Les Houches lectures on laser cooling and trapping
16 – 26 September 2014

Hélène Perrin
helene.perrin[at]univ-paris13.fr

Lecture 2

Doppler cooling and magneto-optical trapping

We have seen in Lecture 1 that a laser beam propagating in a direction opposite to the velocity of the atoms can slow down an atomic beam in a Zeeman slower. In this lecture, the basics of using radiation pressure for reaching very low temperatures is presented.

Doppler cooling was suggested for neutral atoms in 1975 by Hänsch and Schawlow [1], and a similar idea was proposed independently by Wineland and Dehmelt for ions [2]. The idea is to use the Doppler shift $-\mathbf{k}_L.\mathbf{v}$ of the laser frequency due to the atomic velocity $\mathbf{v}$ to make the force velocity dependent. If the laser detuning δ is negative, the radiation pressure is larger for atoms with a velocity opposite to the laser direction, that is if $\mathbf{k}_L.\mathbf{v} < 0$. In this case, the force is opposite to the velocity and the atomic motion is damped.

1 Principle of Doppler cooling

Let use recall the expression of the radiation pressure for a plane wave with wave vector $\mathbf{k}_L$ and saturation parameter s_0:

$$\mathbf{F}_{\rm pr} = \frac{\Gamma}{2}\frac{s_0}{1+s_0}\hbar\mathbf{k}_L \qquad \text{where} \qquad s_0 = \frac{\Omega_1^2/2}{\delta'^2 + \dfrac{\Gamma^2}{4}}. \tag{1}$$

δ' is an effective detuning taking into account the possible frequency shifts (Doppler shift, Zeeman shift...).

1.1 Low velocity limit

If an atom has a non zero velocity $\mathbf{v}$, the detuning δ' entering in the expression of the force is Doppler shifted:

$$\delta' = \delta - \mathbf{k}_L.\mathbf{v} \qquad \text{where we recall that} \quad \delta = \omega_L - \omega_0.$$

The radiation pressure thus depends on the atomic velocity:

$$\mathbf{F}_{\rm pr}(\mathbf{v}) = \frac{\Gamma}{2}\frac{\Omega_1^2/2}{\Omega_1^2/2 + \Gamma^2/4 + (\delta - \mathbf{k}_L.\mathbf{v})^2}\hbar\mathbf{k}_L \tag{2}$$

In the low velocity limit, that is if $|2\delta\mathbf{k}_L.\mathbf{v}| \ll \Omega_1^2/2 + \Gamma^2/4 + \delta^2$, terms in v^2 or higher can be neglected and the expression of the force can be linearized. Note that the condition is true if $|\mathbf{k}_L.\mathbf{v}| \ll \mathrm{Max}\,(\Gamma/2, |\delta|)$.

$$\mathbf{F}_{\rm pr}(\mathbf{v}) \simeq \frac{\Gamma}{2}\frac{\Omega_1^2/2}{\Omega_1^2/2 + \Gamma^2/4 + \delta^2}\hbar\mathbf{k}_L + \frac{\Omega_1^2/2}{(\Omega_1^2/2 + \Gamma^2/4 + \delta)^2}\delta\Gamma\hbar(\mathbf{k}_L.\mathbf{v})\mathbf{k}_L.$$

The first term is the force for zero velocity. The second term is proportional to the component of the velocity in the direction of the laser. If we call $\mathbf{e}_z$ the direction of the wave vector, such that $\mathbf{k}_L = k_L \mathbf{e}_z$, then we simply have $(\mathbf{k}_L.\mathbf{v})\mathbf{k}_L = k_L^2 v_z \mathbf{e}_z$, v_z being the z component of $\mathbf{v}$. In the linear approximation, the radiation pressure is:

$$\mathbf{F}_{\rm pr}(\mathbf{v}) \simeq \mathbf{F}_{\rm pr}(\mathbf{v}=\mathbf{0}) - \frac{\alpha}{2} v_z \mathbf{e}_z.$$

The expression for α is

$$\alpha = -2 \frac{s_0}{(1+s_0)^2} \hbar k_L^2 \frac{\delta\Gamma}{\delta^2 + \Gamma^2/4}. \tag{3}$$

The last term in the expression of the force is a *friction force*, opposite to the direction of the velocity as long as $\alpha > 0$, that is for $\delta < 0$.

The friction coefficient α depends on two independent parameters, either (Ω_1, δ) or (s_0, δ). With this latter choice, α is maximum when $s_0 = 1$, which maximises $s_0/(1+s_0)^2$, and $\delta = -\Gamma/2$, which maximises the term $|\delta|\Gamma/(\delta^2 + \Gamma^2/4)$:

$$\alpha_{\rm max} = \frac{\hbar k_L^2}{2} = M\omega_{\rm rec}$$

where the recoil frequency $\omega_{\rm rec}$ is defined as $E_{\rm rec} = \hbar\omega_{\rm rec}$ (see Lecture 1). Again, the recoil appears as the typical unit for the external motion.

At low values of the saturation parameter $s_0 \ll 1$, the friction coefficient reads

$$\alpha = 2 s_0 \hbar k_L^2 \frac{|\delta|\Gamma}{\delta^2 + \Gamma^2/4} \tag{4}$$

with a maximum value $2s_0\hbar k_L^2 = 4 s_0 \alpha_{\rm max}$ for $\delta = -\Gamma/2$.

1.2 Standing wave configuration

The friction force is able to cool down the velocity in the direction of the laser. However, a single beam cannot cool the atoms as the main component of the force is given by the zero order term $\mathbf{F}_{\rm pr}(\mathbf{v} = \mathbf{0})$, which accelerates the atoms in the direction of the laser, regardless of the (small) atomic velocity.

To obtain a real friction force, a second laser is added in the opposite direction, with a wave vector $\mathbf{k}'_L = -\mathbf{k}_L$. Then, for low saturation parameters s, the two radiation pressure of the two lasers, $\mathbf{F}_+$ for $\mathbf{k}_L$ and $\mathbf{F}_-$ for $-\mathbf{k}_L$, add independently. The second force $\mathbf{F}_-$ can also be expanded at small velocities. The zero order term $\mathbf{F}_-(\mathbf{v} = \mathbf{0})$, proportional to $\mathbf{k}'_L = -\mathbf{k}_L$, is reversed with respect to the corresponding term $\mathbf{F}_+(\mathbf{v} = \mathbf{0})$. On the other hand, the first order term is proportional to $(\mathbf{k}_L.\mathbf{v})\mathbf{k}_L$ and is identical for $\mathbf{k}_L$ and $-\mathbf{k}_L$. The total force then reads:

$$\mathbf{F}(\mathbf{v}) = \mathbf{F}_+(\mathbf{v}=\mathbf{0}) - \frac{\alpha}{2} v_z \mathbf{e}_z - \mathbf{F}_+(\mathbf{v}=\mathbf{0}) - \frac{\alpha}{2} v_z \mathbf{e}_z = -\alpha v_z \mathbf{e}_z.$$

This is exactly the friction force needed for cooling. This configuration is called an *optical molasses* and can be generalised to three dimensions by using a pair of beams in each direction x, y, z of space. The total friction force in 3D is

$$\mathbf{F}(\mathbf{v}) = -\alpha v_x \mathbf{e}_x - \alpha v_y \mathbf{e}_y - \alpha v_z \mathbf{e}_z = -\alpha \mathbf{v}.$$

Recall that the friction coefficient is given by Eq.(4) where s_0 is the saturation parameter for one of the six beams.

Away from the low velocity region, the cooling force deviates from a linear dependence on the velocity. However, there is still a cooling force opposite to the velocity for any value of v, with an amplitude decreasing like $1/v^2$ at large v. Figure 1 gives the force as a function of the atomic velocity, for different choices of the detuning. The friction coefficient is the slope around zero velocity.

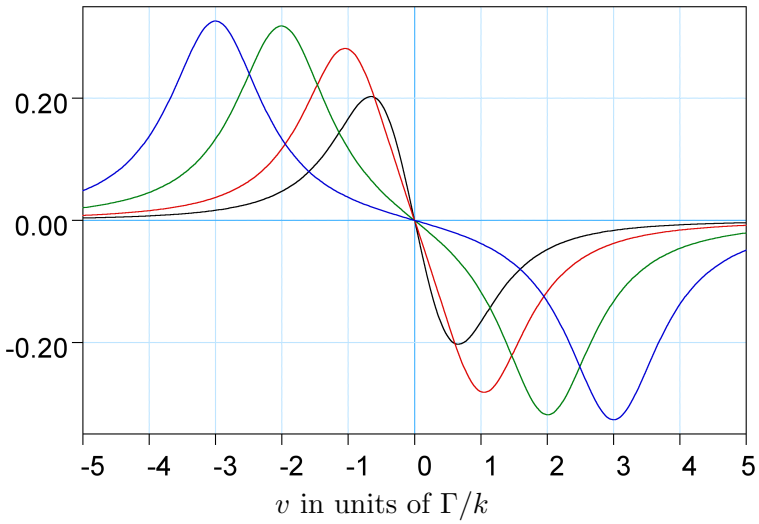

Figure 1: Doppler force in units of $\hbar k_L \Gamma s_0$, as a function of atomic velocity in units of Γ/k_L, for detuning $\delta = -\Gamma/2, -\Gamma, -2\Gamma, -3\Gamma$. Below $\Gamma/2k_L$, the force is almost linear, it is a friction force.

The first optical molasses was obtained experimentally in 1985 by Steven Chu *et al.* [3]. A sodium atomic beam was slowed down by a Zeeman slower and the slow sodium atoms were cooled to a temperature of about 240 μK in the molasses, see Fig. 2.

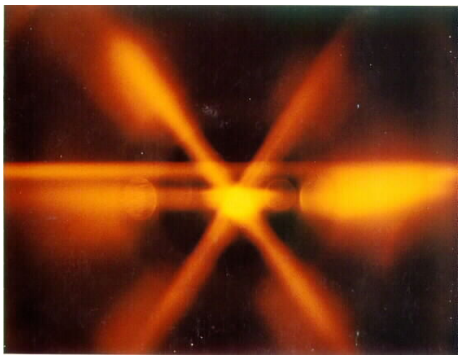

Figure 2: Cold sodium in an optical molasses at the intersection of six laser beams [3].

2 Limit temperature in Doppler cooling

The equation describing the evolution of the atomic velocity in time is Newton's second law, for a classical particle:

$$M\frac{d\mathbf{v}}{dt} = -\alpha\mathbf{v} \quad \Longrightarrow \quad \mathbf{v}(t) = \mathbf{v}_0 e^{-\gamma t}$$

where $\gamma = \alpha/M$. In principle, after a time long as compared to γ^{-1}, the velocity should vanish, and the final temperature should reach $T = 0$. However, this simple model neglects random fluctuations of the force, which give rise to a diffusion in momentum space, and thus to heating. The finite limit temperature is set by the competition between Doppler cooling and this heating process. This is linked to the fluctuation-dissipation theorem, and to Brownian motion. The aim of this section is to evaluate the diffusion coefficient in momentum space and to link it with the limit temperature [4, 5, 6].

2.1 Brownian motion

The theory of Brownian motion gives the link between the fluctuations of the force and the diffusion in momentum space. Let us consider the classical problem of a particle in the presence of both a friction force and a fluctuating force $\mathbf{F}$. The momentum $\mathbf{p}$ obeys the following equation:

$$\frac{d\mathbf{p}}{dt} = -\gamma\mathbf{p} + \mathbf{F}(t). \tag{5}$$

Here, $\gamma = \alpha/M$ and the average value of the fluctuating force over the realisations is zero: $\overline{\mathbf{F}(t)} = 0$. Taking the average of Eq.(5), we find that the average momentum $\bar{\mathbf{p}}$ decreases exponentially to zero:

$$\overline{\mathbf{p}}(t) = \mathbf{p}_0\, e^{-\gamma t}.$$

The average of $\mathbf{F}$ is zero, but the time correlation function of the force is not: $C(t,t') = \overline{\mathbf{F}(t)\cdot\mathbf{F}(t')} \neq 0$. The random force at time $t+\tau$ depends on the force at time t if τ is sufficiently short. If the process is stationary in time, $C(t,t+\tau)$ is a function of τ only, peaked at $\tau = 0$ with a width τ_c, the correlation time. This time gives the scale for the system to lose the memory of the previous value of the force. For light forces, it is given by the typical time for the evolution of the internal degrees of freedom, $t_{\rm int}$. As, again, $t_{\rm int} \ll t_{\rm ext}$ if the broadband condition is valid, the correlation function may be approximated by a delta function:

$$C(t,t') = \overline{\mathbf{F}(t)\cdot\mathbf{F}(t')} = 2D\delta(t-t'). \tag{6}$$

The normalisation coefficient D has an important signification, as we will see. It is related to $\mathbf{F}$ by

$$2D = \int d\tau \overline{\mathbf{F}(t)\cdot\mathbf{F}(t-\tau)}. \tag{7}$$

Let us solve formally the equation of motion. We can write:

$$\mathbf{p}(t) = \mathbf{p}_0 e^{-\gamma t} + \int_0^t dt'\mathbf{F}(t')e^{-\gamma(t-t')} = \overline{\mathbf{p}} + \int_0^t dt'\mathbf{F}(t')e^{-\gamma(t-t')} \tag{8}$$

where $\mathbf{p}_0 = \mathbf{p}(0)$. It is straightforward to check that this expression is indeed solution of Eq.(5). We want to know what is the width of the momentum distribution Δp, to infer the final temperature $k_B T = \Delta p^2/(3M)$, where the factor of 3 arises from the equipartition theorem in three dimensions of space. By definition, $\Delta p^2 = \overline{(\mathbf{p} - \overline{\mathbf{p}})^2}$. We thus have

$$\begin{aligned}
\Delta p^2 &= \overline{\left(\int_0^t dt' \mathbf{F}(t') e^{-\gamma(t-t')} \right)^2} \\
&= \int_0^t dt' \int_0^t dt'' \overline{\mathbf{F}(t') \cdot \mathbf{F}(t'')} e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \\
&\simeq 2D \int_0^t dt' \int_{-\infty}^{+\infty} dt'' \delta(t'-t'') e^{-\gamma(t-t')} e^{-\gamma(t-t'')} \\
&= 2D \int_0^t dt' e^{-2\gamma(t-t')} \\
\Delta p^2 &= \frac{D}{\gamma}(1 - e^{-2\gamma t}). \qquad (9)
\end{aligned}$$

The integration over $[0, t]$ is replaced by an integration over all times, which is justified as soon as $t \gg \tau_c$. At short times, the expression for Δp^2 reduces to a linear increase in time: $\Delta p^2 \simeq 2Dt$. The meaning of D is now clear: it is the diffusion coefficient in momentum space due to the fluctuations of the instantaneous force.

At long times, the variance of the momentum reaches a non zero value $\Delta p^2 = D/\gamma$, corresponding to a limit temperature $k_B T = D/(3M\gamma)$, or equivalently

$$k_B T_{\text{lim}} = \frac{D}{3\alpha}. \qquad (10)$$

2.2 Application to light forces

Coming back to light forces, the diffusion coefficient is linked to the quantum force operator — with quantum fluctuations — as follows [4]:

$$2D = 2\,\text{Re}\left\{ \int_0^\infty d\tau \langle \delta\hat{\mathbf{F}}(t) \cdot \delta\hat{\mathbf{F}}(t-\tau) \rangle \right\}$$

where $\delta\hat{\mathbf{F}} = \hat{\mathbf{F}} - \mathbf{F} = \delta\hat{\mathbf{F}}_L + \delta\hat{\mathbf{F}}_R$, with a part linked to the laser field and the other to vacuum. The cross term in $\delta\hat{\mathbf{F}}_L \cdot \delta\hat{\mathbf{F}}_R$ in the correlation function gives zero, and D is the sum of two independent terms, $D = D_L + D_R$. The calculation of the quantum correlator is beyond the scope of this lecture, and can be found in [4]. We rather give the result and its interpretation in simple cases.

Spontaneous emission: D_R contribution

The contribution to the diffusion coefficient due to the vacuum fluctuation is, in the low saturation limit,

$$D_R = \frac{\Gamma}{4} s \hbar^2 k_L^2 \qquad (11)$$

s being the total saturation parameter ($s = 6s_0$ for six beams). This value can be interpreted in terms of randomness in the direction of the spontaneous emission. The change

in $\mathbf{P}$ due to spontaneous emission after N absorption-emission cycles is

$$\Delta\mathbf{P} = \sum_{n=1}^{N} \hbar\mathbf{k}_n.$$

The random recoils $\mathbf{k}_n$ have the same amplitude k_L, but random directions. As a result, $\overline{\mathbf{k}_n \cdot \mathbf{k}_m} = 0$ for $m \neq n$ and the average squared momentum change is

$$\overline{\Delta\mathbf{P}^2} = \hbar^2 \sum_{n,m=1}^{N} \overline{\mathbf{k}_n \cdot \mathbf{k}_m} = \hbar^2 \sum_{n=1}^{N} \overline{\mathbf{k}_n^2} = N\hbar^2 k_L^2.$$

N is related to the elapsed time t by $N = \frac{\Gamma}{2} s t$. The variance in momentum increases linearly with time as

$$\overline{\Delta\mathbf{P}^2} = \frac{\Gamma}{2} s\hbar^2 k_L^2 t = 2D_R t$$

where we can identify $D_R = \frac{\Gamma}{4} s\hbar^2 k_L^2$.

Fluctuations in the absorption: D_L contribution

This contribution to the diffusion coefficient is due to the fluctuations in the number of absorbed photons in a given time, due to the randomness of the instant of absorption. Its expression is more complex [4]. However, in the case of low saturation parameter $s \ll 1$, it reduces to

$$D_L = \frac{\Gamma}{4} s\,\hbar^2 \left(\frac{\boldsymbol{\nabla}\Omega_1}{\Omega_1}\right)^2 + \frac{\Gamma}{4} s\,\hbar^2 \left(\boldsymbol{\nabla}\phi\right)^2. \tag{12}$$

Let us give an interpretation of this diffusion coefficient in the case of a plane wave, where the first term is zero and the second term gives $D_L = \frac{\Gamma}{4} s\,\hbar^2 k_L^2$. During a time t, the mean number of absorbed photons is $\overline{N} = \frac{\Gamma}{2} s t$. The distribution of photons in the laser is Poissonian for a classical laser source, such that the variance in N is equal to its average: $\overline{\Delta N^2} = \overline{N}$. The change in $\mathbf{P}$ due to absorption is directly proportional to N, all the photons coming from the same plane wave. As a result, $\overline{\Delta\mathbf{P}^2} = \overline{\Delta N^2}\hbar^2 k_L^2 = \frac{\Gamma}{2} s\hbar^2 k_L^2 t$. Again, we can identify D_L with $\frac{\Gamma}{4} s\hbar^2 k_L^2$.

The other term in $\boldsymbol{\nabla}\Omega_1$ appears naturally when looking at the fluctuation of the total force seen by the atom in the dressed state basis. If there is a gradient in Ω_1, two forces $\mathbf{F}_+$ and $\mathbf{F}_-$ appear, being the gradient of the position dependent energies $E_\pm$. The instantaneous force fluctuates between $\mathbf{F}_+$ and $\mathbf{F}_-$ due to the finite lifetime of the states $|\pm\rangle$. The corresponding diffusion coefficient scales as F^2, that is as $(\boldsymbol{\nabla}\Omega_1)^2$.

2.3 Limit temperature for two counter-propagating waves

In the case of two plane waves propagating in opposite directions with orthogonal polarisations, the saturation parameter of the two waves does not depend on position and is twice the saturation parameter of a single wave: $s = 2s_0$. We thus have $D_R = \frac{\Gamma}{2} s_0\hbar^2 k_L^2$. The intensity gradient is zero; the phase gradient is $\pm\mathbf{k}_L$ for each wave of saturation s_0, and we also have $D_L = \frac{\Gamma}{2} s_0\hbar^2 k_L^2$. The total diffusion coefficient is finally $D = \Gamma s_0\hbar^2 k_L^2$.

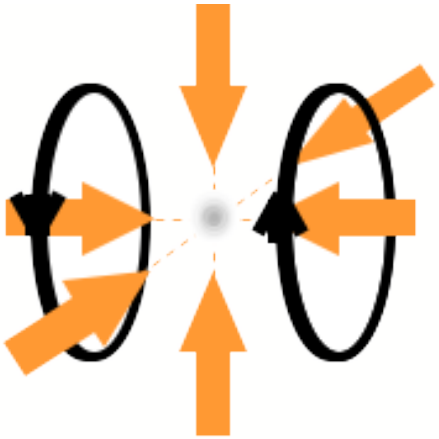

Figure 3: Principle of the magneto-optical trap (MOT). The MOT consists of three pairs of counter-propagating beams with opposite σ^+/σ^- polarisation plus a pair of coils with opposite current for the magnetic field gradient.

In 3D, this may be generalised to $D_{3D} = 3\Gamma s_0 \hbar^2 k_L^2$. Using the expression of α in the limit of low saturation $s_0 \ll 1$, the limit temperature reads:

$$k_B T_{\rm lim} = \frac{D_{3D}}{3\alpha} = \frac{\Gamma s_0 \hbar^2 k_L^2}{2 s_0 \hbar k_L^2} \frac{\delta^2 + \Gamma^2/4}{|\delta|\Gamma} = \frac{\hbar\Gamma}{2} \frac{\delta^2 + \Gamma^2/4}{|\delta|\Gamma}. \tag{13}$$

The temperature is minimum for $\delta = -\Gamma/2$, which corresponds to the largest value of the friction coefficient α. The minimum temperature is called the *Doppler temperature*, and reads:

$$k_B T_D = \frac{\hbar\Gamma}{2}. \tag{14}$$

This temperature is equal to 125 μK for caesium, 140 μK for rubidium, 240 μK for sodium.

For self-consistency, we must check that the typical velocity at this temperature, $v = \sqrt{k_B T/M} = \sqrt{\hbar\Gamma/2M}$ satisfies the small velocity limit that enables to consider the total light force as a friction force. The condition is

$$k_L v \ll \Gamma \Longrightarrow \frac{\hbar k_L^2}{2M} \ll \Gamma \Longleftrightarrow \omega_{\rm rec} \ll \Gamma.$$

We again find the broadband condition!

N.B. In a standing wave configuration, with parallel polarisations of the two contra propagating waves, the total saturation parameter reads $s(z) = 4 s_0 \cos^2 k_L z$. Using Eqs. (11) and (12) we get $D_R = \Gamma s_0 \hbar^2 k_L^2 \cos^2 k_L z$ and $D_L = \Gamma s_0 \hbar^2 k_L^2 \sin^2 k_L z$. The sum is $D = \Gamma s_0 \hbar^2 k_L^2$, as in the case of orthogonal polarisations, and the result is the same.

3 The magneto-optical trap

Doppler cooling enables a quick cooling of the atoms, typically in a few milliseconds. However, the atoms are not trapped and may leave the laser beams and be lost.

The atomic spreading in a given direction Δz follows a diffusive law in real space with a diffusion constant $D_{\rm sp}$:

$$\Delta z^2 = 2D_{\rm sp}\, t \quad \text{with} \quad D_{\rm sp} = \frac{D}{\alpha^2}. \tag{15}$$

For $\delta = -\Gamma/2$, this yields $D_{\rm sp} = \frac{\hbar\Gamma}{2\alpha_{\rm max}} = \Gamma/k_L^2$. In the case of rubidium, $D_{\rm sp} \simeq 1$ mm^2·s^{-1}. In 1 s, an atom has moved typically by 1 mm. This is quite slow, but it means that all the atoms will eventually leave the cooling region in a few seconds. In order to maintain the atoms in the small volume where they are cooled, a magnetic field gradient is applied in addition to the lasers. To understand its effect, we need to take into account the internal structure of the ground and excited states, and depart from the two-level model.

3.1 Magnetic interaction – Zeeman shift

The interaction between an atom with a non zero total spin $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and a position dependent magnetic field $\mathbf{B}(\mathbf{r})$ reads

$$\hat{V}_{\rm mag} = -\hat{\mathbf{J}} \cdot \mathbf{B}(\mathbf{r}). \tag{16}$$

The magnetic sublevels $|J, m_J\rangle$ are shifted by the Zeeman interaction by an amount $m_J g_J \mu_B B$. The quantization axis is chosen along the direction of the magnetic field. g_J is the Landé factor in the state J and μ_B is the Bohr magneton.

Let us consider a transition between a ground state with $J = 0$ and an excited state with $J' = 1$. The excited states $|J', m'\rangle$ are shifted by $m' g_J \mu_B B$, where g_J is the Landé factor in the *excited* state J'. Each transition between $|J = 0, m = 0\rangle$ and $|J', m'\rangle$ has a modified frequency $\omega_0 + m' g_J \mu_B B$. For a magnetic field which depends linearly on the coordinate x as $B = b'x$, where b' is the magnetic gradient, the detuning δ' which enters in the expression of the radiation pressure force Eq.(1) is

$$\delta' = \delta - \frac{m' g_J \mu_B b' x}{\hbar} \tag{17}$$

where we recall that $\delta = \omega_L - \omega_0$. The force now depends on position, and on the internal sublevel.

N.B. Here, the Zeeman interaction will be used to manipulate the detuning between the laser and the atomic transition, which leads to a position dependent *light force*. The magnetic gradients required to achieve an efficient trapping are much lower than those needed to directly trap the atoms with the *magnetic force* in a low-field seeking state. In this last case, the atomic spin remains anti-aligned with the direction of the magnetic field, resulting in a trapping potential $V_{\rm trap} = \mu B(\mathbf{r})$ around the minimum of the magnetic field. These conservative magnetic traps don't require light. They are generally loaded with atoms pre-cooled in a magneto-optical trap or an optical molasses. Bose-Einstein condensation can be achieved in magnetic traps after a phase of evaporative cooling.

3.2 Trapping forces

The idea of the magneto-optical trap (MOT) [7], due to Jean Dalibard, is to add a quadrupole magnetic field to the 3D standing wave configuration of the molasses. The

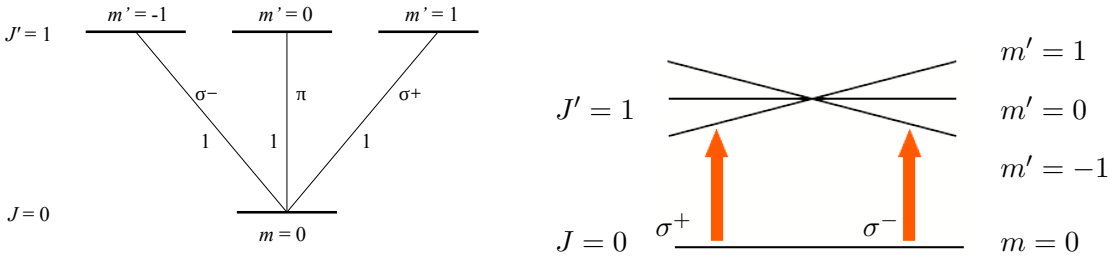

Figure 4: Left: coupling strength for the $J = 0 \to J' = 1$ transition. Right: Position-dependent coupling to the excited states, shifted by the magnetic field gradient (1D scheme).

magnetic field produced by two pairs of coils with the same current in opposite directions, see Fig. 3, can be written

$$\mathbf{B}(\mathbf{r}) = b'(x\mathbf{e}_x + y\mathbf{e}_y - 2z\mathbf{e}_z).$$

In addition, the polarization of two counter-propagating beams is chosen circular with opposite polarization: σ^+ along $+\mathbf{e}_x$ (or $\mathbf{e}_y$) and σ^- along $-\mathbf{e}_x$ (or $-\mathbf{e}_y$), the direction of quantization being $+\mathbf{e}_x$ (or $+\mathbf{e}_y$).[1]

Let us consider the simplest case of a $J = 0 \to J' = 1$ transition, and consider an atom along the x axis for simplicity. Using the gyromagnetic ratio $\gamma_J = g_J\mu_B/\hbar$, we can write the transition frequency between $|J = 0, m = 0\rangle$ and $|J' = 1, m'\rangle$ as $\omega(m') = \omega_0 + m'\gamma_J b'x$.

As shown in Fig. 4, the selection rules of angular momentum conservation imply that the σ^+ polarized laser induces $\Delta m = +1$, transitions only between $J = 0, m = 0$ and $J' = 1, m' = 1$, with a detuning $\delta' = \delta - \gamma_J b'x$, whereas the σ^--polarized laser beam induces transitions only between $J = 0, m = 0$ and $J' = 1, m' = 1$, with a detuning $\delta' = \delta + \gamma_J b'x$.

Moreover, with the choice of the quadrupole magnetic field, the Zeeman interaction shifts the excited levels in such a way that a red-detuned laser is closer to the σ^- $J = 0, m = 0 \to J' = 1, m' = -1$ transition on the right side $x > 0$, and closer to the σ^+ $J = 0, m = 0 \to J' = 1, m' = 1$ transition on the $x < 0$ side where the magnetic field direction is reversed. The strongest force thus comes from the beam which pushes the atoms to the center.

More precisely, the expression of the radiation pressure force exerted by the $\sigma^\pm$ polarized beam propagating along $\pm x$ is, for an atom with zero velocity:

$$\mathbf{F}_\pm(x) = \pm\frac{\Gamma}{2}\frac{\Omega_1^2/2}{\Omega_1^2/2 + \Gamma^2/4 + (\delta \mp \gamma_J b'x)^2}\hbar\mathbf{k}_L. \tag{18}$$

This expression is the analogue of Eq. (2), with the substitution $k_L v_x \leftrightarrow \gamma_J b'x$.

[1] In the z direction where the sign of magnetic field's gradient is opposite, the polarizations have to be reversed.

At low saturation $s_0 \ll 1$, the two forces add, and the expression of the total force, in the vicinity of the center where $|x| \ll |\delta|/(\gamma_J b'), \Gamma/(\gamma_J b')$, is simply

$$\mathbf{F} = -\kappa x \mathbf{e}_x \qquad \text{with} \qquad \kappa = \frac{\gamma_J b'}{k_L}\alpha = -2s_0 \frac{\delta\Gamma}{\delta^2 + \Gamma^2/4}\hbar k_L \gamma_J b'. \tag{19}$$

For a negative detuning $\delta < 0$, this is the expression of a *restoring force*, which pushes the atoms back to the center of the quadrupole field. The restoring coefficient scales with detuning and intensity as the friction coefficient α does, and is maximum with $\kappa_{\max} = 2s_0\hbar k_L \gamma_J b'$ for $\delta = -\Gamma/2$. Its typical value, for rubidium with $s_0 = 0.1$ and a magnetic gradient 0.1 T·m^{-1} (or 10 G·cm^{-1}), is $\kappa_{\max} \simeq 1.5 \times 10^{-18}$ N·m^{-1}.

Taking into account the gradient of the magnetic field, twice as large in the axis z of the coils, the total restoring force in 3D reads:

$$\mathbf{F}(\mathbf{r}) = -\kappa x \mathbf{e}_x - \kappa y \mathbf{e}_y - 2\kappa z \mathbf{e}_z \tag{20}$$

with a correct choice $\sigma^- - \sigma^+$ of the polarizations for the beams propagating along z.

N.B. For an atom close to the center and with a small velocity $\mathbf{v}$, and ignoring the asymmetry of the restoring force for simplicity, the total force is

$$\mathbf{F} = -\alpha \mathbf{v} - \kappa \mathbf{r}.$$

The system is equivalent to a damped oscillator.

N.B. Of course, for values of x larger than $|\delta|/(\gamma_J b')$, the force scales with $\gamma_J b' x$ exactly as the friction force scaled with $k_L v$. Fig. 1 still holds with "x in units of $\Gamma/(\gamma_J b')$" as the label of the horizontal axis.

3.3 Low density regime

The restoring force, arising from the position dependence of the radiation pressure, derives from a harmonic potential $\kappa x^2/2$. When the number of particles in the MOT is very small, a single particle approach gives a good picture of the physics. In particular, the MOT size can be deduced from the equilibrium temperature T:

$$\frac{1}{2}\kappa\langle x^2\rangle = \frac{1}{2}k_B T. \tag{21}$$

If the temperature reaches the Doppler limit, and $\kappa = \kappa_{\max}$, the typical variance of the atomic cloud is thus

$$\langle x^2\rangle \sim \frac{\hbar\Gamma}{2\kappa} = \frac{\Gamma}{4s_0 k_L \gamma_J b'}.$$

With the numbers taken above, this would give $\Delta x = \sqrt{\langle x^2\rangle} = 36$ μm.

This size is limited by the temperature, and independent from the atom number. The cloud has a Gaussian shape, with a size Δz along z smaller by a factor $\sqrt{2}$ due to the larger magnetic gradient. Magneto-optical traps have indeed been observed in this regime of low atom numbers [8]. The atomic density in these traps scale as $N/T^{3/2}$.

3.4 Large density regime

The typical size of a magneto-optical trap of alkali atoms is by far larger than a few tens of micrometers. A size of a few millimeters is very common, and large MOTs are more than one centimeter large. When the atomic density is large, *light induced interactions* come into play and limit the atomic density, which leads to an increased MOT size.

As the atomic sample becomes optically thick, photons scattered by an atom can be reabsorbed by another atom before leaving the cloud. Because of momentum conservation in each elementary process, the two atoms get a kick of one recoil in opposite directions, see Fig. 5. This results in an interatomic repulsion mediated by light.

Figure 5: Photon reabsorption in an optically thick cloud. The photon emitted by the left atom can be reabsorbed by the right atom before leaving the atomic cloud. The recoil of the emitting atom is opposite to the recoil of the absorbing atom.

We can infer from this picture the force corresponding to the reabsorption process. Consider the two atoms of Fig. 5, that we will label by the numbers 1 and 2. First, recall that the photon scattering rate from atom 1, due to the six laser beams of saturation parameter s_0, is

$$\Gamma_{\rm sc} = 6 \times \frac{\Gamma}{2} s_0 = 3\,\Gamma s_0$$

in the low saturation limit. Among these scattered photons, the fraction absorbed by atom 2 is $\sigma/(4\pi r^2)$, where r is the inter-particle distance and σ is the absorption cross section. Its expression is

$$\sigma = \sigma_{\rm res} \frac{\Gamma^2}{4\delta^2 + \Gamma^2} = \frac{3\lambda^2}{2\pi} \frac{\Gamma^2}{4\delta^2 + \Gamma^2}$$

where $\sigma_{\rm res} = 3\lambda^2/(2\pi)$ is the resonant cross section and the other term accounts for a non-zero detuning.

Finally, the momentum change is $\hbar k$ in the direction of the second atom, that it $\hbar k_L \mathbf{r}/r = \hbar k_L \mathbf{u}_r$. The resulting force exerted by atom 1 onto atom 2 is thus

$$\mathbf{F}_{\rm reabs} = 3\Gamma s_0 \frac{\sigma}{4\pi r^2} \hbar k_L \mathbf{u}_r. \tag{22}$$

The expression of this force is formally analogous to a *Coulomb force*, with the mapping

$$\frac{q^2}{\varepsilon_0} = 3\Gamma s_0 \sigma \hbar k_L.$$

The atomic system is thus analogue to a positively charged plasma in an harmonic trap. In this situation, we expect a uniform density in a spherical atomic cloud. Let us assume that the density of charges is indeed uniform in a sphere and equal to n_0. We can show that this corresponds to an equilibrium situation. The forces exerted on a atom at a distance r from the center are central forces:

- The restoring force $-\kappa r$ pointing towards the center.
- The electrostatic force due to the distribution of all the charges. It is equal to $qE(r)$, where the electric field $E(r)$ results from the spherical charge distribution. Only the charges at a distance $r' < r$ contribute to the electric field $E(r)$ at position r. $E(r)$ is equal to the field produced by a single big charge $Q(r)$ placed at the center and corresponding to all the particles within the sphere of radius r:

$$E(r) = \frac{Q(r)}{4\pi\varepsilon_0 r^2} \quad \text{with} \quad Q(r) = q\frac{4\pi}{3}r^3 n_0.$$

  The electrostatic force is thus

$$F = qE(r) = \frac{q^2}{3\varepsilon_0} r n_0.$$

Both forces scale as r, with opposite directions. The plasma is in equilibrium provided that the density satisfies

$$\frac{q^2}{3\varepsilon_0} n_0 = \kappa.$$

Turning back to neutral atoms in the MOT and replacing the effective charge by its expression, we get the equilibrium density in a large MOT:

$$n_0 = \frac{\kappa}{\Gamma s_0 \sigma \hbar k_L} = \frac{4}{3\pi}\frac{|\delta|}{\Gamma}\frac{\gamma_J b'}{\Gamma} k_L^2. \tag{23}$$

The density is uniform in a large MOT. It increases linearly with the detuning and the magnetic field gradient, but is independent of the laser intensity in the low saturation limit.

Neglecting for simplicity the anisotropy of the magnetic field gradient, the cloud radius R is directly deduced from the atom number and the density through

$$R = \left(\frac{3}{4\pi}\frac{N}{n_0}\right)^{1/3}.$$

For typical numbers: $\delta = -\Gamma/2$ and again $b' = 0.1$ T·m^{-1}, with rubidium atoms, we get a density $n_0 = 3 \times 10^9$ cm^{-3}. With the naive low density limit of the last section, this density would be reached for $N = 1500$ atoms already! This means that in typical MOTs, the large density limit applies.

We can calculate the cloud size for a typical atom number, say $N = 10^9$, and find $R = 4$ mm, a much better estimation of what is observed in the experiments.

N.B. Other effects can also come into play. For example, the shadow effect is responsible for an increase in the atomic density: the laser intensity seen by the atoms deep inside the cloud is reduced due to absorption by atoms on the periphery, so that the force is reduced. If the atom is not at the center but, say, on the right side, the force is smaller from the left side where the beam goes through more atoms, which pushes the atom back towards the center and increases the density.

N.B. If the density is large, the interatomic distance is small and molecule formation can occur, enhanced by the presence of red detuned light. This process is called photo-association and leads to atom losses in a MOT.

N.B. The density being limited by reabsorption processes, it does not reach values which could lead to three-body recombination in a MOT, and three-body losses can in general be ignored in this trap.

3.5 Fighting reabsorption

For many applications, like the loading of a conservative trap with a large atomic density to initiate evaporative cooling to Bose-Einstein condensation with good initial conditions, it is relevant to limit the reabsorption processes. This can be done in the following ways.

1. *Dynamical compression.* One could think to simply increase the detuning, as the density scales linearly with $|\delta|$. However, this will also lower the capture efficiency if δ exceeds 3 or 4Γ. The idea is then to do it dynamically, with a time sequence where the atoms are first loaded in a MOT with low density and small detuning, and then compressed to higher density by ramping the detuning to larger absolute value. The same trick can be done also with the magnetic field gradient.

2. *Dark spot.* Reabsorption is naturally suppressed is light is absent... On can thus use laser beam which have a transverse profile with a hole in the center. At the crossing of the six beams, there is no light in a small region where the atoms can accumulate at high densities. As they are not trapped either, they escape from this region after a while and are cooled and trapped again in the laser beams, until they are recycled to the center. This was used for the first time in the group of Wolfgang Ketterle [9].

3. *Dark MOT.* If the atom has a hyperfine structure in the ground state, one typically uses two lasers to obtain a MOT: the cooling laser on the main transition starting from the ground state F_1, and a repumping laser which recycles the atoms lost in the other hyperfine state F_2 back to F_1 through the excited state. Without this repumping beam, the atoms in the state F_2 do not scatter light. We can then use the idea of the dark spot, but only for the repumping beam. The atoms in the central region are depumped in the F_2 state and do not scatter light anymore, which allows to increase the local density. Note that this trick can also be implemented in a dynamical way, by lowering the repumping intensity just before loading the atoms from the MOT to a conservative trap.

4. *Light shift* In a recent experiment led by Florian Schreck, Bose-Einstein condensation of strontium has been reach purely by laser cooling [10]. This is a very important breakthrough. Strontium has both a broad (30 MHz) blue line at 461 nm, the ${}^1S_0 \rightarrow {}^1P_1$ transition, on which usual Doppler cooling is first performed with short time scales in a blue MOT, and a narrow red line ${}^1S_0 \rightarrow {}^3P_1$ (7.4 kHz) at 689 nm, which is used afterwards to cool the atoms even further in a red MOT[11]. In this experiment, the researchers use an auxiliary 'transparency' beam to detune the red light out of resonance near the centre of the red MOT, in order to accumulate the atoms to a high density, see Fig. 6. This transparency laser is tuned between the excited state of the red line and a more excited state, on ${}^3P_1 \rightarrow {}^3S_1$, such that is essentially shifts the ${}^3P_1$ state. As the red line is so narrow, a moderate light shift of the excited state, of about 10 MHz, is enough to put the cooling light completely out of resonance. The atoms in the transparency beam do not feel the repulsive

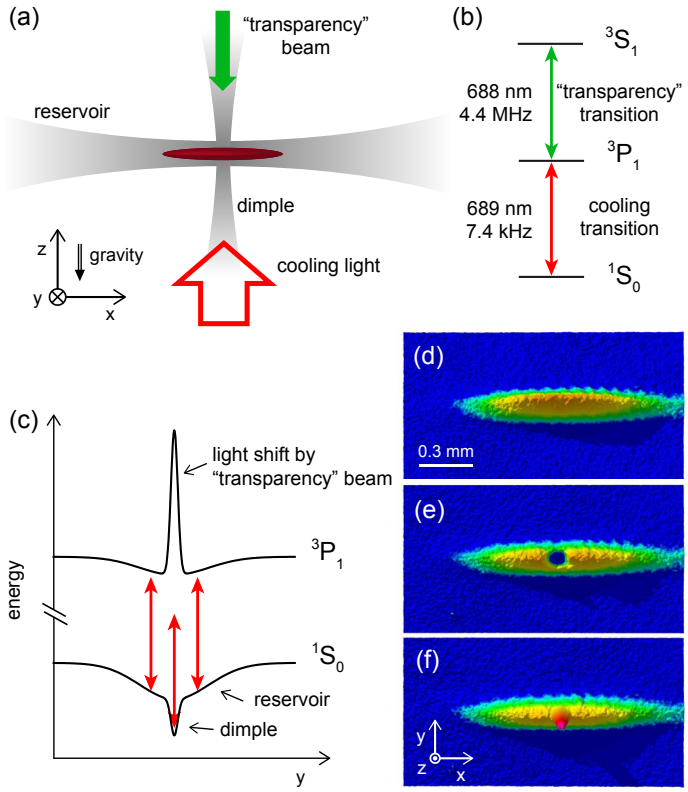

Figure 6: Transparency beam used in the strontium laser cooling to BEC experiment. The light shift in the trap center prevents photon reabsorption and allows very high densities. Figure from Ref. [10], copyright American Physical Society.

force arising from reabsorption, and can accumulate to very large densities in a overlapping dipole trap, where they condense. The advantage of this technique with respect to a dark spot is that the atoms inside the transparency beam can even not absorb light scattered by atoms which are out of the transparency beam, because it is not resonant for their transition.

3.6 Instabilities in large MOTs

In section 3.4, we have shown that the density in the magneto-optical trap is constant as soon as the atom number exceeds a few thousands, and that the cloud size increases with the atom number like $N^{1/3}$. We could wonder to what extend this behaviour holds when N is increased to very large values.

We can first remark that for practical applications, the size of the laser beam is finite in the transverse direction, with a typical size w. It is clear that the MOT size is limited by w, which in turn limits the atom number below about $n_0 w^3$.

On the other hand, the reasoning about the equilibrium in the plasma-like MOT holds only for a restoring force linear in r. However, the trapping force in the MOT is linear only close to the trap center, for distances smaller than $|\delta|/(\gamma_J b')$, see Fig. 1 with the

scale x in units of $\Gamma/(\gamma_J b')$. In particular, the trapping force decreases with r instead of increasing with r at distances larger than about $|\delta|/(\gamma_J b')$. The repulsive interaction is not compensated anymore and the MOT becomes unstable. This behavior has been observed in several experiments [12, 13]. This sets the maximum trap radius to about $R_{\max} = |\delta|/(\gamma_J b')$, and the maximum atom number to

$$N_{\max} \simeq \frac{4\pi}{3} n_0 R_{\max}^3 = \frac{16}{9} \frac{|\delta|}{\Gamma} \frac{\gamma_J b'}{\Gamma} k_L^2 \left(\frac{|\delta|}{\gamma_J b'}\right)^3 = \frac{16}{9} \left(\frac{|\delta|}{\Gamma}\right)^4 \left(\frac{\Gamma k_L}{\gamma_J b'}\right)^2 .$$

For a gradient of 10 G·cm^{-1} and a detuning $\delta = -2\Gamma$, we find $N_{\max} = 3 \times 10^{10}$ rubidium atoms. 10^{10} is indeed a realistic order of magnitude for the atom number in large MOTs [13].

References

[1] T.W. Hänsch and A.L. Schawlow. Cooling of gases by laser radiation. *Opt. Commun.*, 13:393, 1975.

[2] D. Wineland and H. Dehmelt. Proposed $10^{14}\Delta\nu < \nu$ laser fluorescence spectroscopy on TI^+ mono-ion oscillator III. *Bull. Am. Phys. Soc.*, 20:637, 1975.

[3] S. Chu, L. Hollberg, J. E. Bjorkholm, A. Cable, and A. Ashkin. Three-dimensional viscous confinement and cooling of atoms by resonance radiation pressure. *Phys. Rev. Lett.*, 55:48–51, 1985.

[4] C. Cohen-Tannoudji. Atomic motion in laser light. In J. Dalibard, J.-M. Raimond, and J. Zinn-Justin, editors, *Fundamental systems in quantum optics, Les Houches session LIII, July 1990*, pages 1–164. Elsevier, 1992.

[5] V.S. Letokhov, V.G. Minogin, and B.D. Pavlik. Cooling and capture of atoms and molecules by a resonant light field. *Sov. Phys. JETP Lett.*, 45:698, 1977.

[6] D. J. Wineland and Wayne M. Itano. Laser cooling of atoms. *Phys. Rev. A*, 20:1521–1540, Oct 1979.

[7] E.L. Raab, M. Prentiss, A. Cable, S. Chu, and D.E. Pritchard. Trapping of neutral sodium atoms with radiation pressure. *Phys. Rev. Lett.*, 59(23):2631–2634, 1987.

[8] C.G. Townsend, N.H. Edwards, C.J. Cooper, K.P. Zetie, C.J. Foot, A.M. Steane, P. Szriftgiser, H. Perrin, and J. Dalibard. Phase-space density in the magneto-optical trap. *Phys. Rev. A*, 52(2):1423–1440, 1995.

[9] W. Ketterle, D.S Durfee, and D.M Stamper-Kurn. Making, probing and understanding Bose-Einstein condensates. In M. Inguscio, S. Stringari, and C.E. Wieman, editors, *Proceedings of the International School of Physics "Enrico Fermi", Course CXL*, pages 67–176. IOS Press Ohmsha, 1999.

[10] Simon Stellmer, Benjamin Pasquiou, Rudolf Grimm, and Florian Schreck. Laser cooling to quantum degeneracy. *Phys. Rev. Lett.*, 110:263003, Jun 2013.

[11] Thomas H. Loftus, Tetsuya Ido, Andrew D. Ludlow, Martin M. Boyd, and Jun Ye. Narrow line cooling: Finite photon recoil dynamics. *Phys. Rev. Lett.*, 93:073003, Aug 2004.

[12] A. di Stefano, Ph. Verkerk, and D. Hennequin. Deterministic instabilities in the magneto-optical trap. *Eur. Phys. J. D*, 30(2):243–258, 2004.

[13] G. Labeyrie, F. Michaud, and R. Kaiser. Self-sustained oscillations in a large magneto-optical trap. *Phys. Rev. Lett.*, 96:023003, Jan 2006.