c_2|}\int_0^T \xi_2(t,0)e^{-2s\eta_2(t,0)}|\varphi_{2,x}(t,0)|^2 dt,
\end{aligned}
$$

which is indeed the observation integral for our final Carleman estimate.

– **Part D:** Analysing the terms B_{41} and B_{42}.

Next, we will compute the fourth boundary term as follows. First, observe that

$$
\begin{aligned}
|B_{41}| =& \left| s\lambda^2 c_1^2\gamma_1^2 \mathrm{Re}\int_0^T \xi_2(t,1)\psi_2(t,1)\overline{\psi_{1,x}(t,1)}dt \right| \\
\leq& \epsilon s\lambda \int_0^T \xi_2(t,1)|\gamma_1\psi_{1,x}(t,1)|^2 dt + \frac{C}{\epsilon}s\lambda^3 T^4 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt,
\end{aligned}
$$

where $C = C(\gamma_1) > 0$. Here we have used Young's inequality, the estimate $\xi_2 \leq CT^4\xi_2^3$, and the fact that $c_1 = 1$. In the above estimate, we apply (2.29), so that one has

$$\begin{aligned}|B_{41}| \leq &3\epsilon s\lambda \int_0^T \xi_2(t,1) \left|\frac{\gamma_2}{\sigma}\psi_{2,x}(t,1)\right|^2 dt \\ &+ \left(C\epsilon T^4\alpha^2 s\lambda + \frac{C}{\epsilon}s\lambda^3 T^4\right)\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\ =&\frac{3\epsilon s\lambda\gamma_1^2}{c_2^2}\int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt \\ &+ \left(C\epsilon T^4\alpha^2 s\lambda + \frac{C}{\epsilon}s\lambda^3 T^4\right)\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt,\end{aligned} \tag{2.32}$$

where we have used the choice of σ as given by (2.4) and c_2 as (2.5).

Similarly, one can estimate the term B_{42} as follows:

$$\begin{aligned}|B_{42}| \leq &\epsilon s\lambda|c_2|\frac{\gamma_2^2}{\sigma^2}\int_0^T \xi_2(t,1)|\psi_{2,x}(t,1)|^2 dt + \frac{C}{\epsilon}s\lambda^3 T^4\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\ =&\frac{\epsilon s\lambda\gamma_1^2}{|c_2|}\int_0^T \xi_2(t,1)|\psi_{2,x}(t,1)|^2 dt + \frac{C}{\epsilon}s\lambda^3 T^4\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt,\end{aligned} \tag{2.33}$$

finalizing this part.

– **Part E:** Analysis of $B_5 := B_{51} + B_{52}$.

Let us look into the terms of B_5. To do that, these terms can be viewed as

$$\begin{aligned}B_5 =&B_{51} + B_{52} \\ =&s\sum_{j=1}^{2}\nu_j \mathrm{Im}\int_0^T \psi_j(t,1)\overline{\psi_{j,t}(t,1)}\eta_{j,x}(t,1)dt \\ =&-s\lambda\left(c_1\gamma_1 + c_2\frac{\gamma_2}{\sigma}\right)\mathrm{Im}\int_0^T \psi_1(t,1)\overline{\psi_{1,t}(t,1)}\xi_1(t,1)dt,\end{aligned} \tag{2.34}$$

where we have used that $\eta_{j,x} = -\lambda\xi_j c_j$.

It is important to point out that this term is difficult to absorb in terms of the leading terms because of the appearance time derivative term $\overline{\psi_{1,t}(t,1)}$, and this is the main reason why we have chosen the parameter σ as in (1.8) (in other words, (2.4)) and $c_2 = -\kappa$ in (2.5). Thanks to those choices, one readily has

$$\left(c_1\gamma_1 + \frac{c_2\gamma_2}{\sigma}\right) = 0,$$

once $c_1 = 1$, and this makes the quantity B_5 equal zero.

– **Part F:** Estimates to B_{61} and B_{62}.

Finally, we compute the terms B_6. Using that

$$|\eta_{j,t}| \leq CT\xi_j^2 \quad \text{and} \quad \sigma = \frac{\kappa\gamma_2}{\gamma_1} = -\frac{c_2\gamma_2}{\gamma_1},$$

we get

$$\begin{aligned}|B_{62}| \leq &CTs\frac{\gamma_2}{\sigma}\int_0^T |\psi_2(t,1)||\psi_{2,x}(t,1)||\xi_2(t,1)|^2 dt \\ \leq &\frac{\gamma_1^2}{c_2^2}\int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt + Cs^2T^2\int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt.\end{aligned} \tag{2.35}$$

Using the expression of $\gamma_1\psi_{1,x}(t,1)$ from (2.10), we further get

$$
\begin{aligned}
|B_{61}| \leq & CTs\gamma_1 \int_0^T |\psi_2(t,1)||\psi_{1,x}(t,1)|\xi_2^2(t,1)dt \\
\leq & \int_0^T \xi_2(t,1)|\gamma_1\psi_{1,x}(t,1)|^2 dt + Cs^2T^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
\leq & \frac{3\gamma_1^2}{c_2^2} \int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt + C\left(\alpha^2T^4 + s^2T^2\right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt.
\end{aligned} \tag{2.36}
$$

Here, thanks to the quantity (2.29) we can find the estimate for the following term

$$
\int_0^T \xi_2(t,1)|\gamma_1\psi_{1,x}(t,1)|^2 dt.
$$

• **Step 6.** Getting rid of the lower-order boundary integrals.

Finally, thanks to the computation of the boundary terms (step 5), we can add all the parts of this step to have the estimate for B_k for $1 \leq k \leq 6$. Putting together (2.28), (2.30), (2.31), (2.32), (2.33), (2.34), (2.35) and (2.36), we find that

$$
\begin{aligned}
\sum_{k=1}^{6} B_k \geq & s^3\lambda^3\gamma_1^2\left(|c_2|-1\right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
& + \frac{s\lambda\gamma_1^2}{c_2^2}\left(|c_2|-3\right) \int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt \\
& - CT^4 s\lambda\alpha^2 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
& - \frac{\epsilon s\lambda\gamma_1^2}{c_2^2}\left(3+|c_2|\right) \int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt \\
& - \left(C\epsilon T^4\alpha^2 s\lambda + \frac{2C}{\epsilon} s\lambda^3 T^4\right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
& - \frac{4\gamma_1^2}{c_2^2} \int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt \\
& - C\left(\alpha^2T^4 + s^2T^2\right) \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dt \\
& - \frac{s\lambda\gamma_1^2}{|c_2|} s\lambda \int_0^T \xi_2(t,0)e^{-2s\eta_2(t,0)}|\varphi_{2,x}(t,0)|^2 dt.
\end{aligned} \tag{2.37}
$$

In the above we fix $\epsilon > 0$ small enough and choose $\lambda \geq \lambda_0$ and $s \geq s_0 = \mu_0(T+T^2)$ (for $\lambda_0 > 0$, $\mu_0 > 0$ large enough) so that all the lower order boundary integrals can be absorbed by the first two leading integrals of (2.37).

As a consequence, from (2.27), one has

$$
\begin{aligned}
& s^3\lambda^4 \iint_{Q_T} \left(\xi_1^3|\psi_1|^2 + \xi_2^3|\psi_2|^2\right) dxdt + s\lambda^2 \iint_{Q_T} \left(\xi_1|\psi_{1,x}|^2 + \xi_2^3|\psi_{2,x}|^2\right) dxdt \\
& + s^3\lambda^3 \int_0^T \xi_2^3(t,1)|\psi_2(t,1)|^2 dxdt + s\lambda \int_0^T \xi_2(t,1)\left|\psi_{2,x}(t,1)\right|^2 dt \\
\leq & C \iint_{Q_T} (|F_1|^2 + |F_2|^2) dxdt + Cs\lambda \int_0^T e^{-2s\eta_2(t,0)}\xi_2(t,0)|\varphi_{2,x}(t,0)|^2 dt,
\end{aligned}
$$

for all $\lambda \geq \lambda_0$ and $s \geq s_0$. Therefore, the previous inequality gives us the required Carleman estimate (2.8), and the proof of Theorem 2.5 is complete. □

2.2. **Observability inequality and its application.** This section is devoted to proving a suitable observability inequality as a consequence of the Carleman estimate (2.8), which is the key point to deduce the null-controllability of the system (1.2).

Proposition 2.6. *For any $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}$, the associated solution of the adjoint system (2.3)*

$$\varphi := (\varphi_1, \varphi_2) \in C([0,T]; \mathcal{H})$$

satisfies the following boundary observation

$$\|\varphi(0)\|^2_{\mathcal{H}} \leq C \int_0^T |\partial_x \varphi_2(t,0)|^2 dt. \tag{2.38}$$

Proof. Consider $\zeta \in \mathcal{H}$. Thanks to the Carleman inequality, given by Theorem 2.5 to the solution (φ_1, φ_2) of (2.3) with

$$L_1\varphi_1 = \alpha_1\varphi_1 \quad \text{and} \quad L_2\varphi_2 = \frac{\alpha_2\varphi_2}{\sigma} \quad \left(\sigma = \frac{\kappa\gamma_2}{\gamma_1}\right),$$

gives us that

$$\begin{aligned} &s^3\lambda^4 \iint_{Q_T} \left(e^{-2s\eta_1}\xi_1^3|\varphi_1|^2 + e^{-2s\eta_2}\xi_2^3|\varphi_2|^2\right) dxdt \\ &+ s\lambda^2 \iint_{Q_T} \left(e^{-2s\eta_1}\xi_1|\varphi_{1,x}|^2 + e^{-2s\eta_2}\xi_2|\varphi_{2,x}|^2\right) dxdt \\ \leq &C \iint_{Q_T} (e^{-2s\eta_1}|\varphi_1|^2 + e^{-2s\eta_2}|\varphi_2|^2) dxdt + Cs\lambda \int_0^T e^{-2s\eta_2(t,0)}\xi_2(t,0)|\varphi_{2,x}(t,0)|^2 dt. \end{aligned} \tag{2.39}$$

Now, using the fact that $1 \leq 8T^6\xi_j^3$ for $j = 1, 2$, we can easily absorb the first two integrals of the r.h.s. of (2.39) by the associated leading integrals for any $s \geq CT^2$. Thus, we have the following inequality from (2.39),

$$\begin{aligned} &s\lambda^2 \iint_{Q_T} \left(e^{-2s\eta_1}\xi_1|\varphi_{1,x}|^2 + e^{-2s\eta_2}\xi_2|\varphi_{2,x}|^2\right) dxdt \\ &\qquad \leq Cs\lambda \int_0^T \xi_2(t,0)e^{-2s\eta_2(t,0)}|\varphi_{2,x}(t,0)|^2 dt. \end{aligned} \tag{2.40}$$

Moreover, for some $m, M > 0$, we have the following relations

$$e^{-2s\eta_j}\xi_j \geq m \ \text{ in } (T/4, 3T/4) \times (0,1), \quad \text{and} \quad e^{-2s\eta_j}\xi_j \leq M \ \text{ in } Q_T, \quad \text{for } j = 1, 2,$$

which together with (2.40), yields that

$$\int_{T/4}^{3T/4} \int_0^1 \left(|\varphi_{1,x}|^2 + |\varphi_{2,x}|^2\right) dxdt \leq C \int_0^T |\varphi_{2,x}(t,0)|^2 dt.$$

Then, from the estimate given by (A.3), one can deduce that

$$\|\varphi(0)\|^2_{\mathcal{H}} \leq C_T \int_{T/4}^{3T/4} \int_0^1 \left(|\varphi_{1,x}|^2 + |\varphi_{2,x}|^2\right) dxdt \leq C \int_0^T |\varphi_{2,x}(t,0)|^2 dt,$$

and hence the required observability inequality (2.38) follows. □

We are now in a position to prove the first main result.

Proof of Theorem 1.1. The proof follows the classical Hilbert uniqueness method introduced by Lions [25]. Once we have the above observability inequality, then one can prove the existence of a boundary control $h \in L^2(0,T)$ such that (u, v) solutions of (1.2) with boundary conditions (1.3)-(1.4b) satisfies (1.5). □

3. Rapid exponential stabilization

This section is devoted to studying the boundary stabilization issues for (1.2)-(1.3) with a single boundary control force acting on the component v, namely (1.4b). More precisely, we construct a stationary feedback law $h(t)$, of the form $F_\omega(u(t,\cdot), v(t,\cdot))$ such that the solution of the closed-loop system decays exponentially to zero at any prescribed decay rate.

The approach employed in this section was first introduced by Komornik in [20] and has also been studied by Vaste [38] and Urquiza [37], which one will be applied in our context and is the key argument of this section.

3.1. **Gramian method.** Let us consider the abstract control system

$$\begin{cases} \dot{y}(t) = \mathcal{A}y(t) + \mathcal{B}h(t), & t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{3.1}$$

where $y(t) \in \mathcal{H}, y_0 \in \mathcal{H}, h \in L^2(0,T)$, $\mathcal{B}$ is an unbounded operator from $\mathbb{C}$ to $\mathcal{H}$. $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded operator and $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. To employ the method of Urquiza, one needs to take the following assumptions on the operator $\mathcal{A}$ and $\mathcal{B}$

(H1) The skew-adjoint operator $\mathcal{A}$ is an infinitesimal generator of a strongly continuous group $\{e^{t\mathcal{A}}\}_{t\in\mathbb{R}}$ on $\mathcal{H}$.
(H2) The operator $\mathcal{B} : \mathbb{C} \to \mathcal{D}(\mathcal{A}^*)'$ is linear and continuous.
(H3) *(Regularity property)* For every $T > 0$ there exists $C(T) > 0$ such that

$$\int_0^T \left|\mathcal{B}^* e^{-t\mathcal{A}^*} y\right|^2 dt \le C \|y\|_{\mathcal{H}}^2, \quad \forall\, y \in \mathcal{D}(\mathcal{A}^*).$$

(H4) *(Controllability property)* There are two constants $T > 0$ and $c(T) > 0$ such that

$$\int_0^T \left|\mathcal{B}^* e^{-t\mathcal{A}^*} y\right|^2 dt \ge c \|y\|_{\mathcal{H}}^2, \quad \forall\, y \in \mathcal{D}(\mathcal{A}^*).$$

With these hypotheses in hand, the next result holds (for details, see [37], Theorem 2.1). Its proof mainly relies on general results about the algebraic Riccati equation associated with the linear quadratic regulator problem (see [16]).

Theorem 3.1. *Consider operators $\mathcal{A}$ and $\mathcal{B}$ under assumptions (H1)-(H4). For any $\omega > 0$, we have*

(i) *The symmetric positive operator Λ_ω defined by*

$$\langle \Lambda_\omega x, z\rangle_{\mathcal{H}} = \int_0^\infty \left\langle \mathcal{B}^* e^{-\tau(\mathcal{A}+\omega I)^*} x, \mathcal{B}^* e^{-\tau(\mathcal{A}+\omega I)^*} z \right\rangle_{\mathbb{C}} d\tau, \,\forall\, x, z \in \mathcal{H}$$

is coercive and is an isomorphism on $\mathcal{H}$.

(ii) *Let $F_\omega := -\mathcal{B}^*\Lambda_\omega^{-1}$. The operator $\mathcal{A}+\mathcal{B}F_\omega$ with $\mathcal{D}(\mathcal{A}+\mathcal{B}F_\omega) = \Lambda_\omega(\mathcal{D}(\mathcal{A}^*))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.*

(iii) *The closed-loop system (3.1) with $h = F_\omega(y)$ is exponentially stable, that is,*

$$\left\| e^{t(\mathcal{A}+\mathcal{B}F_\omega)} y \right\|_{\mathcal{H}} \le C e^{(-2\omega + g(-\mathcal{A}))t} \|y\|_{\mathcal{H}}, \forall y \in \mathcal{H},$$

where C is a positive constant.

We will use Theorem 3.1 to prove the exponential stabilization of the coupled Schrodinger equation (1.2)-(1.3)-(1.4b) with boundary feedback law. To apply it, we need to verify all the assumptions (H1)–(H4) hold for our system (1.2). Let us do it in the next subsection.

3.2. **Verification of the hypotheses.** It is well-known that a fundamental solution of the Schrödinger system can be obtained by the Fourier expansion, see for instance [12,18]. So, in this way, considering the eigenvalues and the eigenfunctions that form an orthonormal basis of $L^2(\Omega)$, we can define in $\mathcal{H}$ an inner product similarly as in [9].

Notice that we can also find a representation by Fourier series for the solutions of the system (1.2) (see e.g. [18], for one-dimensional Schrödinger equation) and that the operator $\mathcal{A}$, defined by (A.1)-(A.2), is skew-adjoint and hence generates an infinitesimal generator of a group $\{S(t)\}_{t\in\mathbb{R}}$, thus (H1) follows. Also note that $g(-\mathcal{A}) = 0$, where g is the growth bound of the semigroup generated by $\mathcal{A}$. Moreover, comparing the abstract system (3.1) with our system (1.2), the control operator $\mathcal{B} \in \mathcal{L}(\mathbb{C}; \mathcal{D}(\mathcal{A}^*)')$ can be given as follows

$$\langle \mathcal{B}s, (\phi_1, \phi_2)\rangle_{\mathcal{D}(\mathcal{A}^*)', \mathcal{D}(\mathcal{A}^*)} = s\phi_2'(0), \quad s \in \mathbb{C}, \;(\phi_1, \phi_2) \in \mathcal{D}(\mathcal{A}^*),$$

and therefore, (H2) is verified. Additionally, note that the observability inequality (2.38) gives directly (H4).

It remains for us to prove the hypothesis (H3), that is, to prove the trace regularity. This hypothesis follows from the next proposition and is a consequence of Lemma A.4.

Proposition 3.2. *For every $T>0$ there exists $C>0$ such that the following holds*

$$\int_0^T |\varphi_{2,x}(t,0)|^2 dt \le C \left\|(\zeta_1,\zeta_2)\right\|_{\mathcal{H}}, \tag{3.2}$$

for every solution (φ_1,φ_2) of the adjoint problem (2.3) with (ζ_1,ζ_2) lies in sufficiently regular space.

Proof. Choose $m\in C^2(0,1)$ such that $m(1)=0, m(0)>0$, and $m'(1)=1$. Thus, from (A.4), it follows that

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{2}\nu_j\int_0^T|\varphi_{j,x}(t,0)|^2 m(0)dt \le{}& C_1\left\|m\right\|_{L^\infty(0,1)}\Big[\left\|(\varphi_1(T),\varphi_2(T))\right\|^2_{L^2(0,1)}\\&+\left\|(\varphi_1(T),\varphi_2(T))\right\|^2_{\mathcal{H}}+\left\|(\varphi_1(0),\varphi_2(0))\right\|^2_{L^2(0,1)}+\left\|(\varphi_1(0),\varphi_2(0))\right\|^2_{\mathcal{H}}\Big]\\&+C_2\left\|m\right\|_{W^{1,\infty}(0,1)}\left(\int_0^T\left\|(\varphi_1(t),\varphi_2(t))\right\|^2_{\mathcal{H}}dt\right)\\&+C_3\left\|m\right\|_{W^{2,\infty}(0,1)}\left(\int_0^T\left\|(\varphi_1(t),\varphi_2(t))\right\|_{L^2(0,1)}\left\|(\varphi_1(t),\varphi_2(t))\right\|_{\mathcal{H}}dt\right).\end{aligned} \tag{3.3}$$

Note that, using boundary conditions of (2.3), one can prove that the last term of the identity (A.4) is negative. Moreover, using the classical conservation of the energy for the Schrödinger equation we have:

$$\begin{aligned}&\left\|(\varphi_1(t),\varphi_2(t))\right\|_{L^2(0,1)}=\left\|(\varphi_1(0),\varphi_2(0))\right\|_{L^2(0,1)},\forall t\in[0,T],\\&\left\|(\varphi_1(t),\varphi_2(t))\right\|_{\mathcal{H}}=\left\|(\varphi_1(0),\varphi_2(0))\right\|_{\mathcal{H}},\forall t\in[0,T].\end{aligned}$$

Therefore from (3.3), we have the following result (3.2) when (ζ_1,ζ_2) is sufficiently smooth. Using a density argument we get the result when $(\zeta_1,\zeta_2)\in\mathcal{H}$, showing the lemma, and consequently (H3) is achieved. □

3.3. **Proof of Theorem 1.2.** In this section, we employ Urquiza's method to construct the feedback law for our system (1.2) to ensure the exponential decay (1.9).

To do that, let us consider $\Phi_0^1=(\varphi_0,\psi_0)$ and $\Phi_0^2=(r_0,s_0)$ belonging of $\mathcal{H}$, and define the following bilinear form

$$a_w(\Phi_0^1,\Phi_0^2)=\int_0^\infty e^{-2\omega t}\psi_x(t,0)\overline{s_x(t,0)}\,dt,$$

where $\Phi^1=(\varphi,\psi)$ and $\Phi^2=(r,s)$ are the solutions of the following systems respectively

$$\begin{cases}i\varphi_t+\gamma_1\varphi_{xx}-\alpha_1\varphi=0, & \text{in } Q_T,\\ i\sigma\psi_t+\gamma_2\psi_{xx}-\alpha_2\psi=0, & \text{in } Q_T,\\ \varphi(t,1)=\psi(t,1), & \text{in } (0,T),\\ \gamma_1\varphi_x(t,1)+\frac{\gamma_2}{\sigma}\psi_x(t,1)+\alpha\varphi(t,1)=0, & \text{in } (0,T),\\ \varphi(t,0)=0,\ \psi(t,0)=0, & \text{in } (0,T),\\ \varphi(T,x)=\varphi_0(x),\ \psi(T,x)=\psi_0(x), & \text{in } \Omega\end{cases}$$

and

$$\begin{cases}ir_t+\gamma_1 r_{xx}-\alpha_1 r=0, & \text{in } Q_T,\\ i\sigma s_t+\gamma_2 s_{xx}-\alpha_2 s=0, & \text{in } Q_T,\\ r(t,1)=s(t,1), & \text{in } (0,T),\\ \gamma_1 r_x(t,1)+\frac{\gamma_2}{\sigma}s_x(t,1)+\alpha r(t,1)=0, & \text{in } (0,T),\\ r(t,0)=0,\ s(t,0)=0, & \text{in } (0,T),\\ r(T,x)=r_0(x),\ s(T,x)=s_0(x), & \text{in } \Omega.\end{cases}$$

Let us define the operator $\Lambda_\omega:\mathcal{H}\to\mathcal{H}'$ satisfying the following

$$\left\langle\Lambda_\omega\Phi_0^1,\Phi_0^2\right\rangle_{\mathcal{H}',\mathcal{H}}=a_\omega(\Phi_0^1,\Phi_0^2),\quad\forall\Phi_0^1,\Phi_0^2\in\mathcal{H}. \tag{3.4}$$

Next, we see that

$$\begin{aligned} a_w(\Phi_0^1, \Phi_0^2) &= \int_0^\infty e^{-2\omega t} \psi_x(t,0)\overline{s_x(t,0)}\, dt \\ &= \int_0^\infty e^{-2\omega t} \mathcal{B}^*\Phi^1(t)\ \overline{\mathcal{B}^*\Phi^2(t)}\, dt \\ &= \int_0^\infty e^{-2\omega t} \left(\mathcal{B}^* S(T-t)^* S(-T)^* \Phi_0^1\right) \overline{(\mathcal{B}^* S(T-t)^* S(-T)^* \Phi_0^2)}\, dt \\ &= \int_0^\infty e^{-2\omega t} \left(\mathcal{B}^* S(-t)^* \Phi_0^1\right) \overline{(\mathcal{B}^* S(-t)^* \Phi_0^2)}\, dt. \end{aligned}$$

Therefore from (3.4), we have

$$\left\langle \Lambda_\omega \Phi_0^1, \Phi_0^2 \right\rangle_{\mathcal{H}',\mathcal{H}} = \int_0^\infty e^{-2\omega t} \left\langle \mathcal{B}^* \mathbb{T}_{-t}^* \Phi_0^1,\, \mathcal{B}^* \mathbb{T}_{-t}^* \Phi_0^2 \right\rangle_{\mathbb{C}} dt.$$

Thanks to Theorem 3.1, the operator Λ_ω, defined by (3.4), is coercive and isomorphism. Finally, let us define the operator $F_\omega : \mathcal{H} \to \mathbb{C}$ by

$$F_\omega(\mathbf{z}) = -\psi_0'(0),$$

where $\Phi_0^1 = (\varphi_0, \psi_0)$ is the solution of the following Lax-Milgram problem

$$a_\omega(\Phi_0^1, \Phi_0^2) = \left\langle \mathbf{z}, \Phi_0^2 \right\rangle, \ \forall\, \Phi_0^2 \in \mathcal{H}.$$

Hence, we obtain

$$\left\langle \Lambda_\omega \Phi_0^1, \Phi_0^2 \right\rangle_{\mathcal{H}',\mathcal{H}} = \left\langle \mathbf{z}, \Phi_0^2 \right\rangle, \forall\, \Phi_0^2 \in \mathcal{H}.$$

This gives $\Lambda_\omega \Phi_0^1 = \mathbf{z}$. It follows that $\Phi_0^1 = \Lambda_\omega^{-1}\mathbf{z}$. Thus we have $F_\omega = -\mathcal{B}^*\Lambda_\omega^{-1}$. Thanks to Theorem 3.1, rapid exponential stabilization for the system (1.2) is established using the feedback law $h(t) = F_\omega(u(t,\cdot), v(t,\cdot))$. More precisely, we get a positive constant C such that the solution of (1.2) satisfies the estimate (1.9), showing Theorem 1.2. □

4. Further comments and open problems

In this paper, we considered the boundary control problems for coupled Schrödinger equations through the Kirchhoff boundary conditions in a one-dimensional case. The first result is obtained showing a new Carleman estimate with boundary observation. Moreover, with this in hand, together with other hypotheses over the operator $\mathcal{A}$, the second result ensures that the solutions of the system decay exponentially with a decay rate of at least $e^{-2\omega t}$.

Concerning Theorems 1.1 and 1.2 it is natural to ask whether they remain valid in nonlinear framework (1.1). Due to the lack of regularity, we are not able to extend our result for the nonlinear case yet, and this issue remains open.

Finally, we point out that in a forthcoming paper boundary conditions different of (1.3)-(1.4a)/(1.4b) will be considered, which will give a more general view of the boundary control problems, at least, for the system (1.2).

Statements and Declarations

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Appendix A. Auxiliary results

In this appendix, we briefly discuss the well-posedness of the control system (1.2). Additionally, to show that the hypothesis (H3) of Urquiza's approach is verified, we need a key lemma which will be proved in the second part of this appendix.

A.1. **Well-posedness results.** Consider the following operator associated with the control system (1.2), given by

$$\mathcal{A} = \begin{pmatrix} i\gamma_1\partial_{xx} - i\alpha_1\mathbb{I}_d & \mathbf{0} \\ \mathbf{0} & i\frac{\gamma_2}{\sigma}\partial_{xx} - i\frac{\alpha_2}{\sigma}\mathbb{I}_d \end{pmatrix}, \tag{A.1}$$

with

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \big\{(u_1,u_2) \in [H^2(\Omega)]^2 \mid u_1(0) = u_2(0) = 0,\ u_1(1) = u_2(1),\\ \gamma_1 u_{1,x}(1) + \frac{\gamma_2}{\sigma}u_{2,x}(1) + \alpha u_1(1) = 0\big\}.\end{aligned} \tag{A.2}$$

Let us consider the space (1.6) as the completion of $\mathcal{D}(\mathcal{A})$ with the norm (1.7) and the associated inner product in $\mathcal{H}$ is defined by (1.6).With this in hand, the first result shows that the operator in consideration is dissipative.

Proposition A.1. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates a strongly continuous unitary group in $[L^2(\Omega)]^2$.*

Proof. Let us consider $\mathbf{U} = (u,v) \in \mathcal{D}(\mathcal{A})$. A simple computation gives that

$$\begin{aligned}\text{Re}\,\langle \mathcal{A}\mathbf{U},\mathbf{U}\rangle_{[L^2(0,1)]^2} =&\text{Re}\Big[i\gamma_1\int_0^1 u_{xx}\overline{u}dx - i\alpha_1\int_0^1 |u|^2dx + i\frac{\gamma_2}{\sigma}\int_0^1 v_{xx}\overline{v}dx - i\frac{\alpha_2}{\sigma}\int_0^1 |u|^2dx\Big]\\ =&\text{Re}\Big[-i\gamma_1\int_0^1 |u_x|^2dx - i\frac{\gamma_2}{\sigma}\int_0^1 |v_x|^2dx + i\gamma_1 u_x(1)\overline{u(1)} + i\frac{\gamma_2}{\sigma}v_x(1)\overline{v(1)}\Big]\\ =&\text{Re}\Big[i\gamma_1 u_x(1)\overline{u(1)} + i\frac{\gamma_2}{\sigma}v_x(1)\overline{v(1)}\Big]\\ =&\text{Re}\Big[-i\alpha|u(1)|^2\Big] = 0.\end{aligned}$$

By using semigroup theory, $\mathcal{A}$ generates a strongly continuous unitary group on $[L^2(\Omega)]^2$. Moreover it can be easily checked that for all $(u,v) \in \mathcal{D}(\mathcal{A})$,

$$\langle \mathcal{A}(u,v),(u,v)\rangle = -\langle (u,v),\mathcal{A}(u,v)\rangle$$

and also $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$. Therefore $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ is a skew adjoint operator. □

Note that, considering $(u_0,v_0) \in \mathcal{D}(\mathcal{A})$, then (1.2) has a classical solution satisfying $(u,v) \in C([0,T];\mathcal{D}(\mathcal{A})) \cap C^1([0,T];[L^2(\Omega)]^2)$, or, in more general way:

Proposition A.2. *Let $(u_0,v_0) \in [H_0^s(\Omega)]^2$, for $s \geq 0$. Then the linear system (1.2) with boundary conditions (1.3)-(1.4a) (or (1.3)-(1.4b)), with $h(t) = 0$, has a unique solution (u,v) on the space $C([0,T];[H_0^s(\Omega)]^2)$.*

A.2. **Adjoint system.** Let us remember that the adjoint associated with the system (1.2) with (1.3)–(1.4a) or (1.3)–(1.4b), is given by (2.3), with given final data $\zeta := (\zeta_1,\zeta_2)$ from some suitable Hilbert space. Note that, since $\mathcal{A} = -\mathcal{A}^*$ we have that $\mathcal{D}(\mathcal{A})^* = \mathcal{D}(\mathcal{A})$, so, the following result shows the well-posedness for the system (2.3).

Proposition A.3. *For given $\zeta := (\zeta_1,\zeta_2) \in \mathcal{H}$, there exists a unique solution $\varphi := (\varphi_1,\varphi_2) \in C([0,T];\mathcal{H})$ to the adjoint system (2.3) such that it satisfies*

$$\|\varphi\|_{C([0,T];\mathcal{H})} \leq C\|\zeta\|_{\mathcal{H}}, \tag{A.3}$$

for some constant $C > 0$.

Proof. The proof of well-posedness is the same as done in Proposition A.2. □

Additionally to that, if we consider $(g_1,g_2) \in L^1(0,T;\mathcal{H})$, it can be shown that the solution $(\varphi_1,\varphi_2) \in C([0,T],\mathcal{H})$ of (2.2) satisfies

$$\|(\varphi_1(t),\varphi_2(t))\|_{\mathcal{H}} \leq C||(g_1,g_2)||_{L^1(0,T;\mathcal{H})},$$

A.3. **Key lemma.** This second part of this appendix is devoted to presenting an essential lemma that is a key point in ensuring the hypothesis (H3) in Urquiza's approach.

Lemma A.4. *First, consider a function* $m \in C^2(0,1)$. *Then the solution of* (2.3) *satisfies the following identity:*

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{2}\nu_j\int_0^T |\varphi_{j,x}(t,0)|^2 m(0)dt =& \frac{1}{2}\sum_{j=1}^{2}\nu_j\int_0^T |\varphi_{j,x}(t,1)|^2 m(1)dt \\
&-\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^1 \overline{\varphi_j(T,x)}m(x)\varphi_{j,x}(T,x)dx\right) \\
&+\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^1 \overline{\varphi_j(0,x)}m(x)\varphi_{j,x}(0,x)dx\right) \\
&+\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^T \varphi_{j,t}(t,1)m(1)\overline{\varphi_j(t,1)}dt\right) \\
&-\frac{1}{2}\mathrm{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T\int_0^1 \varphi_{j,x}m'(x)\overline{\varphi_{j,x}(t,x)}dxdt\right) \\
&-\frac{1}{2}\mathrm{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T\int_0^1 \varphi_{j,x}m''(x)\overline{\varphi_j(t,x)}dxdt\right) \\
&+\frac{1}{2}\mathrm{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T \varphi_{j,x}(t,1)m'(1)\overline{\varphi_j(t,1)}dt\right), \\
&-\frac{1}{2}\mathrm{Re}\left(\sum_{j=1}^{2}\theta_j\int_0^T m(1)|\varphi_j(t,1)|^2dt\right),
\end{aligned} \tag{A.4}$$

where $\nu_1 = \gamma_1, \nu_2 = \frac{\gamma_2}{\sigma}, \theta_1 = \alpha_1, \theta_2 = \frac{\alpha_2}{\sigma}$.

Proof. Multiply the equations (2.3) by $(m\overline{\varphi_{1,x}} + \frac{1}{2}\overline{\varphi_1}m')$ and $(m\overline{\varphi_{1,x}} + \frac{1}{2}\overline{\varphi_1}m')$ respectively and using integration by parts along with boundary conditions, we have

$$\begin{aligned}
\mathrm{Re}\sum_{j=1}^{2} i\int_0^T\int_0^1 \varphi_{j,t}\left(\frac{1}{2}m'(x)\overline{\varphi_j(t,x)} + m(x)\overline{\varphi_{j,x}(t,x)}\right)dxdt = \\
-\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^1 \overline{\varphi_j(T,x)}m(x)\varphi_{j,x}(T,x)dx\right) \\
+\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^1 \overline{\varphi_j(0,x)}m(x)\varphi_{j,x}(0,x)dx\right) \\
+\frac{1}{2}\mathrm{Im}\left(\sum_{j=1}^{2}\int_0^1 \varphi_{j,t}(t,1)m(1)\overline{\varphi_j(t,1)}dt\right).
\end{aligned}$$

Similarly, we have from the second term of both equations:

$$\begin{aligned}
\operatorname{Re}\sum_{j=1}^{2}\nu_j\int_0^T\int_0^1\varphi_{j,xx}\left(\frac{1}{2}m'(x)\overline{\varphi_j(t,x)}+m(x)\overline{\varphi_{j,x}(t,x)}\right)dxdt &= \frac{1}{2}\sum_{j=1}^{2}\nu_j\int_0^T|\varphi_{j,x}(t,1)|^2m(1)dt\\
&-\frac{1}{2}\operatorname{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T\int_0^1\varphi_{j,x}m'(x)\overline{\varphi_{j,x}(t,x)}dxdt\right)\\
&-\frac{1}{2}\operatorname{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T\int_0^1\varphi_{j,x}m''(x)\overline{\varphi_j(t,x)}dxdt\right)\\
&+\frac{1}{2}\operatorname{Re}\left(\sum_{j=1}^{2}\nu_j\int_0^T\varphi_{j,x}(t,1)m'(1)\overline{\varphi_j(t,1)}dt\right)\\
&-\frac{1}{2}\left(\sum_{j=1}^{2}\nu_j\int_0^T m(0)|\varphi_{j,x}(t,0)|^2dt\right).
\end{aligned}$$

Also, we have

$$\operatorname{Re}\sum_{j=1}^{2}\theta_j\int_0^T\int_0^1\varphi_j\left(\frac{1}{2}m'(x)\overline{\varphi_j(t,x)}+m(x)\overline{\varphi_{j,x}(t,x)}\right)dxdt=\left(\sum_{j=1}^{2}\theta_j\int_0^T m(1)|\varphi_j(t,1)|^2dt\right).$$

Putting together the previous relation (A.4) holds and the lemma is finished. □

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