CONTROLS INSENSITIZING THE NORM OF SOLUTION OF A SCHRÖDINGER TYPE SYSTEM WITH MIXED DISPERSION

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Abstract. The main goal of this manuscript is to prove the existence of insensitizing controls for the fourth-order dispersive nonlinear Schrödinger equation with cubic nonlinearity. To obtain the main result we first prove that insensitizing controls problems are equivalent to obtaining a null controllability property for a coupled fourth-order Schrödinger system of cascade type with zero order coupling. Precisely, by means of new Carleman estimates, we first obtain a null controllability result for the linearized system around zero, then the null controllability for the nonlinear case is extended using an inverse mapping theorem.

1. Introduction

1.1. **Inesitizing controls and presentation of the model.** Inspired by the difficulties in finding data in applications of distributed systems, Lions [33] introduced the topic of insensitizing controls. Precisely, this kind of problem deals with the existence of controls making a functional (depending on the solution) insensible to small perturbations of the initial data. Considering some particular functional, it has been proven that this problem is equivalent to control properties of cascade systems [7, 25].

The insensitivity can be defined in two different ways: An approximate problem or exact problem. Approximate insensitivity is equivalent to approximate controllability of the cascade system, while exact insensitivity is equivalent to its null controllability. Before giving the reader more details about it and a state of the art related to these problems, let us introduce the model which one we want to study.

In this article, we address to the insensitizing problem for the cubic fourth-order Schrödinger equation with mixed dispersion, the so-called fourth-order nonlinear Schrödinger system (4NLS)

$$iu_t + u_{xx} - u_{xxxx} = \lambda |u|^2 u, \tag{1.1}$$

where $x, t \in \mathbb{R}$ and $u(x,t)$ is a complex-valued function. Equation (1.1) has been introduced by Karpman [29] and Karpman and Shagalov [30] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1.1) arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references (see [6, 14, 29, 36, 37] and references therein).

1.2. **Setting of the problem.** Now on, we present the goal of the manuscript. Let $\Omega := (0, L) \subset \mathbb{R}$ be an interval and assume that $T > 0$. We will use the following notations $Q_T = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$ and $q_T = \omega \times (0,T)$, where $\omega \subset \Omega$ is the so-called *control domain.* Let us consider the system

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} - \zeta |u|^2 u = f + 1_\omega h, & \text{in} \quad Q_T, \\ u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u(0,x) = u_0(x) + \tau \hat{u}_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where h stands for the control, u is the state function and $\hat{u}_0(x)$ is an unknown function.

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Consider a functional $J : \mathbb{R} \times L^2(q_T) \to \mathbb{R}$ (called *sentinel functional*) defined by

$$J(\tau, h) := \frac{1}{2} \iint_{\mathcal{O}_T} |u(x,t;\tau,h)|^2 dxdt, \tag{1.3}$$

where $u = u(x,t;\tau,h)$ is the corresponding solution of (1.2) associated to τ, h is the control function and $\mathcal{O}_T = \mathcal{O} \times (0,T)$, where $\mathcal{O}$ is the so-called *observation domain*. Thus, our objective can be expressed in the definition below.

Definition 1 (Insensitizing controls). *Let $u_0 \in L^2(\Omega)$ and $f \in L^2(Q_T)$. We say that a control h insensitizes the functional J, associated with the solution $u(x,t;\tau,h)$ of (1.2), if*

$$\frac{\partial}{\partial \tau} J(\tau,h)\Big|_{\tau=0} = 0, \quad \forall\ \widehat{u}_0 \in L^2(\Omega) \quad \text{with } \|\widehat{u}_0\|_{L^2(\Omega)} = 1. \tag{1.4}$$

The definition 1.3 above can be seen as this: the sentinel does not detect (small) variations of the initial data u_0 provoked by the unknown (small) perturbation $\tau\hat{u}_0$ in the observation domain $\mathcal{O}$ when the system evolve from a time $t = 0$ to a time $t = T$.

1.3. **State of the art.** As mentioned at the beginning of the work, the first time that insensitizing problem was approached was in the early of 90's by Lions [33, 34], where the author studied second and fourth order parabolic equations in limited domains considering a functional with the local L^2 norm of the solution of a system with null initial data and where the control domain (located internally) intersect the observation domain (the set where we want to analyze the functional).

Since then many other issues have been considered as variations of this problem, i.e., to find controls which turn a functional depending of the solution (or some derivative) insensitive to small perturbations of the initial data. We will give a brief state of the art to the reader, precisely, we will present a sample of the insensitizing problems for partial differential equations (PDEs) and some control results to (1.1).

1.3.1. *Insensitizing control problems for PDEs.* The first mathematical results concerned the insensitivity of the L^2-norm of the solution restricted to a subdomain, called the observatory. In [16], the author proves that insensitizing control problems cannot be solved for every initial data. Additionally, de Tereza [19] used a global Carleman estimate approach to get the existence of exact insensitizing controls for a semilinear heat equation. Still with respect to semilinear heat equation with, Bodart *et al.* proved the existence of insensitizing controls for this system with nonlinear boundary Fourier conditions.

Concerning to the variations of the sentinel functional, in [25], the author consider a functional involving the gradient of the state for a linear heat system, in the same way, Guerrero [27] treated the case of the sentinel with the curl of the solution of a Stokes system. Still regard to wave equation, Alabau-Boussouira [3], showed the exact controllability, by a reduced number of controls, of coupled cascade systems of PDE's and the existence of exact insensitizing controls for the scalar wave equation. She gave a necessary and sufficient condition for the observability of abstract-coupled cascade hyperbolic systems by a single observation, the observation operator being either bounded or unbounded.

A variation of the (exact) control strategy was presented in [7], where the author considered an approximated insensitizing problem (called ϵ-insensitizing control) for a nonhomogeneous heat equation. We observe that by smoothing the control strategy it was possible to prove in [18] positive results where the control domain and observation region do not intersect each other. Also in [15] the author proved insensitizing control results on unbounded domains, in [35] the authors treated insensitizing controls for both linear and semilinear heat equation but with a partially unknown domain, finally see [41] for the semilinear parabolic equation with dynamic boundary conditions. It is important to point out that in [28] the second author also treated it with a gradient type sentinel associated with the solutions of a nonlinear Ginzburg-Landau equation.

With respect to the structure/type of the equations/systems many variations were considered. In [12] the author treated insensitizing controls for the Boussinesq systems and in [10] the authors proved a result for a phase field system. In [22] it is considered a Cahn-Hilliard equation of fourth order with superlinear nonlinearity and in [21] the authors proved insensitizing (exact and approximated) controls for a large-scale ocean circulation model.

To finalize this small sample of the state of the art, we cite Bodart *et al.* [8, 9] that studied systems with nonlinearities with certain superlinear growth and nonlinear terms depending on the state and its gradient. For a dispersive problem, we can cite Kumar and Chong [38] that worked with the KdV-Burgers equation. Finally, let us mention a recent work due Lopez-Garcia *et al.* [17]. In this work, the authors presented a control problem for a cascade system of two linear N-dimensional Schrödinger equations. They address the problem of null controllability by means of a control supported in a region not satisfying the classical geometrical control condition. The proof is based on the application of a Carleman estimate with degenerate weights to each one of the equations and a careful analysis of the system in order to prove null controllability with only one control force.

1.3.2. *Control problems for 4NLS.* There are interesting control results in a bounded domain of $\mathbb{R}$ or $\mathbb{R}^n$, which we will summarize on the paragraphs below for the following fourth order linear Schrödinger equation

$$iu_t + u_{xxxx} = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}, \tag{1.5}$$

or

$$iu_t + \Delta^2 u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{1.6}$$

The first result about the exact controllability of the linearized fourth order Schrödinger equation (1.6) on a bounded domain Ω of the $\mathbb{R}^n$ is due to Zheng and Zhongcheng in [44]. In this work, by means of an L^2–Neumann boundary control, the authors proved that the solution is exactly controllable in $H^s(\Omega)$, $s = -2$, for an arbitrarily small time. They used the Hilbert Uniqueness Method (HUM) (see, for instance, [20, 32]) combined with the multiplier techniques to get the main result of the article. More recently, in [43], Zheng proved another interesting problem related with the control theory. To do this, he showed a global Carleman estimate for the fourth order Schrödinger equation posed on a finite domain. The Carleman estimate is used to prove the Lipschitz stability for an inverse problem consisting in retrieving a stationary potential in the Schrödinger equation from boundary measurements.

Still on control theory Wen *et al.*, in two works [39, 40], studied well-posedness and control theory related with the equation (1.6) on a bounded domain of $\mathbb{R}^n$, for $n \geq 2$. In [39], they showed the Neumann boundary controllability with collocated observation. With this result in hand, the exponential stability of the closed-loop system under proportional output feedback control holds. Recently, the authors, in [40], gave positive answers when considering the equation with hinged boundary by either moment or Dirichlet boundary control and collocated observation, respectively.

To get a general outline of the control theory already done for the systems (1.5) and (1.6), two interesting problems were studied recently by Aksas and Rebiai [2] and Peng [23]: Uniform stabilization and stochastic control problem, in a smooth bounded domain Ω of $\mathbb{R}^n$ and on the interval $I = (0, 1)$ of $\mathbb{R}$, respectively. The first work, by introducing suitable dissipative boundary conditions, the authors proved that the solution decays exponentially in $L^2(\Omega)$ when the damping term is effective on a neighborhood of a part of the boundary. The results are established by using multiplier techniques and compactness/uniqueness arguments. Regarding the second work, above mentioned, the author showed a Carleman estimates for forward and backward stochastic fourth order Schrödinger equations which provided to prove the observability inequality, unique continuation property and, consequently, the exact controllability for the forward and backward stochastic system associated with (1.5).

Lastly, the first author worked in this equation with the purpose to obtain controllability results. More precisely, they studied that on torus $\mathbb{T}$, the solution of the associated linear system (1.1) is globally exponential stable, by using certain properties of propagation of compactness and regularity in Bourgain spaces. This property together with the local exact controllability ensures that fourth order nonlinear Schrödinger system (1.1) is globally exactly controllable, see [11] for more details.

We caution that the literature is vastly and one can see the references cited previously for other issues to the 4NLS and also for the existence of the insensitizing controls for other types of PDEs. With the sample in hand, let us present our main results in this manuscript.

1.4. **Main results.** The main goal of this paper is to prove the existence of a control h which insensitizes the functional J defined by (1.3). Precisely, the first result of this article can be read as follows.

Theorem 1.1. *Assume that $\omega\cap\mathcal{O}\neq\emptyset$ and $u_0\equiv 0$. There exists a constant $C>0$ and $\delta>0$ such that for any f satisfying*

$$\|e^{C/t}f\|_{L^2(\Omega)}\leq\delta,$$

one can find a control $h(x,t)=:h\in L^2(Q_T)$ which insensitizes the functional J defined by (1.3), in the sense of Definition 1.

As mentioned at the beginning of this work, the existence of insensitizing controls can be defined in the sense of the null controllability problem, see [7] for a rigorous deduction of this fact. So, in this spirit, due to the choice of J, we will reformulate our goal as a partial null controllability problem to the nonlinear system of cascade type associated to (1.2). In other words:

Problem $\mathcal{A}$: *Can we find a control $h(x,t)=h\in L^2(Q_T)$ such that the solutions (u,v) of the following optimality coupled system*

$$\begin{cases} iu_t+u_{xx}-u_{xxxx}-\zeta|u|^2u=f+1_\omega h, & \text{in} \quad Q_T,\\ iv_t+v_{xx}-v_{xxxx}-\overline{\zeta}\overline{u}^2\overline{v}-2\overline{\zeta}|u|^2v=1_{\mathcal{O}}u, & \text{in} \quad Q_T,\\ u(t,0)=u(t,L)=v(t,0)=v(t,L)=0, & \text{on} \quad t\in(0,T),\\ u_x(t,0)=u_x(t,L)=v_x(t,0)=v_x(t,L)=0, & \text{on} \quad t\in(0,T),\\ u(0,x)=u_0(x), v(T,x)=0, & \text{in} \quad \Omega,\end{cases} \tag{1.7}$$

satisfies, in the time $t=0$, $v|_{t=0}=0$?

The answer to such a question motivates the next theorem, which is the main result of this paper.

Theorem 1.2. *Assume that $\omega\cap\mathcal{O}\neq\emptyset$ and the initial data $u_0\equiv 0$. Then, there exists positive constants C and δ, depending on ω, Ω, $\mathcal{O}$, ζ and T, such that for any f satisfying*

$$\|e^{C/t}f\|_{L^2(\Omega)}\leq\delta,$$

there exists a control $h\in L^2(Q_T)$ such that the corresponding solution (u,v,h) of (1.7) satisfies $v|_{t=0}=0$ in Ω.

An immediate consequence of Theorem 1.2 is the following one.

Corollary 1.3 (Existence of insensitizing controls)**.** *Under the hypothesis of Theorem 1.2, there exists a control h, for system (1.7), such that h insensitizes J, given by (1.3), in the sense of Definition 1.*

1.5. **Heuristic of the manuscript.** To our knowledge, these results are the first with respect to the existence of insensitizing controls for the fourth-order Schrödinger equation, in this way, we believe that this manuscript can open a series of questions, which are discussed at the end of the work (see Section 5).

Let us now explain the ideas to prove the results introduced in the last subsection. The main strategy adopted is based on duality arguments (see, e.g. [20, 32]). Roughly speaking, we prove a suitable observability inequalities for the solutions of an adjoint system, where the main tool is a new Carleman estimate. This Carleman estimate with right-hand side in weight Sobolev spaces will be the key point to deal with the coupling terms of the linear system associated to (1.7).

In details, to prove Theorem 1.2, as mentioned, we will first prove a null controllability result for the linearized system associated to (1.7) around zero, which is given by

$$
\begin{cases}
iu_t + u_{xx} - u_{xxxx} = f^0 + 1_\omega h, & \text{in} \quad Q_T,\\
iv_t + v_{xx} - v_{xxxx} = f^1 + 1_{\mathcal{O}} u, & \text{in} \quad Q_T,\\
u(t,0) = u(t,L) = v(t,0) = v(t,L) = 0, & \text{on} \quad t \in (0,T),\\
u_x(t,0) = u_x(t,L) = v_x(t,0) = v_x(t,L) = 0, & \text{on} \quad t \in (0,T),\\
u(0,x) = u_0(x), v(T,x) = 0, & \text{in} \quad \Omega.
\end{cases} \tag{1.8}
$$

Here, f^0 and f^1 are (small) source terms in appropriated L^p-weighted spaces. In order to prove it, we consider the adjoint system of (1.8), namely,

$$
\begin{cases}
i\varphi_t + \varphi_{xx} - \varphi_{xxxx} = 1_{\mathcal{O}}\psi + g^0, & \text{in} \quad Q_T,\\
i\psi_t + \psi_{xx} - \psi_{xxxx} = g^1, & \text{in} \quad Q_T,\\
\varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = 0, & \text{on} \quad t \in (0,T),\\
\psi(t,0) = \psi(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0, & \text{on} \quad t \in (0,T),\\
\varphi(T,x) = 0, \psi(0,x) = \psi_0 & \text{in} \quad \Omega.
\end{cases} \tag{1.9}
$$

With this in hand, we are able to prove an *observability inequality*, with aspect like the one below,

$$
\iint_{Q_T} \rho_1(|\varphi|^2 + |\psi|^2)dxdt \leq C \iint_{\omega\times(0,T)} \rho_2|\varphi|^2 dxdt + \iint_{Q_T} \rho_3(|g^0|^2 + |g^1|^2)dxdt, \tag{1.10}
$$

where ρ_i, $i = 1, 2, 3$, are appropriate weights functions. Then, by duality approach, the desired partial null controllability property is a direct consequence of the (1.10) and can be read as follows.

Theorem 1.4. *Assuming that $\omega \cap \mathcal{O} \neq \emptyset$ and the initial data $u_0 \equiv 0$, there exists positive constants C, depending on δ, ω, Ω, $\mathcal{O}$ and T, such that for f^0 and f^1, in a suitable weighted spaces, one can find a control h such that the associated solution (u, v) of (1.8) satisfies $v|_{t=0} \equiv 0$ in Ω.*

Lastly, the last step is to use an inverse mapping theorem to extend the previous result to the nonlinear system.

Remarks 1. *Finally, the following comments are now given in order:*

1. *In the Definition 1, it is important to point out that the data u_0 will be taken in a convenient way, precisely, will be taken such that the functional J, given by (1.3), is well-defined.*
2. *On the Theorem 1.2, the smallness of f is related to the fact that we will apply a local inversion argument, that is, we first study this problem when a linearized form of equation (1.7) is considered and then we apply a local inversion mapping theorem.*
3. *It is worth mentioning that, in our work, we need a Carleman estimate with internal observation, differently from what was proven by Zheng [43]. Another interesting point is that on the Zheng's work, he proved the regularity of solution of the 4NLS in the class*
$$C^1([0,T]; L^2(\Omega)) \cap C^0([0,T]; H^3(\Omega) \cap H_0^2(\Omega)),$$
which is also different in our case, we need more regular solutions (see Section 2) to help us to use the inverse mapping theorem.
4. *Finally, observe that our sentinel functional J is defined in the sense of L^2-norm. If we want to insensitize a functional with a norm greater than L^2, for example, $\partial_x^n u$, for $n \geq 1$, then we need a system coupled in the second equation of (1.7) in the form $\partial_x^n(1_{\mathcal{O}}\partial_x^n u)$, this means, the coupling has twice as many derivatives. More details about this kind of problems will be given at Section 5.*

1.6. **Structure of the article.** Our work is outlined in the following way: In Section 2, we study the well-posedness of the fourth-order Schrödinger equation *via* Semigroup Theory. Section 3 is devoted to present a new Carleman estimate which will be the key to prove the main result of this manuscript. In Section 4, we show the null controllability results, that is, the linear case (Theorem 1.4) and the nonlinear one (Theorem 1.2). Finally, Section 5, we present further comments and some open problems which seems to be of interest from the mathematical point of view.

2. Well-posedness

In this section we study the well-posedness of (1.2). The main tools to use are the Semigroup Theory, suitable energy estimates (which was deducted through multipliers) and the Banach Fixed Point Theorem.

2.1. **The linear equation.** We first consider the simplest linear equation with null boundary conditions which is a linearized version of (1.2) around zero. More precisely, we consider the following

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} = f, & \text{in } Q_T, \\ u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = 0 & \text{on } (0,T), \\ u(0,x) = u_0, & \text{in } \Omega. \end{cases} \tag{2.1}$$

The first result is a consequence of the Semigroup Theory. Before present it, let us consider the differential operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ given by

$$Au := iu_{xx} - iu_{xxxx},$$

with domain $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$. Thus, the homogeneous linear system (2.1) takes the from

$$\begin{cases} u_t(t) = Au(t) + if(t), \quad t \in [0,T], \\ u(0) = u_0 \in L^2(\Omega). \end{cases} \tag{2.2}$$

The following proposition guarantees some properties for the operator A. Precisely, the following holds.

Proposition 2.1. *Let $f \in L^2(Q_T)$ and $u_0 \in H_0^2(\Omega)$, then (2.1) has a unique solution*

$$u \in C([0,T]; H_0^2(\Omega)) \cap L^2(0,T; H^4(\Omega)). \tag{2.3}$$

Proof. Consider the linear operator defined by A. This allows us to rewrite (2.1) in the abstract form (2.2). We have that A is self-adjoint operator and A is m-dissipative. Indeed, first, is not difficult to see that

$$(Au, v)_{L^2(\Omega)} = (u, Av)_{L^2(\Omega)},$$

for all $u, v \in D(A)$. That is, A is symmetric. Additionally, $D(A^\star) = D(A)$, so A is self-adjoint. Finally, we have

$$(Au, u)_{L^2(\Omega)} = Re\left(i\int_\Omega (u_{xx} - u_{xxxx})\overline{u}dx\right) = Re\left(i\int_\Omega -(|u_x|^2 + |u_{xx}|^2)\right) = 0,$$

for any $u \in D(A)$, and then A is dissipative. Therefore, A is a m-dissipative operator (e.g. [13, Corollary 2.4.8]) and by the Hille–Yosida–Phillips theorem (e.g. [13, Theorem 3.4.4]) we obtain that A is a generator of a contraction semigroup in $L^2(\Omega)$. Thus, if $u_0 \in D(A)$ and $f \in C^1([0,T]; L^2(\Omega))$, then equation (2.1) has solutions $z \in C([0,T]; H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ (e.g. [13, Proposition 4.1.6]). □

Let us now enunciate two auxiliary lemmas that are necessary to prove some energy estimates to the solutions of (2.1). The first one can be read as follows.

Lemma 2.2. *Let $q = q(x,t) \in C^4(\overline{\Omega} \times (0,T), \mathbb{R})$ with $\overline{\Omega}$ being the closed set of Ω. For every solution of (2.1) with $u_0 \in D(A)$ and $f \in C^1([0,T]; L^2(\Omega))$, the following identity holds:*

$$
\begin{aligned}
&-\frac{Im}{2}\int_{Q_T} u\overline{u}_x q_t dxdt - \int_{Q_T} |u_x|^2 q_x dxdt - 2\int_{Q_T} |u_{xx}|^2 q_x dxdt - \frac{Re}{2}\int_{Q_T} u_x\overline{u} q_{xx} dxdt \\
&+\frac{3}{2}\int_{Q_T} |u_x|^2 q_{xxx} dxdt + \frac{Re}{2}\int_{Q_T} u_x\overline{u} q_{xxxx} dxdt + \frac{Im}{2}\int_\Omega u\overline{u}_x q\Big]_0^T dx + \frac{1}{2}\int_0^T |u_{xx}|^2 q\Big]_{\partial\Omega} dt \\
&-\frac{Im}{2}\int_0^T u\overline{u}_t q\Big]_{\partial\Omega} dt + \frac{Im}{2}\left[u_x\overline{u}q]_0^T\right]_{\partial\Omega} + \frac{1}{2}\int_0^T |u_x|^2 q\Big]_{\partial\Omega} dt + \frac{Re}{2}\int_0^T u_x\overline{u} q\Big]_{\partial\Omega} dt \\
&+\int_0^T \left[-|u_x|^2 q_{xx} + \frac{3}{2}Re(u_{xx}\overline{u}_x)q_x - Re(u_{xxx}\overline{u}_x)q\right]_{\partial\Omega} dt \\
&+\int_0^T \left[-\frac{Re}{2}(u_x\overline{u})q_{xxx} + \frac{Re}{2}(u_{xx}\overline{u})q_{xx} - \frac{Re}{2}(u_{xxx}\overline{u})q_x\right]_{\partial\Omega} dt = \int_Q f(\overline{u}_x q + \frac{1}{2}\overline{u} q_x) dxdt,
\end{aligned}
\tag{2.4}
$$

where $\partial\Omega$ is the boundary of Ω.

Proof. The result is obtained multiplying (2.1) by $g(x,t) := \overline{u}_x q + \frac{1}{2}\overline{u} q_x$, integrating by parts over Q_T and taking the real part. ☐

The final result of this section is related to the conservation laws for the solutions of (2.1), with $f = 0$. More precisely, the result is the following one.

Lemma 2.3. *For any positive time t, the solution u of (2.1), with $f = 0$, satisfies*

$$\|u(t)\|_{L^2(\Omega)} = \|u(0)\|_{L^2(\Omega)}, \tag{2.5}$$

and

$$\|u_x(t)\|_{L^2(\Omega)} + \|u_{xx}(t)\|_{L^2(\Omega)} = \|u_x(0)\|_{L^2(\Omega)} + \|u_{xx}(0)\|_{L^2(\Omega)}. \tag{2.6}$$

Additionally, we have

$$\|u(t)\|_{L^2(\Omega)} + \|u_x(t)\|_{L^2(\Omega)} + \|u_{xx}(t)\|_{L^2(\Omega)} = \|u(0)\|_{L^2(\Omega)} + \|u_x(0)\|_{L^2(\Omega)} + \|u_{xx}(0)\|_{L^2(\Omega)}. \tag{2.7}$$

Proof. Multiplying system (2.1), with $f = 0$, by $i\overline{u}$ and integrating in Ω, we get

$$0 = -\int_\Omega \partial_t u\overline{u} dx + i\int_\Omega \partial_x^2 u\overline{u} dx - i\int_\Omega \partial_x^4 u\overline{u} dx = L_1 + L_2 + L_3.$$

We are now looking for the integral $L_2 + L_3$. Let us, first, rewrite these quantities as follows

$$L_2 + L_3 = i\left(\int_\Omega \partial_x^2 u\overline{u} dx - \int_\Omega \partial_x^4 u\overline{u} dx\right). \tag{2.8}$$

Integrating (2.8) by parts and taking the real part of $L_2 + L_3$, we get that

$$
\begin{aligned}
Re(L_2 + L_3) =& Re\left(i\left(-\int_\Omega |\partial_x u|^2 dx + \partial_x u\overline{u}\Big]_{\partial\Omega}\right)\right) \\
&+ Re\left(i\left(-\int_\Omega |\partial_x^2 u|^2 dx + \partial_x^2 u\partial_x\overline{u}\Big]_{\partial\Omega} - \partial_x^3 u\overline{u}\Big]_{\partial\Omega}\right)\right) \\
=& Re\left(i\partial_x u\overline{u} + \partial_x^2 u\partial_x\overline{u} - \partial_x^3 u\overline{u}\right)\big]_{\partial\Omega} \\
=& 0,
\end{aligned}
\tag{2.9}
$$

thanks to the boundary conditions of (2.1). Thus, (2.5) holds.

We will prove (2.6). Multiplying (2.1), with $f = 0$, by $\overline{u}_t$, integrating on Ω and taking the real part give us

$$Re\left(i\int_\Omega |u_t|^2 dx\right) + Re\left(\int_\Omega u_{xx}\overline{u}_t dx\right) - Re\left(\int_\Omega u_{xxxx}\overline{u}_t dx\right) = 0. \tag{2.10}$$

Integrating (2.10) by parts on Ω and using the boundary conditions of (2.1), yields that

$$\frac{1}{2}\frac{\partial}{\partial_t}\int_\Omega \left(|u_x|^2+|u_{xx}|^2\right)dx=0,$$

which implies (2.6). Finally, adding (2.5) and (2.6), we have (2.7), and so Lemma 2.3 is proved. □

As a consequence of the previous lemmas, we have the following estimates.

Proposition 2.4. *Let $f\in L^2(Q_T)$ and $u_0\in H_0^2(\Omega)$, then solution*

$$u\in C([0,T];H_0^2(\Omega))\cap L^2(0,T;H^4(\Omega))$$

of (2.1) satisfies

$$||u||_{C([0,T];H_0^2(\Omega))\cap L^2(0,T;H^4(\Omega))}\leq C\left(||f||_{L^2(Q_T)}+||u_0||_{H^2(\Omega)}\right), \tag{2.11}$$

for some $C=C(T,L)>0$.

Proof. Let us choose $q(x,t)=x$ in (2.4), we have the following

$$\begin{aligned}&\frac{Im}{2}\int_\Omega u\overline{u}_x x\Big]_0^T dx-\frac{Im}{2}\int_0^T u\overline{u}_t x\Big]_{\partial\Omega}dt+\frac{Im}{2}\left[u_x\overline{u}x]_0^T\right]_{\partial\Omega}-\int_{Q_T}|u_x|^2dxdt\\&+\frac{1}{2}\int_0^T |u_x|^2x\Big]_{\partial\Omega}dt+\frac{Re}{2}\int_0^T u_x\overline{u}x\Big]_{\partial\Omega}dt-2\int_{Q_T}|u_{xx}|^2dxdt+\frac{1}{2}\int_0^T|u_{xx}|^2x\Big]_{\partial\Omega}dt\\&+\int_0^T\left[\frac{3}{2}Re(u_{xx}\overline{v}_x)-Re(u_{xxx}\overline{u}_x)x-\frac{Re}{2}(u_{xxx}\overline{u})\right]_{\partial\Omega}dt=\int_{Q_T}f(\overline{u}_x x+\frac{1}{2}\overline{u})dxdt.\end{aligned}$$

Now, note that thanks to the boundary conditions of (2.1), we get

$$\begin{aligned}\frac{Im}{2}\int_\Omega u\overline{u}_x x\Big]_0^T dx-\int_{Q_T}|u_x|^2dxdt-2\int_Q|u_{xx}|^2dxdt&\\+\frac{L}{2}\int_0^T|u_{xx}(L)|^2dt=\int_{Q_T}f(\overline{v}_x x+\frac{1}{2}\overline{v})dxdt.&\end{aligned}$$

Therefore, thanks to (2.5), we have

$$||u||^2_{L^2(0,T;L^2(\Omega))}+||u||^2_{L^2(0,T;H^2(\Omega))}\leq C\left(||f||^2_{L^2(Q_T)}+||u_0||^2_{L^2(\Omega)}\right). \tag{2.12}$$

On the other hand, in the same way as we did before, multiplying equation (2.1) by $g(x,t)_{xxx}:=g_{xxx}$, where $g(x,t):=\overline{u}_xq+\frac{1}{2}\overline{u}q_x$ and taking $q=1$, performing by integration by parts over Ω, using the boundary conditions of (2.1) and thanks to (2.7) and (2.12), holds that

$$||u_{xx}(t,\cdot)||^2_{L^2(\Omega))}+||u_{xx}||^2_{L^2(0,T;H^2(\Omega))}\leq C||f||^2_{L^2(Q_T)}+||u_0||^2_{H^2(\Omega)}.$$

That is, we infer the existence of $C>0$ such that (2.3) holds true, showing the proposition. □

2.2. **The nonlinear equation.** In this subsection, the full nonlinear equation

$$\begin{cases} iu_t+u_{xx}-u_{xxxx}=\zeta|u|^2u+f & \text{in}\quad Q_T,\\ u(t,0)=u(t,L)=u_x(t,0)=u_x(t,L)=0 & \text{on}\quad t\in(0,T),\\ u(0,x)=u_0(x) & \text{in}\quad \Omega,\end{cases} \tag{2.13}$$

for $\zeta\in\mathbb{R}$ and $u_0\in D(A)$ is studied. A local well-posedness theorem is formulated and a sketch is presented.

Theorem 2.5. *Consider $r>0$. There exists a T^* such that if $u_0\in D(A)$ and $||u_0||_{D(A)}\leq r$, system (2.13) admits a unique solution $u\in C([0,T^*];H_0^2(\Omega))\cap L^2([0,T^*];H^4(\Omega))$. Moreover, the solution u depends on u_0 continuously in the corresponding spaces.*

Proof. First, note that for $s > 1/2$, $H^s(\Omega)$ is a Banach algebra. It follows that there is a constant $C = C(s)$ such that

$$||u|u|^2||_{H^s(\Omega)} \leq C||u||^3_{H^s(\Omega)},$$

when $s = 1, 2, 3, 4$.

Consider $\theta \in (0, T]$ and $v \in C([0, \theta]; H^s(\Omega))$, for $s \in [1/2, 4]$, Proposition 2.4 ensures that system

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} = \zeta|u|^2 u & \text{in} \quad Q_T, \\ u(t, 0) = u(t, L) = u_x(t, 0) = u_x(t, L) = 0 & \text{on} \quad t \in (0, T), \\ u(0, x) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{2.14}$$

admits a unique solution

$$u \in C([0, T]; H^2_0(\Omega)) \cap L^2(0, T; H^4(\Omega)).$$

Moreover, thanks to Lemma 2.3 and (2.11), there exists a constant $C > 0$ independent of θ such that

$$||u||_{C([0,T];H^2_0(\Omega))\cap L^2(0,T;H^4(\Omega))} \leq C\left(||u||^3_{C([0,\theta];H^s(\Omega))} + ||u_0||_{H^2(\Omega)}\right),$$

for some $C = C(s) > 0$. Thus, for any given $u_0 \in D(A)$ system (2.14) defines a nonlinear map Γ from $Y_{s,\theta} := \left\{w \in C([0, \theta]; H^2_0(\Omega)) \cap L^2(0, \theta; H^4(\Omega)\right\}$ to $Y_{s,\theta}$. As well understood argument, similar to the contraction mapping argument in [5, 11, 31], reveals that if $\theta > 0$ is chosen small enough, there exists an $M > 0$ such that

$$\|\Gamma(v_0)\|_{Y_{s,\theta}} \leq M$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{s,\theta}} \leq \frac{1}{2}\|v_1 - v_2\|_{Y_{s,\theta}},$$

for any $v_0, v_1, v_2 \in Y_{s,\theta}$ with

$$\|v_j\|_{Y_{s,\theta}} \leq M, \quad j = 0, 1, 2.$$

Hence, the map Γ is a contraction whose unique fixed point is the desired solution u of (2.14). The proof is complete. □

2.3. **The coupled linearized system.** We are now concerned with the existence of solutions for the coupled linearized system. More precisely, we will prove the well-posedness results to the system,

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} = F^0 & \text{in} \quad Q_T, \\ iv_t + v_{xx} - v_{xxxx} = F^1 + 1_{\mathcal{O}}u & \text{in} \quad Q_T, \\ u(t, 0) = u(t, L) = v(t, 0) = v(t, L) = 0, & \text{on} \quad t \in (0, T), \\ u_x(t, 0) = u_x(t, L) = v_x(t, 0) = v_x(t, L) = 0, & \text{on} \quad t \in (0, T), \\ u(0, x) = u_0(x), v(T, x) = 0, & \text{in} \quad \Omega, \end{cases} \tag{2.15}$$

for given u_0 and (F^0, F^1).

First, define the linear unbounded operators

$$\begin{cases} A_0 u = -iu_{xx} + iu_{xxxx} + iF^0 \in L^2(\Omega) \in H^2_0(\Omega), \\ D(A) = \{u \in H^2_0(\Omega);\ u_{xxxx} \in H^2_0(\Omega)\}, \end{cases}$$

and

$$\begin{cases} A_1 u = -iu_{xx} + iu_{xxxx} + iF^1 \in H^{-2}(\Omega), \\ D(A) = H^2_0(\Omega). \end{cases}$$

Both operators are m-dissipative with dense domains; therefore, they generate the C_0 semigroups of contractions $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively. Now, consider the following spaces:

$$Y_0 = C([0, T]; D(A_0)) \cap C^1([0, T]; H^2_0(\Omega))$$

and

$$Y_1 = C([0, T]; D(A_1)) \cap C^1([0, T]; H^{-2}(\Omega)).$$

Thus, the following theorem ensures the existence of *mild solutions* to the system (2.15).

Theorem 2.6 (Mild solutions). *Consider* $u_0 \in H_0^2(\Omega)$,

$$F^0 \in L^2(0,T;H_0^2(\Omega)) \quad \textit{and} \quad F^1 \in L^2(0,T;H^{-2}(\Omega)).$$

Therefore, system (2.15) *possess a mild solution*

$$(u,v) \in [C([0,T];H_0^2(\Omega)) \cap C([0,T];H^{-2}(\Omega))]^2$$

in the sense of

$$u(t) = \mathcal{S}_0(t)u_0 + \int_0^t \mathcal{S}_0(t-s)F^0(s)\mathrm{d}s \tag{2.16}$$

and

$$v(t) = \int_t^T \mathcal{S}_1(s-t)\left(F^1 + 1_{\mathcal{O}}u\right)(s)\mathrm{d}s. \tag{2.17}$$

Proof. The proof is consequence of (2.3) and Proposition 2.4, and then, will be omitted. □

The next result is dedicated to prove the existence of regular solutions for (2.15)

Theorem 2.7 (Regular solutions). *Assume that* $u_0 \in D(A_0)$,

$$F^0 \in C\left([0,T],H_0^2(\Omega)\right) \cap W^{1,1}\left(0,T;H_0^2\right)$$

and

$$F^1 \in C\left([0,T],H^{-2}(\Omega)\right) \cap W^{1,1}\left(0,T;H^{-2}(\Omega)\right).$$

Then, problem (2.15) *has a unique regular solution in the sense that*

$$\begin{cases} (u,v) \in Y_0 \times Y_1, \\ iu_t + u_{xx} - u_{xxxx} = F^0, \\ iv_t + v_{xx} - v_{xxxx} = F^1 + 1_{\mathcal{O}}u, \\ u|_{t=0} = u_0,\, v|_{t=T} = 0. \end{cases}$$

Proof. Note that, thanks to [13, Proposition 4.1.6], we get that the mild solution (2.16) verifies

$$\begin{cases} u \in Y_0, \\ iu_t + u_{xx} - u_{xxxx} = F^0, \\ u|_{t=0} = u_0. \end{cases} \tag{2.18}$$

Now, it is not difficult to see that

$$\begin{aligned} \int_0^T \left\|\left(u 1_{\mathcal{O}}\right)(s)\right\|^2_{H^{-2}(\Omega)} \mathrm{d}s &\leq \sup_{\zeta \in H_0^2(\Omega), \|\zeta\|_{H_0^2(\Omega)}=1} \int_0^T \int_{\mathcal{O}} |u| \cdot \zeta \mathrm{d}x\mathrm{d}t \\ &\leq \iint_{Q_T} |u|^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

and hence $u 1_{\mathcal{O}} \in C\left([0,T],H^{-2}(\Omega)\right) \cap W^{1,1}\left(0,T;H^{-2}(\Omega)\right)$. Then, applying again [13, Proposition 4.1.6], and we get that the mild solution of (2.17) satisfies

$$\begin{cases} v \in Y_1, \\ iv_t + v_{xx} - v_{xxxx} = F^1 + 1_{\mathcal{O}}u, \\ v|_{t=T} = 0. \end{cases} \tag{2.19}$$

Thus, Theorem 2.7 is is achieved putting together (u,v) satisfying (2.18) and (2.19). □

Remarks 2. *We are now give some remarks in order.*

(i) *The results presented in these subsections can be adapted for the adjoint problem, namely*

$$\begin{cases} i\varphi_t + \varphi_{xx} - \varphi_{xxxx} = 1_{\mathcal{O}}\psi + g^0, & \textit{in} \quad Q_T, \\ i\psi_t + \psi_{xx} - \psi_{xxxx} = g^1, & \textit{in} \quad Q_T, \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = 0, & \textit{on} \quad t \in (0,T), \\ \psi(t,0) = \psi(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0, & \textit{on} \quad t \in (0,T), \\ \varphi(T,x) = 0, \psi(0,x) = \psi_0 & \textit{in} \quad \Omega, \end{cases}$$

without any difficulties. In this way, we will only use the results for the adjoint operator, since the results are analogous to those presented here to the system (2.15).

(ii) *Note that for initial data in* $u_0 \in L^2(\Omega)$ *and* $(F^0, F^1) \in L^2(0,T;H^{-2}(\Omega))$, *we still can prove the existence of mild solutions for* (2.15).

3. Carleman estimates

In this section we prove a new Carleman estimate to the fourth order Schrödinger operator $\mathcal{L} = \partial_x^4 u - \partial_x^2 u$. Initially, for the sake of simplicity, we will consider the operator $\mathcal{L}$ only with the higher term, that is, $\partial_x^4 u$. So, to derive this new Carleman, first, let us introduce the basic weight function $\eta(x) = (x - x_0)$ with $x_0 < 0$. Now, for $\lambda > 1$ and $\mu > 1$ we define the following

$$\theta = e^l, \quad \xi(t,x) = \frac{e^{3\mu\eta(x)}}{t(T-t)} \quad \text{and} \quad l(t,x) = \lambda\frac{e^{3\mu\eta(x)} - e^{5\mu\|\eta\|_\infty}}{t(T-t)}. \tag{3.1}$$

Our result will be derived from a previous result due Zheng [43], that can be seen as follows.

Proposition 3.1. *There exists three constants* $\mu_0 > 1$, $C_0 > 0$ *and* $C > 0$ *such that for all* $\mu > \mu_0$ *and for all* $\lambda \geq C_0(T + T^2)$,

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|u|^2 + \lambda^5\mu^6\xi^5\theta^2|u_x|^2 + \lambda^3\mu^4\xi^3\theta^2|u_{xx}|^2 + \lambda\mu^2\xi\theta^2|u_{xxx}|^2\right) dxdt \\
&\leq C\left(\iint_{Q_T} |\theta P u|^2 dxdt + \lambda^3\mu^3\int_0^T (\xi^3\theta^2|u_{xx}|^2)(t,L)dt + \lambda\mu\int_0^T (\xi\theta^2|u_{xxx}|^2)(t,L)dt\right).
\end{aligned}\tag{3.2}$$

With the previous theorem in hand, we are in position to prove a new Carleman estimate associated to $\mathcal{L}u = \partial_x^4 u$. The result is stated in the following way.

Theorem 3.2. *Let* $\omega, \mathcal{O} \subset \Omega$ *be open subsets such that* $\omega \cap \mathcal{O} \neq \emptyset$. *Then, there exists a positive constant* μ_1, *such that for any* $\mu > \mu_1$, *one can find two positive constants* λ_1 *and* C *depending on* λ, μ, Ω, ω *such that for any* $\lambda > \lambda_1(T + T^2)$ *the following estimate for* φ *and* ψ *of* (3.4) *holds*

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2 + \lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right) dxdt \\
&\quad + \iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right) dxdt \\
&\leq C\left(\iint_{Q_T} \theta^2(|g^0|^2 + |g^1|^2)dxdt + \lambda\mu\iint_{q_T} \xi\theta^2|\varphi|^2 dxdt\right).
\end{aligned}\tag{3.3}$$

Before to prove this result, let us give the idea to derive (3.3). To do it, we split the proof in several steps. The first one, consists in applying the Carleman estimate given by Proposition 3.1 for (φ, ψ) solutions of

$$\begin{cases}
i\varphi_t + \varphi_{xx} - \varphi_{xxxx} = 1_{\mathcal{O}}\psi + g^0, & \text{in} \quad Q_T, \\
i\psi_t + \psi_{xx} - \psi_{xxxx} = g^1, & \text{in} \quad Q_T, \\
\varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = \varphi_x(t,L) = 0 & \text{on} \quad t \in (0,T), \\
\psi(t,0) = \psi(t,L) = \psi_x(t,0) = \psi_x(t,L) = 0, & \text{on} \quad t \in (0,T), \\
\varphi(T,x) = 0, \psi(0,x) = \psi_0 & \text{in} \quad \Omega.
\end{cases}\tag{3.4}$$

The second step concern the estimate for a local integral term of *psi* in terms of a local integral of φ and global integral terms of g^0, g^1, φ, ψ and smaller order terms of ψ. Finally, we will estimate integral terms on the border in terms of global integral of φ , ψ and smaller order integral terms.

Proof of Theorem 3.2. In what follows, remember that $\Omega \subset \mathbb{R}$ is a bounded domain whose boundary $\partial\Omega$ is regular enough. Consider $T > 0$, ω and $\mathcal{O}$ to be two nonempty subsets of Ω. Additionally, as defined at the beginning of this work $Q_T = \Omega \times (0,T)$, $q_T = \omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, $\mathcal{O}_T = \mathcal{O} \times (0,T)$ and denote by C a generic constant which can be different from one computation to another. Thus, let us split the proof in three steps.

Step 1: Applying Carleman estimates (3.2).

Thanks to (3.2) we have, for φ and ψ, solution of (3.4), that

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2+\lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2+\lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right)dxdt\\
&\quad\leq C\left(\iint_{Q_T}\theta^2|\varphi_{xx}|dxdt+\iint_{Q_T}\theta^2|g^0|^2dx+\iint_{\mathcal{O}_T}\theta^2|\psi|^2dxdt\right.\\
&\quad\left.+\lambda^3\mu^3\int_0^T(\xi^3\theta^2|\varphi_{xx}|^2)(t,L)dt+\lambda\mu\int_0^T(\xi\theta^2|\varphi_{xxx}|^2)(t,L)dt\right).
\end{aligned}\tag{3.5}$$

For λ,μ large enough, we obtain

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2+\lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2+\lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right)dxdt\\
&\quad\leq C\left(\iint_{Q_T}\theta^2|g^0|^2dx+\iint_{\mathcal{O}_T}\theta^2|\psi|^2dxdt+\lambda^3\mu^3\int_0^T(\xi^3\theta^2|\varphi_{xx}|^2)(t,L)dt\right.\\
&\quad\left.+\lambda\mu\int_0^T(\xi\theta^2|\varphi_{xxx}|^2)(t,L)dt\right).
\end{aligned}\tag{3.6}$$

Now, applying (3.2) for ψ, we get that

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2+\lambda^5\mu^6\xi^5\theta^2|\psi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2+\lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right)dxdt\\
&\quad\leq C\left(\iint_{Q_T}\theta^2|\psi_{xx}|dxdt+\iint_{Q_T}\theta^2|g^1|^2dx+\iint_{\mathcal{O}_T}|\psi|^2dxdt\right.\\
&\quad\left.+\lambda^3\mu^3\int_0^T(\xi^3\theta^2|\varphi_{xx}|^2)(t,L)dt+\lambda\mu\int_0^T(\xi\theta^2|\varphi_{xxx}|^2)(t,L)dt\right).
\end{aligned}\tag{3.7}$$

Again, by using (3.7) for λ,μ large enough we have the following:

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2+\lambda^5\mu^6\xi^5\theta^2|\psi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2+\lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right)dxdt\\
&\leq C\left(\iint_{Q_T}\theta^2|g^1|^2dx+\lambda^3\mu^3\int_0^T(\xi^3\theta^2|\psi_{xx}|^2)(t,L)dt+\lambda\mu\int_0^T(\xi\theta^2|\psi_{xxx}|^2)(t,L)dt\right).
\end{aligned}\tag{3.8}$$

Note that, putting together (3.6) and (3.8), we obtain the following estimate

$$\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2+\lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2+\lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right)dxdt\\
&+\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2+\lambda^5\mu^6\xi^5\theta^2|\psi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2+\lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right)dxdt\\
&\leq C\left(\iint_{Q_T}\theta^2|g^0|^2dxdt+\iint_{Q_T}\theta^2|g^1|^2dxdt+\iint_{\mathcal{O}_T}\theta^2|\psi|^2dxdt\right.\\
&+\lambda^3\mu^3\int_0^T(\xi^3\theta^2(|\varphi_{xx}|^2+|\psi_x|^2))(t,L)dt+\lambda\mu\int_0^T(\xi\theta^2(|\varphi_{xxx}|^2+|\psi_{xxx}|^2))(t,L)dt.
\end{aligned}\tag{3.9}$$

Step 2: Estimates for the local integral of ψ.

In what follows, we estimate the last term in the right-hand side of (3.9), that is, the local integral term of ψ in $\mathcal{O}_T$. Now, since $\omega\cap\mathcal{O}\neq\emptyset$, there exists $\tilde{\omega}_T\subset\omega\cap\mathcal{O}$. From now on, take a cut-off function $\eta\in C_0^\infty(\omega)$ such that $\eta\equiv 1$ in $\tilde{\omega}_T$. Observe that

$$\psi=-i\varphi_t+\varphi_{xx}-\varphi_{xxxx}-g^0,\quad \text{in } \mathcal{O}_T,$$

so

$$
\begin{aligned}
\iint_{\mathcal{O}_T} \theta^2|\psi|^2 dxdt \le \iint_{\tilde{\omega}_T} \eta\theta^2|\psi|^2 dxdt = \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}\psi dxdt \\
= Re \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}(-i\varphi_t + \varphi_{xx} - \varphi_{xxxx} - g^0)dxdt \\
:= \sum_{i=1}^{4} \Psi_i,
\end{aligned} \tag{3.10}
$$

where Ψ_i, for $i = 1, 2, 3, 4$, are the integrals of the right hand side of (3.10). We now estimate these terms. For Ψ_1, integrating by parts in t we have that

$$
\begin{aligned}
\Psi_1 &= Re \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}(-i\varphi_t)dxdt = Re\left((\eta\theta^2\psi(-i\varphi))\Big|_0^T - \iint_{Q_T} (\eta\theta^2\overline{\psi})_t(-i\varphi)dxdt \right) \\
&= Re\left(-\iint_{\tilde{\omega}_T} (\eta\theta^2)_t\overline{\psi}(-i\varphi)dxdt - \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}_t(-i\varphi)dxdt \right).
\end{aligned} \tag{3.11}
$$

Since,

$$
|(\theta^m)_t| = |(e^{ml})_t| = |e^{ml}(ml)_t| = |me^{ml}(l)_t| \le mC\lambda T\xi^2\theta^m,
$$

then, for $m = 2$, we get that $|(\eta\theta^2)_t| \le C\lambda\xi^2\theta^2$, and by using Young inequality we obtain

$$
\begin{aligned}
Re\left(-\iint_{\tilde{\omega}_T} (\eta\theta^2)_t\overline{\psi}(-i\varphi)dxdt \right) &\le CRe\left(\lambda \iint_{\tilde{\omega}_T} \xi^2\theta^2\overline{\psi}\varphi dxdt \right) \\
&= Re\left(\iint_{\tilde{\omega}_T} \left(\lambda^{\frac{7}{2}}\mu^4\xi^{\frac{7}{2}}\theta\overline{\psi}\right)\left(C\lambda^{-\frac{5}{2}}\mu^{-4}\xi^{-\frac{3}{2}}\varphi\right) dxdt \right) \\
&\le \delta\lambda^7\mu^8 \iint_{Q_T} \xi^7\theta^2|\psi|^2 dxdt + C\lambda^{-5}\mu^{-8} \iint_{\tilde{\omega}_T} \xi^{-3}|\varphi|^2 dxdt.
\end{aligned} \tag{3.12}
$$

Combining (3.11) with (3.12), we get

$$
\begin{aligned}
\Psi_1 = Re\left(-\iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}_t(-i\varphi)dxdt \right) \\
+ \delta\lambda^7\mu^8 \iint_{Q_T} \xi^7\theta^2|\psi|^2 dxdt + C\lambda^{-5}\mu^{-8} \iint_{\tilde{\omega}_T} \xi^{-3}|\varphi|^2 dxdt.
\end{aligned} \tag{3.13}
$$

So, for δ small enough we can absorb the global integral term of (3.13) with the left-hand side of (3.9). Due to the boundary conditions for ψ and φ and since η has compact support on $\tilde{\omega}$ by integrating by parts for space variable, we obtain

$$
\begin{aligned}
\Psi_2 &= Re \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}\varphi_{xx}dxdt = Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2\overline{\psi})_{xx}\varphi dxdt \right) \\
&= Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2)_{xx}\overline{\psi}\varphi dxdt + 2\iint_{\tilde{\omega}_T} (\eta\theta^2)_x\overline{\psi}_x\varphi dxdt + \iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}_{xx}\varphi dxdt \right) \\
&= \sum_{i=1}^{3} \Psi_{2,i}.
\end{aligned} \tag{3.14}
$$

To bound each term of the right hand side of (3.14) first observe that the following estimates, for the derivative in space of the weight function θ, holds true

$$
|(\theta^2)_x| = |(e^{2l})_x| = |e^{2l}(2l)_x| = 2|\theta^2 l_x| \le C\lambda\mu\xi\theta^2
$$

and

$$
\begin{aligned}
|(\theta^2)_{xx}| = |(e^{2l})_{xx}| = |(e^{2l}(2l)_x)_x| &= 2|(\theta^2 l_x)_x| \\
&\leq 4|\theta^2 l_x^2| + 2|\theta^2 l_{xx}| \leq 4\theta^2|l_x|^2 + 2\theta^2|l_{xx}| \\
&\leq C\lambda^2\mu^2\xi^2\theta^2 + C\lambda\mu\xi^2\theta^2 \leq C\lambda^2\mu^2\xi^2\theta^2.
\end{aligned}
$$

Therefore, it yields that

$$
\begin{aligned}
\Psi_{2,1} &= Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2)_{xx}\overline{\psi}\varphi dxdt\right) \leq Re\left(C\lambda^2\mu^2\iint_{\tilde{\omega}_T} \xi^2\theta^2\overline{\psi}\varphi dxdt\right) \\
&= Re\left(\iint_{\tilde{\omega}_T} \left(\lambda^{\frac{7}{2}}\mu^4\xi^{\frac{7}{2}}\theta\overline{\psi}\right)\left(C\lambda^{-\frac{3}{2}}\mu^{-2}\xi^{-\frac{3}{2}}\theta\varphi\right)dxdt\right) \qquad (3.15)\\
&\leq \delta\lambda^7\mu^8\iint_{Q_T} \xi^7\theta^2|\psi|^2 dxdt + C\lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T} \xi^{-3}\theta^2|\varphi|^2 dxdt
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_{2,2} &= 2Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2)_x\overline{\psi}_x\varphi dxdt\right) \leq Re\left(C\lambda\mu\iint_{\tilde{\omega}_T} \xi\theta^2\overline{\psi}\varphi dxdt\right) \\
&= 2Re\left(\iint_{\tilde{\omega}_T} \left(\lambda^{\frac{5}{2}}\mu^3\xi^{\frac{5}{2}}\theta\overline{\psi}_x\right)\left(C\lambda^{-\frac{3}{2}}\mu^{-2}\xi^{-\frac{3}{2}}\theta\varphi\right)dxdt\right) \qquad (3.16)\\
&\leq \delta\lambda^5\mu^6\iint_{Q_T} \xi^5\theta^2|\psi_x|^2 dxdt + C\lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T} \xi^{-3}\theta^2|\varphi|^2 dxdt.
\end{aligned}
$$

Finally, $\Psi_{2,3}$ does not need to be estimate since we use it to obtain the equation for ψ. Now, combining (3.14), (3.15) and (3.16) we get

$$
\begin{aligned}
&\Psi_2 \leq C\lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T} \xi^{-3}\theta^2|\varphi|^2 dxdt \qquad (3.17)\\
&+\delta\left(\lambda^7\mu^8\iint_{Q_T} \xi^7\theta^2|\psi|^2 dxdt + \lambda^7\mu^8\iint_{Q_T} \xi^7\theta^2|\psi|^2 dxdt + \lambda^7\mu^8\iint_{Q_T} \xi^7\theta^2|\psi_{xx}|^2 dxdt\right).
\end{aligned}
$$

For Ψ_3 we use the boundary conditions and integrate with respect to space variable four times to obtain

$$
\begin{aligned}
\Psi_3 =& Re\iint_{\tilde{\omega}_T} \eta\theta^2\overline{\psi}(-\varphi_{xxxx})dxdt = Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2\overline{\psi})_{xxxx}(-\varphi)dxdt\right) \\
=& Re\left(\iint_{\tilde{\omega}_T} (\eta\theta^2)_{xxxx}\overline{\psi}\varphi dxdt + 4\iint_{\tilde{\omega}_T} (\eta\theta^2)_{xxx}\overline{\psi}_x\varphi dxdt + 6\iint_{\tilde{\omega}_T} (\eta\theta^2)_{xx}\overline{\psi}_{xx} dxdt\right) \qquad (3.18)\\
&+ Re\left(4\iint_{\tilde{\omega}_T} (\eta\theta^2)_x\overline{\psi}_{xxx}\varphi dxdt + \iint_{\tilde{\omega}_T} \eta\theta^2(-\overline{\psi}_{xxxx})\varphi dxdt\right) \\
=& \sum_{i=1}^{5}\Psi_{3,i}.
\end{aligned}
$$

Now our task is to estimate these terms. Observe that we have the following estimates in space variable for k-th order derivative in space variable for the weight function θ,

$$
|(\theta^2)_{kx}| \leq 2^k C(\lambda^k\mu^k\xi^k)\theta^2,
$$

thus

$$\begin{aligned}\Psi_{3,1} &= Re\left(\iint_{\tilde{\omega}_T}(\eta\theta^2)_{xxxx}\overline{\psi}\varphi dxdt\right)\\ &\le Re\left(C\lambda^4\mu^4\iint_{\tilde{\omega}_T}\xi^4\theta^2\overline{\psi}\varphi dxdt\right)\\ &= Re\left(\iint_{\tilde{\omega}_T}\left(\lambda^{\frac{7}{2}}\mu^4\xi^{\frac{7}{2}}\theta\overline{\psi}\right)\left(C\lambda^{\frac{1}{2}}\xi^{\frac{1}{2}}\theta\varphi\right)dxdt\right)\\ &\le \delta\lambda^7\mu^8\iint_{Q_T}\xi^7\theta^2|\psi|^2dxdt + C\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt,\end{aligned}\tag{3.19}$$

$$\begin{aligned}\Psi_{3,2} &= Re\left(4\iint_{\tilde{\omega}_T}(\eta\theta^2)_{xxx}\overline{\psi}_x\varphi dxdt\right)\\ &\le Re\lambda^3\mu^3\left(\iint_{\tilde{\omega}_T}\xi^3\theta^2\overline{\psi}_x\varphi dxdt\right)\\ &= Re\left(\iint_{\tilde{\omega}_T}\left(\lambda^{\frac{5}{2}}\mu^3\xi^{\frac{5}{2}}\theta\overline{\psi}_x\right)\left(C\lambda^{\frac{1}{2}}\xi^{\frac{1}{2}}\theta\varphi\right)dxdt\right)\\ &\le \delta\lambda^5\mu^6\iint_{Q_T}\xi^5\theta^2|\psi_x|^2dxdt + C\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt,\end{aligned}\tag{3.20}$$

$$\begin{aligned}\Psi_{3,3} &= Re\left(6\iint_{\tilde{\omega}_T}(\eta\theta^2)_{xx}\overline{\psi}_{xx}dxdt\right)\\ &\le Re\left(C\lambda^2\mu^2\iint_{\tilde{\omega}_T}\xi^2\theta^2\overline{\psi}_{xx}dxdt\right)\\ &= Re\left(\iint_{\tilde{\omega}_T}\left(\lambda^{\frac{3}{2}}\mu^2\xi^{\frac{3}{2}}\theta\overline{\psi}_{xx}\right)\left(C\lambda^{\frac{1}{2}}\xi^{\frac{1}{2}}\theta\varphi\right)dxdt\right)\\ &\le \delta\lambda^3\mu^4\iint_{Q_T}\xi^3\theta^2|\psi_{xx}|^2dxdt + C\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}\Psi_{3,4} &= Re\left(4\iint_{\tilde{\omega}_T}(\eta\theta^2)_x\overline{\psi}_{xxx}\varphi dxdt\right)\\ &\le Re\left(C\lambda\mu\iint_{\tilde{\omega}_T}\xi\theta^2\overline{\psi}_{xxx}\varphi dxdt\right)\\ &= Re\left(\iint_{\tilde{\omega}_T}\left(\lambda^{\frac{1}{2}}\mu\xi^{\frac{1}{2}}\theta\overline{\psi}_{xxx}\right)\left(C\lambda^{\frac{1}{2}}\xi^{\frac{1}{2}}\theta\varphi\right)dxdt\right)\\ &\le \delta\lambda\mu^2\iint_{Q_T}\xi\theta^2|\psi_{xxx}|^2dxdt + C\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt.\end{aligned}\tag{3.22}$$

We do not estimate $\Psi_{3,5}$ since we use this term to obtain the equation for ψ. By putting (3.19), (3.20), (3.21) and (3.22) in (3.18), we conclude

$$\begin{aligned}\Psi_3 \le\ & Re\left(\iint_{\tilde{\omega}_T}\eta\theta^2\overline{\psi}_{xxxx}\varphi dxdt\right) + C\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt\\ &+\delta\left(\iint_{Q_T}\left[\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right]dxdt\right).\end{aligned}\tag{3.23}$$

Finally, for Ψ_4 we get

$$\begin{aligned}\Psi_4 &= Re\iint_{\tilde{\omega}_T}\eta\theta^2\overline{\psi}(-g^0)dxdt\\ &\leq \delta\lambda^7\mu^8\iint_{Q_T}\xi^7\theta^2|\psi|^2dxdt + C\lambda^{-7}\mu^{-8}\iint_{Q_T}\xi^{-7}\theta^2|g^0|^2dxdt.\end{aligned}\tag{3.24}$$

Combining (3.10), (3.13), (3.17), (3.23) and (3.24) we get

$$\begin{aligned}&\iint_{\mathcal{O}_T}\theta^2|\psi|^2dxdt\\ &\leq C\left(\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt + \lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T}\xi^{-3}\theta^2|\varphi|^2dxdt + \lambda^{-5}\mu^{-8}\iint_{\tilde{\omega}_T}\xi^{-3}|\varphi|^2dxdt\right)\\ &+ Re\left(\iint_{\tilde{\omega}_T}\eta\theta^2\varphi(i\overline{\psi}_t + \overline{\psi}_{xx} - \overline{\psi}_{xxxx})dxdt\right) + C\lambda^{-7}\mu^{-8}\iint_{Q_T}\xi^{-7}\theta^2|g^0|^2dxdt\\ &+ \delta\left(\iint_{Q_T}\left[\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right]dxdt\right).\end{aligned}\tag{3.25}$$

Since we get $i\overline{\psi}_t + \overline{\psi}_{xx} - \overline{\psi}_{xxxx} = \overline{-i\psi_t + \psi_{xx} - \psi_{xxxx}} = \overline{g^1}$ in Q_T and

$$Re\left(\iint_{\tilde{\omega}_T}\eta\theta^2\varphi\overline{g^1}dxdt\right) \leq \delta\lambda^7\mu^8\iint_{Q_T}\xi^7\theta^2|\varphi|^2dxdt + C\lambda^{-7}\mu^{-8}\iint_{Q_T}\xi^{-7}\theta^2|g^1|^2dxdt,\tag{3.26}$$

then, combining (3.25) and (3.26), yields that

$$\begin{aligned}&\iint_{\mathcal{O}_T}\theta^2|\psi|^2dxdt \leq C\lambda^{-7}\mu^{-8}\iint_{Q_T}\xi^{-7}\theta^2(|g^0|^2+|g^1|^2)dxdt + \delta\lambda^7\mu^8\iint_{Q_T}\xi^7\theta^2|\varphi|^2dxdt\\ &+ C\left(\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt + \lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T}\xi^{-3}\theta^2|\varphi|^2dxdt + \lambda^{-5}\mu^{-8}\iint_{\tilde{\omega}_T}\xi^{-3}|\varphi|^2dxdt\right)\\ &+ \delta\left(\iint_{Q_T}\left[\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right]dxdt\right).\end{aligned}\tag{3.27}$$

Combining (3.9) with (3.27) we obtain the following

$$\begin{aligned}&\iint_{Q_T}\left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2 + \lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right)dxdt\\ &\quad+\iint_{Q_T}\left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right)dxdt\\ &\leq C\left(\iint_{Q_T}\theta^2(|g^0|^2+|g^1|^2)dx + \lambda^{-7}\mu^{-8}\iint_{Q_T}\xi^{-7}\theta^2(|g^0|^2+|g^1|^2)dxdt\right)\\ &\quad+C\left(\lambda\iint_{\tilde{\omega}_T}\xi\theta^2|\varphi|^2dxdt + \lambda^{-3}\mu^{-4}\iint_{\tilde{\omega}_T}\xi^{-3}\theta^2|\varphi|^2dxdt + \lambda^{-5}\mu^{-8}\iint_{\tilde{\omega}_T}\xi^{-3}|\varphi|^2dxdt\right)\\ &\quad+\delta\left(\iint_{Q_T}\left[\lambda^7\mu^8\xi^7\theta^2(|\varphi|^2+|\psi|^2) + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right]dxdt\right)\\ &\quad+C\left(\lambda^3\mu^3\int_0^T(\xi^3\theta^2(|\varphi_{xx}|^2+|\psi_{xx}|^2))(t,L)dt + \lambda\mu\int_0^T(\xi\theta^2(|\varphi_{xxx}|^2+|\psi_{xxx}|^2))(t,L)dt\right).\end{aligned}$$

Then, for λ, μ large enough and δ small enough we get

$$
\begin{aligned}
&\iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2 + \lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right) dxdt \\
&+ \iint_{Q_T} \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2 + \lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right) dxdt \\
&\le C\left(\iint_{Q_T} \theta^2(|g^0|^2 + |g^1|^2)dxdt + \lambda \iint_{\tilde{\omega}_T} \xi\theta^2|\varphi|^2 dxdt\right) \\
&+ C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2(|\varphi_{xx}|^2 + |\psi_{xx}|^2))(t,L)dt + C\lambda\mu \int_0^T (\xi\theta^2(|\varphi_{xxx}|^2 + |\psi_{xxx}|^2))(t,L)dt \\
&=: \mathcal{I} + \mathcal{B}_1 + \mathcal{B}_2.
\end{aligned} \tag{3.28}
$$

Step 3: Estimates of the boundary terms.

Now, we will find an estimate for the boundary term on the right-hand side of (3.28), precisely, $\mathcal{B}_1$ and $\mathcal{B}_2$. Using trace Theorem (note that $5/2 < 7/2$) we have

$$
\begin{aligned}
\mathcal{B}_1 &= C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2(|\varphi_{xx}|^2 + |\psi_{xx}|^2))(t,L)dt \\
&\le C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\left(\|\varphi\|^2_{H^{\frac{5}{2}}(\tilde{\omega}_T)} + \|\psi\|^2_{H^{\frac{5}{2}}(\tilde{\omega}_T)}\right) dt \\
&\le C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\left(\|\varphi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)} + \|\psi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)}\right) dt
\end{aligned} \tag{3.29}
$$

and

$$
\begin{aligned}
\mathcal{B}_2 &= C\lambda\mu \int_0^T (\xi\theta^2(|\varphi_{xxx}|^2 + |\psi_{xxx}|^2))(t,L)dt \\
&\le C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\left(\|\varphi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)} + \|\psi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)}\right) dt.
\end{aligned} \tag{3.30}
$$

So, putting together (3.29) and (3.30), yields that

$$
\mathcal{B}_1 + \mathcal{B}_2 \le C\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\left(\|\varphi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)} + \|\psi\|^2_{H^{\frac{7}{2}}(\tilde{\omega}_T)}\right) dt. \tag{3.31}
$$

Using interpolation in the Sobolev spaces $H^s(\Omega)$, for $s \ge 0$, yields that

$$
\begin{aligned}
\mathcal{B}_1 + \mathcal{B}_2 \le\ & C_1\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\,\|\varphi\|^{\frac{21}{11}}_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,\|\varphi\|^{\frac{1}{11}}_{L^2(\tilde{\omega}_T)}\,dt \\
&+ C_1\lambda^3\mu^3 \int_0^T (\xi^3\theta^2)(t,L)\,\|\psi\|^{\frac{21}{11}}_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,\|\psi\|^{\frac{1}{11}}_{L^2(\tilde{\omega}_T)}\,dt \\
=\ & C_1\lambda^3\mu^3 \int_0^T (\xi^{\frac{255}{22}}\xi^{-\frac{189}{22}}\theta^{\frac{86}{22}}\theta^{-\frac{42}{22}})(t,L)\,\|\varphi\|^{\frac{21}{11}}_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,\|\varphi\|^{\frac{1}{11}}_{L^2(\tilde{\omega}_T)}\,dt \\
&+ C_1\lambda^3\mu^3 \int_0^T (\xi^{\frac{255}{22}}\xi^{-\frac{189}{22}}\theta^{\frac{86}{22}}\theta^{-\frac{42}{22}})(t,L)\,\|\psi\|^{\frac{21}{11}}_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,\|\psi\|^{\frac{1}{11}}_{L^2(\tilde{\omega}_T)}\,dt \\
\le\ & C_\epsilon\lambda^6\mu^6 \int_0^T (\xi^{255}\theta^{86})(t,L)\,\|\varphi\|^2_{L^2(\tilde{\omega}_T)}\,dt + \epsilon\lambda^{-2}\mu^{-2}\int_0^T (\xi^{-\frac{189}{21}}\theta^{-2})(t,L)\,\|\varphi\|^2_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,dt \\
&+ C_\epsilon\lambda^6\mu^6 \int_0^T (\xi^{255}\theta^{86})(t,L)\,\|\psi\|^2_{L^2(\tilde{\omega}_T)}\,dt + \epsilon\lambda^{-2}\mu^{-2}\int_0^T (\xi^{-\frac{189}{21}}\theta^{-2})(t,L)\,\|\psi\|^2_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,dt,
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
\mathcal{B}_1+\mathcal{B}_2 \leq\ & C_\epsilon \lambda^6\mu^6 \int_0^T (\xi^{255}\theta^{86})(t,L)\,\|\varphi\|^2_{L^2(\tilde{\omega}_T)}\,dt \\
& + \epsilon\lambda^{-2}\mu^{-2}\int_0^T (\xi^{-9}\theta^{-2})(t,L)\,\|\varphi\|^2_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,dt \\
& + C_\epsilon \lambda^6\mu^6 \int_0^T (\xi^{255}\theta^{86})(t,L)\,\|\psi\|^2_{L^2(\tilde{\omega}_T)}\,dt \\
& + \epsilon\lambda^{-2}\mu^{-2}\int_0^T (\xi^{-9}\theta^{-2})(t,L)\,\|\psi\|^2_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,dt \\
=:\ & \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,
\end{aligned} \tag{3.32}
$$

for some positive constant C_ϵ.

In this moment, our goal is to prove integrals $\mathcal{I}_i$, for $i=1,2,3$, can be absolved by the left hand side of (3.28). Let us start with the analysis of $\mathcal{I}_2$, precisely the quantity

$$\int_0^T (\xi^{-9}\theta^{-2})(t,L)\,\|\varphi\|^2_{H^{\frac{11}{3}}(\tilde{\omega}_T)}\,dt.$$

Consider $\varphi_1(t,x) := \xi_1(t)\varphi(t,x)$ with

$$\xi_1(t)=\theta^{-1}\xi^{-\frac{1}{2}}.$$

Then φ_1 satisfies the system

$$
\begin{cases}
-i\varphi_{1t}+\varphi_{1xx}-\varphi_{1xxxx}=f_1:=\xi_{1t}\varphi, & \text{in } Q_T,\\
\varphi_1(t,0)=\varphi_1(t,L)=\varphi_{1x}(t,0)=\varphi_{1x}(t,L)=0, & \text{on } (0,T),\\
\varphi_1(T,x)=0, & \text{in } \Omega.
\end{cases} \tag{3.33}
$$

Now, observe that, since $\varphi_x(t,0)=0$ and $|\xi_{1t}|\leq C\lambda\xi^{\frac{3}{2}}\theta^{-1}$, we have

$$
\begin{aligned}
\|f_1\|^2_{L^2(Q_T)} &\leq C\int_{Q_T}\theta^{-2}\lambda^2\xi^3|\varphi|^2dxdt \\
&\leq C\int_{Q_T}\left\{\lambda^2\xi^3|\varphi|^2+\lambda^3|\varphi_x|^2+\lambda|\varphi_{xx}|^2+\lambda^{-1}|\varphi_{xxx}|^2\right\}\theta^{-2}dxdt,
\end{aligned} \tag{3.34}
$$

for some constant $C>0$ and all $s\geq s_0$. Moreover, thanks to Section 2, $\varphi_1\in L^2(0,T;H^2(\Omega))\cap C([0,T];L^2(\Omega))$. Then, interpolating between $L^2(0,T;H^2(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$, we infer that $\varphi_1\in L^2(0,T;H^{5/3}(\Omega))$ and

$$\|\varphi_1\|_{L^2(0,T;H^{5/3}(\Omega))}\leq C\,\|f_1\|_{L^2(Q_T)}\,. \tag{3.35}$$

Let $\varphi_2(t,x) := \xi_2(t)\varphi(t,x)$ with

$$\xi_2=\theta^{-1}\xi^{-\frac{5}{2}}.$$

Then v_2 satisfies system (3.33) with f_1 replaced by

$$f_2:=\xi_{2t}\xi_1^{-1}\varphi_1.$$

Observe that $\left|\theta_{2t}\theta_1^{-1}\right|\leq C\lambda$. Thus, we obtain

$$\|f_2\|_{L^2(0,T;H^{5/3}(\Omega))}\leq C\lambda\,\|\varphi_1\|_{L^2(0,T;H^{5/2}(\Omega))}\,. \tag{3.36}$$

Now, by using that φ_2 belongs to $L^2(0,T;H^4(\Omega))$ and $L^\infty(0,T;H^2(\Omega))$, thanks to (2.11), and interpolating these two spaces, we have that

$$\varphi_2\in L^2(0,T;H^{11/3}(\Omega))\cap L^\infty(0,T;H^{8/3}(\Omega))$$

with

$$\|\varphi_2\|_{L^2(0,T;H^{11/3}(\Omega))\cap L^\infty(0,T;H^{8/3}(\Omega))}\leq C\,\|f_2\|_{L^2(0,T;H^{5/3}(\Omega))}\,. \tag{3.37}$$

Thus we infer from (3.35)–(3.37), the following

$$\begin{aligned}\|\varphi_2\|^2_{L^2(0,T;H^{11/3}(\Omega))} &\leq C_1\lambda\|f_1\|^2_{L^2(Q_T)}\\ &\leq C_2\int_{Q_T}\left\{\lambda^3\xi^3|\varphi|^2+\lambda^4|\varphi_x|^2+\lambda^2|\varphi_{xx}|^2+|\varphi_{xxx}|^2\right\}\theta^{-2}dxdt.\end{aligned} \tag{3.38}$$

Hence, replacing $\varphi_2=\theta^{-1}\varphi^{-\frac{9}{2}}$ in (3.38), for some constant $C_3>0$, yields that

$$\begin{aligned}&\int_0^T(\xi^{-9}\theta^{-2})(t,L)\,\|\varphi(t,\cdot)\|^2_{H^{11/3}(\tilde\omega_T)}\,dt\\ &\qquad\leq C_3\int_{Q_T}\left\{\lambda^3\xi^3|\varphi|^2+\lambda^4|\varphi_x|^2+\lambda^2|\varphi_{xx}|^2+|\varphi_{xxx}|^2\right\}\theta^{-2}dxdt.\end{aligned} \tag{3.39}$$

Note that analogously we can infer the same relation for ψ, that is,

$$\begin{aligned}&\int_0^T(\xi^{-9}\theta^{-2})(t,L)\,\|\psi(t,\cdot)\|^2_{H^{11/3}(\tilde\omega_T)}\,dt\\ &\qquad\leq C_3\int_{Q_T}\left\{\lambda^3\xi^3|\psi|^2+\lambda^4|\psi_x|^2+\lambda^2|\psi_{xx}|^2+|\psi_{xxx}|^2\right\}\theta^{-2}dxdt.\end{aligned} \tag{3.40}$$

Therefore, adding (3.39) and (3.40), putting in (3.32) and, finally, comparing with (3.28), for λ and μ large enough, yields that,

$$\begin{aligned}&\iint_{Q_T}\left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2+\lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2+\lambda\mu^2\xi\theta^2|\varphi_{xxx}|^2\right)dxdt\\ &+\iint_{Q_T}\left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2+\lambda^5\mu^6\xi^5\theta^2|\psi_x|^2+\lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2+\lambda\mu^2\xi\theta^2|\psi_{xxx}|^2\right)dxdt\\ &\qquad\leq C\left(\iint_{Q_T}\theta^2(|g^0|^2+|g^1|^2)dxdt+\lambda\iint_{\tilde\omega_T}\xi\theta^2|\varphi|^2dxdt\right),\end{aligned} \tag{3.41}$$

since $|(\xi^{255}\theta^{86})(t,L)|$ is bounded in terms of $t\in[0,T]$ due to the choices in (3.1), what guarantees (3.3), and so the Carleman is shown. □

4. Null controllability results

In this section we prove the existence of insensitizing controls for the linearized system (1.8). First we need to obtain an estimate as in Theorem 3.2, with weights that remain bounded as $t\to T$, i.e., have blow-up only in $t=0$. To this purpose we introduce the new weights

$$\begin{aligned}&\sigma=e^m, && \nu(t,x)=\frac{e^{3\mu\eta(x)}}{\gamma(t)} &&\text{and}\quad m(t,x)=\lambda\frac{e^{3\mu\eta(x)}-e^{5\mu\|\eta\|_\infty}}{\gamma(t)}\\ &\sigma^*=e^{m^*}, && \nu^*(t)=\min_{x\in\overline\Omega}\nu(x,t), &&\text{and}\quad m^*(t)=\min_{x\in\overline\Omega}\nu(x,t)\\ &\hat\sigma=e^{\hat m}, && \hat\nu(t)=\max_{x\in\overline\Omega}\nu(x,t) &&\text{and}\quad \hat m(t)=\max_{x\in\overline\Omega}\nu(x,t),\end{aligned} \tag{4.1}$$

where γ is given by

$$\gamma(t)=\begin{cases}t(T-t), & 0\leq t\leq T/2,\\ T^2/4, & T/2<t\leq T.\end{cases} \tag{4.2}$$

Combining Carleman estimate (3.3) with classical energy estimates for the fourth order Schrödinger system, satisfied by φ and ψ, we can prove the following result.

Proposition 4.1. *With the same hypothesis of Proposition 3.2 the solution (φ,ψ) of (3.4) satisfies the following*

$$\begin{aligned}&\|\varphi\|_{L^2(T/2,T;L^2(\Omega))}+\|\varphi_x\|_{L^2(T/2,T;L^2(\Omega))}+\|\varphi_{xx}\|_{L^2(T/2,T;L^2(\Omega))}\\ &\qquad\leq\|(\varphi,\psi)\|_{(L^2(T/4,T/2;L^2(\Omega)))^2}+\|(g^0,g^1)\|_{(L^2(T/2,T;L^2(\Omega)))^2}\end{aligned}$$

and

$$\begin{aligned}||\psi||_{L^2(T/2,T;L^2(\Omega))}+||\psi_x||_{L^2(T/2,T;L^2(\Omega))}+||\psi_{xx}||_{L^2(T/2,T;L^2(\Omega))}\\ \leq ||\psi||_{L^2(T/4,T/2;L^2(\Omega))}+||g^1||_{L^2(T/2,T;L^2(\Omega))}.\end{aligned}$$

Proof. Let us consider $\kappa\in C^1([0,T])$ such that

$$\kappa=\begin{cases}0, & \text{if } t\in[0,T/4],\\ 1, & \text{if } t\in[T/2,T].\end{cases}$$

Note that if (φ,ψ) is a solution for (3.4) , so $(\kappa\varphi,\kappa\psi)$ satisfies the following system

$$\begin{cases} i(\kappa\varphi)_t+(\kappa\varphi)_{xx}-(\kappa\varphi)_{xxxx}=1_{\mathcal{O}}(\kappa\psi)+\kappa g^0+i\kappa_t\varphi & \text{in} \quad Q_T,\\ i(\kappa\psi)_t+(\kappa\psi)_{xx}-(\kappa\psi)_{xxxx}=\kappa g^1+i\kappa_t\psi & \text{in} \quad Q_T,\\ (\kappa\varphi)(t,0)=(\kappa\varphi)(t,L)=(\kappa\varphi)_x(t,0)=(\kappa\varphi)_x(t,L)=0 & \text{on} \quad t\in(0,T),\\ \psi(t,0)=(\kappa\psi)(t,L)=(\kappa\psi)_x(t,0)=(\kappa\psi)_x(t,L)=0, & \text{on} \quad t\in(0,T),\\ (\kappa\varphi)(T,x)=0,(\kappa\psi)(T,x)=0, & \text{in} \quad \Omega.\end{cases} \tag{4.3}$$

Now, since $\kappa,\kappa_t\in C([0,T])$ and $C([0,T])\hookrightarrow L^\infty(0,T)$, moreover, $\kappa\psi,\kappa_t\psi\in L^2(0,T;H_0^2(\Omega))$. Then, for $g^1\in L^2(0,T;H_0^2(\Omega))$ we get that $\kappa\psi$ satisfy a fourth order Schrödinger system equation with null data and right-hand side in $L^2(0,T;H_0^2(\Omega))$. Therefore, we get that

$$\begin{aligned}\int_\Omega|\kappa(t)\psi(t)|dx+\iint_{Q_T}|\kappa\psi_x|dxdt+\iint_{Q_T}|\kappa\psi_{xx}|dxdt\\ \leq C\left(\iint_{Q_T}|\kappa g^1|dxdt+\iint_{Q_T}|\kappa_t\psi|dxdt\right).\end{aligned} \tag{4.4}$$

Multiplying the first equation of (4.3) by $\kappa\varphi$ and integrating over Ω we obtain, after taking the real part and using Young inequality for the integral term of $1_{\mathcal{O}}\kappa\psi\kappa\varphi$, that

$$\begin{aligned}-\frac{1}{2}\frac{d}{dt}\int_\Omega|\kappa(t)\varphi(t)|^2+\int_\Omega|\kappa\varphi_x|^2dx+\int_\Omega|\kappa\varphi_{xx}|^2dx\\ \leq C\left(\int_\Omega|\kappa g^0|^2dx+\int_\Omega|\kappa_t\varphi|^2dx+\int_\Omega|\kappa\psi|^2dx\right)+\delta\int_{\mathcal{O}}|\kappa\varphi|^2dxdt.\end{aligned} \tag{4.5}$$

Finally, integrating (4.5) in $[t,T]$, combining with (4.4) and taking δ small enough, we get

$$\begin{aligned}\int_\Omega|\kappa(t)\varphi(t)|^2+\iint_{Q_T}|\kappa\varphi_x|^2dx+\iint_Q|\kappa\varphi_{xx}|^2dx\\ \leq C\left(\iint_{Q_T}|\kappa|^2(g^0|^2+|g^1|)^2dxdt+\iint_{Q_T}|\kappa_t|^2(\varphi|^2+|\psi|^2)dx\right).\end{aligned} \tag{4.6}$$

Therefore, Proposition 4.1 is consequence of equations (4.4) and (4.6). □

As a consequence of the previous result, and due the definition of (4.1), the following Carleman estimate, with new weight functions σ and ν, can be obtained.

Proposition 4.2. *There exists a constant $C(s,\lambda):=C>0$, such that every solution (φ,ψ) of (3.4) satisfies*

$$\begin{aligned}&\iint_{Q_T}\left(\lambda^7\mu^8\nu^7\sigma^2|\varphi|^2+\lambda^5\mu^6\nu^5\sigma^2|\varphi_x|^2+\lambda^3\mu^4\nu^3\sigma^2|\varphi_{xx}|^2\right)dxdt\\ &\quad+\iint_{Q_T}\left(\lambda^7\mu^8\nu^7\sigma^2|\psi|^2+\lambda^5\mu^6\nu^5\sigma^2|\psi_x|^2+\lambda^3\mu^4\nu^3\sigma^2|\psi_{xx}|^2\right)dxdt\\ &\leq C\left(\iint_{Q_T}\sigma^2(|g^0|^2+|g^1|^2)dxdt+\lambda\iint_{\tilde{\omega}_T}\nu\sigma^2|\varphi|^2dxdt\right).\end{aligned} \tag{4.7}$$

Proof. The result is consequence of Proposition 4.1. Indeed, noting that $\xi = \nu$ and $l = m$, for $t \in [0, T/2]$, and since l is constant in $[T/2, T]$, yields that

$$\begin{aligned}
&\int_0^{T/2}\int_\Omega \left(\lambda^7\mu^8\nu^7\sigma^2|\varphi|^2 + \lambda^5\mu^6\nu^5\sigma^2|\varphi_x|^2 + \lambda^3\mu^4\nu^3\sigma^2|\varphi_{xx}|^2\right) dxdt \\
&\qquad + \int_0^{T/2}\int_\Omega \left(\lambda^7\mu^8\nu^7\sigma^2|\psi|^2 + \lambda^5\mu^6\nu^5\sigma^2|\psi_x|^2 + \lambda^3\mu^4\nu^3\sigma^2|\psi_{xx}|^2\right) dxdt \\
&\qquad = \int_0^{T/2}\int_\Omega \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2 + \lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2\right) dxdt \\
&\qquad + \int_0^{T/2}\int_\Omega \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2\right) dxdt.
\end{aligned} \tag{4.8}$$

Additionally, for $t \in [T/2, T]$, we have that

$$\begin{aligned}
&\int_{T/2}^{T}\int_\Omega \left(\lambda^7\mu^8\xi^7\theta^2|\varphi|^2 + \lambda^5\mu^6\xi^5\theta^2|\varphi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\varphi_{xx}|^2\right) dxdt \\
&\qquad + \int_{T/2}^{T}\int_\Omega \left(\lambda^7\mu^8\xi^7\theta^2|\psi|^2 + \lambda^5\mu^6\xi^5\theta^2|\psi_x|^2 + \lambda^3\mu^4\xi^3\theta^2|\psi_{xx}|^2\right) dxdt \\
&\qquad \le C\left(\int_{T/2}^{T}\int_\Omega (|\varphi|^2 + |\varphi_x|^2 + |\varphi_{xx}|^2 + |\psi|^2 + |\psi_x|^2 + |\psi_{xx}|^2) dxdt\right) \\
&\qquad \le C\left(\int_{T/4}^{T/2}\int_\Omega (|\varphi|^2 + |\psi|^2) dxdt + \int_{T/2}^{T}\int_\Omega (|g^0|^2 + |g^1|^2) dxdt\right) \\
&\qquad \le C\left(\int_{T/4}^{T/2}\int_\Omega \xi^7\theta^2(|\varphi|^2 + |\psi|^2) dxdt + \int_{T/2}^{T}\int_\Omega \sigma^2(|g^0|^2 + |g^1|^2) dxdt\right),
\end{aligned} \tag{4.9}$$

thanks to Proposition 4.1.

Finally, we note that

$$\begin{aligned}
&\iint_{Q_T} \theta^2(|g^0|^2 + |g^1|^2) dxdt + \lambda \iint_{\tilde\omega_T} \xi\theta^2|\varphi|^2 dxdt \\
&\qquad \le C\left(\iint_{Q_T} \sigma^2(|g^0|^2 + |g^1|^2) dxdt + \lambda \iint_{\tilde\omega_T} \nu\sigma^2|\varphi|^2 dxdt\right).
\end{aligned} \tag{4.10}$$

Thus, the result follows from (3.3), (4.8), (4.9) and (4.10). □

Remark 4.1. *We point out that Proposition 4.1 holds true by taking the minimum of the weights in the left-hand side and maximum of the weights in the right-hand side of* (4.4).

4.1. **Null controllability: Linear case.** In what follows we use (4.7), from Proposition 4.2, to deduce the desired null controllability property. Denote $\mathcal{L} = \mathcal{L}^* = i\partial_t + \partial_{xx} - \partial_{xxxx}$ and introduce the following space

$$\begin{aligned}
\mathcal{C} = \Big\{&(u, v, h); (\hat\sigma)^{-1}u \in L^2(Q_T), (\hat\sigma)^{-1}v \in L^2(0, T; H^{-2}(\Omega)), (\hat\nu)^{-\frac12}(\hat\sigma)^{-1}h \in L^2(Q_T), \\
&(\nu^*)^{-\frac72}(\sigma^*)^{-1}(\mathcal{L}u - 1_\omega h) \in L^2(Q_T), (\nu^*)^{-\frac72}(\sigma^*)^{-1}(\mathcal{L}^*v - 1_{\mathcal{O}}u) \in L^2(0, T; H^{-2}(\Omega)), \\
&(\hat\nu)^{-2}\hat\sigma^{-1}u \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H_0^2(\Omega)), \\
&(\hat\nu)^{-2}\hat\sigma^{-1}v \in L^2(0, T; H_0^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\
&v|_{t=T} = 0 \text{ in } \Omega\Big\}.
\end{aligned}$$

Remark 4.2. *It is important to observe that $\mathcal{C}$ is a Banach space endowed with its natural norm. Additionally, as consequence from the definition of the set $\mathcal{C}$, an element $(u, v, h) \in \mathcal{C}$ is such that $v|_{t=0} = 0$ in Ω. This holds since $(\hat\nu)^{-2}\hat\sigma^{-1}v$ belongs to $L^\infty(0, T; L^2(\Omega))$ and $(\hat\nu)^{-2}\hat\sigma^{-1}$ blow-up only at $t = 0$.*

We are now in position to prove the null controllability property for solutions of (1.8). The result can be read as follows.

Theorem 4.3. *Assume the same hypothesis of Proposition 4.2. Additionally, consider*

$$(\nu^*)^{-\frac{7}{2}}(\hat{\sigma})^{-1}f^0 \in L^2(Q_T) \quad \text{and} \quad (\nu^*)^{-\frac{7}{2}}(\hat{\sigma})^{-1}f^1 \in L^2(0,T;H^{-2}(\Omega)).$$

Therefore, we can find a control $h(x,t)=h$ such that the associated solution (u,v) of

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} = f^0 + 1_\omega h, & \text{in} \quad Q_T, \\ iv_t + v_{xx} - v_{xxxx} = f^1 + 1_{\mathcal{O}} u, & \text{in} \quad Q_T, \\ u(t,0) = u(t,L) = v(t,0) = v(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u_x(t,0) = u_x(t,L) = v_x(t,0) = v_x(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u(0,x) = u_0(x), v(T,x) = 0, & \text{in} \quad \Omega, \end{cases} \tag{4.11}$$

satisfies $(u,v,h) \in \mathcal{C}$. In particular, $v|_{t=0} \equiv 0$ in Ω.

Proof. We introduce the following spaces

$$R_0 = \{u \in H_0^2(\Omega);\ u_{xxxx} \in H_0^2(\Omega)\},$$

$$Y_0 = C([0,T];H^4(\Omega)) \cap C^1([0,T];H_0^2(\Omega))$$

and

$$Y_1 = C([0,T];H_0^2(\Omega)) \cap C^1([0,T];H^{-2}(\Omega)).$$

Also, let us consider

$$P_0 = \{(\varphi,\psi)) \in Y_1 \times Y_0 : \mathcal{L}^*v - 1_{\mathcal{O}}u \in L^2(Q_T)\}.$$

Thanks to Theorem (2.7), P_0 is nonempty. Moreover, from now on we will use $\mathcal{L}$ instead of $\mathcal{L}^*$, since both are equal.

Now, define the bilinear form $a : P_0 \times P_0 \to \mathbb{R}$ by

$$\begin{aligned} a((\tilde{\varphi},\tilde{\psi}),(\varphi,\psi)) := Re &\left(\iint_{Q_T} (\hat{\sigma})^2 (\mathcal{L}\hat{\varphi} - 1_{\mathcal{O}}\hat{\psi})(\overline{\mathcal{L}\varphi - 1_{\mathcal{O}}\psi}) dxdt + \iint_{Q_T} (\hat{\sigma})^2 (\mathcal{L}\hat{\psi})(\overline{\mathcal{L}\psi}) dxdt \right) \\ &+ Re\left(\iint_{\tilde{\omega}_T} \hat{\nu}(\hat{\sigma})^2 \hat{\varphi}\overline{\varphi} dxdt \right), \end{aligned}$$

and the linear form $G : P_0 \to \mathbb{R}$ given by

$$\langle G,(\varphi,\psi)\rangle := Re \iint_{Q_T} f^0 \overline{\varphi} dxdt + \int_0^T \left\langle f^1, \overline{\psi} \right\rangle dt,$$

where $\langle \cdot,\cdot \rangle$ denotes the duality between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$. Thanks to Proposition 4.2, the bilinear form over $P_0 \times P_0$, is a sesquilinear, positive and coercive. Let P be the completion of P_0 with the norm induced by $a(\cdot,\cdot)$, in this case, $a(\cdot,\cdot)$ is well-defined, continuous and coercive on $P \times P$. Now, note that for all $(\varphi,\psi) \in P_0$ we have

$$\begin{aligned} \langle G,(\varphi,\psi)\rangle = \ & Re \iint_{Q_T} f^0 \overline{\varphi} dxdt + \int_0^T \left\langle f^1, \overline{\psi} \right\rangle dt \\ \leq & \left(\iint_{Q_T} (\nu^*)^7 (\sigma^*)^2 (|\varphi|^2 + |\psi|^2) dxdt \right)^{1/2} \\ & \times \left(\iint_{Q_T} (\nu^*)^{-7} (\sigma^*)^{-2} |f^0|^2 dxdt + \int_0^T (\nu^*)^{-7} (\sigma^*)^{-2} ||f^1||_{H^{-2}(\Omega)}^2 dt \right)^{1/2} \\ \leq & \ C a((\varphi,\psi),(\varphi,\psi))^{1/2} \left(\iint_{Q_T} (\nu^*)^{-7} (\sigma^*)^{-2} |f^0|^2 dxdt + \int_0^T (\nu^*)^{-7} (\sigma^*)^{-2} ||f^1||_{H^{-2}(\Omega)}^2 dt \right)^{1/2}, \end{aligned}$$

where we use Young inequality on the first inequality. Therefore, G is a bounded functional on P_0 and we can extend it continuously to a bounded functional on P due to the Hahn-Banach theorem. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$\left\{\begin{array}{l}\text{Find } (\hat{\varphi},\hat{\psi}) \in P \text{ such that, } \forall(\varphi,\psi)\in P,\\ a((\hat{\varphi},\hat{\psi}),(\varphi,\psi)) = \langle G,(\varphi,\psi)\rangle\end{array}\right. \tag{4.12}$$

has a unique solution $(\hat{\varphi},\hat{\psi}) \in P\times P$.

Let us define $(\hat{u},\hat{v},\hat{h})$ by

$$\left\{\begin{array}{ll}\hat{u} = (\hat{\sigma})^2(\mathcal{L}\hat{\varphi} - 1_{\mathcal{O}}\hat{\psi}), & \text{in } Q_T,\\ \hat{v} = (\hat{\sigma})^2\mathcal{L}\hat{\psi}, & \text{in } Q_T,\\ \hat{h} = -\hat{\nu}(\hat{\sigma})^2\hat{\varphi}, & \text{in } Q_T,\end{array}\right. \tag{4.13}$$

remembering that $\mathcal{L}^\star = \mathcal{L}$. Thanks to (4.12) and (4.13), we have that

$$\iint_{Q_T} (\hat{\sigma})^{-2}\left(|\hat{u}|^2 + |\hat{v}|^2 + (\hat{\nu})^{-1}|\hat{h}|^2\right)dxdt = a\left((\hat{\varphi},\hat{\psi}),(\hat{\varphi},\hat{\psi})\right) < \infty.$$

Considering $(\tilde{u},\tilde{v})$ be a weak solution of

$$\left\{\begin{array}{lll}i\tilde{u}_t + \tilde{u}_{xx} - \tilde{u}_{xxxx} = f^0 + 1_\omega\tilde{h} & \text{in} & Q_T,\\ i\tilde{v}_t + \tilde{v}_{xx} - \tilde{v}_{xxxx} = f^1 + 1_{\mathcal{O}}\tilde{u} & \text{in} & Q_T,\\ \tilde{u}(t,0) = \tilde{u}(t,L) = \tilde{v}(t,0) = \tilde{v}(t,L) = 0, & \text{on} & t\in(0,T),\\ \tilde{u}_x(t,0) = \tilde{u}_x(t,L) = \tilde{v}_x(t,0) = \tilde{v}_x(t,L) = 0, & \text{on} & t\in(0,T),\\ \tilde{u}(0,x) = 0, \tilde{v}(T,x) = 0, & \text{in} & \Omega,\end{array}\right. \tag{4.14}$$

with control $h = \hat{h}$ and source terms f^0 and f^1, since $\tilde{h} \in L^2(Q_T)$, we have, from well-posed result given by Theorem 2.7, that $(\tilde{u},\tilde{v})$ are well defined. In the following we prove that the weak solution $(\hat{u},\hat{u})$ is a solution by transposition. In fact, for every $(\varphi,\psi)\in P_0$, it holds from (4.12) and (4.13) that

$$\begin{aligned}&Re\iint_{Q_T} f^0\overline{\varphi}dxdt + \int_0^T \langle f^1,\overline{\psi}\rangle_{H^{-2}\times H_0^2}dt + Re\iint_{\tilde{\omega}} \hat{h}\overline{\varphi}dxdt\\ &= Re\iint_{Q_T}\hat{u}(\overline{\mathcal{L}\varphi - 1_{\mathcal{O}}\psi})dxdt + \int_0^T\langle\hat{v},\overline{\mathcal{L}\psi}\rangle_{H^{-2}\times H_0^2}dt.\end{aligned} \tag{4.15}$$

From (4.15), we get that

$$\begin{aligned}Re\iint_{Q_T}\hat{u}\overline{g^0}dxdt + \int_0^T\left\langle\hat{v},\overline{g^1}\right\rangle_{H^{-2}\times H_0^2}dt &= Re\iint_{\tilde{\omega}}\hat{h}\overline{\varphi}dxdt\\ &+ Re\iint_{Q_T} f^0\overline{\varphi}dxdt + \int_0^T\langle f^1,\overline{\psi}\rangle_{H^{-2}\times H_0^2}dt,\end{aligned}$$

for all $(g^0,g^1)\in L^2(0,T;H_0^1(\Omega))$, that is, $(\hat{u},\hat{v}) = (\tilde{u},\tilde{v})$.

Now on, we prove that solutions $\hat{u}$ and $\hat{v}$ of (4.14) are, in fact, more regular. Let us start defining the functions

$$u_* := (\hat{\nu})^{-2}(\hat{\sigma})^{-1}\hat{u}, \qquad v_* := (\hat{\nu})^{-2}(\hat{\sigma})^{-1}\hat{v},$$

$$f_*^0 := (\hat{\nu})^{-2}(\hat{\sigma})^{-1}(f^0 + \hat{h}1_\omega) \quad \text{and} \quad f_*^1 := (\hat{\nu})^{-2}(\hat{\sigma})^{-1}f^1.$$

It follows, from (4.11), that u_*, v_*, f_*^1 and f_*^2 satisfies the following system

$$\left\{\begin{array}{ll}i(u_*)_t + (u_*)_{xx} - (u_*)_{xxxx} = f_*^0 + i\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}\right)_t\hat{u}, & \text{in } Q_T,\\ i(v_*)_t + (v_*)_{xx} - (v_*)_{xxxx} = f_*^1 + 1_{\mathcal{O}}u_* + i\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}\right)_t\hat{v}, & \text{in } Q_T,\\ u_* = v_* = 0, & \text{in } \Sigma,\\ (u_*)_{t=0} = 0, (v_*)|_{t=T} = 0, & \text{in } \Omega.\end{array}\right. \tag{4.16}$$

Now, since $\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}\right)_t \leq CT^2 s(\hat{\sigma})^{-1}$ we get that $f_*^0 + i\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}\right)_t \hat{u} \in L^2(Q)$ and also $f_*^1 + i\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}\right)_t \hat{v} \in L^2(0,T;H^{-2}(\Omega))$. Now, using the results of Section 2, for (4.16), we obtain

$$u_* \in L^2(0,T;H^4(\Omega)) \cap L^\infty(0,T;H_0^2(\Omega))$$

and

$$v_* \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).$$

This finishes the proof of Theorem 4.3. □

4.2. **Null controllability: Nonlinear case.** In this section we use an inverse mapping theorem to obtain the existence of insensitizing controls for the fourth order nonlinear Schrödinger equation (1.7). We invite the reader to see the result below as well as additional comments on [4].

Theorem 4.4 (Inverse Mapping Theorem)**.** *Let B_1 and B_2 be two Banach spaces and let*

$$\mathcal{Y} : B_1 \to B_2$$

satisfying $\mathcal{Y} \in C^1(B_1, B_2)$. Assume that $b_1 \in B_1$, $\mathcal{Y}(b_1) = b_2$ and

$$\mathcal{Y}'(b_1) : B_1 \to B_2$$

is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in B_2$ satisfying

$$||b' - b_2||_{B_2} < \delta,$$

there exists a solution of the equation

$$\mathcal{Y}(b) = b', \quad b \in B_1.$$

Finally, we will give the proof of the main result of this manuscript.

Proof of Theorem 1.2. Consider, in Theorem 4.4, the following

$$B_1 = \mathcal{C} \quad \text{and} \quad B_2 = L^2((\hat{\nu})^{-6}(\hat{\sigma})^{-3}(0,T); L^2(\Omega)) \times L^2((\hat{\nu})^{-6}(\hat{\sigma})^{-3}(0,T); H^{-2}(\Omega)).$$

Define the operator

$$\mathcal{Y} : B_1 \to B_2$$

such that

$$\mathcal{Y}(u,v,h) := (\mathcal{L}u - \zeta|u|^2 u - 1_\omega h\,,\, \mathcal{L}v - \overline{\zeta}\overline{u}^2\overline{v} - \overline{\zeta}|u|^2 v - 1_{\mathcal{O}} u).$$

Claim 1. *Operator $\mathcal{Y}$ belongs to $C^1(B_1, B_2)$.*

Indeed, first note that all terms of $\mathcal{Y}$ are linear except: $|u|^2u$, $\overline{u}^2\overline{v}$ and $|u|^2v$. So, the Claim 1 is equivalent to prove that the trilinear operator given by

$$((u_1,v_1,h_1),(u_1,v_1,h_1),(u_1,v_1,h_1)) \mapsto u_1u_2u_3$$

and

$$((u_1,v_1,h_1),(u_1,v_1,h_1),(u_1,v_1,h_1)) \mapsto u_1u_2v_3,$$

are continuous maps from $\mathcal{C}^3$ to $L^2((\hat{\nu})^{-6}(\hat{\sigma})^{-1}(0,T); L^2(\Omega))$. However, $(u_i, v_i, h_i) \in \mathcal{C}$, thus we get that

$$(\hat{\nu})^{-2}(\hat{\sigma})^{-1}u \in L^2(0,T;H^4((\Omega)) \cap L^\infty(0,T;H_0^2(\Omega)) \hookrightarrow L^6(Q_T)$$

and

$$(\hat{\nu})^{-2}(\hat{\sigma})^{-1}u \in L^2(0,T;H^2((\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \hookrightarrow L^6(Q_T),$$

since we are working in an unidimensional case. At this point, we have fixed λ and μ such that

$$B_2 \subset L^2((\nu^*)^{-\frac{7}{2}}(\sigma^*)^{-1}(0,T); L^2(\Omega)) \times L^2((\nu^*)^{-\frac{7}{2}}(\sigma^*)^{-1}(0,T); H^{-2}(\Omega))^{1}$$

[1]Note that if necessary we could have taken λ and μ large enough in such a way that this inclusion is still satisfied.

holds. Therefore, by using Hölder inequality, we get that

$$
\begin{aligned}
\left|\left|(\hat{\nu})^{-6}(\hat{\sigma})^{-3}u_1u_2u_3\right|\right|_{L^2(Q_T)} &= \left|\left|\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}u_1\right)\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}u_2\right)\left((\hat{\nu})^{-2}(\hat{\sigma})^{-1}u_3\right)\right|\right|_{L^2(Q_T)} \\
&\leq C\prod_{k=1}^{3}\left|\left|(\hat{\nu})^{-2}(\hat{\sigma})^{-1}u_i\right|\right|_{L^6(Q_T)} \leq C\prod_{k=1}^{3}||(u_i,v_i,z_i)||_{\mathcal{C}}
\end{aligned} \tag{4.17}
$$

and analogously, we have

$$
\left|\left|(\hat{\nu})^{-6}(\hat{\sigma})^{-3}u_1u_2v_3\right|\right|_{L^2(Q_T)} \leq C\prod_{k=1}^{3}||(u_i,v_i,z_i)||_{\mathcal{C}}\,,
$$

showing the Claim 1.

Claim 2. $\mathcal{Y}'(0,0,0)$ *is surjective.*

First, note that $\mathcal{Y}(0,0,0)=(0,0)$. By the other hand, observe that $\mathcal{Y}'(0,0,0):B_1\to B_2$ is given by

$$
\mathcal{Y}'(0,0,0)(u,v,h)=(iu_t+u_{xx}-u_{xxxx}-1_\omega h, iv_t+v_{xx}-v_{xxxx}-1_{\mathcal{O}}u),
$$

for $(u,v,h)\in B_1$. Invoking the null controllability result for linear system (1.8), that is, thanks to Theorem 4.3, $\mathcal{Y}'(0,0,0)$ is surjective, proving the Claim 2.

Finally, by taking $b_1=(0,0,0)$, $b_2=(0,0)$ and using Theorem 4.4, there exists $\delta>0$ such that if $||(f^0,f^1)||_{B_2}<\delta$, then we can find a control h such that the triple $(u,v,h)\in B_1$ satisfies $\mathcal{Y}(u,v,h)=(f^0,f^1)$. By a particular choice of $f^0=f\in L^2((\hat{\nu})^{-6}(\hat{\sigma})^{-1}(0,T);L^2(\Omega))$ and $f^1\equiv 0$, Theorem 1.2 is showed since a triple $(u,v,h)\in B_1$ satisfies $v(0)=0$ in Ω and solves (1.7). □

5. Further Comments and open issues

In this work we proved the existence of solution and existence of insensitizing controls for the fourth-order nonlinear Schrödinger system (4NLS) (1.2). As it is well known, this kind of problem is equivalent to prove a partial null controllability result for a cascade system associated with (1.2) which is given by (1.7) and that is what we have done.

In the first part of the work we presented the existence of mild and regular solutions associated to (1.7). In the second part, we started proving the desired controllability for the linearized system around zero of (1.7), namely (1.8). In this part, we used new global Carleman estimates combined with some energy estimates and duality arguments, results given by Theorem 3.2, Proposition 4.1 and Theorem 4.3, respectively. Finally, for the nonlinear case we used inverse mapping arguments.

Our work, in our knowledge, is the first one in the literature to treat insensitizing controls for 4NLS. We believe that there are prospects to find various interesting open problems. In this spirit, we will give some of that, and also, present some further comments on our work.

5.1. **Null condition of the initial data.** In this point, we discuss the necessity to assume the null condition of the initial data in Theorem 1.2. In [16], the author proves that under some suitable conditions the existence of insensitizing controls may or may not hold, which indicates that this kind of problem cannot be solved for every initial data. In this way, we also have the same drawback in our result. To overcome this difficulty, we believe that the techniques used for Heat equation, due De Tereza [16], can be adapted for our case. Precisely, the idea consists in using the fundamental solution to construct an explicit solution where the observability inequality does not hold.

5.2. **About the nonlinear terms.** Note that if we change the cubic term $|u|^2u$ by a more general term $|u|^{p-2}u$, with $p\geq 3$, then one must prove a partial null controllability for the following system

$$
\begin{cases}
iu_t+u_{xx}-u_{xxxx}-\zeta|u|^{p-2}u=f+1_\omega h & \text{in} \quad Q,\\
iv_t+v_{xx}-v_{xxxx}-\overline{\zeta}p|u|^p\overline{u}^2\overline{v}-(p+1)\overline{\zeta}|u|^{p-2}\overline{v}=1_{\mathcal{O}}u & \text{in} \quad Q,\\
u(t,0)=u(t,L)=v(t,0)=v(t,L)=0, & \text{on} \quad t\in(0,T),\\
u_x(t,0)=u_x(t,L)=v_x(t,0)=v_x(t,L)=0, & \text{on} \quad t\in(0,T),\\
u(0,x)=u_0(x), v(T,x)=0, & \text{in} \quad \Omega.
\end{cases}
$$

If the structure of the problem is still the same and we only change the nonlinearity, the main difficult here is to obtain well-posedness results which gives enough regularity for the solutions to obtain the analogous Hölder estimate as in (4.17). In fact, to solve it one must have valid embedding from the state spaces into $L^{2p-2}(Q)$, for $p \geq 3$, which is possible since the following estimate holds

$$\left|\left||u|^{p-2}u\right|\right|_{H^s(\Omega)} \leq C||u||^{p-1}_{H^s(\Omega)},$$

when $s \geq \frac{1}{2}$ and $p \geq 3$, see [31] for well-posedness of the general nonlinear problem.

Additionally, if we change to a general type nonlinearity g, we obtain the following optimal system

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} + g(u) = f + 1_\omega h, & \text{in} \quad Q, \\ iv_t + v_{xx} - v_{xxxx} + g'(u)v = 1_{\mathcal{O}}u, & \text{in} \quad Q, \\ u(t,0) = u(t,L) = v(t,0) = v(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u_x(t,0) = u_x(t,L) = v_x(t,0) = v_x(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u(0,x) = u_0(x), v(T,x) = 0, & \text{in} \quad \Omega. \end{cases}$$

It is expected that most of these problems have no solution, i.e., it is not possible to insensitize the functional unless we impose some conditions on g. To exemplify the comments above, some of these issues were already considered for the case of nonlinearities with superlinear growth at infinity. In [8], the authors dealing with a semilinear heat equation proved positive result of existence of insensitizing controls considering $g \in C^1$ a nonlinear function verifying $g'' \in L^\infty_{loc}(\mathbb{R})$, $g(0) = 0$ and

$$\lim_{|s|\to\infty} \frac{g'(s)}{\ln(1+|s|)} = 0,$$

furthermore, the result is also valid for nonlinearities g of the form

$$|g(s)| = |p_1(s)| \ln^\alpha(1 + |p_2(s)|),$$

for all $|s| \geq s_0 > 0$, with $\alpha \in [0,1)$ and p_i, $i = 1,2$, are affine functions. Moreover, they proved negative results of existence considering a nonlinearity g verifying the conditions above, that is, taking g as

$$g(s) = \int_0^{|s|} \ln^\alpha(1 + |\sigma|^2)d\sigma, \quad \text{for all } s \in \mathbb{R},$$

but choosing $\alpha > 2$. Similar results are proved in [42] for a class of nonlinear Ginzburg-Landau equation.

Thus, in the case of fourth-order nonlinear Schrödinger equation, this kind of situation, that is, introduce a function g with certain properties and to prove existence of insensitizing controls is still an open issue.

5.3. **About the sentinel functional.** One way to solve the problems of nonlinearity is to change the structure of the functional. Due to the lack of regularity of the characteristic function, if we change it to a more regular function then one can still prove result for more general nonlinearity $|u|^{p-2}u$, with $p \geq 3$, considering a functional of the form

$$J(\tau, h) = \frac{1}{2}\iint_{Q_T} \mathcal{R}(x)|u(x,t)|^2 dxdt,$$

where $\mathcal{R} \in C^\infty(\Omega)$ is a smooth function with $Supp(\mathcal{R}) \subset \mathcal{O}$.

We note that there exists an uncountable insensitizing controls problem as we change the sentinel functional. In fact, by the equivalent formulation in a cascade system with double the equations of the original system, controllability problems with less control forces then equations are not fully understood in PDEs, so they can also be interesting from the Control Theory point of view. Some of the motivations for these problems arise from physical phenomena, thus typically we focus our attention in functionals which have "physical" meanings: If the functional is the local L^2-norm of the the solution then we are looking for controls that locally preserve the energy (kinetic or potential, depending on the modeling) of the system, and if we change to a first derivative (or gradient in the $N-$dimensional case) the problem consists in finding controls that locally preserves the mean value of the energy.

In this perspective, let D be a derivative operator such as $Du = u_x$ or $Du = u_{xx}$. An interesting – and difficult – problem is to analyze the existence of insensitizing controls when the sentinel functional take the form

$$J(\tau, h) := \frac{1}{2}\int_{Q_T} |Du(x,t)|^2 dxdt.$$

In such, the optimal system become

$$\begin{cases} iu_t + u_{xx} - u_{xxxx} - \zeta|u|^2u = f + 1_\omega h & \text{in} \quad Q_T, \\ iv_t + v_{xx} - v_{xxxx} - \overline{\zeta}\overline{u}^2\overline{v} - 2\overline{\zeta}|u|^2v = D(1_{\mathcal{O}}Du) & \text{in} \quad Q_T, \\ u(t,0) = u(t,L) = v(t,0) = v(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u_x(t,0) = u_x(t,L) = v_x(t,0) = v_x(t,L) = 0, & \text{on} \quad t \in (0,T), \\ u(0,x) = u_0(x), v(T,x) = 0, & \text{in} \quad \Omega. \end{cases}$$

Again, it is not expected to obtain positive results of existence of insensitizing controls for every differential operator D in virtue of the lack of regularity provoked by coupling term $D(1_{\mathcal{O}}Du)$, since (again) the characteristic function is not regular. Despite that, Guerrero [25] dealt with a parabolic equation. The author proved a positive result of existence considering a functional depending on the gradient of the solution. Since the equation was linear, with constant coefficients, the argument consisted in considering a global Carleman estimate with different exponents, not for the equation, but for the equation satisfied by the Laplacian of the solutions to then recover information using the equation with the coupling. This is not the case when dealing with a nonlinear problem since deriving the equation would give us many other terms. In [27], the same author proved a similar result considering a linear Stokes equation with constant coefficients but with for the *curl* of the solution. Finally, we cite the work of the second author [28], where the authors proved positive results of insensitizing controls considering a functional depending on the gradient of the solution for the cubic nonlinear Ginzburg-Landau equation. The result arose by proving a new suitable Carleman estimate for the Ginzburg-Landau equation.

In this spirit, there are many alternatives to define the sentinel functional related with the insensitizing controls problems for 4NLS. Thus, we expect that these three works together with the results on this paper, open prospect to prove similar results considering a sentinel functional with the gradient of the solution. Moreover, since the Carleman estimate (3.3) has third order terms, maybe it is possible, at some point, to adapt the arguments to consider a functional with the Laplacian of the solution of the 4NLS, but clearly, to prove it is necessary new arguments of those that were applied here, at least proving a new Carleman estimate for the fourth-order Schrödinger equation, as was done in [24, Theorem 1.1] for the Cahn-Hilliard type equation. The readers are invited to read the recent and interesting work by Imanuvilov and Yamamoto [26], which proves a Carleman estimate for a fourth-order parabolic equation in general dimensions.

5.4. $N-$**dimensional case.** When we consider the $N-$dimensional fourth-order nonlinear Schrödinger equation, Zheng and Zhou [43] studied the boundary controllability of the 4NLS in a bounded domain $\Omega \subset \mathbb{R}^n$. Using a L^2-Neumann boundary control, the authors proved that the solution of 4NLS is exactly controllable in $H^{-2}(\Omega)$ using the Hilbert Uniqueness Method and multiplier techniques.

In the sense of existence of insensitizing controls, we conjecture that the Carleman inequality shown here can be extended to the $N-$dimensional case. Thus, if we consider the sentinel functional as defined in (1.3), our result remains valid, for this case. However, the main issue here is when we consider functional like the one mentioned in the Subsection 5.3 or other types of functional associated with the nonlinear problem. This type of problem looks interesting and still is an open for the fourth-order nonlinear Schrödinger equation.

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