GLOBAL CONTROL ASPECTS FOR LONG WAVES IN NONLINEAR DISPERSIVE MEDIA

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Abstract. A class of models of long waves in dispersive media with coupled quadratic nonlinearities on a periodic domain $\mathbb{T}$ are studied. We used two distributed controls, supported in $\omega \subset \mathbb{T}$ and assumed to be generated by a linear feedback law conserving the *"mass" (or "volume")*, to prove global control results. The first result, using spectral analysis, guarantees that the system in consideration is locally controllable in $H^s(\mathbb{T})$, for $s \geq 0$. After that, by certain properties of Bourgain spaces we show a property of global exponential stability. This property together with the local exact controllability ensures for the first time in the literature that long waves in nonlinear dispersive media are globally exactly controllable in large time. Precisely, our analysis relies strongly on the *bilinear estimates* using the Fourier restriction spaces in two different dispersions that will guarantee a global control result for coupled systems of the Korteweg–de Vries type. This result, of independent interest in the area of control of coupled dispersive systems, provides a necessary first step for the study of global control properties to the coupled dispersive systems in periodic domains.

1. Introduction

Nonlinear dispersive wave equations arise in a number of important application areas. Because of this, and because their mathematical properties are interesting and subtle, they have seen enormous development since the 1960s when they first came to the fore[1]. The theory for a single nonlinear dispersive wave equation is well developed by now, though there are still interesting open issues, however the theory for coupled systems of such equations is much less developed, though they, too, arise as models of a range of physical phenomena.

Considered here is a class of such systems, namely coupled Korteweg–de Vries (KdV) equations. The systems we have in mind take the form

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x P(u,v) = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \partial_x Q(u,v) = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \end{cases} \tag{1.1}$$

which comprise two linear Korteweg–de Vries equations coupled through their nonlinearity. Here, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions of variables $(x,t) \in \mathbb{T} \times \mathbb{R}$ and the nonlinearities are taken to be homogeneous quadratic polynomials

$$\begin{cases} P(u,v) = Au^2 + Buv + Cv^2, \\ Q(u,v) = Dv^2 + Evu + Fu^2, \end{cases} \tag{1.2}$$

where A, B, C, D, E and F are constants will be employed when convenient.

As far as we know, there are no studies of the *global control properties* of this kind of coupled systems in a periodic domain. Thus, in this article, the goal is to fill this gap focusing on the global exact controllability and global asymptotic behavior to the solutions of the coupled system of KdV equations (1.1) when we add two control inputs in each equation and considering initial conditions $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ belonging in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, for any $s \geq 0$.

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[1]See [32] for a sketch of the early history of the subject.

1.1. **Models under consideration.** Such systems arise as models for wave propagation in physical systems where both nonlinear and dispersive effects are important. Moreover, their close relatives arise as models for waves in a number of situations. Before to present details about the problem that we will study let us start to list a few specializations of systems (1.1)-(1.2) that appeared in the literature.

1.1.1. *The coupled KdV system.* The classical Boussinesq systems were first derived by Boussinesq in [7], to describe the two-way propagation of small amplitude, long wavelength gravity waves on the surface of water in a canal. These systems and their higher-order generalizations also arise when modeling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. Recently Bona *et al.*, in [2], derived a four-parameter family of Boussinesq systems to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. More precisely, they studied a family of systems of the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{1.3}$$

In (1.3), η is the elevation from the equilibrium position, and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c, d, that one might choose in a given modeling situation, are required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1-\theta^2) \geq 0, \quad \theta \in [0,1],$$

where $\theta \in [0,1]$ specifies which horizontal velocity the variable w represents (cf. [2]). Consequently,

$$a + b + c + d = \frac{1}{3}.$$

As it has been proved in [2], the initial value problem for the linear system associated with (1.3) is well-posed on $\mathbb{R}$ if either C_1 or C_2 is satisfied, where

$$\begin{aligned} &(C_1) \qquad b, d \geq 0, \ a \leq 0, \ c \leq 0; \\ &(C_2) \qquad b, d \geq 0, \ a = c > 0. \end{aligned}$$

When $b = d = 0$ and (C_2) is satisfied, then necessarily $a = c = 1/6$. Nevertheless, the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives an system equivalent to (1.3) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + w_x + w_{xxx} + (\eta w)_x = 0, \\ w_t + \eta_x + \eta_{xxx} + w w_x = 0, \end{cases} \tag{1.4}$$

which is the so-called *Boussinesq system of KdV-KdV type.*

1.1.2. *Gear-Grimshaw system.* In [17] a complex system of equations was derived by Gear and Grimshaw as a model to describe the strong interaction of two-dimensional, weakly nonlinear, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid, where the two waves correspond to different modes. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been the object of intensive research in recent years. The system can be read as follows

$$\begin{cases} u_t + u u_x + u_{xxx} + a v_{xxx} + a_1 v v_x + a_2 (uv)_x = 0, \\ c v_t + r v_x + v v_x + a b u_{xxx} + v_{xxx} + a_2 b u u_x + a_1 b (uv)_x = 0, \end{cases} \tag{1.5}$$

where $a_1, a_2, a, b, c, r \in \mathbb{R}$ are physical constants and we may assume that

$$1 - a^2 b > 0 \quad \text{and} \quad b, c > 0.$$

1.1.3. *Majda-Biello system.* The following coupled system

$$\begin{cases} u_t + u_{xxz} = -vv_x, \\ v_t + \alpha v_{xxx} = -(uv)_x, \end{cases}$$

when $\alpha \in (0,1)$[2], was proposed by Majda and Biello in [31] as a reduced asymptotic model to study the nonlinear resonant interactions of long wavelength equatorial Rossby waves and barotropic Rossby waves.

1.1.4. *Hirota-Satsuma system.* In the eighties, Hirota and Satsuma introduced in [25] the set of two coupled KdV equations, namely

$$\begin{cases} u_t + au_{xxx} = 6auu_x + bvv_x, \\ v_t + v_{xxx} = -3uv_x, \end{cases}$$

with $a \neq 0$, where $a, b \in \mathbb{R}$ are constants that appear in the model deduction. This model describes the interaction of two long waves with different dispersion relations.

We caution that this is only a small sample of the extant equations with the similar structure to the system (1.1)-(1.2). For an extensive review of the physical meanings of these equations, as well as local and global well-posedness results, the authors suggest the following nice two references [3, 43].

1.2. **Setting of the problem.** Since any solution (u,v) of system (1.1) has its components with invariant mean value, we can introduce the numbers $[u] := \beta$ and $[v] := \gamma$. Setting $\widetilde{u} = u - \beta$ and $\widetilde{v} = v - \gamma$, we obtain $[\widetilde{u}] = [\widetilde{v}] = 0$ and $(\widetilde{u}, \widetilde{v})$ solves

$$\begin{cases} \partial_t \widetilde{u} + \partial_x^3 \widetilde{u} + (2\beta A + \gamma B)\partial_x \widetilde{u} + (\beta B + \gamma C)\partial_x \widetilde{v} + \partial_x P(\widetilde{u}, \widetilde{v}) = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t \widetilde{v} + \alpha \partial_x^3 \widetilde{v} + (\beta B + \gamma C)\partial_x \widetilde{u} + (2\gamma D + \beta C)\partial_x \widetilde{v} + \partial_x Q(\widetilde{u}, \widetilde{v}) = 0, & x \in \mathbb{T},\ t \in \mathbb{R}. \end{cases} \tag{1.6}$$

Throughout the paper, we will denote $\mu := 2\beta A + \gamma B$, $\eta := \beta B + \gamma C$, $\zeta := 2\gamma D + \beta C$ which are real constants. Thus, as mentioned before, this article presents for the first time the global control results for a class of models of long waves with coupled quadratic nonlinearities. Precisely, thanks to (1.6) we will study the following system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v + \partial_x P(u,v) = p(x,t), & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u + \partial_x Q(u,v) = q(x,t), & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}, \end{cases} \tag{1.7}$$

with quadratic nonlinearities

$$\begin{cases} P(u,v) = Au^2 + Buv + \frac{C}{2}v^2, \\ Q(u,v) = Dv^2 + Cvu + \frac{B}{2}u^2, \end{cases} \tag{1.8}$$

where α, A, B, C and D real constants, from a control point of view with forcing terms $p = p(x,t)$ and $q = q(x,t)$ added to the equation as two control inputs on the periodic domain. Therefore, the following classical issues related with control theory are considered in this work.

Problem 1.1 (Exact controllability)**.** *Given an initial state (u_0, v_0) and a terminal state (u_1, v_1) in a certain space, can one find two appropriate control inputs p and q so that the equation (1.7) admits a solution (u,v) which satisfies $(u(\cdot,0), v(\cdot,0)) = (u_0,v_0)$ and $(u(\cdot,T), v(\cdot,T)) = (u_1,v_1)$?*

Problem 1.2 (Stabilizability)**.** *Can one find some (linear) feedback controls $p = K_1(u,v)$ and $q = K_2(u,v)$ such that the resulting closed-loop system (1.7) is stabilized, i.e., its solution (u,v) tends to zero in an appropriate space as $t \to \infty$?*

[2]The parameter $\alpha > 0$ depends upon the Rossby wave in question and it typically has a value near 1.

Note that system (1.7) has the *mass (or volume)* and the *energy* conserved, which are

$$M_1(u,v)=\int_{\mathbb{T}} u(x,t)dx, \qquad M_2(u,v)=\int_{\mathbb{T}} v(x,t)dx,$$

and

$$E(u,v)=\frac{1}{2}\int_{\mathbb{T}}(u^2(x,t)+v^2(x,t))dx,$$

respectively. In order to keep the mass M_1 and M_2 conserved, the two control inputs $p(x,t)$ and $q(x,t)$ will are chosen to be of the form $Gf(x,t)$ and $Gh(x,t)$, respectively, where this operator is defined by

$$(G\ell)(x,t):=g(x)\left(\ell(x,t)-\int_{\mathbb{T}} g(y)\ell(y,t)dy\right), \tag{1.9}$$

where f and h are considered the new control inputs, and $g(x)$ is a given nonnegative smooth function such that $\{g>0\}=\omega\subset\mathbb{T}$ and

$$2\pi[g]=\int_{\mathbb{T}} g(x)dx=1.$$

Due to such a choice of g, it is easy to see that for any solution (u,v) of (1.7) with $p=Gf$ and $q=Gh$ we have

$$\frac{d}{dt}M_1(u,v)=\int_{\mathbb{T}} Gf(x,t)dx=0$$

and

$$\frac{d}{dt}M_2(u,v)=\int_{\mathbb{T}} Gh(x,t)dx=0,$$

that is, the *mass* of the system is indeed conserved.

In order to stabilize system (1.7) we want to employ two feedback control laws that help make the energy of the system decreasing, that is,

$$\frac{d}{dt}E(u,v)\leq 0.$$

We will see that this is possible, and so makes sense to show global answers to the Problems 1.1 and 1.2, mentioned before. Before it, let us give a state of the arts of control theory for KdV type systems.

1.3. **State of the art.** The study of the controllability and stabilization to the KdV equation started with the works of Russell and Zhang [38, 39] for the system

$$u_t+uu_x+u_{xxx}=f, \tag{1.10}$$

with periodic boundary conditions and an internal control f. Since then, both controllability and stabilization problems have been intensively studied. In particular, exact boundary controllability of KdV on a finite domain was investigated in e.g. [13, 14, 18, 36, 45] and the internal boundary controllability was studied in [8, 10].

Equation (1.10) is known to possess an infinite set of conserved integral quantities, of which the first three are

$$I_1(t)=\int_{\mathbb{T}} u(x,t)dx, \qquad I_2(t)=\int_{\mathbb{T}} u^2(x,t)dx \quad\text{and}\quad I_3(t)=\int_{\mathbb{T}}\left(u_x^2(x,t)-\frac{1}{3}u^3(x,t)\right)dx.$$

From the historical origins of the KdV equation involving the behavior of water waves in a shallow channel [6, 12, 32, 24], it is natural to think of I_1 and I_2 as expressing conservation of *volume (or mass)* and *energy*, respectively. The Cauchy problem for equation (1.10) has been intensively studied for many years (see [5, 21, 22, 40] and the references therein).

The first work of Russell and Zhang [38] is purely linear. In fact, they had to wait for several years to extend their results to the nonlinear systems [39] until Bourgain [5] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain, thanks to which Bourgain was able to show that the Cauchy problem (1.10) is well-posed in the space $H^s(\mathbb{T})$, for any $s \geq 0$. This novelty discovered the smoothing property of the KdV equation has played a crucial role in the proofs of the results in [39].

Precisely, in [39] the authors studied equation (1.10) from a control point of view with a forcing term $f = f(x,t)$ added to the equation as a control input:

$$u_t + uu_x + u_{xxx} = f, \quad x \in \mathbb{T}, \qquad t \in \mathbb{R}, \tag{1.11}$$

where f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. In order to keep the mass $I_1(t)$ conserved, the control input $f(x,t)$ is chosen to be of the form

$$f(x,t) = [Gh](x,t) := g(x)\left(h(x,t) - \int_{\mathbb{T}} g(y)\, h(y,t)\, dy\right), \tag{1.12}$$

where h is considered as a new control input, and $g(x)$ is a given non-negative smooth function such that $\{g > 0\} = \omega$ and

$$2\pi\,[g] = \int_{\mathbb{T}} g(x)\, dx = 1.$$

For the chosen g, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)\, dx = \int_{\mathbb{T}} f(x,t)\, dx = 0,$$

for any $t \in \mathbb{R}$ and for any solution $u = u(x,t)$ of the system

$$u_t + uu_x + u_{xxx} = Gh.$$

Thus, the mass of the system is indeed conserved. With this in hand Russell and Zhang were able to show the local exact controllability and local exponential stabilizability for the system (1.10). Indeed, the results presented in [39] are essentially linear; they are more or less small perturbation of the linear results. After these works, two natural questions arise, now with global character. These questions have already been cited in this work.

Question 1: *Can one still guide the system by choosing appropriate control input h from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have large amplitude?*

Question 2: *Do the large amplitude solutions of the closed-loop system (1.11) decay exponentially as $t \to \infty$?*

Laurent *et al.* [28] gave the positive answers to these questions. These answers are established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the associated linear system of (1.11).

We have to mention that there are other works in the literature that deal with the models having similar structure as the system (1.1) in periodic domains. Micu *et al.*[33] gave a rather complete picture of the control properties of (1.3) on a periodic domain with a locally supported forcing term. According to the values of the four parameters a, b, c, and d, the linearized system may be controllable in any positive time, or only in large time, or it may not be controllable at all.

Recently, Capistrano-Filho *et al.* [11] considered the problem of controlling pointwise, by means of a time dependent Dirac measure supported by a given point, a coupled system of two Korteweg–de Vries equations (1.5) on the unit circle. More precisely, by means of spectral analysis and Fourier expansion they proved, under general assumptions on the physical parameters of the system, a pointwise observability inequality which leads to the pointwise controllability by using two control functions. In addition, with a uniqueness property proved for the linearized system without control, they are able to show pointwise controllability when only one control function acts internally.

There are two important points to say about the results shown in [33] and [11]. The first one is that the results presented in [33] are purely local (controllability and stability), the authors did not use propagation of singularities, provided by the Bourgain spaces, to obtain more general results. In fact, one of the problems left in [33] is to prove global results for systems like (1.7). With respect to the results proved in [11], the results are purely linear, and extensions to the non-linear system are only possible in regular spaces.

1.4. **Notation and Main results.** Let us introduce some notation and present the main results of the manuscript.

We denote $\mathcal{D}(\mathbb{T})$ the space of periodic distributions whose dual space is $C^\infty(\mathbb{T})$. The Fourier series of periodic distributions is given by

$$\mathcal{F}f(k) = \widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx, \quad k \in \mathbb{Z}$$

and the inverse Fourier series by

$$\mathcal{F}^{-1}f(x) = \sum_{k\in\mathbb{Z}} e^{ikx}f(x).$$

For $s>0$, we use the operator $D^s = (-\Delta)^{\frac{1}{2}}$ given on the Fourier side as

$$\widehat{D^s f}(k) = |k|^s \widehat{f}(\xi).$$

Similarly, we have the operators J^s given on the Fourier side as

$$\widehat{J^s f}(k) = \langle k\rangle^s \widehat{f}(k)$$

where $\langle k\rangle := (1+|k|) \sim (1+|k|^2)^{\frac{1}{2}}$. Here we define the $H^s(\mathbb{R})$ Sobolev spaces, for $s\in\mathbb{R}$

$$H^s(\mathbb{T}) = \{f\in\mathcal{D}(\mathbb{T}) \ : \ \|f\|_s := \|J^s f\| < \infty\}$$

For the Cartesian spaces $H^s(\mathbb{T})\times H^s(\mathbb{T})$ we define $\|(u,v)\|_s := \|(u,v)\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})} = \|u\|_s + \|v\|_s$. Throughout this paper we will denote the norm $\|(\cdot,\cdot)\|_{L^2(\mathbb{T})\times L^2(\mathbb{T})}$ simply by $\|(\cdot,\cdot)\|$. Let X be one of the previously defined spaces, we will denote X_0 the function space belong in X with media-value null, i.e., $X_0 := \{u\in X \ : \ [u]=0\}$.

The aim of this manuscript is to address the control and stabilization (global) issues. Precisely, we want to give answers for both questions (see Problems 1.1 and 1.2) presented at the beginning of this introduction. As first result we will to analyse the exact controllability for the following linear system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v = Gf, & x\in\mathbb{T}, \ t\in\mathbb{R}, \\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x v + \eta\partial_x u = Gh, & x\in\mathbb{T}, \ t\in\mathbb{R}, \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)), & x\in\mathbb{T}. \end{cases} \tag{1.13}$$

Here, f and g are defined as two control inputs and the operator G is given by (1.9). We have established the following.

Theorem 1.3. *Let $T>0$ and $s\geq 0$ be given. Then for any (u_0,v_0), $(u_1,v_1)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$, there exists a pair of control functions $(f,h)\in L_0^2(\mathbb{T})\times L_0^2(\mathbb{T})$, such that system (1.13) has a solution in the class*

$$(u,v)\in C([0,T];H_0^s(\mathbb{T}))\times C([0,T];H_0^s(\mathbb{T}))$$

satisfying

$$(u(x,0),v(x,0)) = (u_0,v_0) \quad \text{and} \quad (u(x,T),v(x,T)) = (u_1,v_1).$$

Taking advantage of the results obtained by Bourgain [5], we are able to extend the previous local result to the nonlinear system, which is represented by,

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v + \partial_x P(u,v) = Gf, & x\in\mathbb{T}, \ t\in\mathbb{R}, \\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x v + \eta\partial_x u + \partial_x Q(u,v) = Gh, & x\in\mathbb{T}, \ t\in\mathbb{R}, \\ (u(x,0),v(x,0)) = (u_0(x),v_0(x)), & x\in\mathbb{T}, \end{cases} \tag{1.14}$$

where $P(u,v)$, $Q(u,v)$ are defined by (1.8), G is represented by (1.9), with f and g are control inputs. Thus, our second result deals with the asymptotic behavior of the solutions of (1.7). In order to stabilize system (1.14), choose the two feedback controls

$$f = -G^*L_{1,\mu,\lambda}^{-1}u \quad \text{and} \quad h = -G^*L_{\alpha,\zeta,\lambda}^{-1}v,$$

in (1.14), to transform it in a resulting closed-loop system reads as follows

$$(1.15)\qquad \begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v + \partial_x P(u,v) = -K_{1,\mu,\lambda} u, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x u + \eta \partial_x u + \partial_x Q(u,v) = -K_{\alpha,\zeta,\lambda} v, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}, \end{cases}$$

with the damping mechanism defined by

$$K_{\beta,\gamma,\lambda} := GG^* L_{\beta,\gamma,\lambda}^{-1}.$$

Here, $L_{\beta,\gamma,\lambda}$ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$, $s \geq 0$, for details see Section 3. So, as for Problem 1.2, we have the following affirmative answer.

Theorem 1.4. *Let $s \geq 0$ and $\gamma \in \mathbb{R}$ be given. There exists a constant $\kappa > 0$ such that for any $u_0, v_0 \in H_0^s(\mathrm{T})$ the corresponding solution (u,v) of the system (1.15) satisfies*

$$\|(u,v)\|_s \leq a_{s,\gamma}\left(\|(u_0,v_0)\|_0\right) e^{-\kappa t} \|(u_0,v_0)\|_s \,,$$

for all $t \geq 0$. Here $a_{s,\gamma} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s and γ.

To finalize, observe that Theorem 1.3 is purely linear. Thanks to Theorem 1.4 we guarantee a global controllability for long waves, thus responding to Problem 1.1. The result can be read as follows.

Theorem 1.5. *Let $s \geq 0$ and $R_0 > 0$ be given. There exists a time $T > 0$ such that if (u_0, v_0), $(u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ are such that*

$$\|(u_0,v_0)\|_s \leq R_0, \quad \|(u_1,v_1)\|_s \leq R_0,$$

then one can find two controls input $f, g \in L^2(0,T; H_0^s(\mathbb{T}))$ such that system (1.14) admits a solution

$$(u,v) \in C([0,T]; H_0^s(\mathbb{T})) \times C([0,T]; H_0^s(\mathbb{T}))$$

satisfying

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \ \text{ and } \ (u(x,T), v(x,T)) = (u_1(x), v_1(x)).$$

Remark 1.6. It is important to point out that Theorems 1.4 and 1.5 are valid for the case when we consider in the systems above mentioned $\alpha < 0$ and $|\mu| + |\zeta|$ is small enough. This restriction is necessary due to the fact that we need estimates for non-linear terms (see Lemmas 3.7 and 3.8 in the Section 3) which needs to be verified when $|\mu| + |\zeta| << 1$, $\alpha < \frac{1}{4}$ and $\frac{1}{\alpha} < \frac{1}{4}$, simultaneously. However, if $B = C = 0$, i.e. $\eta = 0$ (see system (1.7)), we have two KdV-type systems coupled only in the nonlinear terms. Thus, $\eta = 0$ ensures that all the results presented in this manuscript remain valid without any restriction in the constants α, μ and ζ.

1.5. **Heuristic and organization of the manuscript.** In this article, our goal is to give answers for two control problems (global results) mentioned in this introduction. Observe that the results obtained so far are concentrated in a single KdV equation (or, as in the case of [33] and [11], linear results or local results for coupled KdV systems). It is important to point out that in [33] the authors left some open problems with respect to the global controllability. In this spirit, our work is dedicated to cover this lack of results, that is, when presenting global results the article intends to give the first step to understand global control problems in periodic domains for systems like (1.7) with quadratic nonlinearities. Let us describe briefly the main arguments of the proof of our results.

First, thanks to the spectral problem, we prove that the system (1.13) is controllable, precisely, Theorem 1.3 holds. The second part of the work, due to the global well-posedness results in Bourgain spaces, we are able to prove that the singularities of the operator associated to (1.13), with $f = g = 0$, can be propagated of a subset ω to $\mathbb{T}$. These propagation results help us to prove that the solutions of (1.15) decay exponentially, that is, the solutions tend to zero as $t \to \infty$, showing Theorem 1.4. Finally, the control result for large data (Theorem 1.5) will be a combination of the global stabilization result (Theorem 1.4) and local control result (Theorem 5.1), as is usual in control theory.

We will give some considerations about the importance of these results. The system (1.1), differently from what happens for the KdV or for a pair of KdV which is decoupled only by the linear part, admits two families of eigenvalues associated with two different families of eigenfunctions. In this way, we proceed carefully to guarantee some spectral properties essential for obtaining a gap condition that satisfies an Ingham type theorem (see Theorem 2.1 below). Another new and important fact, and itself interesting, is that the orthonormal basis for the space $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ formed by the eigenfunctions is not a pair compound for two identical copies of the basis $\{\frac{1}{\sqrt{2\pi}}e^{ikx}\}_{k\in\mathbb{Z}}$, as usual in this kind of problem, which makes the controls obtained for each equations different but comparable to each other.

In addition to the previous paragraph, global results presented in Theorems 1.4 and 1.5 are truly nonlinear and, which is more important, are global properties, which means that the initial and final data are controlled in a ball with no size restrictions. The novelty is that for the first time *Fourier restriction spaces* introduced by Bourgain [5] are used in two different dispersion to ensure the existence of solutions to (1.15) (see [43] and Section3 bellow). So, propagation of compactness and regularity can be proved in a good way to achieve the results of global stability and controllability.

Lastly, it is important to mention that the propagation results has been successfully applied in control theory in several systems represented by single equations, such as wave equation [16], after that for the Schrödinger equation [26], for the Benjamin-Ono equation [30] , KdV equation [28], the Kawahara equation [44], biharmonic Schrödinger equation [9], for the Benney-Luke equation [20] and, finally, for the Benjamin equation [34].

To end the introduction, we present the outline of the manuscript: In the Section 2 we established the spectral analysis necessary to prove the exact controllability result for the linear system associated to (1.7). Next, Section 3, we present the Bourgain spaces and its property. Precisely, thanks to linear and nonlinear estimates we are able to prove the global well-posedness results for the system (1.15). In Sections 4 and 5, the reader will find the proofs of the main theorems of the article. Section 6 is devoted to presenting the conclusion of the work and some open problems. Finally, on Appendix A, we collect results of propagation for solutions of the linear operator associated to the system (1.7), which were used throughout the paper.

2. The linear problem

This section is devoted to studying the spectral properties of the linear system associated to (1.13). Precisely, we will present the Ingham type theorem, spectral analysis and well-posedness results that will guarantee the exact controllability for this system.

2.1. Ingham type theorem.

Given a family $\Omega = (\omega_k)_{k\in K} := \{\omega_k : k \in K\}$ of real numbers, we consider functions of the form

$$\sum_{k\in K} c_k e^{i\omega_k t}$$

with square summable complex coefficients $(c_k)_{k\in K} := \{c_k : k \in K\}$, and we investigate the relationship between the quantities

$$\int_I |\sum_{k\in K} c_k e^{i\omega_k t}|^2 \, dt \quad \text{and} \quad \sum_{k\in K} |c_k|^2$$

where I is some given bounded interval. In this work, the following version of Ingham type theorem will be used.

Theorem 2.1. *Let $\{\lambda_k\}$ be a family of real numbers, satisfying the uniform gap condition*

$$\gamma = \inf_{k\neq n} |\lambda_k - \lambda_n| > 0$$

and set

$$\gamma' = \sup_{A\subset K} \inf_{k,n\in K\setminus A} |\lambda_k - \lambda_n| > 0$$

where A rums over the finite subsets of K. If I is a bounded interval of length $|I| \geq \frac{2\pi}{\gamma'}$, then there exist positive constants A and B such that

$$A \sum_{k\in K} |c_k|^2 \leq \int_I |f(t)|^2 dt \leq B \sum_{k\in K} |c_k|^2 \tag{2.1}$$

for all functions given by the sum $f(t) = \sum_{k\in K} c_k e^{i\lambda_k t}$ with square-summable complex coefficients c_k.

Proof. See Theorem 4.6 in [23], page 67. □

We recall the definition of the *upper density* $D^+ = D^+(\Omega)$ of Ω. For each $\ell > 0$ we denote by $n^+(\ell)$ the largest number of exponents ω_k that we may find in an interval of length ℓ, and then we set

$$D^+ := \inf_{\ell>0} \frac{n^+(\ell)}{\ell} \in [0, \infty].$$

It can be shown (see, e.g., [1, p. 57] or [23, p. 174]) that

$$D^+ = \lim_{\ell\to\infty} \frac{n^+(\ell)}{\ell}.$$

It follows from the definition that D^+ is subadditive:

$$D^+(\Omega_1 \cup \Omega_2) \leq D^+(\Omega_1) + D^+(\Omega_2)$$

for any families Ω_1 and Ω_2. If Ω is *uniformly separated*, i.e., if

$$\gamma = \gamma(\Omega) = \inf\{|\omega_k - \omega_n| \ : \ k \neq n\} > 0,$$

then $D^+ \leq 1/\gamma$, and hence $D^+ < \infty$.

2.2. **Spectral properties.** Now on, consider the following operator

$$L = \begin{pmatrix} -\partial^3 - \mu\partial & -\eta\partial \\ -\eta\partial & -\alpha\partial^3 - \zeta\partial \end{pmatrix} \tag{2.2}$$

with domain $\mathcal{D}(L) = H^3(\mathbb{T}) \times H^3(\mathbb{T})$. The operator defined above has the following properties.

Proposition 2.2. *Consider the operator L defined as in (2.2). If $\alpha < 0$ and $\zeta - \mu > 0$ then L generates a strongly continuous group $S(t)$ in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. Moreover, the eigenfunctions are defined by $e^{-ikx} Z_k^{\pm}$, with $k \in \mathbb{Z}$ and form an orthogonal basis in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ satisfying*

$$Z_k^{\pm} \longrightarrow Z^{\pm}, \qquad \text{as } k \to \pm\infty,$$

where $Z^+ := (0, 0)$ and $Z^- := (0, 2(1 - \alpha))$.

Proof. A simple calculation shows that $L^* = -L$ and

$$\langle Lu, u\rangle = -\langle u, Lu\rangle = 0.$$

Thus L and L^* are dissipative. Since $\mathcal{D}(L)$ is dense on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ follows from [35, Corollary 4.4] that L is an infinitesimal generator of a strongly continuous group of contractions on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$.

We claim that, for each fixed $k \in \mathbb{Z}$, $e^{-ikx}(\sigma_k, \tau_k)$ is an eigenvector of L with eigenvalue $i\omega_k$ if and only if

$$\begin{cases} (k^3 - \mu k - \omega_k)\sigma_k - \eta k \tau_k = 0, \\ -\eta k \sigma_k + (\alpha k^3 - \zeta k - \omega_k)\tau_k = 0. \end{cases} \tag{2.3}$$

That is, there exist non-trivial solutions if and only if

$$\begin{vmatrix} k^3 - \mu k - \omega_k & -\eta k \\ -\eta k & \alpha k^3 - \zeta k - \omega_k \end{vmatrix} = 0,$$

or equivalently,

$$\omega_k^2 + \omega_k(\zeta + \mu - (1 + \alpha)k^2)k + \alpha k^6 - k^4(\zeta + \alpha\mu) - k^2(\eta^2 - \mu\zeta) = 0.$$

Hence, we have two possible exponents, given by the formula

$$\begin{aligned} 2\omega_k^{\pm} &= k((1+\alpha)k^2 - (\mu+\zeta)) \pm \sqrt{k^2(\mu+\zeta-(1+\alpha)k^2)^2 - 4k^2(\alpha k^4 - k^2(\alpha\mu+\zeta) - (\eta^2-\mu\zeta))} \\ &= k^3(1+\alpha-(\mu+\zeta)k^{-2}) \pm k^3\sqrt{[(1-\alpha)+k^{-2}(\zeta-\mu)]^2 + 4k^{-4}\eta^2}, \end{aligned}$$

that is,

$$2\omega_k^{\pm} = k^3\left[(1+\alpha)-(\zeta+\mu)k^{-2} \pm \sqrt{[(1-\alpha)+k^{-2}(\zeta-\mu)]^2+4k^{-4}\eta^2)}\right]. \tag{2.4}$$

If $k \neq 0$, with $\eta \neq 0$, then $\omega_k^- \neq \omega_k^+$ and two corresponding non-zero eigenvectors are given by the formula

$$Z_k^{\pm} = (\sigma_k, \tau_k) = \ 2k^{-3}\left(\eta k, k^3 - \mu k - \omega_k^{\pm}\right). \tag{2.5}$$

If $k = 0$, then both eigenvalues are equal to zero and two linearly independent eigenvectors are given for example by

$$Z_0^{\pm} = (\sigma_0, \tau_0) = \left(2\eta, (1-\alpha) \mp \sqrt{(1-\alpha)^2+4\eta^2}\right) \tag{2.6}$$

A direct calculation show that

$$Z_k^+ \cdot Z_k^- = 0,$$

for all $k \in \mathbb{Z}$ and $Z_k^{\pm} \longrightarrow Z^{\pm}$ as $k \longrightarrow \pm\infty$. Thus,

$$(\phi_k^{\pm}, \psi_k^{\pm}) = e^{-ikx} \cdot Z_k^{\pm},$$

where $Z_k^{\pm} = (\sigma_k^{\pm}, \tau_k^{\pm})$ is defined as in (2.5)-(2.6), form an orthogonal basis in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ with the eigenvalues given by (2.4), showing the proposition. □

Lemma 2.3. *Let $\omega_k^{\pm}$ be as in (2.4). We have*

$$\lim_{k\to\pm\infty}(\omega_{k+1}^+ - \omega_k^+) = +\infty \quad \textit{and} \quad \lim_{k\to\pm\infty}(\omega_{k+1}^- - \omega_k^-) = -\infty$$

Consequently, we have that

$$D^+(\{\omega_k^{\pm}\}) = 0.$$

Proof. Since $\omega_{-k}^+ = -\omega_k^+$, it suffices to consider the case $k \to +\infty$. One denotes

$$T^{\pm}(k) = (1+\alpha) - (\zeta+\mu)k^{-2} \pm \sqrt{[(1-\alpha)+(\zeta-\mu)k^{-2}]^2+4k^{-4}\eta^2}$$

Thus,

$$T^+(k) = 2 + O(k^{-2}) \quad \text{as} \quad k \to \infty$$

and

$$\omega_k^+ = \frac{1}{2}k^3T^+(k).$$

Hence

$$\omega_{k+1}^+ - \omega_k^+ = (k+1)^3 - k^3 + O(k) = 3k^2+3k+1+O(k) \to +\infty, \ \text{as } k \to +\infty.$$

In the similar way

$$\omega_{k+1}^- - \omega_k^- = \alpha[(k+1)^3 - k^3] + O(k) \to -\infty, \ \text{as } k \to +\infty,$$

where the last convergence is due to the fact that $\alpha < 0$. Now, as a consequence of these converges and by definition of $D^+ \leq 1/\gamma$, where

$$\gamma = \gamma(\Omega) = \inf\{|\omega_k - \omega_n| \ : \ k \neq n\} > 0,$$

we have that $D^+(\{\omega_k^{\pm}\}) = 0$. □

We now need to order our orthonormal basis, let us do it as follows. Consider $(\phi_k,\psi_k)=e^{-ikx}(\sigma_k,\tau_k)$, so

$$(\phi_k,\psi_k):=\begin{cases}(\phi_k^+,\psi_k^+)=e^{-ikx}(\sigma_k^+,\tau_k^+)=e^{-ikx}Z_k^+, & \text{if} \quad k=2k' \text{ for all } k'\in\mathbb{Z},\\ (\phi_k^-,\psi_k^-)=e^{-ikx}(\sigma_k^-,\tau_k^-)=e^{-ikx}Z_k^-, & \text{if} \quad k=2k'+1 \text{ for all } k'\in\mathbb{Z}.\end{cases} \tag{2.7}$$

Therefore, any vector $(u,v)\in H^s(\mathbb{T})\times H^s(\mathbb{T})$ can be represented by

$$(u,v)=\left(\sum_{k\in\mathbb{Z}}a_k\phi_k,\sum_{k\in\mathbb{Z}}b_k\psi_k\right),$$

with the coefficients a_k and b_k are defined by

$$a_k=\langle u,\phi_k\rangle \text{ and } b_k=\langle v,\psi_k\rangle,$$

where $\langle\cdot,\cdot\rangle$ denoting the inner product in $L^2(\mathbb{T})$. Consider, also, the following

$$\omega_k=\begin{cases}\omega_k^+, & \text{if} \quad k=2k' \text{ for all } k'\in\mathbb{Z},\\ \omega_k^-, & \text{if} \quad k=2k'+1 \text{ for all } k'\in\mathbb{Z}.\end{cases} \tag{2.8}$$

With these notions in hand, the following lemma gives the behavior of $\omega_k^\pm$ and concludes that the upper density of the set $\{\omega_k^\pm\}$ is zero. It is important to notice that to use Theorem 2.1 we need the following uniform gap condition

$$\gamma=\inf_{k\neq n}|\omega_k-\omega_n|>0,$$

where ω_k is defined by (2.8). The next proposition will give us such information.

Proposition 2.4 (Gap condition). *Let ω_k be as in (2.8). Thus,*

$$\lim_{|k|,|r|\to+\infty}|\omega_k-\omega_r|=+\infty.$$

Proof. Start noting that Lemma 2.3 ensures the result for k and n both odd or both even. Now, we need guarantee that the same is true for the other cases of k and n. Consider without loss of generality $r=2k'$ and $k=2(k'+k'')+1$ for any $k'\in\mathbb{Z}$ and k'' is a fixed positive integer. Using the notation of Lemma 2.3, follows that

$$\omega_{2(k'+k'')+1}-\omega_{2k'}=8(\alpha-1)k'^3+\alpha[12k'^2(2k''+1)+6k'(2k''+1)^2+(2k''+1)^3]+O(k').$$

Thus,

$$\lim_{|k'|\to+\infty}|\omega_{2(k'+k'')+1}-\omega_{2k'}|=+\infty,$$

and then the proposition is proved. □

Remark 2.5. Thanks to Theorem 2.1 and Proposition 2.4 there exists a subset $\mathbb{K}\subset\mathbb{Z}$ such that $\overline{span\{e^{-i\omega_k t}\}_{k\in\mathbb{K}}}^{L^2(0,T)}$ has a unique biorthogonal Riesz basis $\{q_k\}\subset L^2(0,T)$, where

$$\mathbb{K}=\{k\in\mathbb{Z}\ ;\ \omega_k\neq\omega_r \text{ for all } k\neq r\}. \tag{2.9}$$

2.3. **Exact controllability: Linear result.** With these previous information that concern the spectral properties of the operator L, in this section, we will analyse the exact controllability for the following linear system

$$\begin{cases}\partial_t u+\partial_x^3u+\mu\partial_x u+\eta\partial_x v=Gf, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u=Gh, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T}.\end{cases} \tag{2.10}$$

Here, f and g are defined as two control inputs and the operator G is given by

$$(G\ell)(x,t):=g(x)\left(\ell(x,t)-\int_{\mathbb{T}}g(y)\ell(y,t)dy\right). \tag{2.11}$$

Precisely, given an initial state (u_0, v_0) and a terminal state (u_1, v_1) in a certain space, we will study the existence of two control functions f and g such that the system (2.10) admits a solution (u, v) which satisfies $(u(x,T), v(x,T)) = (u_1(x), v_1(x))$.

Before to present the main result of this section, let us first consider some properties of the following linear initial value problem (IVP)

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}. \end{cases} \tag{2.12}$$

It is well know, thanks to Proposition 2.2, that (2.12) has solution on the Sobolev space $H^s(\mathbb{T})$, for $s \in [0,3]$, which is given by

$$(u(t), v(t)) = (S(t)u_0, S(t)v_0) := \left(\sum_k e^{-i(\omega_k t + kx)} \widehat{u}_0, \sum_k e^{-i(\omega_k t + kx)} \widehat{v}_0 \right). \tag{2.13}$$

Additionally, using Semigroup Theory, see for instance [35, Theorems 1.1 and 1.4], we have that the open loop control system (2.10) has a unique solution in

$$C([0,T]; H^3(\mathbb{T})) \cap C^1([0,T]; L^2(\mathbb{T})) \times C([0,T]; H^3(\mathbb{T})) \cap C^1([0,T]; L^2(\mathbb{T})).$$

Remark 2.6. Operator G defined as in (2.11) from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$ is linear, bounded and self-adjoint. Actually, was proved in [29, Remark 2.1] (see also [33, Lemma 2.20]) that operator G is a linear bounded operator from $L^2(0,T; H^s(\mathbb{T}))$ into $L^2(0,T; H^s(\mathbb{T}))$, for any $s \geq 0$.

Now on, we dedicate the rest of this section to prove the exact controllability result for the system (2.10), precisely, Theorem 1.3.

Proof of Theorem 1.3. Since the functions (ϕ_k, ψ_k), defined by (2.7), form an orthonormal basis on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and the space $L^2_0(\mathbb{T}) \times L^2_0(\mathbb{T})$ is a closed space, we can represent the initial and terminal states like expansions, which are convergent in $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$, as follows

$$\begin{aligned} u_j &= \sum_{k \in \mathbb{Z}} u_{k,j} \phi_k, \quad u_{k,j} = \int_{\mathbb{T}} u_j(x) \overline{\phi_k(x)} dx, \qquad \text{for } j = 0,1, \\ v_j &= \sum_{k \in \mathbb{Z}} v_{k,j} \psi_k, \quad v_{k,j} = \int_{\mathbb{T}} v_j(x) \overline{\psi_k(x)} dx, \qquad \text{for } j = 0,1. \end{aligned} \tag{2.14}$$

The homogeneous (adjoint) system is given by (2.12) and the corresponding solutions can be expressed by

$$(u_k(x,t), v_k(x,t)) = (e^{-i\omega_k t} \phi_k(x), e^{-i\omega_k t} \psi_k(x)),$$

where ω_k are the eigenvalues defined in (2.8).

Pick smooth functions (f, h) on $\mathbb{T} \times \mathbb{T}$. Multiplying (2.10) by $(\overline{u_k(x,t)}, \overline{v_k(x,t)})^T$ and using integration by parts on $\mathbb{T} \times (0,T)$, we obtain

$$\begin{aligned} \int_{\mathbb{T}} u(x,T) \overline{u_k(x,T)} dx - \int_{\mathbb{T}} u(x,0) \overline{u_k(x,0)} dx &= \int_0^T \int_{\mathbb{T}} Gf(x,t) \overline{u_k(x,t)} dx dt, \\ \int_{\mathbb{T}} v(x,T) \overline{v_k(x,T)} dx - \int_{\mathbb{T}} v(x,0) \overline{v_k(x,0)} dx &= \int_0^T \int_{\mathbb{T}} Gh(x,t) \overline{v_k(x,t)} dx dt, \end{aligned} \tag{2.15}$$

with the previous equality valid for f, $h \in L^2([0,T]; H^s_0(\mathbb{T}))$, for any $s \geq 0$, where (u, v) satisfies (2.10).

Observe that $(\overline{u_k}, \overline{v_k}) = (e^{i\omega_k t} \overline{\phi_k(x)}, e^{i\omega_k t} \overline{\psi_k(x)})$. Moreover, thanks to (2.15), we get that

$$\int_{\mathbb{T}} u(x,T) e^{i\omega_k T} \overline{\phi_k(x)} dx - \int_{\mathbb{T}} u_0(x) \overline{\phi_k(x)} dx = \int_0^T \int_{\mathbb{T}} Gf(x,t) e^{i\omega_k t} \overline{\phi_k(x)} dx$$

and

$$\int_{\mathbb{T}} v(x,T) e^{i\omega_k T} \overline{\psi_k(x)} dx - \int_0^T v_0(x) \overline{\psi_k(x)} dx = \int_0^T \int_{\mathbb{T}} Gh(x,t) e^{i\omega_k t} \overline{\psi_k(x)} dx.$$

Evaluation of the integrals in (2.15) with

$$w_k = \int_{\mathbb{T}} u(x,T)\overline{\phi_k(x)}dx \quad \text{and} \quad z_k = \int_{\mathbb{T}} v(x,T)\overline{\psi_k(x)}dx \tag{2.16}$$

gives that

$$\begin{aligned} w_k - u_{k,0}e^{-i\omega_k T} &= \int_0^T e^{-i\omega_k(T-t)}\int_{\mathbb{T}} Gf(x,t)\overline{\phi_k(x)}dxdt, \qquad \forall k\in\mathbb{Z},\\ z_k - v_{k,0}e^{-i\omega_k T} &= \int_0^T e^{-i\omega_k(T-t)}\int_{\mathbb{T}} Gh(x,t)\overline{\psi_k(x)}dxdt, \qquad \forall k\in\mathbb{Z}. \end{aligned} \tag{2.17}$$

Let us take our control functions f and h in the following way

$$\begin{aligned} f(x,t) &= \sum_{j\in\mathbb{Z}} f_j q_j(t) G\phi_j(x),\\ h(x,t) &= \sum_{j\in\mathbb{Z}} h_j q_j(t) G\psi_j(x). \end{aligned} \tag{2.18}$$

Here the coefficients f_j and h_j must be determined so that, among other things, the series (2.18) is appropriately convergent. Substituting (2.18) into (2.17) yields,

$$w_k - u_{k,0}e^{-i\omega_k T} = e^{-i\omega_k T}\sum_{j\in\mathbb{Z}} f_j \int_0^T e^{i\omega_k t}q_j(t)dt\int_{\mathbb{T}} GG\phi_j(x)\overline{\phi_k(x)}dx \tag{2.19}$$

and

$$z_k - v_{k,0}e^{-i\omega_k T} = e^{-i\omega_k T}\sum_{j\in\mathbb{Z}} h_j \int_0^T e^{i\omega_k t}q_j(t)dt\int_{\mathbb{T}} GG\psi_j(x)\overline{\psi_k(x)}dx. \tag{2.20}$$

Thanks to the fact that $\{q_k\}_{k\in\mathbb{K}}$ is a biorthogonal Riesz basis to $\{e^{-i\omega_k t}\}_{k\in\mathbb{K}}$ in $L^2_0(0,T)$, for $\mathbb{K}$ defined by (2.9), and due to the Remark 2.6 we can get that

$$\begin{aligned} w_k - u_{k,0}e^{-i\omega_k T} &= e^{-i\omega_k T} f_k \int_{\mathbb{T}} G\phi_k(x)\overline{G\phi_k(x)}dx = e^{-i\omega_k T} f_k\|G\phi_k\|^2,\\ z_k - v_{k,0}e^{-i\omega_k T} &= e^{-i\omega_k T} h_k \int_{\mathbb{T}} G\psi_k(x),\overline{G\psi_k(x)}dx = e^{-i\omega_k T} h_k\|G\psi_k\|^2, \end{aligned} \tag{2.21}$$

for all $k_j \in \mathbb{Z}\setminus\cup_{j=1}^{\ell}\mathbb{K}_j$, where $\mathbb{K}_j := \{k\in\mathbb{Z}\ ;\ \omega_k = \omega_{k_j} \text{ and } k\neq k_j\}$. By the definition of G, see (2.11), yield that

$$\|G\phi_k\|^2 = \int_{\mathbb{T}}\left|g(x)\left(\phi_k(x) - \int_{\mathbb{T}} g(y)\phi_k(y)dy\right)\right|^2 dx = |\sigma_k|^2\beta_k \tag{2.22}$$

and

$$\|G\psi_k\|^2 = \int_{\mathbb{T}}\left|g(x)\left(\psi_k(x) - \int_{\mathbb{T}} g(y)\psi_k(y)dy\right)\right|^2 dx = |\tau_k|^2\beta_k, \tag{2.23}$$

where

$$\beta_k := \left\|G\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right)\right\|^2.$$

Since $[g] = \frac{1}{2\pi}$ it is easy to see that $\beta_0 = 0$. The fact that $g(x)$ is real valued shows that $g(x)\frac{e^{-ikx}}{\sqrt{2\pi}}$ cannot be a constant multiple of $g(x)$ on any interval. Thus, follows that $\beta_k \neq 0$, $k>0$ and

$$\lim_{k\to\infty}\beta_k = \int_{\mathbb{T}} g(x)^2dx \neq 0.$$

Its implies that there is a $\delta>0$ such that

$$|\beta_k| > \delta, \qquad \text{for } k\neq 0. \tag{2.24}$$

Due to the fact that $\sigma_k \neq 0$ and $\tau_k \neq 0$, for all k, we can putting $f_0 = h_0 = 0$ and

$$f_k = \frac{u_{k,1}e^{i\omega_k T} - u_{k,0}}{|\sigma_k|^2\beta_k} \quad \text{and} \quad h_k = \frac{v_{k,1}e^{i\omega_k T} - v_{k,0}}{|\tau_k|^2\beta_k}, \tag{2.25}$$

for all $k \in \mathbb{Z}^* \setminus \cup_{j=1}^n \mathbb{K}_j$. So we get, from (2.21), that

$$w_k = u_{k,1} \quad \text{and} \quad z_k = v_{k,1},$$

where $u_{k,1}$ and $v_{k,1}$ are given by (2.14)[3].

Since ω_k is given by a polynomial of degree 3, each set $\mathbb{K}_j$ has at most three elements. So, we can consider $k_{j,i} \in \cup_{j=1}^{\ell} \mathbb{K}_j$ for $i = 0, 1, 2$. In this case, from (2.19)-(2.20) follows that

$$\begin{aligned} w_{k_{j,i}} - u_{k_{j,i},0}e^{-i\omega_{k_{j,0}}T} &= \sigma_{k_{j,i}}e^{-i\omega_{k_{j,0}}T}\sum_{\ell=0}^{2} f_{k_{j,\ell}}\sigma_{k_{j,\ell}}M_{k_{j,\ell}k_{j,i}}, \\ z_{k_{j,i}} - v_{k_{j,i},0}e^{-i\omega_{k_{j,0}}T} &= \tau_{k_{j,i}}e^{-i\omega_{k_{j,0}}T}\sum_{\ell=0}^{2} h_{k_{j,\ell}}\tau_{k_{j,\ell}}M_{k_{j,\ell}k_{j,i}}, \end{aligned} \tag{2.26}$$

where

$$M_{k_{j,\ell}k_{j,i}} := \frac{1}{2\pi}\int_{\mathbb{T}} GG\left(e^{-ik_{j,\ell}x}\right)\overline{e^{-ik_{j,i}x}}dx.$$

In other words, $f_{k_{j,\ell}}$ and $h_{k_{j,\ell}}$, for each $j = 1, 2, \cdots, n$ and $\ell = 0, 1, 2$, must be satisfy the following matrix identities

$$\begin{pmatrix} \sigma_{k_{j,0}}M_{k_{j,0},k_{j,0}} & \sigma_{k_{j,0}}M_{k_{j,1},k_{j,0}} & \sigma_{k_{j,0}}M_{k_{j,2},k_{j,0}} \\ \sigma_{k_{j,1}}M_{k_{j,0},k_{j,1}} & \sigma_{k_{j,1}}M_{k_{j,1},k_{j,1}} & \sigma_{k_{j,1}}M_{k_{j,2},k_{j,1}} \\ \sigma_{k_{j,2}}M_{k_{j,0},k_{j,2}} & \sigma_{k_{j,2}}M_{k_{j,1},k_{j,2}} & \sigma_{k_{j,2}}M_{k_{j,2},k_{j,2}} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{k_{j,0}}f_{k_{j,0}} \\ \sigma_{k_{j,1}}f_{k_{j,1}} \\ \sigma_{k_{j,2}}f_{k_{j,2}} \end{pmatrix} = \begin{pmatrix} w_{k_{j,0}}e^{i\omega_{k_{j,0}}T} - u_{k_{j,0},0} \\ w_{k_{j,1}}e^{i\omega_{k_{j,0}}T} - u_{k_{j,1},0} \\ w_{k_{j,2}}e^{i\omega_{k_{j,0}}T} - u_{k_{j,2},0} \end{pmatrix}$$

and

$$\begin{pmatrix} \tau_{k_{j,0}}M_{k_{j,0},k_{j,0}} & \tau_{k_{j,0}}M_{k_{j,1},k_{j,0}} & \tau_{k_{j,0}}M_{k_{j,2},k_{j,0}} \\ \tau_{k_{j,1}}M_{k_{j,0},k_{j,1}} & \tau_{k_{j,1}}M_{k_{j,1},k_{j,1}} & \tau_{k_{j,1}}M_{k_{j,2},k_{j,1}} \\ \tau_{k_{j,2}}M_{k_{j,0},k_{j,2}} & \tau_{k_{j,2}}M_{k_{j,1},k_{j,2}} & \tau_{k_{j,2}}M_{k_{j,2},k_{j,2}} \end{pmatrix} \cdot \begin{pmatrix} \tau_{k_{j,0}}h_{k_{j,0}} \\ \tau_{k_{j,1}}h_{k_{j,1}} \\ \tau_{k_{j,2}}h_{k_{j,2}} \end{pmatrix} = \begin{pmatrix} z_{k_{j,0}}e^{i\omega_{k_{j,0}}T} - v_{k_{j,0},0} \\ z_{k_{j,1}}e^{i\omega_{k_{j,0}}T} - v_{k_{j,1},0} \\ z_{k_{j,2}}e^{i\omega_{k_{j,0}}T} - v_{k_{j,2},0} \end{pmatrix}.$$

In order to achieve the result, we will need to prove the following two claims.

Claim 1. The previous systems have a unique solution $(f_{k_{j,0}}, f_{k_{j,1}}, f_{k_{j,2}})$ and $(h_{k_{j,0}}, h_{k_{j,1}}, h_{k_{j,2}})$, for each $j = 1, 2, \cdots, n$.

Indeed, note that the determinant of the above matrices are given by $\sigma_{k_{j,0}} \times \sigma_{k_{j,1}} \times \sigma_{k_{j,2}} \times \det M_j$ and $\tau_{k_{j,0}} \times \tau_{k_{j,1}} \times \tau_{k_{j,2}} \times \det M_j$, respectively, with M_j defined by

$$M_j := \begin{pmatrix} m_{k_{j,0},k_{j,0}} & m_{k_{j,0},k_{j,1}} & m_{k_{j,0},k_{j,2}} \\ m_{k_{j,1},k_{j,0}} & m_{k_{j,1},k_{j,1}} & m_{k_{j,1},k_{j,2}} \\ m_{k_{j,2},k_{j,0}} & m_{k_{j,2},k_{j,1}} & m_{k_{j,2},k_{j,2}} \end{pmatrix}.$$

Since $\sigma_{k_{j,0}} \times \sigma_{k_{j,1}} \times \sigma_{k_{j,2}} \neq 0$ and $\tau_{k_{j,0}} \times \tau_{k_{j,1}} \times \tau_{k_{j,2}} \neq 0$, we only have show that the hermitian matrices M_j are invertible for all $j = 1, \cdots, \ell$. For fixed j, let us consider Σ_2 the space spanned by $\Upsilon_{k_{j,i}} = e^{-ik_{j,i}}$, $i = 0, 1, 2$. Let $\rho_{k_{j,\ell}}$ be the projection of $GG(\Upsilon_{k_{j,\ell}})$ onto the space Σ_2, that is,

$$\rho_{k_{j,\ell}} = \sum_{i=0}^{2} M_{k_{j,\ell}k_{j,i}}\Upsilon_{k_{j,i}}.$$

Now, it suffices to show that $\rho_{k_{j,\ell}}$, $\ell = 0, 1, 2$, is a linearly independent subset of Σ_2. Assume that there exist scalars λ_ℓ, $\ell = 0, 1, 2$, such that

$$\sum_{\ell=0}^{2} \lambda_\ell \rho_{k_{j,\ell}}(x) = 0 \iff \sum_{\ell,i=0}^{2} \lambda_\ell M_{k_{j,\ell},k_{j,i}}\Upsilon_{k_{j,i}}(x) = 0$$

[3]Note that clearly w_0 and z_0 must be zero.

Then, it yields that

$$\sum_{i=0}^{2}\sum_{\ell=0}^{2}\left\langle \lambda_\ell G\Upsilon_{k_{j,\ell}}, G\Upsilon_{k_{j,i}}\right\rangle \Upsilon_{k_{j,i}} = \sum_{i=0}^{2}\left\langle GG\left(\sum_{\ell=0}^{2}\lambda_\ell \Upsilon_{k_{j,\ell}}\right), \Upsilon_{k_{j,i}}\right\rangle \Upsilon_{k_{j,i}} = 0$$

Since $\Upsilon_{k_{j,i}}$ is a basis of Σ_2, follows that

$$\left\langle GG\left(\sum_{\ell=0}^{2}\lambda_\ell \Gamma_{k_{j,\ell}}\right), \Gamma_{k_{j,i}}\right\rangle = 0,$$

for each $i = 0, 1, 2$. As consequence of the last equality, we get

$$0 = \left\langle GG\left(\sum_{\ell=0}^{2}\lambda_\ell \Upsilon_{k_{j,\ell}}\right), \sum_{i=0}^{2}\lambda_i \Upsilon_{k_{j,i}}\right\rangle \iff \sum_{\ell=0}^{2}\lambda_\ell \Upsilon_{k_{j,\ell}} = 0 \iff \lambda_\ell = 0,$$

for $\ell = 0, 1, 2$, showing the Claim 1.

Claim 2. The functions f and h defined by (2.18) and (2.25) belongs to $L^2([0,T]; H_0^s(\mathbb{T}))$ provided that $(u_0, v_0), (u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$.

In fact, let us write $G\phi_j(x)$ and $G\psi_j(x)$ as follows

$$G\phi_j(x) = \sum_{k\in\mathbb{Z}} a_{jk}\phi_k \quad \text{and} \quad G\psi_j(x) = \sum_{k\in\mathbb{Z}} b_{jk}\psi_k, \tag{2.27}$$

where

$$a_{jk} = \int_{\mathbb{T}} G\phi_j \overline{\phi_k(x)}dx \quad \text{and} \quad b_{jk} = \int_{\mathbb{T}} G\psi_j \overline{\psi_k(x)}dx, \ k \in \mathbb{Z}.$$

Therefore, we can see that

$$f(x,t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f_j a_{jk} q_j(t)\phi_k(x)$$

and

$$h(x,t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} h_j b_{jk} q_j(t)\psi_k(x).$$

Consequently, this yields that

$$\|f\|^2_{L^2([0,T];H_0^s(\mathbb{T}))} = \int_0^T \sum_{k\in\mathbb{Z}}(1+|k|)^{2s}\left|\sum_{j\in\mathbb{Z}} a_{jk} f_j q_j(t)\right|^2 dt = \sum_{k\in\mathbb{Z}}(1+|k|)^{2s}\int_0^T\left|\sum_{j\in\mathbb{Z}} a_{jk} f_j q_j(t)\right|^2 dt.$$

As $\{q_k\}_{k\in\mathbb{K}}$ is a Bessel sequence and $\mathbb{Z}\setminus\mathbb{K}$ is a finite set, from the previous identity holds that

$$\|f\|_{L^2([0,T];H_0^s(\mathbb{T}))} \leq c\sum_{j\in\mathbb{Z}}|f_j|^2\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|a_{jk}|^2. \tag{2.28}$$

Analogously, we can obtain the following estimate for h, that is,

$$\|h\|_{L^2([0,T];H_0^s(\mathbb{T}))} \leq c\sum_{j\in\mathbb{Z}}|h_j|^2\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|b_{jk}|^2. \tag{2.29}$$

To finish the proof of Claim 2, let us prove that the right hand side of (2.28) and (2.29) are bounded. For this, note that

$$|a_{jk}| = |\langle G\phi_j(x), \phi_k(x)\rangle| \leq \frac{1}{\sqrt{2\pi}}|\sigma_{k-j}||\sigma_j||\sigma_k||\langle g, \phi_{k-j}\rangle| + |\sigma_k||\sigma_j||\langle g, \phi_k\rangle||\langle g, \phi_j\rangle|$$

and, in a similar way,

$$|b_{jk}| \leq \frac{1}{\sqrt{2\pi}}|\tau_{k-j}||\tau_j||\tau_k||\langle g, \psi_{k-j}\rangle| + |\tau_k||\tau_j||\langle g, \psi_k\rangle||\langle g, \psi_j\rangle|.$$

Hence,

$$|a_{jk}|^2 \leq 2|\sigma_j|^2(|\sigma_{k-j}|^2|\sigma_k|^2|\langle g, \phi_{k-j}\rangle|^2 + |\sigma_k|^2|\langle g, \phi_k\rangle|^2|\langle g, \phi_j\rangle|^2)$$

and

$$|b_{jk}|^2 \leq 2|\tau_j|^2(|\tau_{k-j}|^2|\tau_k|^2| \langle g, \psi_{k-j}\rangle |^2 + |\tau_k|^2| \langle g, \psi_k\rangle |^2| \langle g, \psi_j\rangle |^2).$$

Using the last inequalities we can estimate

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|a_{jk}|^2 \leq& 2|\sigma_j|^2 \left[(1+|j|)^{2s}\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}| \langle g, \phi_k\rangle |^2 + | \langle g.\phi_j\rangle |^2 \sum_{k\in\mathbb{Z}}(1+|k|)^{2s}| \langle g, \phi_k\rangle |^2\right]\\ \leq& 2|\sigma_j|^2 \left[(1+|j|)^{2s} + | \langle g, \phi_j\rangle |^2\right] \|g\|_s^2,\end{aligned}$$

and analogously, we have

$$\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|b_{jk}|^2 \leq 2|\tau_j|^2 \left[(1+|j|)^{2s} + | \langle g, \psi_j\rangle |^2\right] \|g\|_s^2.$$

Therefore, (2.28) and (2.29) together the previous inequality results

$$\|f\|^2_{L^2([0,T];H^s_0(\mathbb{T}))} \leq 2C_0 \sum_{j\in\mathbb{Z}} \frac{|u_{j,1}e^{i\omega_j T} - u_{j,0}|^2}{|\sigma_j|^2|\beta_j|^2}[(1+|j|)^{2s} + | \langle g, \phi_j\rangle |^2]\|g\|_s^2,$$

where $C_0 = \max_{j=1,\cdots,n}\{1, \|M_j^{-1}\|^2\}$ and $\|M_j^{-1}\|$ denote the Euclidean norms of the Matrices M_j^{-1}. An analogous inequality is obtained for $\|h\|^2_{L^2([0,T];H^s_0(\mathbb{T}))}$. Putting all these inequality together and using the relation (2.24), we get

$$\begin{aligned}\|(f,h)\|^2_{L^2([0,T];H^s_0(\mathbb{T}))\times L^2([0,T];H^s_0(\mathbb{T}))} \leq& C_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})} \sum_{j\in\mathbb{Z}}(1+|j|)^{2s} \frac{|\sigma_j|^2(|\widetilde{u_{j,1}}|^2 + |\widetilde{u_{j,0}}|^2)}{|\sigma_j|^2}\\ &+ C_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})} \sum_{j\in\mathbb{Z}}(1+|j|)^{2s} \frac{|\tau_j|^2(|\widetilde{v_{j,1}}|^2 + |\widetilde{v_{j,0}}|^2)}{|\tau_j|^2},\end{aligned}$$

where $\widetilde{u_{j,i}}$ and $\widetilde{v_{j,i}}$ denote the Fourier coefficients with respect to orthonormal base $\left\{\frac{e^{-ijx}}{\sqrt{2\pi}}\right\}_{j\in\mathbb{Z}}$. So,

$$\|(f,h)\|^2_{L^2([0,T];H^s_0(\mathbb{T}))\times L^2([0,T];H^s_0(\mathbb{T}))} \leq K_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})}(\|(u_0,v_0)\|^2_{H^s_0(\mathbb{T})\times H^s_0(\mathbb{T})} + \|(u_1,v_1)\|^2_{H^s_0(\mathbb{T})\times H^s_0(\mathbb{T})})$$

completing the proof of Claim 2 and showing Theorem 1.3. □

As a consequence of Theorem 1.3 we have the next result, which will be important to extend the result to the nonlinear system.

Corollary 2.7. *Equations (2.18), (2.25) and (2.27) define, for $s \geq 0$, two bounded operators Φ and Ψ*

$$\Phi(u_0, v_0) = f \quad \text{and} \quad \Psi(u_1, v_1) = h$$

from $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$ to $L^2([0,T]; H^s_0(\mathbb{T}))$ such that

$$S(T)(u_0, v_0) + \int_0^T S(T-\tau)(G\Phi(u_0,u_1), G\Psi(v_0,v_1))(\cdot,\tau)d\tau = (u_1, v_1),$$

for any (u_0,u_1), $(v_0,v_1) \in H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$. Moreover, there exists a constant $C_{T,g} := C(T,g)$ such that the following inequality is verified

$$\|(\Phi(u_0,u_1), \Psi(v_0,v_1))\|_{[L^2([0,T];H^s_0(\mathbb{T}))]^2} \leq C_{T,g}\left(\|(u_0,v_0)\|_s + \|(u_1,v_1)\|_s\right).$$

3. Well-posedness theory in Bourgain spaces

In this section we present the definition and some properties of the Bourgain spaces which are used to prove the exact controllability and stabilizability of the nonlinear systems. Precisely, is well known that the Bourgain in [5] discovered a subtle smoothing property of solutions of the KdV equation posed on Torus, thanks to which he was able to show that the KdV equation is well-posed in the space $H^s(\mathbb{T})$, for any $s \geq 0$. Due to this fact, we will present below the smoothing properties to the IVP (2.12), which are the key to prove the global control results in this manuscript.

3.1. **Fourier restriction space.** Let us define the appropriate spaces for the following system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}. \end{cases} \tag{3.1}$$

Precisely, to present the space, we will see the previous system as follows

$$\begin{cases} \begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} u_{xxx} \\ v_{xxx} \end{pmatrix} + \begin{pmatrix} \mu & \eta \\ \eta & \zeta \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x \in \mathrm{T}, t \in \mathbb{R}, \\ \begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s(\mathbb{T}) \times H^s(\mathbb{T}). \end{cases}$$

Note that we want to find an appropriated way to define the $X_{s,b}$ for the targeted system (3.1) which one contains linear terms, so, let us consider the following equivalent system

$$\begin{cases} \partial_t v + \beta \partial_x^3 v + \gamma \partial_x v = 0, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ v(0) = v_0, & \in H^s(\mathbb{T}). \end{cases} \tag{3.2}$$

The solution to (3.2) is given explicitly by

$$v(x,t) = \sum_{k \in \mathbb{Z}} e^{ikx} e^{i\phi^{\beta,\gamma}(k)t} \widehat{v_0}(k) := S^{\beta,\gamma}(t) v_0 \tag{3.3}$$

with

$$\phi^{\beta,\gamma}(k) := \beta k^3 - \gamma k.$$

For convenience, $\phi^{1,0}$ will be written as ϕ.

Remark 3.1. With the notation (3.3), note that when we put $\eta = 0$ in (2.2), the operator L remains an infinitesimal generator of a strongly continuous group of contraction on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ which is given by

$$S(t) = e^{-tL} = \begin{pmatrix} e^{-t(\partial_x^3 + \mu\partial_x)} & 0 \\ 0 & e^{-t(\alpha\partial_x^3 + \zeta\partial_x)} \end{pmatrix}$$

Hence, $S(t)(u_0, v_0) = (S^{1,\mu}(t)u_0, S^{\alpha,\zeta(t)}v_0)$. In this way, the Corollary 2.7 also is obtained for $\eta = 0$.

Definition 3.2. [4] For any $\beta, \gamma, s, b \in \mathbb{R}$, the Fourier restriction space $X^{\beta,\gamma}_{s,b}$ is defined to be the completion of the Schwartz space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ with respect to the norm

$$\|v\|_{X^{\beta,\gamma}_{s,b}} := \left\| \langle k \rangle^s \left\langle \tau - \phi^{\beta,\gamma}(k) \right\rangle^b \widetilde{v}(k,\tau) \right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})},$$

where $\widetilde{v}$ refers to the space-time Fourier transform of v. In addition, for any $T > 0$,

$$X^{\beta}_{s,b}([0,T]) := X^{\beta,\gamma,T}_{s,b}$$

denotes the restriction of $X^{\beta,\gamma}_{s,b}$ on the domain $\mathbb{T} \times [0,T]$ which is a Banach space when equipped with the usual quotient norm.

As well known (see e. g. [22]), for the periodic KdV equation, one needs to take $b = \frac{1}{2}$. But, this space barely fails to be in $C(\mathbb{R}_t; H^s_x)$. To ensure the continuity of the time flow of the solution, will be used the norm $Y^{\beta,\gamma}_{s,b}$ given by

$$\|v\|_{Y^{\beta,\gamma}_{s,b}} = \left\| \langle k \rangle^s \langle \tau - \phi_{\beta,\gamma} \rangle^b \widetilde{v}(k,\tau) \right\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}$$

and the companion spaces will be defined as

$$Z^{\beta,\gamma}_{s,b} = X^{\beta,\gamma}_{s,b} \cap Y^{\beta,\gamma}_{s,b-\frac{1}{2}}, \quad b, s \in \mathbb{R},$$

[4] We infer for more details the two references [5, 22].

and be endowed with the norm

$$\|v\|_{Z^{\beta,\gamma}_{s,b}} = \|v\|_{X^{\beta,\gamma}_{s,b}} + \|v\|_{Y^{\beta,\gamma}_{s,b-\frac{1}{2}}}.$$

Due the fact that the second term $\|\langle k\rangle^s\widehat{v}(k,\tau)\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}$ has already dominated the $L^\infty_t H^s_x$ norm of v, it follows that $Z^{\beta,\gamma}_{s,\frac{1}{2}} \subset C(\mathbb{R}_t; H^s_x)$ continuously. Lastly, the spaces

$$Z^{\beta,\gamma}_{s,b}([0,T]) := Z^{\beta,\gamma,T}_{s,b}$$

denotes the restriction of $Z^{\beta,\gamma}_{s,b}$ on the domain $\mathbb{T}\times[0,T]$ which is a Banach space when equipped with the usual quotient norm.

Remark 3.3. When $b=-\frac{1}{2}$, the companion spaces $Z^{\beta,\gamma}_{s,-\frac{1}{2}}$ *via* the norm previously defined is so introduced to control the $Z^{\beta,\gamma}_{s,\frac{1}{2}}$-norm of the integral term from the Duhamel principle (see Lemma 3.4)

$$\|v\|_{Z^{\beta,\gamma}_s} = \|v\|_{X^{\beta,\gamma}_{s,-\frac{1}{2}}} + \left\|\frac{\langle k\rangle^s\widehat{v}(k,\tau)}{\langle\tau-\phi^{\beta,\gamma}(k)\rangle}\right\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}.$$

3.2. **Linear and nonlinear estimates.** To obtain global well-posedness result for the following system

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u=-\eta\partial_x v+\partial_x P(u,v)=0, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v=-\eta\partial_x u+\partial_x Q(u,v)=0, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases} \tag{3.4}$$

where P and Q are polynomials defined by

$$\begin{cases}P(u,v)=Au^2+Buv+\frac{C}{2}v^2,\\ Q(u,v)=Dv^2+Cvu+\frac{B}{2}u^2,\end{cases} \tag{3.5}$$

with A,B,C and D are real constants and $\alpha<0$, we will need some estimates related with linear and nonlinear IVP associated to (3.4).

Let us first recall some classic results in the literature for dispersive systems.

Lemma 3.4. [5] *Let s, $b\in\mathbb{R}$ and $T>0$ be given. There exists a constant $C_0>0$ such that:*

(i) *For any $w\in H^s(\mathbb{T})$,*

$$\|S(t)^{\beta,\gamma}w\|_{X^{\beta,\gamma,T}_{s,b}} \le C_0\|w\|_s;$$

$$\|S(t)^{\beta,\gamma}w\|_{Z^{\beta,\gamma,T}_{s,b}} \le C_0\|w\|_s;$$

(ii) *For any $f\in X^{\beta,\gamma}_{s,b-1}$,*

$$\left\|\int_0^t S^{\beta,\gamma}(t-\tau)f(\tau)d\tau\right\|_{X^{\beta,\gamma,T}_{s,b}} \le C_0\|f\|_{X^{\beta,\gamma,T}_{s,b-1}}$$

provided that $b>\frac{1}{2}$;

(iii) *For any $f\in Z^{\beta,T}_{s,-\frac{1}{2}}$,*

$$\left\|\int_0^t S^{\beta,\gamma}(t-\tau)f(\tau)d\tau\right\|_{Y^{\beta,\gamma,T}_s} \le C_0\|f\|_{Z^{\beta,\gamma,T}_s}.$$

[5]For details about this lemma the authors suggest the following references [15, 42, 43].

Remark 3.5. Observe that the Bourgain spaces associated to (3.2) will be $X^{1,\mu}_{s,b}$ and $X^{\alpha,\zeta}_{s,b}$ ($Z^{1,\mu}_{s}$ and $Z^{\alpha,\zeta}_{s}$, respectively). In our case, it is important to see that $\sup_{k\in\mathbb{Z}}|\phi^{\mu}-\phi^{\alpha,\zeta}|=\infty$, which results that the norms $\|\cdot\|_{X^{1,\mu}_{s,b}}$ and $\|\cdot\|_{X^{\alpha,\zeta}_{s,b}}$ never will be equivalent (see for instance, [19, Remark 1.1]). To overcome this difficulty we need appropriate lemmas which were introduced for the first time in a general context by Yang and Zhang in [43].

Consider $X^{\beta_i,\gamma_i}_{s,b}$ for β_i and γ_i, $i=1$ and 2. The next lemma is presented in [43, Lemma 3.10] for the case $b=\frac{1}{2}$. Here, we are able to extend the result for $b\in(\frac{1}{3},\frac{1}{2}]$ and, in these cases, the lemma takes the following form.

Lemma 3.6. *Let $\beta_1\neq\beta_2$, $s\in\mathbb{R}$, $\frac{1}{3}<b\leq\frac{1}{2}$ and $0<T<1$. There exist constants $\epsilon=\epsilon(\beta_1,\beta_2)$, $C_1=C_1(\beta_1,\beta_2)$ and $\theta>0$ such that for any γ_1,γ_2 with $|\gamma_1|+|\gamma_2|<\epsilon$*

$$\|\partial_x w\|_{Z^{\beta_2,\gamma_2,T}_{s,b-1}}\leq C_1T^{\theta}\|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}} \tag{3.6}$$

is verified for any $w\in X^{\beta_1,\gamma_1,T}_{s,b}$.

Proof. We must to prove that

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2,T}_{s,b-1}}\leq C_1T^{\epsilon}\|w\|^{\beta_1,\gamma_1,T}_{X_{s,b},}\quad\text{and}\quad \|\partial_x w\|_{Y^{\beta_2,\gamma_2,T}_{s,b-\frac{3}{2}}}\leq C_1T^{\epsilon}\|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}}. \tag{3.7}$$

Thus, it is sufficient to show the following estimates

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}}\leq C_1\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}\quad\text{and}\quad \|\partial_x w\|_{Y^{\beta_2,\gamma_2,T}_{s,b-\frac{3}{2}}}\leq C_1\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}, \tag{3.8}$$

here b^- denote $b-\widetilde{\epsilon}$ for $\widetilde{\epsilon}\ll 1$.

We will start by showing that the first inequality of (3.8) holds. Using duality approach and Plancherel theorem, we get that

$$\begin{aligned}\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}}&=\sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}}\leq 1}\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}ik\widetilde{w}(k,\tau)\widetilde{g}(k,\tau)d\tau\right|\\&=\sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}}\leq 1}\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}H(k,\tau)\widetilde{W}(k,\tau)\widetilde{G}(k,\tau)d\tau\right|,\end{aligned}$$

where

$$H(k,\tau)=\frac{ik}{\langle\tau-\phi^{\beta_1,\gamma_1}(k)\rangle^{b^-}\langle\tau-\phi^{\beta_2,\gamma_2}(k)\rangle^{1-b}},$$

$$\widetilde{W}(k,\tau)=\langle k\rangle^{s}\left\langle\tau-\phi^{\beta_1,\gamma_1}(k)\right\rangle^{b^-}\widetilde{w}(k,\tau)$$

and

$$\widetilde{G}(k,\tau)=\langle k\rangle^{-s}\left\langle\tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle^{1-b}\widetilde{g}(k,\tau).$$

The following claim shows that the function $H(k,\tau)$ is bounded.

Claim: For some constant $C_1>0$, which depends only of β_1,β_2, we have that

$$\sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}}|H(k,\tau)|\leq C_1.$$

In fact, if $|k|\leq 1$ is immediate. If $|k|>1$ note that

$$\left\langle\tau-\phi^{\beta_1,\gamma_1}(k)\right\rangle\left\langle\tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle\geq\left|\phi^{\beta_1,\gamma_1}(k)-\phi_{\beta_2,\gamma_2(k)}\right|=|(\beta_1-\beta_2)k^3-(\gamma_1-\gamma_2)k|.$$

Since $\beta_1\neq\beta_2$ we can choose $\epsilon\ll 1$ such that $|\gamma_1|+|\gamma_2|\leq\epsilon$ and consequently

$$|\gamma_1-\gamma_2||k|\leq\frac{1}{2}|\beta_1-\beta_2||k|^3.$$

So, using this previous inequality, yields that

$$\left\langle \tau-\phi^{\beta_1,\gamma_1}(k)\right\rangle \left\langle \tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle \geq |\beta_1-\beta_2||k|^3-|\gamma_1-\gamma_2||k| \geq \frac{1}{2}|\beta_1-\beta_2||k|^3.$$

Thus, we obtain

$$|H(k,\tau)| \leq \frac{C_1(\beta_1,\beta_2)|k|}{|k|^{3(b^-)}\left\langle \tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle^{(1-2b)^+}} \leq \frac{C_1(\beta_1,\beta_2)}{|k|^{3(b^-)-1}} \leq C_1(\beta_1,\beta_2)$$

where we use the fact the $b\in(\frac{1}{3},\frac{1}{2}]$ in the second and third inequality, respectively. This ends the proof of the claim.

With this in hand, we infer that

$$\begin{aligned}\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}} &\leq C_1 \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}}\leq 1} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \left|\widetilde{W}(k,\tau)\right|\left|\widetilde{G}(k,\tau)\right| d\tau \\ &\leq C_1 \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}}\leq 1} \left\|\widetilde{W}\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\left\|\widetilde{G}\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \\ &\leq C_1\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}.\end{aligned}$$

Consequently, for $\frac{1}{3}<b\leq\frac{1}{2}$, there exists $\theta>0$ such that

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2,T}_{s,b-1}} \leq C_1\|w\|_{X^{\beta_1,\gamma_1,T}_{s,b^-}} \leq C_1T^{\theta}\|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}},$$

reaching estimate (3.7).

Now, to prove the second inequality in (3.8), note that by duality we have

$$\begin{aligned}\|\partial_x w\|_{Y^{\beta_2,\gamma_2}_{s,b-\frac{3}{2}}} &= \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\\ a_k\neq 0}} \sum_{k\in\mathbb{Z}} a_k\int_{\mathbb{R}} \frac{ik\left\langle k\right\rangle^s|\widetilde{w}(k,\tau)|}{\left\langle \tau-\phi_{\beta_2,\gamma_2}(k)\right\rangle^{\frac{3}{2}-b}}d\tau \\ &\leq \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\\ a_k\neq 0}} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} H(k,\tau)\frac{a_k}{\left\langle \tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle^{\frac{1}{2}^+}}\widetilde{W}(k,\tau)d\tau\end{aligned}$$

where

$$H(k,\tau)=\frac{|k|}{\left\langle \tau-\phi^{\beta_1,\gamma_1}(k)\right\rangle^{b^-}\left\langle \tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle^{(1-b)^-}}$$

and

$$\widetilde{W}(k,\tau)=\left\langle k\right\rangle^s\left\langle \tau-\phi^{\beta_1,\gamma_1}(k)\right\rangle^{b^-}|\widetilde{w}(k,\tau)|.$$

The claim proved before give us

$$\begin{aligned}\|\partial_x w\|_{Y^{\beta_2,\gamma_2}_{s,-b-\frac{1}{2}}} \leq& C_1 \sup_{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1}\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}\left\|\frac{|a_k|}{\left\langle \tau-\phi^{\beta_2,\gamma_2}(k)\right\rangle^{\frac{1}{2}^+}}\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \\ \leq& C_1 \sup_{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1}\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}\|a_k\|_{\ell^2(\mathbb{Z})} \leq C_1\|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}},\end{aligned}$$

showing the second estimate of (3.8), and consequently, Lemma 3.6 is proved. □

The next lemma was borrowed from [43, Lemmas 4.1 and 4.2] and concerns with the *bilinear estimates* in Bourgain spaces for the term $\partial_x(uv)$ when the functions u and v belong in $X^{\beta_i,\gamma_i}_{s,b}$ for β_i and γ_i, $i=1$ and 2, distinct. In fact, the authors in [43] showed the result for general cases on domain $\mathbb{T}_\lambda\times\mathbb{R}$, for $\lambda\geq 1$. Here, we will revisit the result proving in a simpler way the *bilinear estimates* on $\mathbb{T}\times[0,T]$ for any $s\geq 0$, which will be used for obtaining our future results.

Lemma 3.7. *Let $s \geq 0$, $T \in (0,1)$ and $\beta_1, \beta_2 \in \mathbb{R}^*$, with $\beta_1 \neq \beta_2$*[6]*. Also consider that $\frac{\beta_1}{\beta_2} < \frac{1}{4}$. Let u and v functions such that with $[u] = [v] = 0$. There exist constants $\theta > 0$, $\epsilon = \epsilon(\beta_1, \beta_2) > 0$ and $C_2 = C_2(\beta_1, \beta_2) > 0$, independent of T, u and v, such that if $|\gamma_1| + |\gamma_2| < \epsilon$, we have:*

a) *If $u \in X_{s,\frac{1}{2}}^{\beta_1,\gamma_1,T}$ and $v \in X_{s,\frac{1}{2}}^{\beta_2,\gamma_2,T}$, then*

$$\|\partial_x(uv)\|_{Z_{s,-\frac{1}{2}}^{\beta_2,\gamma_2,T}} \leq C_2 T^{\theta} \|u\|_{X_{s,\frac{1}{2}}^{\beta_1,\gamma_1,T}} \|v\|_{X_{s,\frac{1}{2}}^{\beta_2,\gamma_2,T}}. \tag{3.9}$$

b) *If $u, v \in X_{s,\frac{1}{2}}^{\beta_2,\gamma_2,T}$ then*

$$\|\partial_x(uv)\|_{Z_{s,-\frac{1}{2}}^{\beta_1,\gamma_1,T}} \leq C_2 T^{\theta} \|u\|_{X_{s,\frac{1}{2}}^{\beta_2,\gamma_2,T}} \|v\|_{X_{s,\frac{1}{2}}^{\beta_2,\gamma_2,T}}. \tag{3.10}$$

Proof. We will prove the estimate (3.9). The proof of (3.10) is shown similarly and we omit its demonstration. Let $u \in X_{s,\frac{1}{2}}^{\beta_1,\gamma_1}$ and $v \in X_{s,\frac{1}{2}}^{\beta_2,\gamma_2}$, with $[u] = [v] = 0$. Necessarily, we must to prove

$$\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}^{\beta_2,\gamma_2}} \leq C_2 T^{\theta} \|u\|_{X_{s,\frac{1}{2}}^{\beta_1,\gamma_1}} \|v\|_{X_{s,\frac{1}{2}}^{\beta_2,\gamma_2}} \tag{3.11}$$

and

$$\|\partial_x(uv)\|_{Y_{s,-1}^{\beta_2,\gamma_2}} \leq C_2 T^{\theta} \|u\|_{X_{s,\frac{1}{2}}^{\beta_1,\gamma_1}} \|v\|_{X_{s,\frac{1}{2}}^{\beta_2,\gamma_2}}, \tag{3.12}$$

for some $\theta > 0$.

Firstly, we will show (3.11). For this end from Plancherel theorem, duality approach and convolution properties, yields that

$$\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}^{\beta_2,\gamma_2}} = \sup_{\|g\|_{X_{-s,\frac{1}{2}}^{\beta_2,\gamma_2}} \leq 1} \sum_{\Gamma} \int_{\Lambda} \frac{|k_3| \langle k_3 \rangle^s}{\langle k_1 \rangle^s \langle k_2 \rangle^s} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i(k_i, \tau_i) \rangle^{\frac{1}{2}}} d\Lambda. \tag{3.13}$$

Here $L_i(k_i, \tau_i) = \tau_i - \phi^{\beta_i,\gamma_i}(k_i)$, for $i = 1, 2$, $L_3(k_3, \tau_3) = \tau_3 - \phi^{\beta_2,\gamma_2}(k_3)$, Γ and Λ given by

$$\Gamma := \left\{ (k_1, k_2, k_3) \in \mathbb{Z}^3 \quad : \quad \sum_{i=1}^{3} k_i = 0 \right\} \quad \text{and} \quad \Lambda := \left\{ (k_1, k_2, k_3) \in \mathbb{R}^3 \quad : \quad \sum_{i=1}^{3} \tau_i = 0 \right\},$$

respectively, with

$$f_1(k_1, \tau_1) = \langle k_1 \rangle^s \left\langle \tau_1 - \phi^{\beta_1,\gamma_1}(k_1) \right\rangle^{\frac{1}{2}} \widetilde{u}(k_1, \tau_1),$$

$$f_2(k_2, \tau_2) = \langle k_2 \rangle^s \left\langle \tau_2 - \phi^{\beta_2,\gamma_2}(k_2) \right\rangle^{\frac{1}{2}} \widetilde{v}(k_2, \tau_2)$$

and

$$f_3(k_3, \tau_3) = \langle k_3 \rangle^{-s} \left\langle \tau_3 - \phi^{\beta_2,\gamma_2}(k_3) \right\rangle^{\frac{1}{2}} \widetilde{g}(k_3, \tau_3).$$

The condition $[u] = [v] = 0$ together with the fact that $(k_1, k_2, k_3) \in \mathbb{Z}^3$ ensure us that we only need to consider the case $|k_i| \geq 1$ for $i = 1, 2, 3$. In addition, as $(k_1, k_2, k_3) \in \Gamma$, we have $\frac{\langle k_3 \rangle}{\langle k_1 \rangle \langle k_2 \rangle} \leq 1$. Thus,

$$\frac{|k_3| \langle k_3 \rangle^s}{\langle k_1 \rangle^s \langle k_2 \rangle^s} \leq (|k_1||k_2||k_3|)^{\frac{1}{2}},$$

for all $s \geq 0$. Define

$$H(k_1, k_2, k_3) := \phi^{\beta_1,\gamma_1}(k_1) + \phi^{\beta_2,\gamma_2}(k_2) + \phi^{\beta_2,\gamma_2}(k_3). \tag{3.14}$$

[6]The cases $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$ is known be true and can be seen in [15, Proposition 5].

Claim. Let $\frac{\beta_1}{\beta_2} < \frac{1}{4}$. There exist $\epsilon = \epsilon(\beta_1, \beta_2)$ and $\delta > 0$ such that if $|\mu| + |\zeta| < \epsilon$ then the function H defined in (3.14) is δ-significant on $\mathbb{Z}$, i.e.,

$$\langle H(k_1, k_2, k_3)\rangle \geq \delta \prod_{i=1}^{3} |k_i| \quad \text{for any} \quad (k_1, k_2, k_3) \in \Gamma.$$

Assume that the claim holds true. Thus,

$$\prod_{i=1}^{3} |k_i| \lesssim \langle H(k_1, k_2, k_3\rangle = \left\langle \sum_{i=1}^{3} L_i(k_i, \tau_i) \right\rangle.$$

Using the last inequality we obtain

$$\begin{aligned} \sum_{\Gamma} \int_{\Lambda} \frac{|k_3| \langle k_3 \rangle^s}{\langle k_1 \rangle^s \langle k_2 \rangle^s} \prod_{i=1}^{3} \frac{|f_i(k_i, \tau_i)|}{\langle L_i(k_i, \tau_i)\rangle^{\frac{1}{2}}} d\Lambda &\lesssim \sum_{\Gamma} \int_{\Lambda} \left\langle \sum_{j=1}^{3} L_i(k_i, \tau_i) \right\rangle^{\frac{1}{2}} \prod_{i=1}^{3} \frac{f_i(k_i, \tau_i)}{\langle L_i(k_i, \tau_i)\rangle^{\frac{1}{2}}} d\Lambda \\ &\lesssim \sum_{j=1}^{3} \sum_{\Gamma} \int_{\Lambda} \frac{\langle L_j(k_j, \tau_j)\rangle^{\frac{1}{2}} \prod_{i=1}^{3} f_i(k_i, \tau_i)}{\langle L_i(k_i, \tau_i)\rangle^{\frac{1}{2}}} d\Lambda. \end{aligned} \tag{3.15}$$

Let us estimate each term of the right hand side of (3.15). For simplicity, we will present the estimate corresponding to $j = 1$. The other terms will be estimated in the similar way. For this case, we have

$$\sum_{\Gamma} \int_{\Lambda} |f_1(k_1, \tau_1)| \frac{|f_2(k_2, \tau_2)|}{\langle L_2(k_2, \tau_2)\rangle^{\frac{1}{2}}} \frac{|f_3(k_3, \tau_2)|}{\langle L_2(k_3, \tau_3)\rangle^{\frac{1}{2}}} = \sum_{\Gamma} \int_{\Lambda} \widetilde{g_1}(k_1, \tau_1) \widetilde{g_2}(k_2, \tau_2) \widetilde{g_3}(k_3, \tau_3)$$

with

$$\widetilde{g_1} = |f_1(k_1, \tau_1)| \quad \text{and} \quad \widetilde{g_i} = \frac{|f_i(k_i, \tau_i)|}{\langle L_i(k_i, \tau_i)\rangle^{\frac{1}{2}}} \quad \text{for} \quad i = 2, 3.$$

Thus,

$$\begin{aligned} \sum_{\Gamma} \int_{\Lambda} |f_1(k_1, \tau_1)| \frac{|f_2(k_2, \tau_2)|}{\langle L_2(k_2, \tau_2)\rangle^{\frac{1}{2}}} \frac{|f_3(k_3, \tau_2)|}{\langle L_2(k_3, \tau_3)\rangle^{\frac{1}{2}}} &\lesssim \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \widetilde{g_1}(-k_1, -\tau_1) \widetilde{g_2 g_3}(k_1, \tau_1) d\tau_1 \\ &\lesssim \|g_1\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \|g_2\|_{\ell^4(\mathbb{Z})L^4(\mathbb{R})} \|g_3\|_{\ell^4(\mathbb{Z})L^4(\mathbb{R})} \\ &\lesssim \|u\|_{X^{\beta_1, \gamma_1}_{s, \frac{1}{2}}} \|v\|_{X^{\beta_2, \gamma_2}_{s, \frac{1}{3}}} \|g\|_{X^{\beta_2, \gamma_2}_{-s, \frac{1}{3}}}, \end{aligned}$$

where we have used that $X^{\beta, \gamma}_{0, \frac{1}{3}}$ is continuously imbedded in the space $\ell^4(\mathbb{Z})L^4(\mathbb{R})$[7]. Replacing the last inequality in (3.15) we conclude from (3.13) that

$$\|\partial_x(uv)\|_{X^{\beta_2, \gamma_2}_{s, -\frac{1}{2}}} \lesssim \left(\|u\|_{X^{\beta_1, \gamma_1}_{s, \frac{1}{2}}} \|v\|_{X^{\beta_2, \gamma_2}_{s, \frac{1}{3}}} + \|u\|_{X^{\beta_1, \gamma_1}_{s, \frac{1}{3}}} \|v\|_{X^{\beta_2, \gamma_2}_{s, \frac{1}{2}}} \right),$$

which implies that for any $T \in (0, 1)$, there exists a positive constant C_2, independent of T, such that

$$\|\partial_x(uv)\|_{X^{\beta_2, \gamma_2, T}_{s, -\frac{1}{2}}} \leq C_2 T^{\frac{1}{6}} \|u\|_{X^{\beta_1, \gamma_1, T}_{s, \frac{1}{2}}} \|v\|_{X^{\beta_2, \gamma_2, T}_{s, \frac{1}{2}}},$$

showing (3.11).

Before presenting the proof of (3.12), let us prove the claim. In fact, note that the function H defined by (3.14) can be rewrite as

$$H(k_1, k_2, k_3) = -3\beta_2 k_1^3 h\left(\frac{k_2}{k_1}\right) - (\gamma_1 - \gamma_2)k_1,$$

where $h(x) = x^2 + x + \frac{1}{3}(1 - \frac{\beta_1}{\beta_2})$. Since $\frac{\beta_1}{\beta_2} < \frac{1}{4}$, so h does not have real roots. Thus, there exists $\delta_1 > 0$ such that $h(x) \geq \delta_1(x^2 + 1)$ for all $x \in \mathbb{R}$. In addition, we can take ϵ sufficiently enough such

[7]See [28, Lemma 3.2] or [43, Lemma 3.9].

that $|\beta_1-\beta_2||k_1| \leq \frac{1}{2}+\delta_1|\beta_2||k_1|^3$ for any $k_2 \in \mathbb{Z}^*$. Hence, for $k_1, k_2, k_3 \in \mathbb{Z}^*$ satisfying $\sum_{i=1}^3 k_i = 0$ we have

$$\begin{aligned}\langle H(k_1,k_2,k_3)\rangle \geq& 1+3|\beta_2||k_1|^3 h\left(\frac{k_2}{k_1}\right) - |\gamma_1-\gamma_2||k_1| \\ \geq& \frac{1}{2}+3\delta_1|\beta_2||k_1||k_1|^2\left(\frac{k_2^2}{k_1^2}+1\right)-\delta_1|\beta_2||k_1|^3 \\ \geq& \delta\left(1+|k_1|\sum_i^3 |k_i|^2\right) \\ \geq& \delta\prod_{i=1}^3 |k_i|,\end{aligned}$$

where δ is a positive constant which depends on β_1, β_2 and the claim is verified.

To prove (3.12) using Cauchy-Schwarz inequality and for arguments similar to the one used previously, follows that

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \leq I \times \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1\\ a_k\neq 0}} \left\|\frac{\chi_{\Omega(k_3)}(L_3)a_{k_3}}{\langle L_3\rangle^{1-a}}\right\|_{\ell^2_{k_3}(\mathbb{Z})L^2_{\tau_3}(\mathbb{R})} \tag{3.16}$$

with

$$I = \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \sum_\Gamma \int_\Lambda \frac{|k_3|\langle k_3\rangle^s}{\langle k_2\rangle^s\langle k_1\rangle^s}\frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{1}{2}}}\frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}}\frac{|\widetilde{f}_3(k_3,\tau_3)|}{\langle L_3(k_3,\tau_3)\rangle^{a}}d\Lambda$$

and f_i, L_i, for $i=1,2,3$, defined as in (3.13). Here the characteristic function $\chi_{\Omega(k_3)}(L_3)$ will be chosen so that

$$\left\|\frac{\chi_{\Omega(k_3)}(L_3)}{\langle L_3\rangle^{1-a}}\right\|_{L^2_{\tau_3}(\mathbb{R})} \lesssim 1$$

uniformly in the parameter k_3,$|k_i|\geq 1$, for $i=1,2,3$, and $a>0$ to be chosen conveniently.

Now, define $MAX := \max\{\langle L_1(k_1,\tau_1)\rangle, \langle L_2(k_2,\tau_2)\rangle, \langle L_3(k_3,\tau_3)\rangle\}$. Since H is δ-significant, we have

$$\prod_{i=1}^3 |k_i| \lesssim H(k_1,k_2,k_3) = \sum_{i=1}^3 L_i(k_i,\tau_i) \lesssim \sum_i^3 \langle L_i(k_i,\tau_i)\rangle \lesssim MAX = \langle L_1(k_1,\tau_1)\rangle.$$

The rest of the proof will be split in two cases.

Case 1: $MAX = \langle L_1(k_1,\tau_1)\rangle$ or $MAX = \langle L_2(k_2,\tau_2)\rangle$.

Assume without loss of generality $MAX = \langle L_1(k_1,\tau_1)\rangle$. Take $a = \frac{1}{2}^- = \frac{1}{2}-\epsilon'$ with $12\epsilon' = \frac{1}{100}$. In this case, we can put $\Omega(k_3)=\mathbb{R}$. Thus, from (3.16) we have

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1}\sum_\Gamma\int_\Lambda |f_1(k_1,\tau_1)|\frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}}\frac{|\widetilde{f}_3(k_3,\tau_3)|}{\langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}^-}}d\Lambda.$$

Similarly to $X^{\beta_2,\gamma_2}_{s,-\frac{1}{2}}$-norm, we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}}\|f_3\|_{X^{\beta_2,\gamma_2}_{0,-\frac{1}{6}+}} \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}}. \tag{3.17}$$

Case 2: $MAX = \langle L_3(k_3,\tau_3)\rangle$

This case will be divided in two parts.

Part I. $\langle L_3(k_3,\tau_3)\rangle^{\frac{1}{100}} \leq \delta\langle L_1(k_1,\tau_1)\rangle\langle L_2(k_2,\tau_2)\rangle$.

Observe that

$$\left(\prod_{i=1}^{3}|k_i|\right)^{\frac{1}{2}} \lesssim \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}} = \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}-\epsilon'} \langle L_1(k_1,\tau_1)\rangle^{\frac{1}{12}} \langle L_2(k_2,\tau_2)\rangle^{\frac{1}{12}}. \tag{3.18}$$

Therefore, choosing a and $\Omega(k_3)$ as in the Case 1, from (3.16) results

$$\begin{aligned}\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim & \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\le 1} \sum_{\Gamma}\int_{\Lambda} \frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{5}{12}}} \frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{12}}} |\widetilde{f}_3(k_3,\tau_3)| d\Lambda \\ \lesssim & \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{5}{12}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{5}{12}}}.\end{aligned}$$

Part II. $\langle L_1(k_1,\tau_1)\rangle \langle L_2(k_2,\tau_2)\rangle \ll \delta \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{100}}$

Note that

$$\langle L_3(k_3,\tau_3)+H(k_1,k_2,k_3)\rangle = \langle L_1+L_2\rangle \ll \delta \langle L_3\rangle^{\frac{1}{100}}.$$

Thus, $|H(k_1,k_2,k_3)| \sim |L_3(k_3,\tau_3)|$ and

$$\langle L_3(k_3,\tau_3)+H(k_1,k_2,k_3)\rangle \ll \delta \langle H(k_1,k_2,k_3)\rangle^{\frac{1}{100}}. \tag{3.19}$$

Define for any $k_3 \in \mathbb{Z}^*$ the set

$$\begin{aligned}\Omega^{\delta}(k_3) := \Big\{ & \tau\in\mathbb{R} \ : \ \exists\, k_1,k_2,\in\mathbb{Z} \ \text{ such that } \sum_{i=1}^{3}k_i=0 \ \text{ and} \\ & \langle L_3(k_3,\tau_3)+H(k_1,k_2,k_3)\rangle \ll \delta \langle H(k_1,k_2,k_3)\rangle^{\frac{1}{100}} \Big\}.\end{aligned} \tag{3.20}$$

From (3.19) follows that $L_3(k_3,\tau_3)\in\Omega^{\delta}(k_3)$. Taking $a=\frac{1}{2}$ and $\Omega(k_3)=\Omega^{\delta}(k_3)$ defined by (3.20) we have[8]

$$\left\| \frac{\chi_{\Omega(k_3)}(L_3)}{\langle L_3\rangle^{\frac{1}{2}}} \right\|_{L^2_{\tau_3(\mathbb{R})}} \lesssim 1$$

uniformly in k_3 implying

$$\sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\le 1 \\ a_k\neq 0}} \left\| \frac{a_{k_3}}{\langle L_3\rangle^{\frac{1}{2}}} \right\|_{\ell^2_{k_3}(\mathbb{Z})L^2_{\tau_3}(\mathbb{R})} \lesssim 1.$$

Hence, using the first inequality in (3.18) we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\le 1} \sum_{\Gamma}\int_{\Lambda} \frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{1}{2}}} \frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}} |\widetilde{f}_3(k_3,\tau_3)| \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{3}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}}.$$

Then, in both situation we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim T^{\frac{1}{12}} \|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}},$$

showing (3.12) and, consequently, finishing the demonstration of the lemma. □

Finally, to finish this section we will prove nonlinear estimates associated with the solutions of (3.4). To do it, we introduce the following notation

$$\mathcal{Z}_i := Z^{\beta_i,\gamma_i,T}_{s,-\frac{1}{2}},$$

for $i=1,2$ and

$$\mathcal{Z} := Z^{\beta_1,\gamma_1,T}_{s,-\frac{1}{2}} \times Z^{\beta_2,\gamma_2,T}_{s,-\frac{1}{2}}.$$

[8] This prove is extremely technical and can be found in [43, Lemma 6.2 case 5.2.2. and Lemma 6.3] which in turn was inspired by [15, Lemma 7.4]

Lemma 3.8. *Let (u,v) and (w,z) belong to $\mathcal{Z}$ with $[u]=[v]=0$. Consider s,β_i,γ_i, $i=1,2$, as in Lemma 3.7 satisfying $\frac{\beta_2}{\beta_1}<0$, P and Q defined by (3.5). Then, there exist constants $\theta>0$, $\epsilon=\epsilon(\beta_1,\beta_2)>0$ and $C_3=C_3(\beta_1,\beta_2)>0$, independent of T, u and v, such that if $|\gamma_1|+|\gamma_2|<\epsilon$, the following estimates are satisfied*

$$\|\partial_x(P(u,v),Q(u,v))\|_{\mathcal{Z}} \le C_3T^{\theta}\|(u,v)\|^2_{X^{\beta_1,\gamma_1,T}_{s,\frac12}\times X^{\beta_2,\gamma_2,T}_{s,\frac12}}, \tag{3.21}$$

$$\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \le C_3T^{\theta}\|(u,v)-(w,z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right) \tag{3.22}$$

and

$$\|\partial_x(Q(u,v)-Q(w,z))\|_{\mathcal{Z}_2} \le C_3T^{\theta}\|(u,v)-(w,z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right). \tag{3.23}$$

Proof. First of all, (3.21) is a direct consequence of Lemma 3.7. Just applying (3.9) for P and Q provides us the existence of positive constants C_3, θ and ϵ such that

$$\begin{aligned}\|\partial_x(P(u,v),Q(u,v))\|_{\mathcal{Z}} \le{}& C_2T^{\theta}\Big(|A|\|u\|_{\mathcal{Z}_1}+|B|\|u\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}+\frac{|C|}{2}\|v\|_{\mathcal{Z}_2}+|D|\|v\|_{\mathcal{Z}_2}\\ &+|C|\|u\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}+\frac{|B|}{2}\|u\|_{\mathcal{Z}_1}\Big)\\ \le{}& C_3T^{\theta}\|(u,v)\|_{X^{\beta_1,\gamma_1,T}_{s,\frac12}\times X^{\beta_2,\gamma_2,T}_{s,\frac12}},\end{aligned}$$

whenever $|\gamma_1|+|\gamma_2|<\epsilon$, where $C_3=C_2\cdot 2\max\{|A|,|B|,|C|,|D|\}$.

Let us now prove (3.22). As the proof of (3.23) is analogous we will omit it. Note that we can write

$$P(u,v)-P(w,z)=A(u-w)(u+w)+B(u-w)v+B(v-z)w+\frac{C}{2}(v-z)(v+z)$$

Thus, gain by (3.9), we get that

$$\begin{aligned}\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \le{}& C_3T^{\theta}\left(\|(u-w)\|_{\mathcal{Z}_1}\|u+w\|_{\mathcal{Z}_1}+\|u-w\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}\right.\\ &\left.+\|v-z\|_{\mathcal{Z}_2}\|w\|_{\mathcal{Z}_1}+\|v-z\|_{\mathcal{Z}_2}+\|v+z\|_{\mathcal{Z}_2}\right),\end{aligned}$$

which implies

$$\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \le C_3T^{\theta}\|(u,v)-(w-z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right).$$

Therefore, (3.22) and (3.23) is verified and the proof of the lemma is complete. □

3.3. **Local well-posedness.** Throughout the article, from now on, we will consider the following notations

$$\mathcal{Z}_{s,b}:=Z^{1,\mu,T}_{s,b}\times Z^{\alpha,\zeta,T}_{s,b},\quad \mathcal{Z}^1_{s,b}:=Z^{1,\mu,T}_{s,b},\quad \mathcal{Z}^{\alpha}_{s,b}:=Z^{\alpha,\zeta,T}_{s,b},$$

and

$$\mathcal{X}_{s,b}:=X^{1,\mu,T}_{s,b}\times X^{\alpha,\zeta,T}_{s,b},\quad \mathcal{X}^1_{s,b}:=X^{1,\mu,T}_{s,b},\quad \mathcal{X}^{\alpha}_{s,b}:=X^{\alpha,\zeta,T}_{s,b}.$$

Additionally, when $b=\frac12$, we will denote

$$\mathcal{Z}_{s}:=Z^{1,\mu,T}_{s,\frac12}\times Z^{\alpha,\zeta,T}_{s,\frac12},\quad \mathcal{Z}^1_{s}:=Z^{1,\mu,T}_{s,\frac12},\quad \mathcal{Z}^{\alpha}_{s}:=Z^{\alpha,\zeta,T}_{s,\frac12},$$

and

$$\mathcal{X}_{s}:=X^{1,\mu,T}_{s,\frac12}\times X^{\alpha,\zeta,T}_{s,\frac12},\quad \mathcal{X}^1_{s}:=X^{1,\mu,T}_{s,\frac12},\quad \mathcal{X}^{\alpha}_{s}:=X^{\alpha,\zeta,T}_{s,\frac12}.$$

Let us now consider the following IVP

$$\begin{cases}\partial_t u+\partial_x^3u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=Gf, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x u+\eta\partial_x u+\partial_x Q(u,v)=Gh, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases} \tag{3.24}$$

where the quantities P, Q and G were defined in the previous sections and $\alpha<0$. For given $\lambda>0$, let us define

$$L_{\beta,\gamma,\lambda}\phi=\int_0^1 e^{-2\lambda\tau}S^{\beta,\gamma}(-\tau)GG^*S^*(-\tau)\phi d\tau,$$

for any $\phi\in H^s(\mathbb{T})$ and $s\geq 0$. Clearly, L_λ is a bounded linear operator from $H_0^s(\mathbb{T})$ to $H_0^s(\mathbb{T})$. Moreover, $L_{\beta,\gamma,\lambda}$ is a self-adjoint positive operator on $L_0^2(\mathbb{T})$, and so is its inverse $L_{\beta,\gamma,\lambda}^{-1}$. Therefore $L_{\beta,\gamma,\lambda}$ is an isomorphism from $L_0^2(\mathbb{T})$ onto itself, and the same is true on $H_0^s(\mathbb{T})$, with $s\geq 0$ (see, for instance, [28, Lemma 2.4]).

With these information in hand, choose the two feedback controls

$$f=-G^*L_{\beta,\gamma,\lambda}^{-1}u \quad \text{and} \quad h=-G^*L_{\beta,\gamma,\lambda}^{-1}v,$$

in (3.24), to transform this system in a resulting closed-loop system reads as follows

$$\begin{cases}\partial_t u+\partial_x^3u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=-K_{1,\mu,\lambda}u, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x u+\eta\partial_x u+\partial_x Q(u,v)=-K_{\alpha,\zeta,\lambda}v, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases} \tag{3.25}$$

with $K_{\beta,\gamma,\lambda}:=GG^*L_{\beta,\gamma,\lambda}^{-1}$ for $\beta=1$ and $\gamma=\mu$, and for $\beta=\alpha$ and $\gamma=\eta$. If $\lambda=0$, we have $K_0=GG^*$.

We will prove that the IVP (3.25) is well-posed in the spaces $H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$, for $s\geq 0$. To prove it, we will borrow the following lemma shown in [28, Lemma 4.2] for the case $\beta=1$ and $\gamma=\mu>0$. The proof in the general case, presented below, is similar to the one made there and will be omitted.

Lemma 3.9. *For any $\widetilde{\epsilon}>0$ and $\phi\in Z_{s,\frac12}^{\beta,\gamma,T}$ there exists a positive constant $C(\widetilde{\epsilon})>0$ such that*

$$\left\|\int_0^t S^{\beta,\gamma}(t-\tau)(K_\lambda\phi)(\tau)d\tau\right\|_{Z_{s,\frac12}^{\beta,\gamma,T}}\leq C(\widetilde{\epsilon})T^{1-\widetilde{\epsilon}}\|\phi\|_{Z_{s,\frac12}^{\beta,\gamma,T}}.$$

The next local well-posedness result is a consequence of Lemmas 3.4, 3.8 and 3.9.

Theorem 3.10. *Let $\lambda\geq 0$ and $s\geq 0$ be given. Then, there exists $\epsilon=\epsilon(\alpha)$ with $|\mu|+|\zeta|<\epsilon$ such that for $T>0$, small enough, and any $(u_0,v_0)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$ there exists a unique solution (u,v) of (3.25) in the class*

$$(u,v)\in\mathcal{X}:=\mathcal{Z}_s^1\cap C([0,T];L_0^2(\mathbb{T}))\times\mathcal{Z}_s^\alpha\cap C([0,T];L_0^2(\mathbb{T})). \tag{3.26}$$

Furthermore, the following estimate holds

$$\|(u,v)\|_{\mathcal{Z}_s}\leq a_{T,s}(\|(u_0,v_0)\|)\|(u_0,v_0)\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})}, \tag{3.27}$$

where $a_{s,T}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a nondecreasing continuous function depending only of T, s and constants α,μ,ζ.

In addition, for any $T_0\in(0,T)$ there exists a neighborhood U_0 of (u_0,v_0) such that the application $(u_0,v_0)\longmapsto(u,v)$ from U_0 into $\mathcal{X}$ is Lipschitz.

Proof. Let $(u_0,v_0)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$. By Duhamel's principle we can rewrite the solution of the system (3.25) in its integral form

$$u(t)=S^{1,\mu}(t)u_0-\int_0^t S^{1,\mu}(t-\tau)(\partial_x v(x,\tau))d\tau-\int_0^t S^{1,\mu}(t-\tau)(\partial_x P(u,v))(x,\tau)d\tau$$
$$-\int_0^t S^{1,\mu}(t-\tau)(K_{1,\mu,\lambda}u)(\tau)d\tau$$

and

$$v(t) = S^{\alpha,\zeta}(t)v_0 - \int_0^t S^{\alpha,\zeta}(t-\tau)(\partial_x u(x,\tau))d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)(\partial_x Q(u,v))(x,\tau)d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)(K_{\alpha,\zeta,\lambda}v)(\tau)d\tau.$$

For given (u_0, v_0), take $\mathcal{X}$ as in (3.26) and define the map $\Gamma : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ by

$$\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v)),$$

with

$$\Gamma_1(u,v) = S^{1,\mu}(t)u_0 - \int_0^t S^{1,\mu}(t-\tau)(\partial_x v(x,\tau))d\tau - \int_0^t S^{1,\mu}(t-\tau)(\partial_x P(u,v))(x,\tau)d\tau - \int_0^t S^{1,\mu}(t-\tau)(K_{1,\mu,\lambda}u)(\tau)d\tau, \tag{3.28}$$

and

$$\Gamma_2(u,v) = S^{\alpha,\zeta}(t)v_0 - \int_0^t S^{\alpha,\zeta}(t-\tau)(\partial_x u(x,\tau))d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)(\partial_x Q(u,v))(x,\tau)d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)(K_{\alpha,\zeta,\lambda}v)(\tau)d\tau. \tag{3.29}$$

Thanks to the Lemmas 3.4, 3.8 and 3.9, by an application of Banach's fixed point theorem, we have there exists a unique solution (u, v) of (3.28) in the class

$$(u,v) \in Z_s^1 \cap L^2(0,T;L_0^2(\mathbb{T})) \times Z_s^\alpha \cap L^2(0,T;L_0^2(\mathbb{T})).$$

Finally, to prove (3.27) observe that $\mathcal{Z}_s \subset C(0,T;H_0^s(\mathbb{T})) \times C(0,T;H_0^s(\mathbb{T}))$ for any $s \in \mathbb{R}$, so there exists a constant $C_4 > 0$ such that

$$\|(u,v)\|_{L^\infty(0,T;H_0^s(\mathbb{T}))\times L^\infty(0,T;H_0^s(\mathbb{T}))} \le C_4\|(u,v)\|_{\mathcal{Z}_s} \le 2C_0C_4\|(u_0,v_0)\|_s.$$

This ends the proof of Theorem 3.10. □

3.4. **Global well-posedness.** We check that the system (3.25) is globally well-posed in the space $H^s(\mathbb{T})$, for any $s \ge 0$. Precisely, the result can be read as follows.

Theorem 3.11. *Let $(u_0, v_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$, for any $s \ge 0$. Then the solution $(u, v) \in \mathcal{X}$ given in Theorem 3.10 can be extended for any $T > 0$ and still satisfies (3.27).*

Proof. Assume first $s = 0$. Multiplying the first equation of (3.25) by u and the second one by v, integrating on $\mathbb{T} \times (0, t)$, for $t \ge 0$, we have

$$\|(u(\cdot,t),v(\cdot,t))\|^2 \le 2\|G\|^2\| \left[L_{1,\mu,\lambda}^{-1}\| + \|L_{\alpha,\zeta,\lambda}^{-1}\|\right] \int_0^t \|(u,v)(\cdot,\tau)\|^2,$$

since G and $L_{\beta,\gamma,\lambda}^{-1}$ are continuous in $L^2(\mathbb{T})$ and

$$\int_{\mathbb{T}} \partial_x P(u,v)u + \partial_x Q(u,v)v = \frac{2}{3}\int_{\mathbb{T}} \frac{d}{dx}\left[(Au^3 + Bv^3) + Bu^2v + Cuv^2\right] = 0. \tag{3.30}$$

Using Grönwall's inequality holds that

$$\|(u(\cdot,t),v(\cdot,t))\|^2 \le \|(u_0,v_0)\|^2 e^{C_5 t}, \tag{3.31}$$

with $C_5 = 2\|G\|^2\left[\|L_{1,\mu,\lambda}^{-1}\| + \|L_{\alpha,\zeta,\lambda}^{-1}\|\right]$. In particular, for $\lambda = 0$, from the energy identity, we get

$$\frac{1}{2}\frac{d}{dt}\|(u(\cdot,t),v(\cdot,t))\|^2 = -\|(Gu,Gv)\|^2 \le 0 \tag{3.32}$$

and

$$\|(u(\cdot,t),v(\cdot,t))\|^2 \le \|(u_0,v_0)\|^2,$$

which ensures that (3.25) is globally well-posed in $L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$.

Next, we show that (3.25) is globally well-posed in the space $H_0^3(\mathbb{T}) \times H_0^3(\mathbb{T})$. For a smooth solution (u, v) of (3.25), let $(\widetilde{u}, \widetilde{v}) = (\partial_t u, \partial_t v)$. Then

$$\begin{cases} \partial_t \widetilde{u} + \partial_x^3 \widetilde{u} + \mu \partial_x \widetilde{u} + \eta \widetilde{v} + \partial_x \widetilde{P}(\widetilde{u}, \widetilde{v}) = -K_\lambda \widetilde{u}, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t \widetilde{v} + \alpha \partial_x^3 \widetilde{v} + \zeta \partial_x \widetilde{v} + \eta \widetilde{u} + \partial_x \widetilde{Q}(\widetilde{u}, \widetilde{v}) = -K_\lambda \widetilde{v}, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (\widetilde{u}(x,0), \widetilde{v}(x,0)) = (\widetilde{u}_0, \widetilde{v}_0), & x \in \mathbb{T}, \end{cases}$$

where

$$\begin{aligned} \widetilde{P}(\widetilde{u}, \widetilde{v}) &= 2Au\widetilde{u} + B\widetilde{u}v + Bu\widetilde{v} + Cv\widetilde{v}, \\ \widetilde{Q}(\widetilde{u}, \widetilde{v}) &= 2Dv\widetilde{v} + C\widetilde{v}u + Cv\widetilde{u} + Bu\widetilde{u}, \\ \widetilde{u}_0 &= -K_\lambda u_0 - u_0''' - \mu u_0' - \eta v_0' - P'(u_0, v_0), \\ \widetilde{v}_0 &= -K_\lambda v_0 - v_0''' - \zeta v_0' - \eta u_0' - Q'(u_0, v_0), \end{aligned}$$

with " $'$ " denoting here the derivative with respect to variable x. Observe that

$$\begin{aligned} \|P'(u_0, v_0), Q'(u_0, v_0)\| \leq & \frac{2C_3}{C_2} \|(u_0, v_0)\| \|(u_0', v_0')\|_{L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})} \\ \leq & \frac{2C_3}{C_2} \|(u_0, v_0)\| C_5 \|(u_0, v_0)\|^{\frac{1}{2}} \|(\partial_x^3 u_0, \partial_x^3 v_0)\|^{\frac{1}{2}} \\ \leq & \frac{2C_3 C_5}{C_2} \|(u_0, v_0)\|^{\frac{3}{2}} \|(u_0, v_0)\|_3^{\frac{1}{2}}, \end{aligned}$$

where C_5 is due to the Gagliardo-Nirenberg inequality. So, we have

$$\begin{aligned} \|(\widetilde{u}_0, \widetilde{v}_0)\| \leq & \|(K_\lambda u_0, K_\lambda v_0)\| + C_6 \|(u_0, v_0)\|_3 + 2(|\mu| + |\eta| + |\zeta|) \|(u_0, v_0)\|_1 \\ & + \|P'(u_0, v_0), Q'(u_0, v_0)\| \\ \leq & \|(u_0, v_0)\| + 2(|\mu| + |\eta| + |\zeta|) \|(u_0, v_0)\|_1 + (1 + C_6) \|(u_0, v_0)\|_3 \\ & + \left(\frac{C_3 C_5}{C_2} \right)^2 \|(u_0, v_0)\|^3. \end{aligned} \tag{3.33}$$

where $C_6 = \max\{1, |\alpha|\}$. Since,

$$\|\partial_x (\widetilde{P}(\widetilde{u}, \widetilde{v}), \widetilde{Q}(\widetilde{u}, \widetilde{v}))\|_{\mathcal{Z}_s} \leq 2C_3 T^\theta \|(u, v)\|_{\mathcal{Z}_s} \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_s},$$

with $|\mu| + |\zeta| < \epsilon$, for some ϵ that depends only α, we obtain from (3.33) that

$$\|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0} \leq C_0 \|(\widetilde{u}_0, \widetilde{v}_0)\| + (C(\widetilde{\epsilon}) T^{1-\widetilde{\epsilon}} + C_1 T^\theta) \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0} + 2C_0 C_3 T^\theta \|(u, v)\|_{\mathcal{Z}_0} \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0}.$$

Choosing T as

$$C(\epsilon) T^{1-\widetilde{\epsilon}} + C_1 T^\theta + 2C_0 C_3 T^\theta d < \frac{1}{2},$$

we have

$$\|(u, v)\|_{\mathcal{Z}_0} \leq d = 2C_0 \|(u_0, v_0)\|$$

and consequently

$$\|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0} \leq C_0 \|(\widetilde{u}_0, \widetilde{v}_0)\| + (C(\epsilon) T^{1-\epsilon} + C_1 T^\theta + 4C_0^2 C_3 T^\theta \|(u_0, v_0)\|) \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0}.$$

Hence, for T_1 satisfying

$$C(\epsilon) T_1^{1-\epsilon} + C_1 T^\theta + 4C_0^2 C_3 T_1^\theta \|(u_0, v_0)\| < \frac{1}{2},$$

we obtain

$$\|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0} \leq 2C_0 \|(\widetilde{u}_0, \widetilde{v}_0)\|.$$

Therefore, for $T_0 = \min\{T, T_1\}$, we see that

$$\|(\widetilde{u}, \widetilde{v})\|_{L^\infty(0,T_0; L^2(\mathbb{T}) \times L^\infty(0,T_0; L^2(\mathbb{T})} \leq C_7 \|(\widetilde{u}, \widetilde{v})\|_{\mathcal{Z}_0} \leq 2C_0 C_7 \|(\widetilde{u}_0, \widetilde{v}_0)\|.$$

From the following equations

$$\begin{cases}\partial_x^3 u = -\widetilde{u} - \mu\partial_x u - \eta\partial_x v - \partial_x P(u,v) - K_\lambda u,\\ \partial_x^3 v = -\widetilde{v} - \zeta\partial_x v - \eta\partial_x u - \partial_x Q(u,v) - K_\lambda v,\end{cases}$$

we infer that

$$\begin{aligned}\|(\partial_x^3 u, \partial_x^3 v)\| \leq& \|(\widetilde{u},\widetilde{v})\| + \|(u,v)\| + \left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)\|(\partial_x u, \partial_x v)\|_{L^\infty(\mathbb{T})\times L^\infty(\mathbb{T})}\\ \leq& \|(\widetilde{u},\widetilde{v})\| + \|(u,v)\| + C_6\left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)\|(u,v)\|^{\frac12}\|(\partial_x^3 u, \partial_x^3 v)\|^{\frac12}\\ \leq& \|(\widetilde{u},\widetilde{v})\| + \frac12\|(\partial_x^3 u, \partial_x^3 v)\| + \left[1 + \frac{C_6^2}{2}\left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)^2\right]\|(u,v)\|,\end{aligned}$$

for $0 < t < T_0$ and $C_7 = 2(|\mu| + |\eta| + |\zeta|)$. Consequently, since $(\widetilde{u}_0, \widetilde{v}_0)$ satisfies (3.33) , we get that

$$\|(u,v)\|_{L^\infty(0,T;H_0^3(\mathbb{T}))\times L^\infty(0,T;H_0^3(\mathbb{T}))} \leq a_{T,3}(\|(u_0,v_0)\|)\|(u_0,v_0)\|_3.$$

Combining to (3.31), this shows that $(u,v) \in C(\mathbb{R}^+; H_0^3(\mathbb{T})) \times C(\mathbb{R}^+; H_0^3(\mathbb{T}))$ and (3.27) holds true for $s = 3$. A similar result can be obtained for any $s \in 3\mathbb{N}^*$. Note that for other values of s, the global well-posedness follows by nonlinear interpolation as done in [4]. This achieves the result. □

4. Stabilization results: Nonlinear problems

In this section, we are concerned with stabilizability of the closed loop system (3.25).

4.1. **A local result.** Our first result is local in the sense that the initial data need to be in a small ball in the energy space to ensure that the solution of the system goes to zero exponentially, for t sufficiently large. The result is the following one.

Theorem 4.1. *Let $0 < \lambda' < \lambda$ and $s \geq 0$ be given. There exists $\delta > 0$ such that for any $(u_0, v_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ and $\|(u_0,v_0)\|_s \leq \delta$, the corresponding solution (u,v) of (3.25) satisfies*

$$\|(u(\cdot,t), v(\cdot,t))\|_s \leq Ce^{-\lambda' t}\|(u_0,v_0)\|_s, \quad \forall t \geq 0,$$

where $C > 0$ is a constant independent of (u_0, v_0).

Proof. Since K_λ is a bounded operator, the solution of (3.25) can be rewritten in its integral form

$$\begin{aligned} u(t) &= S_\lambda^{1,\mu}(t)u_0 - \eta\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau - \int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x P(u,v)(\tau)d\tau \\ v(t) &= S_\lambda^{\alpha,\zeta}(t)v_0 - \eta\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau - \int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x Q(u,v)(\tau)d\tau, \end{aligned} \tag{4.1}$$

where $S_\lambda^{\beta,\gamma}(t) = e^{-t(\beta\partial_x^3 + \gamma\partial_x + K_\lambda)}$ is the group to the linear system associated to (3.25).

Now, let us consider α, μ, ζ satisfying the hypothesis of the Lemma 3.8 and 3.9. Next, using Lemma 3.4 twice, Lemmas 3.8 and 3.9 and finally, the fact that $L_{\beta,\gamma,\lambda}$ is a bounded linear operator from $H_0^s(\mathbb{T})$ to $H_0^s(\mathbb{T})$, for all $s \geq 0$, we can guarantee the existence of a constant $c > 0$ such that the following inequalities are verified

$$\|S_\lambda^{\beta,\gamma}(t)\phi\|_{Z^{\beta,\gamma,T}_{s,\frac12}} \leq c\|\phi\|_s, \tag{4.2}$$

for any $\phi \in H_0^s(\mathbb{T})$,

$$\left\|\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau\right\|_{\mathcal{Z}_s^1} \leq c\|v\|_{\mathcal{Z}_s^\alpha}, \tag{4.3}$$

$$\left\|\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)(\tau)d\tau\right\|_{\mathcal{Z}_s^\alpha} \leq c\|u\|_{\mathcal{Z}_s^1} \tag{4.4}$$

for any $(u,v)\in \mathcal{Z}_s$,

$$\left\|\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x P(u,v)(\tau)d\tau\right\|_{\mathcal{Z}_s^1}\leq c\|(u,v)\|_{\mathcal{Z}_s}^2, \tag{4.5}$$

and

$$\left\|\int_0^t S_\lambda(t-\tau)\partial_x Q(u,v)(\tau)d\tau\right\|_{\mathcal{Z}_s^\alpha}\leq c\|(u,v)\|_{\mathcal{Z}_s}^2, \tag{4.6}$$

for any $(u,v)\in \mathcal{Z}_s$ with $[u]=[v]=0$.

Now, thanks to the ideas introduced by [41, Theorem 2.1] and [28, Proposition 2.5], for given $s\geq 0$, there exists some positive constant M_s such that

$$\|(S_\lambda^{1,\mu}(t)u_0, S_\lambda^{\alpha,\zeta}(t)v_0)\|_s\leq M_s e^{-\lambda t}\|(u_0,v_0)\|_s,\ \forall\ t\geq 0,$$

where $S_\lambda^{\beta,\gamma}$ with $\beta=1$ and $\gamma=\mu$, and for $\beta=\alpha$ and $\gamma=\eta$ are the groups associated to the linear system (3.25). Pick $T>0$ such that

$$2M_s e^{-\lambda T}\leq e^{-\lambda' T}.$$

We seek a solution (u,v) to the integral equations (4.1), as a fixed point of the following map

$$\Gamma_\lambda(w,z)=(\Gamma_\lambda^1(w,z),\Gamma_\lambda^2(w,z)),$$

defined by

$$\Gamma_\lambda^1(w,z)=S_\lambda^{1,\mu}(t)u_0-\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau-\int_0^t S_\lambda^{1,\mu}(t-\tau)(\partial_x P(w,z))(\tau)d\tau$$

and

$$\Gamma_\lambda^2(w,z)=S_\lambda^{\alpha,\zeta}(t)v_0-\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau-\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)(\partial_x Q(w,z))(\tau)d\tau,$$

in some closed ball

$$B_R(0)\subset Z_s^1\cap L^2(0,T;L_0^2(\mathbb{T}))\times Z_s^\alpha\cap L^2(0,T;L_0^2(\mathbb{T}))$$

for the $\|(w,z)\|_{\mathcal{Z}_s}$-norm. This will be done provided that

$$\|(u_0,v_0)\|_s\leq\delta,$$

where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and R will be chosen in such a way that

$$\|(u(T),v(T))\|_s\leq e^{-\lambda' T}\|(u_0,v_0)\|_s.$$

Since Γ_λ is a contraction in $B_R(0)$, by a fixed point argument, its unique fixed point $(u,v)\in B_R(0)$ fulfills

$$\|(u(T),v(T))\|=\|\Gamma_\lambda(u,v)\|_s\leq e^{-\lambda' T}\delta.$$

Finally, assume that $\|(u_0,v_0)\|_s<\delta$. Changing δ into $\delta':=\|(u_0,v_0)\|_s$ and R into $R'=\left(\frac{\delta'}{\delta}\right)^{\frac{1}{2}}R$, we infer that

$$\|(u(T),v(T))\|_s\leq e^{-\lambda' T}\|(u_0,v_0)\|_s$$

and by induction yields

$$\|(u(nT),v(nT))\|_s\leq e^{-\lambda' nT}\|(u_0,v_0)\|_s,$$

for any $n\geq 0$. As $\mathcal{Z}_s\cap L^2(0,T;L_0^2(\mathbb{T}))\subset C([0,T];H_0^s(\mathbb{T}))$, we infer by the semigroup property that there exists some constant $C'>0$ such that

$$\|(u(t),v(t))\|_s\leq C'e^{-\lambda' t}\|(u_0,v_0)\|_s,$$

provided that $\|(u_0,v_0)\|_s\leq\delta$ and the proof of Theorem 4.1 is completed. □

4.2. **A global result.** As previously mentioned the stability result presented in Theorem 4.1 is purely local. Now we are in position to extend it to a global stability.

Proof of Theorem 1.4. Theorem 1.4 is a direct consequence of the following observability inequality.

Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\rho > 0$ such that for any $(u_0, v_0) \in L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ satisfying $\|(u_0, v_0)\| \le R_0$, the corresponding solution (u, v) of (3.25) satisfies

$$\|(u_0, v_0)\|^2 \le \rho \int_0^T \|(Gu, Gv)\|^2(t)dt. \tag{4.7}$$

In fact, if (4.7) holds, the energy estimate give us

$$\|u(\cdot, t)\|^2 = \|(u, v)(\cdot, 0)\|^2 - \int_0^t \|(Gu, Gv)\|^2(\tau)d\tau, \quad \forall t \ge 0.$$

The last equality ensures that

$$\|(u, v)(\cdot, T)\|^2 \le (1 - \rho^{-1}) \|(u_0, v_0)\|^2.$$

Thus,

$$\|(u, v)(\cdot, mT)\|^2 \le (1 - \rho^{-1})^m \|(u_0, v_0)\|^2,$$

which yields

$$\|(u, v)(\cdot, t)\| \le Ce^{-\rho t} \|(u_0, v_0)\|, \forall t > 0.$$

Finally, we obtain a constant γ independent of R_0 by noticing that for $t > c(\|(u_0, v_0)\|)$, the L^2 norm of $u(\cdot, t)$ is smaller than 1, so that we can take the γ corresponding to $R_0 = 1$, proving the result. □

Before presenting the proof of the observability inequality (4.7), we will need an auxiliary lemma.

Lemma 4.2. *Let $(w_n, z_n) \in \mathcal{X}_s$ be bounded sequences. Define*

$$b_n := \int_{\mathbb{T}} g(y)w_n(y, t)dy \quad \text{and} \quad c_n := \int_{\mathbb{T}} g(y)z_n(y, t)dy.$$

If $(w_n, z_n) \rightharpoonup (0, 0)$ in $\mathcal{X}_s$ then

$$b_n, c_n \longrightarrow 0 \text{ in } L^2(0, T). \tag{4.8}$$

Proof. By Cauchy-Schwarz inequality we have

$$\begin{aligned}\|(b_n, c_n)\|^2_{L^2_T \times L^2_T} &\le \int_0^T \|g\|^2_{L^2(\mathbb{T})} \|(w_n(\cdot, t), z_n(\cdot, t))\|^2_{L^2(\mathbb{T})} dt \\ &\le C\|g\|^2_{L^2(\mathbb{T})} \|(w_n, z_n)\|_{\mathcal{X}_{0,0}},\end{aligned}$$

for some constant $C > 0$. By hypothesis

$$(w_n, z_n) \rightharpoonup (0, 0) \text{ in } \mathcal{X}_0,$$

since $\mathcal{X}_0$ is compactly embedded in $\mathcal{X}_{0,0}$ the result is proved. □

Proof of the observability inequality (4.7). Suppose that (4.7) does not occur. Thus, for any $n \in \mathbb{N}$, there exists $(u_{n,0}, v_{n,0}) := (u_n(0), v_n(0)) \in L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ such that the solution $(u_n, v_n) \in X$ of IVP (3.25), given by Theorem 3.11, satisfies

$$\|(u_{n,0}, v_{n,0})\| \le R_0 \tag{4.9}$$

and

$$\int_0^T \|(Gu_n, Gv_n)\|^2(t)dt < \frac{1}{n}\|(u_{n,0}, v_{n,0})\|^2. \tag{4.10}$$

Since $a_n := \|(u_{n,0}, v_{n,0})\|$ is a bounded sequence in $\mathbb{R}$, we can choose a subsequence of $\{a_n\}$, still denoted by $\{a_n\}$, such that

$$\lim_{n\to\infty} a_n = a.$$

So, there are two possibilities for the limit, which will be divided in the following cases:

$$i.\ \ a > 0 \quad \text{and} \quad ii.\ \ a = 0.$$

i. Case $a > 0$.

Note that the sequence (u_n, v_n) is bounded in $L^\infty(0,T;L^2(\mathbb{T})) \times L^\infty(0,T;L^2(\mathbb{T}))$ and, also, in $\mathcal{X}_0$. Thus, applying Lemma 3.7 in each term of P and Q we have that $\partial_x P(u_n, v_n)$ and $\partial_x Q(u_n, v_n)$ are bounded in $\mathcal{Z}^1_{0,-\frac{1}{2}}$ and $\mathcal{Z}^\alpha_{0,-\frac{1}{2}}$, respectively, with $|\zeta| + |\mu| < \epsilon$, for some $\epsilon \ll 1$ and $\alpha < 0$. In particular, is bounded in $\mathcal{X}^1_{0,-\frac{1}{2}}$ and $\mathcal{X}^\alpha_{0,-\frac{1}{2}}$, respectively. Additionally, Bourgain spaces are reflexive and have the following compact embedding

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_{0,-1}.$$

Therefore, we can extract a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \mathcal{X}_0,$$
$$(u_n, v_n) \longrightarrow (u, v) \text{ in } \mathcal{X}_{0,-1}$$

and

$$(\partial_x P(u_n, v_n), \partial_x Q(u_n, v_n)) \rightharpoonup (f, g) \text{ in } \mathcal{X}_{0,-\frac{1}{2}},$$

where $(u, v) \in \mathcal{X}_0$ and $(f, g) \in \mathcal{X}_{0,-\frac{1}{2}}$. Moreover, since $\mathcal{X}_0$ is continuously embedded in $L^4((0,T)\times\mathbb{T})$, we have

$$\|u_n v_n\|_{L^2((0,T)\times\mathbb{T})} \leq \|u_n\|^2_{L^4((0,T)\times\mathbb{T})} \|v_n\|^2_{L^4((0,T)\times\mathbb{T})} \lesssim \|(u_n, v_n)\|^4_{\mathcal{X}_0},$$

which implies that $(P(u_n, v_n), Q(u_n, v_n))$ is bounded in $L^2((0,T) \times \mathbb{T}) \times L^2((0,T) \times \mathbb{T})$. Hence, it follows that $\partial_x(P(u_n, v_n), Q(u_n, v_n))$ is bounded in

$$L^2(0,T;H^{-1}(\mathbb{T})) \times L^2(0,T;H^{-1}(\mathbb{T})) = \mathcal{X}_{-1,0}.$$

Interpolating the spaces $\mathcal{X}_{0,-\frac{1}{2}}$ and $\mathcal{X}_{-1,0}$, we obtain that $\partial_x(P(u_n, v_n), Q(u_n, v_n))$ is bounded in $\mathcal{X}_{-\theta,-\frac{1}{2}(1-\theta)}$ for any $\theta \in [0,1]$. As $0 < \theta < 1$, it follows that $\mathcal{X}_{-\theta,-\frac{1}{2}(1-\theta)}$ is compactly embedded in $\mathcal{X}_{-1,-\frac{1}{2}}$. Thus, we can extract a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that

$$\partial_x(P(u_n, v_n), Q(u_n, v_n)) \longrightarrow (f, g) \text{ in } \mathcal{X}_{-1,-\frac{1}{2}}. \tag{4.11}$$

Thanks to (4.10) and the continuity of G, we ensures that

$$\int_0^T \|(Gu_n, Gv_n)\|^2 dt \longrightarrow \int_0^T \|(Gu, Gv)\|^2 dt = 0. \tag{4.12}$$

This convergence means that $(Gu, Gv) = (0, 0)$. Besides, since g is positive on $\omega \subset \mathbb{T}$, we have from definition (2.11) that

$$u(x,t) = \int_{\mathbb{T}} g(y) u(y,t) dy = c_1(t), \quad \text{on } \omega \times (0,T)$$

and

$$v(x,t) = \int_{\mathbb{T}} g(y) v(y,t) dy = c_2(t), \quad \text{on } \omega \times (0,T).$$

Thus, letting $n \longrightarrow \infty$, we obtain from (3.25) that

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v = f, & \text{on } \mathbb{T} \times (0,T), \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u = g, & \text{on } \mathbb{T} \times (0,T), \\ u = c_1(t), & \text{on } \omega \times (0,T), \\ v = c_2(t), & \text{on } \omega \times (0,T). \end{cases}$$

Consider $w_n = u_n - u$, $z_n = v_n - v$, $P_n = -\partial_x P(u_n, v_n) + f - K_0 u_n$ and $Q_n = -\partial_x Q(u_n, v_n) + g - K_0 v_n$. Thus,

$$(w_n, z_n) \rightharpoonup (0,0) \quad \text{in} \quad \mathcal{X}_0$$

and satisfies

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n + \mu \partial_x w_n + \eta \partial_x z_n = P_n, \\ \partial_t z_n + \alpha \partial_x^3 z_n + \zeta \partial_x z_n + \eta \partial_x w_n = Q_n. \end{cases}$$

Now, note that using the linearity of G we can rewrite

$$\int_0^T \|(Gw_n, Gz_n)\|^2 dt = \int_0^T \|(Gu_n, Gv_n)\|^2 dt + \int_0^T \|(Gu, Gv)\|^2 dt - 2\int_0^T \langle Gu_n, Gu\rangle \, dt - 2\int_0^T \langle Gv_n, Gv\rangle \, dt$$

From (4.12), we obtain

$$\int_0^T \|(Gw_n, Gz_n)\|^2 dt \longrightarrow 0. \tag{4.13}$$

On the other hand,

$$\int_0^T \|(Gw_n, Gz_n)\|^2 dt = I + II + III, \tag{4.14}$$

where,

$$I = \int_0^T \int_{\mathbb{T}} g^2(x) \left[w_n^2(x,t) + z_n^2(x,t)\right] dx dt,$$

$$II = \int_0^T \left(\int_{\mathbb{T}} g^2(x) dx\right) \left(\int_{\mathbb{T}} g(y) w_n(y,t) dy\right)^2 dt + \int_0^T \left(\int_{\mathbb{T}} g^2(x) dx\right) \left(\int_{\mathbb{T}} g(y) z_n(y,t) dy\right)^2 dt$$

and

$$\begin{aligned} III = & -2\int_0^T \left(\int_{\mathbb{T}} g^2(x) w_n(x,t) dx\right) \left(\int_{\mathbb{T}} g(y) w_n(y,t) dy\right) dt \\ & -2\int_0^T \left(\int_{\mathbb{T}} g^2(x) z_n(x,t) dx\right) \left(\int_{\mathbb{T}} g(y) z_n(y,t) dy\right) dt. \end{aligned}$$

Let us prove that each previous term tends to zero as $n \longrightarrow \infty$. First, a direct application of Lemma 4.2 implies that $II \longrightarrow 0$. Now, we can estimate III as follows

$$\begin{aligned} |III| &\leq \|b_n\|_{L^2(0,T)} \left\| \int_{\mathbb{T}} g^2 w_n(x,t) dx \right\|_{L^2(0,T)} + \|c_n\|_{L^2(0,T)} \left\| \int_{\mathbb{T}} g^2 z_n(x,t) dx \right\|_{L^2(0,T)} \\ &\leq \|g\|_{L^4(\mathbb{T})}^2 \|(b_n, c_n)\|_{L^2(0,T)\times L^2(0,T)} \|(w_n, z_n)\|_{\mathcal{X}_0} \\ &\leq C\|(b_n, c_n)\|_{L^2(0,T)\times L^2(0,T)}, \end{aligned}$$

where C is a positive constant. So, it follows that $III \longrightarrow 0$, when $n \longrightarrow \infty$, again thanks to the Lemma 4.2. Lastly, combining the last two convergences with (4.13), we infer by (4.14) that $I \longrightarrow 0$.

We claim that

$$\|(w_n, z_n)\|_{L^2(0,T;L^2(\widetilde{\omega}))\times L^2(0,T;L^2(\widetilde{\omega}))} \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.15}$$

In fact, it is sufficient to observe that

$$\begin{aligned} \|(w_n, z_n)\|_{L^2(0,T;L^2(\widetilde{\omega}))\times L^2(0,T;L^2(\widetilde{\omega}))} &= \int_0^T \int_{\widetilde{\omega}} |g(x)|^{-2} \left[|g(x)|^2 w_n^2(x,t) + |g(x)|^2 z_n^2(x,t)\right] dx dt \\ &\leq \frac{4}{\|g\|_{L^\infty(\mathbb{T})}^2} \cdot \int_0^T \int_{\mathbb{T}} g^2(x) \left[w_n^2(x,t) + z_n^2(x,t)\right] dx dt, \end{aligned}$$

where $\widetilde{\omega} := \left\{ g(x) > \frac{\|g\|_{L^\infty(\mathbb{T})}}{2} \right\}$ and using this previous inequality (4.15) follows. Additionally, note that

$$\begin{aligned}\|(P_n,Q_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} \leq& \|GG^*(u_n,v_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} + \|(\partial_x P(u_n,v_n)-f, \partial_x Q(u_n,v_n)-g)\|_{\mathcal{X}_{-1,-\frac{1}{2}}}\\ \leq& C\|(Gu_n,Gv_n)\|_{\mathcal{X}_{0,0}} + \|(\partial_x P(u_n,v_n)-f, \partial_x Q(u_n,v_n)-g)\|_{\mathcal{X}_{-1,-\frac{1}{2}}}.\end{aligned}$$

From (4.11), (4.12) and the previous inequality we obtain

$$\|(P_n,Q_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Applying Proposition A.1 with $b'=0$ and $b=\frac{1}{2}$ yields that

$$(w_n,z_n) \longrightarrow 0 \text{ in } L^2_{loc}(0,T;L^2(\mathbb{T})) \times L^2_{loc}(0,T;L^2(\mathbb{T})). \tag{4.16}$$

Consequently,

$$(P(u_n,v_n),Q(u_n,v_n)) \longrightarrow (P(u,v),Q(u,v)) \text{ in } L^1_{loc}(0,T;L^1(\mathbb{T})) \times L^1_{loc}(0,T;L^1(\mathbb{T})) \tag{4.17}$$

and

$$(\partial_x P(u_n,v_n), \partial_x Q(u_n,v_n)) \longrightarrow (\partial_x P(u,v), \partial_x Q(u,v)),$$

in the distributional sense. Therefore, $(f,g) = (\partial_x P(u,v), \partial_x Q(u,v))$ and $(u,v) \in \mathcal{X}_0$ satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v + \partial_x P(u,v) = 0, & \text{on } \mathbb{T}\times(0,T),\\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x v + \eta\partial_x u + \partial_x Q(u,v) = 0, & \text{on } \mathbb{T}\times(0,T),\\ (u,v) = (c_1(t),c_2(t)), & \text{on } \omega\times(0,T). \end{cases}$$

From Corollary A.5, we infer that $(u,v)=(0,0)$ on $\mathbb{T}\times(0,T)$, which combined with (4.16) yields that $(u_n,v_n)\to(0,0)$ in $L^2_{loc}((0,T);L^2(\mathbb{T}))\times L^2_{loc}((0,T);L^2(\mathbb{T}))$. We can pick some time $t_0\in[0,T]$ such that $(u_n(t_0),v_n(t_0)) \longrightarrow (0,0)$ in $L^2(\mathbb{T})\times L^2(\mathbb{T})$. Since

$$\|(u_n(0),v_n(0))\|^2 = \|(u_n(t_0),v_n(t_0))\|^2 + \int_0^{t_0} \|(Gu_n,Gv_n)\|^2 dt,$$

it is inferred that $a_n = \|(u_n(0),v_n(0))\| \longrightarrow 0$ which is a contradiction to the assumption $a>0$.

ii. Case $a=0$.

Note first that $a_n>0$ for all n. Let $(w_n,z_n) = \left(\frac{u_n}{a_n}, \frac{v_n}{a_n}\right)$ for all $n\geq 1$. Then

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n + \mu\partial_x w_n + \eta\partial_x z_n + K_0 w_n + a_n\partial_x P(w_n,z_n) = 0,\\ \partial_t z_n + \alpha\partial_x^3 z_n + \zeta\partial_x z_n + \eta\partial_x w_n + K_0 z_n + a_n\partial_x Q(w_n,z_n) = 0, \end{cases}$$

$$\int_0^T \|(Gw_n,Gz_n)\|^2 dt < \frac{1}{n} \tag{4.18}$$

and

$$\|(w_n(0),z_n(0))\| = \frac{\|u_n(0)\|}{\|(u_{0,n},v_{0,n})\|} + \frac{\|v_n(0)\|}{\|(u_{0,n},v_{0,n})\|} = 1. \tag{4.19}$$

So, we obtain that the sequence $\{(w_n,z_n)\}$ which are bounded in both spaces $L^\infty(0,T;L^2(\mathbb{T}))\times L^\infty(0,T;L^2(\mathbb{T}))$ and $\mathcal{X}_0$. Indeed, $\|(w_n(t),z_n(t))\|$ is a nonincreasing function of t and since a_n is bounded, we have

$$\|(w_n,z_n)\|_{\mathcal{X}_0} \leq C_0 + \frac{(C(\widetilde{\epsilon})T^{1-\widetilde{\epsilon}} + C_1T^\theta)}{a_n}\|(u_n,v_n)\|_{\mathcal{X}_0} + \frac{C_0C_3T^\theta}{a_n^2}\|(u_n,v_n)\|_{\mathcal{X}_0}.$$

We can extract a subsequence of $\{(w_n,z_n)\}$, still denoted by $\{(w_n,z_n)\}$, such that

$$(w_n,z_n) \rightharpoonup (w,z) \text{ in } \mathcal{X}_0,$$

$$(w_n,z_n) \longrightarrow (w,z) \text{ in } \mathcal{X}_{-1,-\frac{1}{2}}$$

and

$$(w_n, z_n) \longrightarrow (w, z) \text{ in } \mathcal{X}_{-1,0},$$

as $n \to \infty$. Moreover, the sequence $\{(\partial_x P(w_n, z_n), \partial_x Q(w_n, z_n))\}$ is bounded in the space $\mathcal{X}_{0,-\frac{1}{2}}$, and therefore

$$a_n(\partial_x P(w_n, z_n), \partial_x Q(w_n, z_n)) \longrightarrow 0 \text{ in } \mathcal{X}_{0,-\frac{1}{2}},$$

when $n \to \infty$. Finally,

$$\int_0^T \|(Gw_n, Gz_n)\|^2 dt = 0.$$

Thus, (w, z) is solution of

$$\begin{cases} \partial_t w + \partial_x^3 w + \mu \partial_x w + \eta \partial_x z = 0, & \text{on } \mathbb{T} \times (0, T), \\ \partial_t z + \alpha \partial_x^3 z + \zeta \partial_x z + \eta \partial_x w = 0, & \text{on } \mathbb{T} \times (0, T), \\ w = c_1(t), \quad z = c_2(t), & \text{on } \omega \times (0, T). \end{cases}$$

Using Holmgren's uniqueness theorem, we can deduce that $w(x,t) = c_1(t) = c_1$ and $z(x,t) = c_2(t) = c_2$. However, as $[w] = [z] = 0$, we infer that $c_1 = c_2 = 0$.

According to (4.18)

$$\int_0^T \|(Gw_n, Gz_n)\|^2 dt \longrightarrow 0, \qquad \text{as } n \to \infty,$$

and consequently $(K_0 w_n, K_0 z_n)$ converges strongly to $(0,0)$ in $\mathcal{X}_{-1,-\frac{1}{2}}$. Applying again Proposition A.1, as in the case $a > 0$, it follows that

$$(w_n, z_n) \longrightarrow (0,0) \text{ in } L^2_{loc}(0,T; L^2(\mathbb{T})) \times L^2_{loc}(0,T; L^2(\mathbb{T})).$$

Thus we can take $t_0 \in (0,T)$ such that $(w_n(t_0), z_n(t_0))$ converges to $(0,0)$ strongly in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. Since

$$\|(w_n(0), z_n(0))\|^2 = \|(w_n(t_0), z_n(t_0))\|^2 + \int_0^{t_0} \|(Gw_n, Gz_n)\|^2 dt,$$

we infer from (4.18) that $\|(w_n(0), z_n(0))\| \to 0$ which contradicts (4.19). Therefore, the observability inequality is shown. □

5. Controllability result: Nonlinear problem

Let us now consider the controllability properties for the nonlinear open loop control system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v + \partial_x P(u,v) = Gf, & x \in \mathbb{T}, \ t \in \mathbb{R}, \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u + \partial_x Q(u,v) = Gh, & x \in \mathbb{T}, \ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}, \end{cases} \tag{5.1}$$

where $P(u,v)$, $Q(u,v)$ are defined by (3.5), G is represented by (2.11), $\alpha < 0$, $|\mu| + |\zeta| < \epsilon = \epsilon(\alpha)$, for some $\epsilon \ll 1$ and f, g are control inputs. The following result is local.

Theorem 5.1. *Let $T > 0$ and $s \geq 0$ be given. Then there exists a $\delta > 0$ such that for any (u_0, v_0), $(u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ and*

$$\|(u_0, v_0)\|_s \leq \delta, \quad \|(u_1, v_1)\|_s \leq \delta,$$

one can find two control inputs $(f, h) \in L^2([0,T]; H_0^s(\mathbb{T})) \times L^2([0,T]; H_0^s(\mathbb{T}))$ such that equation (5.1) has a solution

$$(u, v) \in C([0,T]; H_0^s(\mathbb{T})) \times C([0,T]; H_0^s(\mathbb{T}))$$

satisfying

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{and} \quad (u(x,T), v(x,T)) = (u_1(x), v_1(x)).$$

Proof. It is well known that the solution (u, v) of IVP associated to (5.1) with initial data (u_0, v_0) is given by

$$u(t) = S^{1,\mu}(t)u_0 + \int_0^t S^{1,\mu}(t-\tau)(Gf)(\tau) - \eta \int_0^t S^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau - \int_0^t S^{1,\mu}(t-\tau)\partial_x(P(u,v))(\tau)d\tau \tag{5.2}$$

and

$$v(t) = S^{\alpha,\zeta}(t)v_0 + \int_0^t S^{\alpha,\zeta}(t-\tau)(Gh)(\tau) - \eta \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x(Q(u,v))(\tau)d\tau. \tag{5.3}$$

We define

$$w(T,(u,v)) := \int_0^T S^{1,\mu}(T-\tau)\partial_x P(u,v)(\tau)d\tau + \eta \int_0^t S^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau$$

and

$$z(T,(u,v)) := \int_0^T S^{\alpha,\zeta}(T-\tau)\partial_x Q(u,v)(\tau)d\tau + \eta \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau.$$

According to Corollary 2.7 and Remark 3.1 , for given (u_0, v_0), $(u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$, if one chooses

$$f = \Phi(u_0, u_1 + w(T,(u,v)) \quad \text{and} \quad h = \Psi(v_0, v_1 + z(T,(u,v)),$$

in the equations (5.2) and (5.3), we get that

$$\begin{aligned} u(t) &= S^{1,\mu}(t)u_0 + \int_0^t S^{1,\mu}(t-\tau)(G\Phi(u_0, u_1 + w(T,(u,v)))(\tau) - \eta \int_0^t S^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau \\ &\quad - \int_0^t S^{1,\mu}(t-\tau)\partial_x(P(u,v))(\tau)d\tau, \\ v(t) &= S^{\alpha,\zeta}(t)v_0 + \int_0^t S^{\alpha,\zeta}(t-\tau)(G\Psi(v_0, v_1 + z(T,(u,v)))(\tau) - \eta \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau \\ &\quad - \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x(Q(u,v))(\tau)d\tau, \end{aligned} \tag{5.4}$$

and

$$(u(0), v(0)) = (u_0, v_0) \quad \text{and} \quad (u(T), v(T)) = (u_1, v_1),$$

due to the definitions of the operators Φ and Ψ (see Corollary 2.7 for more details).

Now, consider the following map

$$\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$$

where $\Gamma_1(u,v)$ and $\Gamma_2(u,v)$ are defined by

$$\Gamma_1(u,v) = S^{1,\mu}(t)u_0 + \int_0^t S^{1,\mu}(t-\tau)(G\Phi(u_0, u_1 + w(T,(u,v)))(\tau) - \int_0^t S^{1,\mu}(t-\tau)\partial_x v(\tau)d\tau - \int_0^t S^{1,\mu}(t-\tau)\partial_x(P(u,v))(\tau)d\tau$$

and

$$\Gamma_2(u,v) = S^{\alpha,\zeta}(t)v_0 + \int_0^t S^{\alpha,\zeta}(t-\tau)(G\Psi(v_0, v_1 + z(T,(u,v)))(\tau) \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)d\tau - \int_0^t S^{\alpha,\zeta}(t-\tau)\partial_x(Q(u,v))(\tau)d\tau,$$

respectively.

As Γ is a contraction map in the space $L^2(\mathbb{R}; H_0^s(\mathbb{T})) \cap \mathcal{Z}_s^1 \times L^2(\mathbb{R}; H_0^s(\mathbb{T})) \cap \mathcal{Z}_s^\alpha$, then its fixed point (u, v) is a solution of IVP associated to (5.1) with initial data (u_0, v_0), $(f, h) = (\Phi(u_0, u_1 + w(T, (u, v)), \Psi(v_0, v_1 + z(T, (u, v)))$ and satisfies $(u(x, T), v(x, T)) = (u_1(x), v_1(x))$, showing Theorem 5.1. □

6. Further comments and open problems

This article deals by the first time with the global aspect of control problems for the long waves in dispersive media, precisely systems like (1.1) with the coupled nonlinearity (1.8). From the perspective of considering the functions with null mean, the presence of the terms involving the constants B and C in the nonlinearities make the system studied coupled in the linear part (as well as in the nonlinear part) as can be seen in (1.6), that is, precisely the system with the terms $\eta\partial_x v$ and $\eta\partial_x u$, in the first and second equations of the system of (1.7). Unless that η can be considered 0, when $[u] = [v] = \beta$ and $B = -C$, the matrix of the operator L defined by (2.2) is not a diagonal matrix. For this reason, the arguments used in general for a singular dispersive equation (e.g. as the KdV case [28, 39]) in the study of the existence of solutions, controllability and stabilization can not be directly applied here.

This work opens a series of situations that can be studied to understand the well-posedness theory and global controllability problems for long waves in dispersive media. We will now detail below the novelties of this work and open issues that seem interesting from a mathematical point of view.

6.1. **Control problems.** The problems in this work were solved requiring some conditions over the constants of the system, namely α, μ and ζ. Under the conditions $\alpha < 0$ and $\zeta - \mu > 0$, we find eigenfunctions associated to the operator L that define an orthonormal basis in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$, so through a spectral analysis we were able to show, using the *moment method* [37], that linear system (1.13) is exactly controllable.

In addition, the global control problems are also verified thanks to the soothing properties of the Bourgain spaces. This is the main novelty in this work, since it is not to our knowledge that systems like (1.7) have global control properties (see [11, 33] for local results). In fact, the main tool used here is Bourgain spaces in different dispersions. With the smoothing properties of the Bourgain spaces in hand, the propagation of singularities showed in [28], for the single KdV equation, can be extended to the coupled KdV system defined by the operator L and together with the *bilinear estimates* and a *unique continuation property* we can achieve the global control results. However, there is a drawback in this method, we are able to solve these global control problems only with two controls input. The use of a control in one of the equations is still an open issue.

6.2. **Well-posedness theory.** As mentioned before, the Bourgain spaces are the key point in showing global results in this article. These spaces have been applied with success in the literature for global control results in less regular spaces (see, for instance, [9, 26, 27, 28, 34]).

The Bourgain spaces related to the linear system (3.1) can be defined *via* the norm

$$\|W(-t)u\|_{H_x^s H_t^b} =: \|u\|_{X_{s,b}}$$

where $W(t) := e^{-tL}$ is the strongly continuous group generated by the operator L. However, in the aspect of the nonlinear problem, it is necessary to make bilinear estimates and there are no studies on such estimates with Bourgain spaces of this nature. To get around this situation, we use two Bourgain spaces associated with each dispersion present in (3.1). This strategy was used by Zhang and Yang in [43] for the study of the well-posedness of KdV-KdV type systems. Thus, the terms $-\eta\partial_x v$ and $-\eta\partial_x u$ are treated as part of nonlinearity in both studies: the well-posedness theory and in the control problem that concerns the asymptotic behavior of solutions to the problem (3.25).

Since such spaces do not have equivalent norms, a natural question that arises is whether we can estimate the components of the first equation with Bourgain's norm regarding the dispersion of the second and the converse. In this sense, lemmas 3.6 and 3.7 provide answers to this problem and are essential for this work.

Lemma 3.6 is an extension of Lemma 3.10 presented in [43]. In this work we verify that the result is still valid in the range $\frac{1}{3} < b < \frac{1}{3}$. Since $\beta_1, \beta_2, \gamma_1$ and γ_2 are fixed constants, the lemma is still true if we change the hypothesis $|\gamma_1| + |\gamma_2| < \epsilon$ for $\nu := |\beta_2 - \beta_1| - |\gamma_1 - \gamma_2| \geq \widetilde{\delta} > 0$, but the condition $\beta_2 \neq \beta_1$ is still required.

In turn, the key point of Lemma 3.7 is that the function H defined by (3.14) is δ-significant, and for it the condition $|\eta| + |\mu| < \epsilon$ is required. Furthermore, due to the nature of the nonlinearity given in (1.8) it is also essential that α is strictly negative. To extend the results presented here to a larger class of constants $\alpha, \mu, \zeta, \eta \in \mathbb{R}^*$ it is necessary to find conditions for which the function H is δ-significant. So, the well-posedness theory, global controllability and global stabilization properties for the system (1.6) where ζ and μ are not small enough is still an open problem, as well as the case when $\alpha > 0$.

Appendix A. Propagation of singularities and unique continuation property

In this appendix we will give some results of propagation of singularities for the operator

$$L = \begin{pmatrix} -\partial^3 - \mu\partial & -\eta\partial \\ -\eta\partial & -\partial^3 - \zeta\partial \end{pmatrix},$$

which were used throughout the paper. The main ingredient is pseudo-differential analysis together with microlocal analysis on $\mathbb{T}$.

A.1. Propagation of compactness and regularity.

The first result of this appendix shows that we can propagate, due to the smoothing effect of the Bourgain spaces, the compactness of a $\omega \subset \mathbb{T}$ for the entire space $\mathbb{T}$. The result can be read as follows.

Proposition A.1 (Propagation of compactness)**.** *Let $T > 0$ and $0 \leq b' < b \leq 1$ be given. Suppose that $(u_n, v_n) \in \mathcal{X}_{0,b}$ and $(f_n, g_n) \in \mathcal{X}_{-2+2b,-b}$ satisfies*

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n + \mu\partial_x u_n + \eta\partial_x v_n = f_n, & \text{on } \mathbb{T}\times(0,T), \\ \partial_t v_n + \alpha\partial_x^3 v_n + \zeta\partial_x v_n + \eta\partial_x u_n \ = g_n, & \text{on } \mathbb{T}\times(0,T), \end{cases}$$

for $n \in \mathbb{N}$. Assume that there exists a constant $C > 0$ such that

$$\|(u_n, v_n)\|_{\mathcal{X}_{0,b}} \leq C, \quad \forall n \geq 1, \tag{A.1}$$

and that

$$\begin{aligned} &\|(u_n, v_n)\|_{\mathcal{X}_{-2+2b,-b}} \longrightarrow 0, \ \text{as } n \longrightarrow \infty, \qquad \|(f_n, g_n)\|_{\mathcal{X}_{-2+2b,-b}} \longrightarrow 0, \ \text{as } n \longrightarrow \infty, \\ &\qquad\qquad \|(u_n, v_n)\|_{\mathcal{X}_{-1+2b,-b}} \longrightarrow 0, \ \text{as } n \longrightarrow \infty. \end{aligned} \tag{A.2}$$

In addition, assume that for some nonempty open set $\omega \subset \mathbb{T}$ it holds

$$(u_n, v_n) \longrightarrow (0,0) \text{ in } L^2(0,T;L^2(\omega)) \times L^2(0,T;L^2(\omega)).$$

Then,

$$(u_n, v_n) \longrightarrow (0,0) \text{ in } L^2_{loc}(0,T;L^2(\mathbb{T})) \times L^2_{loc}(0,T;L^2(\mathbb{T})).$$

Proof. Pick $\phi \in C^\infty(\mathbb{T})$ and $\psi \in C_0^\infty((0,T))$ real valued and set

$$\Phi = \phi(x)D^{-2} \quad \text{and} \quad \Psi = \psi(t)B,$$

where D is defined by

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \hat{u}(k) & \text{if } k \neq 0, \\ \hat{u}(0) & \text{if } k = 0. \end{cases} \tag{A.3}$$

Since

$$\int_0^T \int_{\mathbb{T}} \Psi u(x,t) v(x,t) dx dt = \int_0^T \int_{\mathbb{T}} u(x,t)\phi(t) D^{-2}(\phi(x)v(x,t)) dx dt.$$

we have

$$\Psi^* = \psi(t)D^{-2}\phi(x)$$

For any $\epsilon > 0$, let $\Psi_\epsilon = \Phi e^{\epsilon \partial_x^2} = \psi(t)\Phi_\epsilon$ be regularization of Ψ. We define

$$\alpha_\epsilon^1(u_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_1]u_n, u_n\rangle_{L^2(\mathbb{T}\times(0,T))}, \qquad \alpha_\epsilon^2(v_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_2]v_n, v_n\rangle_{L^2(\mathbb{T}\times(0,T))},$$
$$\alpha_\epsilon^3(u_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_3]u_n, v_n\rangle_{L^2(\mathbb{T}\times(0,T))},$$

where

$$\mathcal{L}_1 = \partial_t + \partial_x^3 + \mu\partial_x, \quad \mathcal{L}_2 = \partial_t + \alpha\partial_x^3 + \zeta\partial_x \quad \text{and} \quad \mathcal{L}_3 = \eta\partial_x.$$

Note that

$$[\Psi_\epsilon, \partial_t]w_n(x,t) = -\psi'(t)\phi(x)D^{-2}w_n(x,t).$$

Denoting $\alpha_{n,\epsilon} := \alpha_\epsilon^1(u_n, u_n) + \alpha_\epsilon^3(v_n, u_n) + \alpha_\epsilon^2(v_n, v_n) + \alpha_\epsilon^3(u_n, v_n)$, we have

$$\begin{aligned}\alpha_{n,\epsilon} = &\left\langle [\Psi_\epsilon, \partial_x^3 + \mu\partial_x]u_n, u_n\right\rangle - \left\langle \psi'(t)\Phi_\epsilon u_n, u_n\right\rangle + \left\langle [\Psi_\epsilon, \alpha\partial_x^3 + \zeta\partial_x]v_n, v_n\right\rangle \\ &- \left\langle \psi'(t)\Phi_\epsilon v_n, v_n\right\rangle + \langle [\Psi_\epsilon, \eta\partial_x]u_n, v_n\rangle + \langle [\Psi_\epsilon, \eta\partial_x]v_n, u_n\rangle .\end{aligned}$$

On the other hand,

$$\begin{aligned}\alpha_\epsilon^1(u_n, u_n) + \alpha_\epsilon^3(v_n, u_n) &= \langle \Psi_\epsilon(\mathcal{L}_1 u_n), u_n)\rangle + \langle \Psi_\epsilon(\mathcal{L}_3 v_n), u_n\rangle - \langle \mathcal{L}_1(\Psi_\epsilon u_n), u_n\rangle - \langle \mathcal{L}_3(\Psi_\epsilon v_n), u_n\rangle \\ &= \langle f_n, \Psi_\epsilon^* u_n\rangle + \langle \Psi_\epsilon u_n, \mathcal{L}_1 u_n\rangle + \langle \Psi_\epsilon v_n, \mathcal{L}_3 u_n\rangle ,\end{aligned}$$

since $\mathcal{L}_1 u_n + \mathcal{L}_3 v_n = f_n$, $\mathcal{L}_1^* = -\mathcal{L}_1$ and $\mathcal{L}_3^* = -\mathcal{L}_3$. Similarly

$$\alpha_\epsilon^2(v_n, v_n) + \alpha_\epsilon^3(u_n, v_n) = \langle g_n, \Psi_\epsilon^* v_n\rangle + \langle \Psi_\epsilon v_n, \mathcal{L}_2 v_n\rangle + \langle \Psi_\epsilon u_n, \mathcal{L}_3 v_n\rangle .$$

Thus,

$$\alpha_{n,\epsilon} = \langle f_n, \Psi_\epsilon^* u_n\rangle + \langle g_n, \Psi_\epsilon^* v_n\rangle + \langle \Psi_\epsilon u_n, f_n\rangle + \langle \Psi_\epsilon v_n, g_n\rangle .$$

Now, following the ideas almost as done in [28, Proposition 3.5] the result is so achieved. Here, to sake of completeness, we express only the details that need special attention and are different from those presented in [28, Proposition 3.5].

As in [28], we obtain

$$\lim_{n\to\infty} \sup_{0<\epsilon\leq 1} |\alpha_{n,\epsilon}| = 0$$

and

$$\lim_{n\to\infty} \sup_{0<\epsilon\leq 1} \left| \left\langle \psi'(t)\Phi_\epsilon u_n, u_n\right\rangle + \left\langle \psi'(t)\Phi_\epsilon v_n, v_n\right\rangle \right| = 0.$$

Therefore, this yields that

$$\lim_{n\to\infty} \left\langle [\Psi, \partial_x^3 + \mu\partial_x]u_n, u_n\right\rangle + \left\langle [\Psi, \alpha\partial_x^3 + \zeta\partial_x]v_n, v_n\right\rangle + \langle [\Psi, \eta\partial_x]u_n, v_n\rangle + \langle [\Psi, \eta\partial_x]v_n, u_n\rangle = 0$$

We can rewrite

$$\begin{aligned}&\left\langle\, [\Psi, \partial_x^3 + \mu\partial_x]u_n, u_n\right\rangle + \left\langle [\Psi, \alpha\partial_x^3 + \zeta\partial_x]v_n, v_n\right\rangle + \langle [\Psi, \eta\partial_x]u_n, v_n\rangle + \langle [\Psi, \eta\partial_x]v_n, u_n\rangle \\ &:= I + II + III + IV,\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}I &= -3\left\langle \psi(t)\partial_x\phi(x)\partial_x^2 D^{-2}u_n, u_n\right\rangle - 3\alpha\left\langle \psi(t)\partial_x\phi(x)\partial_x^2 D^{-2}v_n, v_n\right\rangle ,\\ II &= -3\left\langle \psi(t)\partial_x^2\phi(x)\partial_x D^{-2}u_n, u_n\right\rangle - 3\alpha\left\langle \psi(t)\partial_x^2\phi(x)\partial_x D^{-2}v_n, v_n\right\rangle ,\\ III &= -\left\langle \psi(t)(\partial_x^3\phi(x) + \mu\partial_x\phi(x))D^{-2}u_n, u_n\right\rangle + \left\langle \psi(t)(\alpha\partial_x^3\phi(x) + \zeta\partial_x\phi(x))D^{-2}v_n, v_n\right\rangle\end{aligned}$$

and

$$IV = \left\langle \eta\psi(t)\partial_x\phi(x)D^{-2}v_n, u_n\right\rangle - \left\langle \eta\psi(t)\partial_x\phi(x)D^{-2}u_n, v_n\right\rangle .$$

Following [28], we can prove that $|II|, |III| \to 0$. Hence, if $|IV| \to 0$ the same argument applied in [28] can be applied here and we finish the proof.

The main point here is to prove the fact that $|IV| \to 0$. Firstly, note that

$$\|u_n\|_{\mathcal{X}^1_{\frac{2b'-1}{2},\frac{b-b'}{2}}} \leq \|u_n\|_{\mathcal{X}^1_{0,b}} \|u_n\|_{\mathcal{X}^1_{-1+2b',-b'}} \leq C\|u_n\|_{\mathcal{X}^1_{-1+2b',-b'}} \longrightarrow 0,$$

as $n\to\infty$. On the other hand

$$\begin{aligned}\left\langle\psi(t)\partial_x\phi(x)D^{-2}v_n,u_n\right\rangle &\leq \sup_{t\in(0,T)}|\psi(t)|\|\partial_x\phi(x)D^{-2}v_n\|_{\mathcal{X}^{\alpha}_{\frac{1}{2}-b',0}}\|u_n\|_{\mathcal{X}^{1}_{\frac{2b'-1}{2},0}}\\ &\leq \sup_{t\in(0,T)}|\psi(t)|\|v_n\|_{\mathcal{X}^{\alpha}_{-\frac{3}{2}-b',0}}\|u_n\|_{\mathcal{X}^{1}_{\frac{2b'-1}{2},\frac{b-b'}{2}}}\\ &\leq \sup_{t\in(0,T)}|\psi(t)|\|v_n\|_{\mathcal{X}^{\alpha}_{0,b}}\|u_n\|_{\mathcal{X}^{1}_{\frac{2b'-1}{2},\frac{b-b'}{2}}}.\end{aligned}$$

Similarly, we obtain an analogous inequality to the other term in IV. Thus, we conclude that $|IV|\longrightarrow 0$, when $n\to\infty$, and the proof is complete. □

The following result concerns the propagation of regularity. Precisely, the result guarantees that if we have gain of derivatives in the spatial space in a subset ω of $\mathbb{T}$, then this is also valid in the whole space $\mathbb{T}$.

Proposition A.2 (Propagation of regularity). *Let $T>0$, $0\leq b<1$, $r\in\mathbb{R}$ and $(p,q)\in\mathcal{X}_{r,-b}$. Let $(u,v)\in\mathcal{X}_{r,b}$ be a solution of*

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v=p, & on\ \mathbb{T}\times(0,T),\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u=q, & on\ \mathbb{T}\times(0,T).\end{cases}$$

If there exists a nonempty open set ω of $\mathbb{T}$ such that

$$(u,v)\in L^2_{loc}(0,T;H^{r+\rho}(\omega))\times L^2_{loc}(0,T;H^{r+\rho}(\omega)),$$

for some ρ satisfying

$$0<\rho\leq\min\left\{1-b,\frac{1}{2}\right\},$$

then $(u,v)\in L^2_{loc}(0,T;H^{r+\rho}(\mathbb{T}))\times L^2_{loc}(0,T;H^{r+\rho}(\mathbb{T}))$.

Proof. Set $s=r+\rho$ and for $n\in\mathbb{N}$ consider

$$\begin{cases}u_n=e^{\frac{1}{n}\partial_x^2}u=:\Xi_n u, & p_n=\Xi_n p=\mathcal{L}_1u_n+\mathcal{L}_3v_n,\\ v_n=e^{\frac{1}{n}\partial_x^2}v=:\Xi_n v, & q_n=\Xi_n q=\mathcal{L}_2v_n+\mathcal{L}_3u_n.\end{cases}$$

There exists a constant $C>0$ such that

$$\|(u_n,v_n)\|_{\mathcal{X}_{r,b}}\leq C\quad\text{and}\quad\|(p_n,q_n)\|_{\mathcal{X}_{r,-b}}\leq C,\quad\forall\, n\in\mathbb{N}.$$

Pick $\phi\in C^\infty(\mathbb{T})$ and $\psi\in C_0^\infty((0,T))$ as in the proof of Proposition A.1, and set

$$\Phi=D^{2s-2}\phi(x)\quad\text{and}\quad\Psi=\psi(t)B,$$

where D is defined by (A.3). We have

$$\begin{aligned}&\langle\mathcal{L}_1u_n+\mathcal{L}_3v_n,\Psi^*u_n,\rangle+\langle\Psi u_n,\mathcal{L}_1u_n+\mathcal{L}_3v_n\rangle\\ &\quad=\left\langle[\Psi,\partial_x^3+\mu\partial_x]u_n,u_n\right\rangle+\langle[\Psi,\eta\partial_x]v_n,u_n\rangle-\left\langle\psi'(t)\Phi u_n,u_n\right\rangle,\end{aligned}$$

and, similarly

$$\begin{aligned}&\langle\mathcal{L}_2v_n+\mathcal{L}_3u_n,\Psi^*v_n,\rangle+\langle\Psi v_n,\mathcal{L}_2v_n+\mathcal{L}_3u_n\rangle\\ &\quad=\left\langle[\Psi,\alpha\partial_x^3+\zeta\partial_x]v_n,v_n\right\rangle+\langle[\Psi,\eta\partial_x]u_n,v_n\rangle-\left\langle\psi'(t)\Phi v_n,v_n\right\rangle,\end{aligned}$$

where $\mathcal{L}_i$, for $i=1,2,3$, were defined on Proposition A.1. With this in hand, the result follows in a similar way as proved in [28, Proposition 3.6]. □

A.2. **Unique continuation property.** We borrow the following auxiliary lemmas that have been proven in [28, Corollary 3.7], as a consequence of Proposition A.2 and [30, Lemma 2.9], respectively.

Lemma A.3. *Let $(u,v) \in \mathcal{X}_0$ be a solution of*

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v + \partial_x P(u,v) = 0 & \text{on } \mathbb{T}\times(0,T), \\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x v + \eta\partial_x u + \partial_x Q(u,v) = 0 & \text{on } \mathbb{T}\times(0,T). \end{cases} \tag{A.5}$$

Assume that $(u,v) \in C^\infty(\omega\times(0,T))\times C^\infty(\omega\times(0,T))$, where ω is a nonempty open set in $\mathbb{T}$. Then $(u,v)\in C^\infty(\mathbb{T}\times(0,T))\times C^\infty(\mathbb{T}\times(0,T))$.

Lemma A.4. *Let $s\in\mathbb{R}$ and $f(x)=\sum_{k\geq 0}\widehat{f}_k e^{ikx}$ be such that $f\in H^s(\mathbb{T})$ and $f=0$ on $\omega\subset\mathbb{T}$. Then $f\equiv 0$ on $\mathbb{T}$.*

We are in position to prove the unique continuation property for our dispersive operator L.

Corollary A.5. *Let ω be a nonempty set in $\mathbb{T}$ and*

$$(u,v)\in \mathcal{X}_0^1\cap C([0,T];L_0^2(\mathbb{T}))\times\mathcal{X}_0^\alpha\cap C([0,T];L_0^2(\mathbb{T}))$$

be solution of (3.25). Suppose that (u,v) satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v + \partial_x P(u,v) = 0, & \text{on } \mathbb{T}\times(0,T), \\ \partial_t v + \partial_x^3 v + \zeta\partial_x v + \eta\partial_x u + \partial_x Q(u,v) = 0, & \text{on } \mathbb{T}\times(0,T), \\ (u(x,t),v(x,t)) = (c_1(t),c_2(t)), & \text{for a.e. } (x,t)\in\omega\times(0,T), \end{cases}$$

where c_1, $c_2\in L^2(0,T)\times L^2(0,T)$. Then $(u(x,t),v(x,t))\equiv(0,0)$ for a.e. $(x,t)\in\mathbb{T}\times(0,T)$.

Proof. Since $(u(x,t),v(x,t))=(c_1(t),c_2(t))$ for a.e. $(x,t)\in\omega\times(0,T)$, we have that

$$\partial_t u = c_1'(t) = 0 \quad \text{and} \quad \partial_t v = c_2'(t) = 0. \tag{A.6}$$

Pick a time $t\in(0,T)$ as above and define $(p,q):=(\partial_x^3 u(\cdot,t),\partial_x^3 v(\cdot,t))$. Thus, it holds that

$$(p,q)\in H^{-3}(\mathbb{T})\times H^{-3}(\mathbb{T}) \quad \text{with} \quad (p,q)=(0,0) \text{ in } \omega.$$

Decompose p and q as

$$p(x)=\sum_{k\in\mathbb{Z}}\widehat{p}_k e^{ikx} \text{ and } q(x)=\sum_{k\in\mathbb{Z}}\widehat{q}_k e^{ikx}.$$

the convergence of the Fourier series being in $H^{-3}(\mathbb{T})$. Since p and q are real-valued functions, we also have that $\widehat{p_{-k}}=\widehat{p_k}$ for all k and the same is true for q. Then,

$$0=p(x)=\sum_{k>0}\widehat{p}_k e^{ikx} \quad \text{and} \quad 0=q(x)=\sum_{k>0}\widehat{q}_k e^{ikx},$$

for each $x\in\omega$. Applying Lemma A.4 to p and q we obtain $(p,q)\equiv(0,0)$ on $\mathbb{T}$. It follows, $\partial_x^3 u=\partial_x^3 v=\partial_x u=\partial_x v=0$ on $\mathbb{T}$ for a.e $t\in(0,T)$. Hence,

$$(u(x,t),v(x,t))=(c_1(t),c_2(t)) \text{ for a.e. } (x,t)\in\mathbb{T}\times(0,T).$$

As in (A.6) we deduce that for a.e. $(x,t)\in\mathbb{T}\times(0,T)$ we have $(c_1'(t),c_2'(t))=(0,0)$ for c_1, $c_2\in\mathbb{R}$, which, combined with the fact that $[u_0]=[v_0]=0$, gives that $c_1=c_2=0$. The proof of Corollary A.5 is complete. □

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