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Well-Posedness and Long-Time Dynamics of a Water-Waves Model With Time-Varying Boundary Delay

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Abstract
A higher-order nonlinear Boussinesq system with a time-dependent boundary delay is considered. Sufficient conditions are presented to ensure the well-posedness of the problem by utilizing Kato's variable norm technique and the Fixed Point Theorem. More significantly, the energy decay for the linearized problem is demonstrated using the energy method.

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Keywords
higher-order Boussinesq system, stabilization, delay, decay rate, Lyapunov approach

1 Introduction

1.1 Background

The Boussinesq system comprises a set of nonlinear partial differential equations that model wave dynamics in fluids with small amplitude and long wavelengths. Originally formulated by the French mathematician Joseph Boussinesq in the 19th century to describe shallow water waves (Boussinesq, 1871), since then, the system has been recognized as a model for various physical phenomena, including ocean currents, atmospheric circulation, and heat transfer in fluids. Consequently, the Boussinesq system remains an essential tool in numerous fluid dynamics, with broad applications in fields such as meteorology, oceanography, and engineering.

In more recent studies, Bona et al. (2002, 2004) introduced a four-parameter family of Boussinesq systems to describe the motion of small-amplitude long waves on the surface of an ideal fluid under gravity, particularly in scenarios where the motion is predominantly two-dimensional. In particular, Bona et al. (2002, 2004) investigated the following system:

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} - b\eta_{txx} + a_1\omega_{xxxxx} + b_1\eta_{txxxx} \\ \qquad\qquad = -(\eta\omega)_x + b(\eta\omega)_{xxx} - \alpha'(\eta\omega_{xx})_x, \\ \omega_t + \eta_x + c\eta_{xxx} - d\omega_{txx} + c_1\eta_{xxxxx} + d_1\omega_{txxxx} \\ \qquad\qquad = -\omega\omega_x - c(\omega\omega_x)_{xx} - (\eta\eta_{xx})_x + \beta'\omega_x\omega_{xx} + \rho\,\omega\omega_{xxx}. \end{cases} \tag{1.1}$$

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In this context, η represents the elevation of the fluid surface from its equilibrium position, while $\omega = \omega_\theta$ denotes the horizontal velocity of the flow at a height θh, where h is the undisturbed depth of the fluid and θ is a constant within the interval $[0, 1]$. The variables x and t correspond to space and time, respectively, and the physical parameters $a, b, c, d, a_1, c_1, b_1, d_1$ must satisfy the following relationships:

$$\begin{cases} a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1-\theta^2), \\ a_1 - b_1 = -\frac{1}{2}\left(\theta^2 - \frac{1}{3}\right) b + \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2, \\ c_1 - d_1 = \frac{1}{2}(1-\theta^2)c + \frac{5}{24}(1-\theta^2)\left(\theta^2 - \frac{1}{5}\right), \\ \alpha' = a + b - \frac{1}{3}, \ \beta' = c + d - 1, \ \rho = c + d. \end{cases}$$

Stabilization results for the higher-order system (1.1) on the periodic domain were established in Bautista and Pazoto (2020b) under the conditions $a_1 = c_1 = 0$, with general damping applied to each equation. Furthermore, the local exact controllability of system (1.1) was investigated in Bautista et al. (2021), where the control is localized within the interior of the domain and influences only one equation.

Negative controllability results are explored in Bautista and Pazoto (2020a) and Sierra Fonseca and Pazoto (2022) when the third- and fifth-order Korteweg–de Vries (KdV) terms are removed from the system mentioned above, that is, (1.1) with $a = a_1 = c = c_1 = 0$. In this case, the system consists of two coupled Benjamin–Bona–Mahony-type equations. The authors demonstrated that the linear model is approximately controllable but not spectrally controllable. This implies that although any state can be brought arbitrarily close to another, no finite linear combination of eigenfunctions, other than zero, can be driven to zero.

Let us now consider $b = d = b_1 = d_1 = 0$ and make a scaling argument to obtain the fifth-order Boussinesq system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = -(\eta\omega)_x - \alpha'(\eta\omega_{xx})_x, \\ \omega_t + \eta_x + c\eta_{xxx} + c_1\eta_{xxxxx} = -\omega\omega_x - c(\omega\omega_x)_{xx} - (\eta\eta_{xx})_x + \beta'\omega_x\omega_{xx} + \rho\omega\omega_{xxx}. \end{cases} \tag{1.2}$$

In the above system, we note that $c, a_1 \geq 0$. Thus, we consider the following case:

$$a = c > 0, \ \text{ and } \ c_1 = a_1 > 0. \tag{1.3}$$

It is important to highlight that, to the best of our knowledge, there are no existing results that combine a damping mechanism with a boundary time-varying delay to achieve stabilization of the higher-order Boussinesq system associated with (1.2). This gap forms the main motivation for the present work. It is also worth emphasizing the practical prevalence of time delays in control systems, a phenomenon that is virtually unavoidable due to factors such as the lag between sensors, actuators, and data processing. In light of this, significant efforts have been made to mitigate or eliminate the effects of constant time delays. For example, stability results under smallness conditions on the domain length L and the initial data have been established in Baudouin et al. (2019) and Valein (2022) for the KdV equation and in Capistrano-Filho et al. (2023) for the Kawahara equation. In the case of time-dependent delays, similar results have been obtained in Parada et al. (2023) for the KdV equation and Capistrano-Filho et al. (2024) for the coupled KdV–KdV system.

1.2 Notations and Main Results

This article is concerned with the following system:

$$
\begin{cases}
\eta_t + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = -(\eta\omega)_x - \alpha'(\eta\omega_{xx})_x, & \text{in } \mathbb{R}^+ \times (0,L),\\
\omega_t + \eta_x + c\eta_{xxx} + c_1\eta_{xxxxx} = -\omega\omega_x - c(\omega\omega_x)_{xx} & \\
\qquad\qquad -(\eta\eta_{xx})_x + \beta'\omega_x\omega_{xx} + \rho\omega\omega_{xxx}, & \text{in } \mathbb{R}^+ \times (0,L),\\
\eta(t,0) = \eta(t,L) = \eta_x(t,0) = \eta_x(t,L) = \eta_{xx}(t,0) = 0, & t \in \mathbb{R}^+,\\
\omega(t,0) = \omega(t,L) = \omega_x(t,0) = \omega_x(t,L) = 0, & t \in \mathbb{R}^+,\\
\omega_{xx}(t,L) = \alpha\eta_{xx}(t,L) + \beta\eta_{xx}(t-\tau(t),L), & t > 0,\\
\eta_{xx}(t-\tau(0),L) = z_0(t-\tau(0)) \in L^2(0,1), & 0 < t < \tau(0),\\
(\eta(0,x),\omega(0,x)) = \big(\eta_0(x),\omega_0(x)\big) \in X_0, & x \in (0,L),
\end{cases}
\tag{1.4}
$$

where the parameters a, c, a_1, c_1 verify (1.3). Moreover, we assume that there exist positive constants τ_0, M, and $d < 1$ such that the time-dependent delay function $\tau(t)$ satisfies the following standard conditions:

$$
\begin{cases}
0 < \tau_0 \le \tau(t) \le M, \quad \dot{\tau}(t) \le d < 1, & \forall t \ge 0,\\
\tau \in W^{2,\infty}([0,T]), & T > 0.
\end{cases}
\tag{1.5}
$$

Finally, the feedback gains $\alpha > 0$ and β must obey the following constraint:

$$
\text{The matrix } \Phi_{\alpha,\beta} = \begin{pmatrix} -2a_1\alpha + |\beta| & -a_1\beta \\ -a_1\beta & |\beta|(d-1) \end{pmatrix} \text{ is negative definite.} \tag{1.6}
$$

The condition (1.6) ensures the dissipation of system (1.4). It is worth mentioning that a similar condition is used for other types of delayed dispersive systems (see, for instance, Capistrano-Filho et al., 2024; Parada et al., 2023). Furthermore, recalling that $0 \le d < 1$ and $a_1 > 0$, one can readily check that (1.6) is fulfilled if, for instance, the feedback gains α and β satisfy

$$
\alpha > \frac{|\beta|}{2a_1}\left(\frac{a_1^2 + 1 - d}{1-d}\right).
$$

Next, let $X_0 := L^2(0,L) \times L^2(0,L)$, and the state space

$$
H := X_0 \times L^2(0,1)
$$

equipped with the inner product

$$
\langle(\eta,\omega,z),(\tilde{\eta},\tilde{\omega},\tilde{z})\rangle_t = \langle(\eta,\omega),(\tilde{\eta},\tilde{\omega})\rangle_{X_0} + |\beta|\tau(t)\,\langle z,\tilde{z}\rangle_{L^2(0,1)},
$$

for any $(\eta,\omega;z),(\tilde{\eta},\tilde{\omega};\tilde{z}) \in H$. Moreover, we shall consider the space

$$
\mathcal{B} := C([0,T],X_0) \cap L^2(0,T,[H_0^2(0,L)]^2),
$$

whose norm is

$$
\|(\eta,\omega)\|_{\mathcal{B}} = \sup_{t\in[0,T]} \|(\eta(t),\omega(t))\|_{X_0} + \|(\eta,\omega)\|_{L^2(0,T,[H_0^2(0,L)]^2)}.
$$

To present our first result, let us introduce the following space:

$$
X_3 := \left\{(\eta,\omega) \in \left[H^3(0,L) \cap H_0^2(0,L)\right]^2 \mid \eta_{xx}(0) = 0\right\}. \tag{1.7}
$$

The first result of this manuscript ensures the local well-posedness of system (1.4).

Theorem 1.1. *Let $T > 0$ and suppose the parameters a, c, a_1, c_1 verify (1.3). Then, there exists $\theta = \theta(T) > 0$ such that, for every $(\eta_0, \omega_0; z_0) \in X_3 \times L^2(0,1)$ satisfying*

$$\left\| \left(\eta_0, \omega_0\right) \right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} < \theta,$$

the system (1.4) admits a unique solution $(\eta, \omega) \in C([0,T]; X_3)$. Moreover

$$\|(\eta,\omega)\|_{C([0,T]:[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq C \left\| \left(\eta_0, u_0\right) \right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2},$$

for some positive constant, $C = C(T)$.

Our second result is closely related to the total energy associated with system (1.4) that is defined in H by

$$E(t) = \frac{1}{2}\int_0^L (\eta^2(t,x) + \omega^2(t,x))\,\mathrm{d}x + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_{xx}^2(t-\tau(t)\rho, L)\,\mathrm{d}\rho. \tag{1.8}$$

Indeed, the second result of the article guarantees that the energy $E(t)$ associated with the following system:

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = 0, & \text{in } \mathbb{R}^+ \times (0,L), \\ \omega_t + \eta_x + c\eta_{xxx} + c_1\eta_{xxxxx} = -0, & \text{in } \mathbb{R}^+ \times (0,L), \\ \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \eta_x(t,L) = \eta_{xx}(t,0) = 0, & t \in \mathbb{R}^+, \\ \omega(t,0) = \omega(t,L) = \omega_x(t,0) = \omega_x(t,L) = 0, & t \in \mathbb{R}^+, \\ \omega_{xx}(t,L) = \alpha\eta_{xx}(t,L) + \beta\eta_{xx}(t-\tau(t),L), & t > 0, \\ \eta_{xx}(t-\tau(0),L) = z_0(t-\tau(0)) \in L^2(0,1), & 0 < t < \tau(0), \\ (\eta(0,x),\omega(0,x)) = \left(\eta_0(x),\omega_0(x)\right) \in X_0, & x \in (0,L), \end{cases} \tag{1.9}$$

decays exponentially, even in the presence of delay, and provides an estimate of the decay rate. The result is expressed as follows:

Theorem 1.2. *Let the parameters a, c, a_1, c_1 verify (1.3) and $0 < L < \sqrt{(5a_1/3a)}\pi$. Suppose also that the time-dependent delay function satisfies (1.5). Then, there exist two positive constants*

$$\zeta = \frac{1 + \max\{\mu_1 L, \mu_2\}}{1 - \max\{\mu_1 L, \mu_2\}}, \tag{1.10}$$

and

$$\lambda \leq \min\left\{ \frac{\mu_1\pi^2\left(5a_1\pi^2 - 3aL^2\right)}{L^4(1+\mu_1 L)}, \frac{\mu_2(1-d)}{M(1+\mu_2)} \right\}, \tag{1.11}$$

such that the energy $E(t)$ given by (1.8) associated with system (1.9) satisfies

$$E(t) \leq \zeta E(0)e^{-\lambda t}, \quad \text{for all } t \geq 0.$$

Here μ_1 and μ_2 are two positive constants small enough to be well-chosen.

The main contribution of this work is to establish the local well-posedness of system (1.2) and prove the exponential stability of system (1.9). These results extend and refine those obtained in Capistrano-Filho et al. (2024) and Parada et al. (2023) in several significant directions. More specifically, unlike Capistrano-Filho et al. (2024), where the nonlinear coupling appears only in one equation through the term $(\eta\omega)_x$, our model introduces an additional nonlinear coupling term of the form $(\eta\omega_{xx})_x$. Furthermore, while Capistrano-Filho et al. (2024) feature a single uncoupled nonlinear term, our system includes four additional uncoupled nonlinear terms of higher order, making the analysis more intricate and requiring careful handling of the extra terms in the computations. In particular, in our case, the higher-order spatial derivatives (of order three and five) appear with positive signs, leading to conflicts between the H^1 and H^2-norm terms during integration by parts. Compared to Parada et al. (2023), the situation is even more complex: the problem in Parada et al. (2023) involves a single uncoupled equation with only one nonlinearity and a highest-order derivative of three. Finally, and importantly, in contrast to Capistrano-Filho et al. (2024) and Parada et al. (2023), we employ the transposition method (Capistrano-Filho et al., 2019) to address the well-posedness of our system.

1.3 Outline

The structure of the paper is as follows. In Section 2, we establish the well-posedness of the nonlinear problem (1.4), namely, we show Theorem 1.1 starting with an analysis of the linear system (1.9) using the *variable norm technique of Kato*, followed by the application of the Fixed Point Theorem to prove well-posedness of the full nonlinear problem. Section 3 focuses on the stability result presented in Theorem 1.2, along with a discussion of the optimal decay rate. Finally, we conclude the paper with further remarks in Section 4.

2 Well-Posedness Results

In the sequel, we will assume $a = c > 0$ and $a_1 = c_1 > 0$ in (1.4) as well as in (1.9). We will first examine the well-posedness of the linear system (1.9) and subsequently analyze the properties of the nonlinear problem (1.4) in suitable spaces.

2.1 Linear Problem

Consider the following linear Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}U(t) = A(t)U(t), & t > 0, \\ U(0) = U_0, & t > 0, \end{cases} \tag{2.1}$$

where $A(t) : D(A(t)) \subset H \to H$ is densely defined, and $D(A(t))$ is independent of time t, that is, $D(A(t)) = D(A(0))$, for all $t > 0$. The next theorem ensures the existence and uniqueness of the Cauchy problem (2.1).

Theorem 2.1 (Kato, 1970)**.** *Assume that:*

(1) $\mathcal{Z} = D(A(0))$ *is a dense subset of* H *and* $D(A(t)) = D(A(0))$*, for all* $t > 0$.
(2) $A(t)$ *generates a strongly continuous semigroup on* H*. Moreover, the family* $\{A(t) : t \in [0, T]\}$ *is stable with stability constants* C *and* m *independent of* t.
(3) $\partial_t A(t)$ *belongs to* $L^\infty_*([0, T], B(\mathcal{Z}, H))$*, the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0, T]$ *into the set* $B(\mathcal{Z}, H)$ *of bounded operators from* $\mathcal{Z}$ *into* H.

Then, problem (2.1) has a unique solution $U \in C([0, T], \mathcal{Z}) \cap C^1([0, T], H)$ *for any initial datum in* $\mathcal{Z}$.

The task ahead is to apply the above result to ensure the existence of solutions for the linear system (1.9). Arguing as in Xu et al. (2006), Nicaise and Pignotti (2006), and Nicaise et al. (2009), let us define the auxiliary variable

$$z(t, \rho) = \eta_{xx}(t - \tau(t)\rho, L),$$

which satisfies the transport equation:

$$\begin{cases} \tau(t)z_t(t, \rho) + (1 - \dot{\tau}(t)\rho)z_\rho(t, \rho) = 0, & t > 0, \rho \in (0, 1), \\ z(t, 0) = \eta_{xx}(t, L),\ z(0, \rho) = z_0(-\tau(0)\rho), & t > 0,\ \rho \in (0, 1). \end{cases} \tag{2.2}$$

Now, we pick up $U = (\eta, \omega; z)^{\mathrm{T}}$ and consider the time-dependent operator

$$A(t) : D(A(t)) \subset H \to H,$$

given by

$$A(t)\,(\eta,\omega,z) := \left(-\omega_x - a\omega_{xxx} - a_1\omega_{xxxxx}, -\eta_x - a\eta_{xxx} - a_1\eta_{xxxxx}, \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho\right), \tag{2.3}$$

with a domain defined by

$$D(A(t)) = \left\{ \begin{array}{r} (\eta,\omega,z) \in H; (\eta,\omega) \in \left[H^5(0,L) \cap H_0^2(0,L)\right]^2,\ z \in H^1(0,1), \\ \eta_{xx}(0) = 0, z(0) = \eta_{xx}(L), \omega_{xx}(L) = \alpha\eta_{xx}(L) + \beta z(1) \end{array} \right\}. \tag{2.4}$$

This allows us to write problem (1.9) in the abstract form (2.1) by using (2.2)–(2.4). Additionally, it is noteworthy that $D(A(t))$ is independent of time t since $D(A(t)) = D(A(0))$.

Now, taking the triplet $\{A, H, \mathcal{Z}\}$, with $A = \{A(t) : t \in [0,T]\}$, for some $T > 0$ fixed and $\mathcal{Z} = D(A(0))$, we can state and prove the well-posedness result of (2.1) related to $\{A, H, \mathcal{Z}\}$.

Theorem 2.2. *Let the parameters a, c, a_1, c_1 verify (1.3). Assume that α and β are real constants such that (1.6) holds. Taking $U_0 \in H$, there exists a unique solution $U \in C([0,+\infty), H)$ to (2.1) whose operator is defined by (2.3)–(2.4). Moreover, if $U_0 \in D(A(0))$, then $U \in C([0,+\infty), D(A(0))) \cap C^1([0,+\infty), H)$.*

Proof. The result will be proved in a standard way (see, for instance, Nicaise et al., 2009). First, it is not difficult to see that $\mathcal{Z} = D(A(0))$ is a dense subset of H and $D(A(t)) = D(A(0))$, for all $t > 0$. Thus, the requirement (1) of Theorem 2.1 is fulfilled.

Concerning the condition (2) of Theorem 2.1, let us note that simple integrations by parts together with the boundary conditions yield

$$\langle A(t)U, U\rangle_t - \kappa(t)\,\langle U, U\rangle_t \le \frac{1}{2}\left(\eta_{xx}(t,L), \eta_{xx}(t-\tau(t),L)\right)\Phi_{\alpha,\beta}\left(\eta_{xx}(t,L), \eta_{xx}(t-\tau(t),L)\right)^{\mathrm{T}},$$

where

$$\kappa(t) = \frac{(\dot{\tau}(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)}.$$

Owing to (1.6), it follows that

$$\langle A(t)U, U\rangle_t - \kappa(t)\,\langle U, U\rangle_t \le 0.$$

Consequently, $\tilde{A}(t) = A(t) - \kappa(t)I$ is dissipative.

Now, we claim the following:

Claim 1. For all $t \in [0,T]$, the operator $A(t)$ is maximal, or equivalently, we have that $\lambda I - A(t)$ is surjective, for some $\lambda > 0$.

In fact, let us fix $t \in [0,T]$. Given $(f_1, f_2, h)^{\mathrm{T}} \in H$, we seek a solution $U = (\eta,\omega,z)^{\mathrm{T}} \in D(A(t))$ of the equation $(\lambda I - A(t))U = (f_1, f_2, h)$, that is,

$$\begin{cases} \lambda\eta + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = f_1, \\ \lambda\omega + \eta_x + a\eta_{xxx} + a_1\eta_{xxxxx} = f_2, \\ \lambda z + \left(\dfrac{1 - \dot{\tau}(t)\rho}{\tau(t)}\right) z_\rho = h, \\ \eta(0) = \eta(L) = \eta_x(0) = \eta_x(L) = \eta_{xx}(0) = 0, \\ \omega(0) = \omega(L) = \omega_x(0) = \omega_x(L) = 0, \\ \omega_{xx}(L) = \alpha\eta_{xx}(L) + \beta z(1), z(0) = \eta_{xx}(L). \end{cases} \tag{2.5}$$

One can readily verify that z is given by

$$z(\rho)=\begin{cases}\eta_{xx}(L)e^{-\lambda\tau(t)\rho}+\tau(t)e^{-\lambda\tau(t)\rho}\displaystyle\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)\,\mathrm{d}\sigma, & \text{if } \dot{\tau}(t)=0,\\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\rho)}\left[\eta_{xx}(L)+\displaystyle\int_0^\rho\frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}\,\mathrm{d}\sigma\right], & \text{if } \dot{\tau}(t)\neq 0.\end{cases}$$

Thereby, $z(1)=\eta_{xx}(L)g_0(t)+g_h(t)$, in which

$$g_0(t)=\begin{cases}e^{-\lambda\tau(t)}, & \text{if } \dot{\tau}(t)=0,\\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}, & \text{if } \dot{\tau}(t)\neq 0,\end{cases}$$

and

$$g_h(t)=\begin{cases}\tau(t)e^{-\lambda\tau(t)}\displaystyle\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)\mathrm{d}\sigma, & \text{if } \dot{\tau}(t)=0,\\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}\displaystyle\int_0^1\frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}\mathrm{d}\sigma, & \text{if } \dot{\tau}(t)\neq 0.\end{cases}$$

Combining the latter with (2.5), it follows that η and ω are solutions of the system

$$\begin{cases}\lambda\eta+\omega_x+a\omega_{xxx}+a_1\omega_{xxxxx}=f_1,\\ \lambda\omega+\eta_x+a\eta_{xxx}+a_1\eta_{xxxxx}=f_2,\end{cases}\tag{2.6}$$

and satisfy the boundary conditions

$$\begin{cases}\eta(0)=\eta(L)=\eta_x(0)=\eta_x(L)=\eta_{xx}(0)=0,\\ \omega(0)=\omega(L)=\omega_x(0)=\omega_x(L)=0,\\ \omega_{xx}(L)=(\alpha+\beta g_0(t))\eta_{xx}(L)+\beta g_h(t).\end{cases}$$

Now, let $\phi_1\in C^\infty([0,L])$ be a function such that $\phi_1(0)=\phi_1(L)=\phi_{1,x}(0)=\phi_{1,x}(L)=0$ and $\phi_{1,xx}(L)=1$. Next, we define a function $\psi(x,\cdot)=\phi_1(x)\beta g_h(\cdot)\in C^\infty([0,L])$ and let $\hat{\omega}:=\omega-\psi$. This, together with (2.6), implies that η and ω satisfy

$$\begin{cases}\lambda\eta+\omega_x+a\omega_{xxx}+a_1\omega_{xxxxx}=f_1-\left(\psi_x+\psi_{xxx}+\psi_{xxxxx}\right)=:\tilde{f}_1,\\ \lambda\omega+\eta_x+a\eta_{xxx}+a_1\eta_{xxxxx}=f_2-\lambda\psi=:\tilde{f}_2,\end{cases}$$

as well as the boundary conditions

$$\begin{cases}\eta(0)=\eta(L)=\eta_x(0)=\eta_x(L)=\eta_{xx}(0)=0,\\ \omega(0)=\omega(L)=\omega_x(0)=\omega_x(L)=0,\\ \omega_{xx}(L)=(\alpha+\beta g_0(t))\eta_{xx}(L).\end{cases}$$

Let us mention that for the sake of simplicity, we still use ω after translation. Then, we can verify that $0<g_0(t)<1$ (see, for instance, Capistrano-Filho et al., 2024). Thus, thanks to (1.6), we deduce that $\tilde{\alpha}:=\alpha+\beta g_0(t)>0$. Consequently, showing the Claim 1 is equivalent to proving that $\lambda I-\hat{A}$ is surjective, where $\hat{A}$ is given by

$$\hat{A}(\eta,\omega)=(-\omega_x-a\omega_{xxx}-a_1\omega_{xxxxx},-\eta_x-a\eta_{xxx}-a_1\eta_{xxxxx}),$$

with a dense domain

$$D(\hat{A}):=\left\{(\eta,\omega)\in\left[H^5(0,L)\cap H_0^2(0,L)\right]^2:\eta_{xx}(0)=0,\ \omega_{xx}(L)=\tilde{\alpha}\eta_{xx}(L)\right\}\subset X_0.$$

Now, observe that adjoint of $\hat{A}$, denoted by $\hat{A}^*$, is defined by

$$\hat{A}^*(u,v) = (u_x + au_{xxx} + a_1 u_{xxxxx}, v_x + av_{xxx} + a_1 v_{xxxxx}),$$

with

$$D(\hat{A}^*) := \left\{(u,v) \in \left[H^5(0,L) \cap H_0^2(0,L)\right]^2 : v_{xx}(0) = 0,\ u_{xx}(L) = -\tilde{\alpha} v_{xx}(L)\right\}.$$

Since

$$\left\langle \hat{A}(\eta,\omega),(\eta,\omega)\right\rangle_{X_0} = -a_1\tilde{\alpha}\eta_{xx}^2(L),$$

and

$$\left\langle \hat{A}^*(u,v),(u,v)\right\rangle_{X_0} = -a_1\tilde{\alpha}v_{xx}^2(L),$$

we can claim that the operators $\hat{A}$ and $\hat{A}^*$ are dissipative. Therefore, the desired result follows from the Lummer–Phillips theorem (see, e.g., Pazy, 1983). This shows the Claim 1. Consequently, $\tilde{A}(t)$ generates a strongly semigroup on H and $\tilde{A} = \{\tilde{A}(t), t \in [0,T]\}$ is a stable family of generators in H, whose stability constant is independent of t. Thus, the condition (2) of Theorem 2.1 is satisfied.

Lastly, since $\tau \in W^{2,\infty}([0,T])$ for all $T > 0$, we reach that

$$\dot{\kappa}(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)\left(\dot{\tau}(t)^2+1\right)^{1/2}} - \frac{\dot{\tau}(t)\left(\dot{\tau}(t)^2+1\right)^{1/2}}{2\tau(t)^2}$$

is bounded on $[0,T]$ for all $T > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)U = \left(0, 0, \frac{\ddot{\tau}(t)\tau(t)\rho - \dot{\tau}(t)(\dot{\tau}(t)\rho - 1)}{\tau(t)^2} z_\rho\right).$$

Moreover, the coefficient of z_ρ is bounded on $[0,T]$, and the regularity (3) of Theorem 2.1 is satisfied.

To sum up, we verified the assumptions of Theorem 2.1 and hence for each $U_0 \in D(A(0))$, the Cauchy problem

$$\begin{cases} \tilde{U}_t(t) = \tilde{A}(t)\tilde{U}(t), & t > 0,\\ \tilde{U}(0) = U_0, \end{cases}$$

has a unique solution $\tilde{U} \in C([0,\infty), H)$ and $\tilde{U} \in C([0,\infty), D(A(0))) \cap C^1([0,\infty), H)$. Thus, the solution of (2.1) is explicitly given by $U(t) = e^{\int_0^t \kappa(s)\mathrm{d}s}\tilde{U}(t)$. □

We also have the following result.

Proposition 2.3. *Let the parameters a, c, a_1, c_1 verify (1.3). Suppose α and β are real constants such that (1.6) holds. Then, for any mild solution of (2.1), the energy $E(t)$ defined by (1.8) is nonincreasing and*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \frac{1}{2}\begin{pmatrix}\eta_{xx}(t,L)\\ \eta_{xx}(t-\tau(t),L)\end{pmatrix}^{\mathrm{T}} \Phi_{\alpha,\beta}\begin{pmatrix}\eta_{xx}(t,L)\\ \eta_{xx}(t-\tau(t),L)\end{pmatrix}, \tag{2.7}$$

where matrix $\Phi_{\alpha,\beta}$ is given by (1.6).

Proof. The proof is straightforward and hence omitted. □

We now proceed to prove the Kato smoothing property, along with several a priori estimates. These results are crucial for establishing the well-posedness of the system (1.4). In the following, $(S_t(s))_{s\geq 0}$ represents the two-parameter semigroup of contractions associated with the operator $A(t)$. We are now prepared to state the following result:

Proposition 2.4. *Let the parameters a, c, a_1, c_1 verify (1.3) and α and β are real constant such that (1.6) holds. Then, the following estimate holds:*

$$\|(\eta,\omega)\|^2_{X_0} + |\beta|\|z\|^2_{L^2(0,1)} \leq \|(\eta_0,\omega_0)\|^2_{X_0} + |\beta|\|z_0(-\tau(0)\cdot)\|^2_{L^2(0,1)}, \tag{2.8}$$

Furthermore, for every initial condition $(\eta_0, \omega_0, z_0) \in H$, we have that

$$\|\eta_{xx}(\cdot,L)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)} \leq \|(\eta_0,\omega_0)\|^2_{X_0} + \|z_0(-\tau(0)\cdot)\|^2_{L^2(0,1)}. \tag{2.9}$$

On the other hand, for the initial datum, we have the following estimates:

$$\|(\eta_0,\omega_0)\|^2_{X_0} \leq \frac{1}{T}\|(\eta,\omega)\|^2_{L^2(0,T;X_0)} + (2\alpha+|\beta|)\|\eta_{xx}(\cdot,L)\|^2_{L^2(0,T)} + |\beta|\|z(\cdot,1)\|^2_{L^2(0,1)}, \tag{2.10}$$

and

$$\|z_0(-\tau(0)\cdot)\|^2_{L^2(0,1)} \leq C_1(d,M)\left(\|z(T,\cdot)\|_{L^2(0,1)} + \|z(\cdot,1)\|^2_{L^2(0,T)}\right). \tag{2.11}$$

Finally, for $0 < L < \sqrt{(5a_1/3a)}\pi$, the Kato smoothing effect is verified

$$\int_0^T\int_0^L \left(\eta^2_{xx}+\omega^2_{xx}\right)\,\mathrm{d}x\,\mathrm{d}t \leq C(L,T,\alpha)\left(\|(\eta_0,\omega_0)\|^2_{X_0} + \|z_0(-\tau(0)\cdot)\|^2_{L^2(0,1)}\right), \tag{2.12}$$

and the map

$$(\eta_0,\omega_0;z_0) \in H \mapsto (\eta,\omega;z) \in \mathcal{B}\times C(0,T;L^2(0,1))$$

is well-defined and continuous.

Proof. Using (2.7) and the fact that $\Phi_{\alpha,\beta}$ is a symmetric negative definite matrix, we deduce the existence of a positive constant C, such that

$$E'(t) = \frac{1}{2}\begin{pmatrix}\eta_{xx}(t,L)\\ z(t,1)\end{pmatrix}^{\mathrm{T}}\Phi_{\alpha,\beta}\begin{pmatrix}\eta_{xx}(t,L)\\ z(t,1)\end{pmatrix} \leq -C\left(\eta^2_{xx}(t,L)+z^2(t,1)\right).$$

Thus, it follows from the above estimate that

$$E'(t) + \eta^2_{xx}(t,L) + z^2(t,1) \leq 0. \tag{2.13}$$

Integrating (2.13) in $[0,s]$, for $0 \leq s \leq T$, we get

$$E(s) + \int_0^s \eta^2_{xx}(t,L)\,\mathrm{d}t + \int_0^s z^2(t,1)\,\mathrm{d}t \leq E(0),$$

and (2.8) is obtained. Taking $s = T$ and since $E(t)$ is a nonincreasing function (see Proposition 2.3), the estimate (2.9) holds.

Secondly, the proof of estimates (2.10) and (2.11) is analogous to that of Capistrano-Filho et al. (2024), and we will omit the details.

Now, we show the inequality (2.12) provided that $0 < L < \sqrt{(5a_1/3a)}\pi$. Initially, multiplying the first equation of (1.9) by $x\omega$ and the second one by $x\eta$. Next, adding the results, then integrating by parts over $(0,L)\times(0,T)$ and invoking (2.8) and (2.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^T\int_0^L\left(\eta^2+\omega^2\right)\,\mathrm{d}x\,\mathrm{d}t - \frac{3a}{2}\int_0^T\int_0^L\left(\eta^2_x+\omega^2_x\right)\,\mathrm{d}x\,\mathrm{d}t + \frac{5a_1}{2}\int_0^T\int_0^L\left(\eta^2_{xx}+\omega^2_{xx}\right)\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{a_1L}{2}\int_0^T\left(\eta^2_{xx}(t,L)+\omega^2_{xx}(t,L)\right)\mathrm{d}t - \int_0^L x\left(\eta(t,x)\omega(t,x)-\eta_0(x)\omega_0(x)\right)\mathrm{d}x\\
&\leq C(L,\alpha,a_1)\left(\|(\eta_0,\omega_0)\|^2_{X_0} + \|z_0(-\tau(0)\cdot)\|^2_{L^2(0,1)}\right),
\end{aligned} \tag{2.14}$$

for some positive constant $C(L,\alpha,a_1)$. Since $0 < L < \sqrt{(5a_1/3a)}\pi$, from Poincaré inequality, there exists $C_L = (1/2)(5a_1\pi^2 - 3aL^2) > 0$, such that

$$C_L \int_0^T \int_0^L \left(\eta_{xx}^2 + \omega_{xx}^2\right) \mathrm{d}x\,\mathrm{d}t \leq -\frac{3a}{2} \int_0^T \int_0^L \left(\eta_x^2 + \omega_x^2\right) \mathrm{d}x\,\mathrm{d}t + \frac{5a_1}{2} \int_0^T \int_0^L \left(\eta_{xx}^2 + \omega_{xx}^2\right) \mathrm{d}x\,\mathrm{d}t. \tag{2.15}$$

Thus, from (2.14) and (2.15), we obtain (2.12). □

The next result ensures the existence of solutions to the fifth-order KdV–KdV system with sufficient regular source terms.

Theorem 2.5. *Suppose that (1.5) and (1.6) hold. Let $U_0 = (\eta_0, \omega_0, z_0) \in H$ and the source terms $(f_1, f_2) \in L^1(0,T;X_0)$. Then, if the parameters a, a_1 verify (1.3), there exists a unique solution $U = (\eta, \omega, z) \in C([0,T], H)$ to*

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = f_1, & t > 0, x \in (0,L), \\ \omega_t + \eta_x + a\eta_{xxx} + a_1\eta_{xxxxx} = f_2, & t > 0, x \in (0,L), \end{cases}$$

with boundary conditions as in (1.4). Moreover, for $T > 0$, there exists a positive constant C such that the following estimates hold

$$\begin{cases} \|(\eta,\omega;z)\|_{C([0,T],H)} \leq C\left(\|(\eta_0,\omega_0,z_0)\|_H + \|(f,g)\|_{L^1(0,T,X_0)}\right), \\ \|(\eta_{xx}(\cdot,L), z(\cdot,1))\|^2_{[L^2(0,T)]^2} \leq C\left(\|(\eta_0,\omega_0,z_0)\|_H^2 + \|(f,g)\|^2_{L^1(0,T,X_0)}\right), \end{cases}$$

and, for $0 < L < \sqrt{(5a_1/3a)}\pi$,

$$\|(\eta,\omega)\|_{L^2(0,T,H_0^2(0,L))} \leq C\left(\|(\eta_0,\omega_0,z_0)\|_H + \|(f,g)\|_{L^1(0,T,X_0)}\right).$$

Proof. This proof is analogous to that of Capistrano-Filho et al. (2024, Theorem 2.5), and hence we omit it. □

2.2 Nonlinear Problem

In this subsection, we show the well-posedness of the nonlinear problem (1.4) by using the approach of Capistrano-Filho et al. (2019), where the solutions are obtained via the transposition method and the existence and uniqueness by using the Riesz-representation theorem.

To prove the well-posedness result for system (1.4), we consider the nonhomogeneous system

$$\begin{cases} \eta_t + \omega_x + a\omega_{xxx} + a_1\omega_{xxxxx} = h_1, & \text{in } (0,T)\times(0,L), \\ \omega_t + \eta_x + a\eta_{xxx} + a_1\eta_{xxxxx} = h_2, & \text{in } (0,T)\times(0,L), \\ \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \eta_x(t,L) = \eta_{xx}(t,0) = 0, & t \in (0,T), \\ \omega(t,0) = \omega(t,L) = \omega_x(t,0) = \omega_x(t,L) = 0, & t \in (0,T), \\ \omega_{xx}(t,L) = f(t) & t \in (0,T), \\ (\eta(0,x), \omega(0,x)) = \left(\eta_0(x), \omega_0(x)\right), & x \in (0,L), \end{cases} \tag{2.16}$$

where the parameters a, a_1 verify (1.3). Remember the definition of X_3 given by (1.7), and also consider the following set:

$$\tilde{X}_3 := \left\{ (\varphi,\psi) \in \left[H^3(0,L) \cap H_0^2(0,L)\right]^2 | \varphi_{xx}(0) = \psi_{xx}(L) = 0 \right\}.$$

We define a solution by transposition as follows, see Lions and Magenes (1968, 1986), to justify the choice of formula (2.17) below:

Definition 1 (Solution by transposition)**.** Let $T > 0$, $(\eta_0, \omega_0) \in X_3, f \in L^2(0,T)$ and

$$(h_1, h_2) \in L^2(0,T,[H^{-2}(0,L)]^2).$$

A solution of problem (2.16) is a function $(\eta, \omega) \in C(0, T; X_3)$ such that, for all $\sigma \in [0, T]$ and $(\varphi_\sigma, \psi_\sigma) \in \tilde{X}_3$ the following identity holds

$$\begin{aligned} \left\langle (\eta(\sigma), \omega(\sigma)), \left(\varphi_\sigma, \psi_\sigma\right) \right\rangle_{[H^3(0,L)\cap H_0^2(0,L)]^2} &= \left\langle \left(\eta_0, \omega_0\right), (\varphi(0), \psi(0)) \right\rangle_{[H^3(0,L)\cap H_0^2(0,L)]^2} \\ &+ \int_0^\sigma f(t)\, \varphi_{xx}(t, L) \mathrm{d}t + \int_0^\sigma \left\langle \left(h_1(t), h_2(t)\right), (\varphi(t), \psi(t)) \right\rangle_{(H^{-2}, H_0^2)^2} \mathrm{d}t, \end{aligned} \tag{2.17}$$

where the pair (φ, ψ) is the solution of

$$\begin{cases} \varphi_t + \psi_x - a\psi_{xxx} + a_1\psi_{xxxxx} = 0, & \text{in } (0, L) \times (0, \sigma), \\ \psi_t + \varphi_x - a\varphi_{xxx} + a_1\varphi_{xxxxx} = 0, & \text{in } (0, L) \times (0, \sigma), \\ \varphi(0, t) = \varphi(L, t) = \varphi_x(0, t) = \varphi_x(L, t) = \varphi_{xx}(0, t) = 0, & \text{on } (0, \sigma), \\ \psi(0, t) = \psi(L, t) = \psi_x(0, t) = \psi_x(L, t) = \psi_{xx}(L, t) = 0, & \text{on } (0, \sigma), \\ \varphi(x, \sigma) = \varphi_\sigma, \quad \psi(x, \sigma) = \psi_\sigma, & \text{on } (0, L). \end{cases} \tag{2.18}$$

Thanks to Capistrano-Filho et al. (2019, Corollary 2.5 and Proposition 2.6), the following well-posedness result for system (2.18) is established:

Proposition 2.6. *For all $(\varphi_\sigma, \psi_\sigma) \in \tilde{X}_3$, system (2.18) admits a unique solution $(\varphi, \psi) \in C([0, \sigma]; \tilde{X}_3)$, which satisfies*

$$\|(\varphi(t), \psi(t))\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} \le C\|(\varphi_\sigma, \psi_\sigma)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2}, \quad \forall t \in [0, \sigma]. \tag{2.19}$$

Additionally, we have that

$$\int_0^\sigma |\varphi_{xx}(L, t)|^2 + |\psi_{xx}(0, t)|^2 \,\mathrm{d}t \le C\|(\varphi_\sigma, \psi_\sigma)\|^2_{[H_0^2(0,L)]^2}. \tag{2.20}$$

The following result gives us the existence and uniqueness of the solution for system (2.16).

Lemma 2.7. *Let $T > 0$, $(\eta_0, \omega_0) \in X_3$, $(h_1, h_2) \in L^2(0, T; [H^{-2}(0, L)]^2)$ and $f \in L^2(0, T)$. There exists a unique solution $(\eta, \omega) \in C([0, T]; X_3)$ of system (2.16). Moreover, there exists a positive constant C_T, such that*

$$\|(\eta(\sigma), \omega(\sigma))\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} \le C_T \left(\left\| \left(\eta_0, \omega_0\right) \right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \|f\|_{L^2(0,T)} \left\| \left(h_1, h_2\right) \right\|_{L^2\left(0,T:[H_0^2(0,L)]^2\right)} \right), \tag{2.21}$$

for all $\sigma \in [0, T]$.

Proof. Let us define Δ as the linear functional given by the right-hand side of (2.17), that is

$$\begin{aligned} \Delta\left(\varphi_\sigma, \psi_\sigma\right) = &\left\langle \left(\eta_0, \omega_0\right), (\varphi(0), \psi(0)) \right\rangle_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \int_0^\sigma f(t)\, \varphi_{xx}(t, L) \mathrm{d}t \\ &+ \int_0^\sigma \left\langle \left(h_1(t), h_2(t)\right), (\varphi(t), \psi(t)) \right\rangle_{(H^{-2}, H_0^2)^2} \mathrm{d}t. \end{aligned}$$

We infer from (2.19), (2.20), and the Cauchy–Schwarz inequality that

$$\begin{aligned} |\Delta\left(\varphi_\sigma, \psi_\sigma\right)| \le &\|(\eta_0, \omega_0)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} \|(\varphi_\sigma, \psi_\sigma)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \|f\|_{L^2(0,T)} \|\varphi_{xx}(L)\|_{L^2(0,T)} \\ &+ C\|(\varphi_\sigma, \psi_\sigma)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} \|(h_1, h_2)\|_{L^1(0,T:H^{-2}(0,L)} \\ \le &C_T \left(\|(\eta_0, \omega_0)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \|f\|_{L^2(0,T)} + \|(h_1, h_2)\|_{L^1(0,T:H^{-2}(0,L)} \right) \|(\varphi_\sigma, \psi_\sigma)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2}, \end{aligned}$$

and we obtain that $\Delta \in \mathcal{L}([H^3(0,L) \cap H_0^2(0,L)]^2; \mathbb{R})$. Thus, from the Riesz-representation theorem, there exists one and only one $(\eta_\sigma, \omega_\sigma) \in [H^3(0,L) \cap H_0^2(0,L)]^2$ such that

$$\begin{cases} \Delta\left(\varphi_\sigma, \psi_\sigma\right) = \left\langle \left(\eta_\sigma, \omega_\sigma\right), \left(\varphi_\sigma, \psi_\sigma\right)\right\rangle_{[H^3(0,L)\cap H_0^2(0,L)]^2} \\ \text{with} \quad \|\Delta\|_{\mathcal{L}([H^3(0,L)\cap H_0^2(0,L)]^2;\mathbb{R})} = \|\left(\eta_\sigma, \omega_\sigma\right)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2}, \end{cases} \tag{2.22}$$

and we obtain the uniqueness of the solution to problem (2.16). Now, in order to prove the estimate (2.21), we define the map $(\eta, \omega) : [0,T] \to [H^3(0,L) \cap H_0^2(0,L)]^2$ as

$$(\eta(\sigma), \omega(\sigma)) := (\eta_\sigma, \omega_\sigma) \text{ for all } \sigma \in [0,T],$$

and we conclude from (2.22) that

$$\begin{aligned} \|(\eta(\sigma), \omega(\sigma))\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} &= \|\Delta\|_{\mathcal{L}([H^3(0,L)\cap H_0^2(0,L)]^2;\mathbb{R})} \\ &\le C_T\left(\left\|\left(\eta_0, \omega_0\right)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \|f\|_{L^2(0,T)} + \left\|\left(h_1, h_2\right)\right\|_{L^2\left(0,T;[H_0^2(0,L)]^2\right)}\right). \end{aligned}$$

Finally, the fact that $(\eta, \omega) \in C([0,T]; X_3)$ was already proved in Capistrano-Filho and Gallego (2018) and Capistrano-Filho et al. (2019), and hence we omit the details. □

Now, we pass to show the well-posedness of the nonhomogeneous feedback linear system associated with (2.16)

Lemma 2.8. *Let $T > 0$. Then, for every (η_0, ω_0) in X_3 and (h_1, h_2) in $L^2(0,T; [H^{-2}(0,L)]^2)$, there exists a unique solution (η, ω) of system (2.16) such that $(\eta, \omega) \in C([0,T]; X_3)$, with $f(t) = \alpha\eta_{xx}(t,L) + \beta\eta_{xx}(t-\tau(t),L)$, where α and β belong to $\mathbb{R}$. Moreover, for some positive constant $C = C(T)$, we have*

$$\|(\eta(t), \omega(t))\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} \le C\left(\left\|\left(\eta_0, \omega_0\right)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \left\|\left(h_1, h_2\right)\right\|_{L^2\left(0,T;[H^{-2}(0,L)]^2\right)}\right),$$

for all $t \in [0,T]$.

Proof. Note that if $(\eta, \omega) \in C([0,T]; X_3)$, from the trace theorems, it follows that

$$f(t) = \alpha\eta_{xx}(t,L) + \beta\eta_{xx}(t-\tau(t),L) \in L^2(0,T).$$

We claim that: there exists a positive constant $C_{\alpha,\beta}$ such that

$$\|f\|_{L^2(0,T)} \le C_{\alpha,\beta}T^{1/2}\|(\eta,\omega)\|_{C([0,T];[H^3(0,L)\cap H_0^2(0,L)]^2)}. \tag{2.23}$$

Indeed, note that

$$\begin{aligned} \|f\|_{L^2(0,T)}^2 &\le |\alpha|^2 CT\|\eta\|_{C([0,T];H^3(0,L))}^2 + |\beta|^2\int_0^T |\eta(t-\tau(t),L)|^2 \mathrm{d}t \\ &\le |\alpha|^2 CT|\eta\|_{C([0,T];H^3(0,L))}^2 + |\beta|^2\int_0^{T-\tau(T)} |\eta(s,L)|^2\frac{1}{1-\dot{\tau}(t)}\mathrm{d}s. \end{aligned}$$

By using the conditions in (1.5), we deduce the existence of some positive constantC_M such that

$$\|f\|_{L^2(0,T)}^2 \le \left(|\alpha|^2 C + |\beta|^2\frac{C_M}{1-d}\right)T\,\|\eta\|_{C([0,T];H^3(0,L))}^2,$$

giving the claim.

Now, let $0 < \gamma \leq T$ to be determined later. For each $(\eta_0, \omega_0) \in X_3$, consider the map

$$\begin{aligned}\Gamma : C([0,\gamma]; [H^3(0,L) \cap H_0^2(0,L)]^2) &\longrightarrow C([0,\gamma]; [H^3(0,L) \cap H_0^2(0,L)]^2)\\ (\eta,\omega) &\longmapsto \Gamma(\eta,\omega) = (w,v),\end{aligned}$$

where (w, v) is the solution of (2.16) with $f(t) = \alpha\eta_{xx}(t, L) + \beta\eta_{xx}(t - \tau(t), L)$. By Lemma 2.7 and (2.23), the linear operator Γ is well defined. Furthermore, there exists a positive constant C_γ such that

$$\begin{aligned}\|\Gamma(\eta,\omega)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq & C_\gamma \Big(\left\|(\eta_0,\omega_0)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \left\|(h_1,h_2)\right\|_{L^2(0,\gamma:[H^{-2}(0,L)]^2)} \\ & + \|\alpha\eta_{xx}(L) + \beta\eta_{xx}(\cdot - \tau(\cdot), L)\|_{L^2(0,\gamma)}\Big) .\end{aligned}$$

From (2.23), it follows that

$$\begin{aligned}\|\Gamma(\eta,\omega)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq & C_\gamma \left(\left\|(\eta_0,\omega_0)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \left\|(h_1,h_2)\right\|_{L^2(0,\gamma:[H^{-2}(0,L)]^2)} \right) \\ & + C_{\alpha,\beta}\gamma^{1/2}\|(\eta,\omega)\|_{C([0,T];[H^3(0,L)\cap H_0^2(0,L)]^2)}.\end{aligned}$$

Let $(\eta, \omega) \in B_R(0)$, where

$$B_R(0) := \left\{ (\eta,\omega) \in C\left([0,\gamma]; [H^3(0,L) \cap H_0^2(0,L)]^2\right) \ : \|(\eta,\omega)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq R \right\},$$

and

$$R = 2C_T \left(\left\|(\eta_0,\omega_0)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \left\|(h_1,h_2)\right\|_{L^2(0,T:[H^{-2}(0,L)]^2)} \right).$$

Choosing γ such that

$$C_{\alpha,\beta}\gamma^{1/2} \leq \frac{1}{2},$$

it follows that

$$\|\Gamma(\eta,\omega)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq R,$$

and

$$\|\Gamma(\eta_1,\omega_1) - \Gamma(\eta_2,\omega_2)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq \frac{1}{2}\|(\eta_1,\omega_1) - (\eta_2,\omega_2)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)}.$$

Hence, $\Gamma : B_R(0) \to B_R(0)$ is a contraction, and by Banach Fixed Point Theorem, we obtain a unique $(\eta, \omega) \in B_R(0)$ such that $\Gamma(\eta, \omega) = (\eta, \omega)$ and

$$\|(\eta,\omega)\|_{C([0,\gamma];[H^3(0,L)\cap H_0^2(0,L)]^2)} \leq 2C_T \left(\left\|(\eta_0,\omega_0)\right\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} + \left\|(h_1,h_2)\right\|_{L^2(0,T:[H^{-2}(0,L)]^2)} \right).$$

Since γ is independent of (η_0, ω_0), the standard continuation extension argument yields that the solution (η, ω) belongs to $C([0, T]; [H^3(0, L) \cap H_0^2(0, L)]^2)$, and the proof ends. □

The first main result of the article ensures the existence of local solutions to (1.4) and is proved below.

Proof of Theorem 1.1. Let $T > 0$ and $\|(\eta_0, \omega_0)\|_{[H^3(0,L)\cap H_0^2(0,L)]^2} < \theta$, where $\theta > 0$ will be determined later. We know from Capistrano-Filho et al. (2019) that for $(\eta, \omega) \in C([0, T]; [H^3(0, L) \cap H_0^2(0, L)]^2)$, there exists a positive constant C_1 such that the following inequalities hold true:

$$\begin{aligned}\|\eta\omega_x\|_{L^2(0,T;L^2(0,L))} &\leq C_1T^{1/2}\|(\eta,\omega)\|^2_{C(0,T;[H^3(0,L)\cap H_0^2(0,L)]^2)},\\ \|\eta_x\omega_{xx}\|_{L^2(0,T;L^2(0,L))} &\leq C_1T^{1/2}\|(\eta,\omega)\|^2_{C(0,T;[H^3(0,L)\cap H_0^2(0,L)]^2)},\end{aligned}$$

and

$$\|\eta\omega_{xxx}\|_{L^2(0,T;L^2(0,L))} \leq C_1 T^{1/2}\|(\eta,\omega)\|^2_{C(0,T;[H^3(0,L)\cap H^2_0(0,L)]^2)}.$$

Thus, the nonlinearities

$$(h_1,h_2) := \left(-(\eta\omega)_x - \alpha'(\eta\omega_{xx})_x, -\omega\omega_x - c(\omega\omega_x)_{xx} - (\eta\eta_{xx})_x + \beta'\omega_x\omega_{xx} + \rho\omega\omega_{xxx}\right)$$

belong to $L^2(0,T;[L^2(0,L)]^2)$, and

$$\begin{cases} \|h_1\|_{L^2(0,T;L^2(0,L))} \leq (2+2|\alpha'|)C_1T^{1/2}\|(\eta,\omega)\|^2_{C(0,T;[H^3(0,L)\cap H^2_0(0,L)]^2)}, \\ \|h_2\|_{L^2(0,T;L^2(0,L))} \leq (3+4|c|+|\beta'|+|\rho|)C_1T^{1/2}\|(\eta,\omega)\|^2_{C(0,T;[H^3(0,L)\cap H^2_0(0,L)]^2)}. \end{cases} \tag{2.24}$$

Taking this into consideration, we define the following map:

$$\begin{aligned} \Gamma : C\left([0,T];\left[H^3(0,L)\cap H^2_0(0,L)\right]^2\right) &\longrightarrow C\left([0,T];\left[H^3(0,L)\cap H^2_0(0,L)\right]^2\right) \\ (\eta,\omega) &\longmapsto \Gamma(\eta,\omega) = (\bar{\eta},\bar{\omega}), \end{aligned}$$

where $(\bar{\eta},\bar{\omega})$ is the solution of (2.16) with

$$(h_1,h_2) \in L^2(0,T;[L^2(0,L)]^2) \subset L^2(0,T;[H^{-2}(0,L)]^2),$$

as defined above, and with $f(t) = \alpha\eta_{xx}(t,L) + \beta\eta_{xx}(t-\tau(t),L)$. From Lemma 2.8, we find that Γ is well defined and there exists a positive constant C_T such that

$$\|\Gamma(\eta,\omega)\|_{C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2)} \leq C_T\left(\left\|\left(\eta_0,\omega_0\right)\right\|_{[H^3(0,L)\cap H^2_0(0,L)]^2} + \left\|\left(h_1,h_2\right)\right\|_{L^2\left(0,T:[H^2_0(0,L)]^2\right)}\right).$$

On the other hand, from the inequalities (2.24), we have that

$$\|\Gamma(\eta,\omega)\|_{C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2)} \leq C_T\left\|\left(\eta_0,\omega_0\right)\right\|_{[H^3(0,L)\cap H^2_0(0,L)]^2} + C_T 13KC_1T^{1/2}\|(\eta,\omega)\|^2_{C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2)}, \tag{2.25}$$

where $K = \max\{1,|c|,|\alpha'|,|\beta'|,|\rho|\}$. Now, we consider the ball

$$B_R(0) = \{(\eta,\omega) \in C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2) : \|(\eta,\omega)\|_{C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2)} \leq R\},$$

with

$$R = 2C_T\|(\eta_0,\omega_0)\|_{[H^3(0,L)\cap H^2_0(0,L)]^2}.$$

The inequality (2.25) leads to

$$\|\Gamma(\eta,\omega)\|_{C([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2)} \leq \frac{R}{2} + C_T 13KC_1T^{1/2}R^2 \leq \frac{R}{2} + C_T^2 26KC_1T^{1/2}\theta R,$$

for all $(\eta,\omega) \in B_R(0)$. By choosing θ such that

$$C_T^2 26KC_1T^{1/2}\theta < \frac{1}{4},$$

we obtain that $\Gamma(B_R(0)) \subset B_R(0)$. Finally, following the same argument as done in Lemma 2.8, we can conclude that Γ is a contraction in $B_R(0)$, then, the Banach Fixed Point Theorem guarantees the existence of a unique $(\eta,\omega) \in B_R(0)$ such that $\Gamma(\eta,\omega) = (\eta,\omega)$ and

$$\|(\eta,\omega)\|_{C\left([0,T]:[H^3(0,L)\cap H^2_0(0,L)]^2\right)} \leq 2C_T\left\|\left(\eta_0,u_0\right)\right\|_{[H^3(0,L)\cap H^2_0(0,L)]^2},$$

achieving the proof. □

Remark 1. In (1.5), the time-dependent delay is assumed to be positive for all $t \geq 0$. This requirement is relaxed in Nicaise et al. (2011) since τ is allowed to degenerate. Notwithstanding, the problem in Nicaise et al. (2011) is linear and hence simpler than ours. The key idea of the proof in Nicaise et al. (2011) is to consider a new delay τ_ϵ defined by

$$\tau_\epsilon(t) := \epsilon + \tau_\epsilon(t),$$

where $0 < \epsilon < \overline{\epsilon}$, for some $\overline{\epsilon} > 0$. Therefore, τ_ϵ satisfy (1.5) and hence problem (2.1) has a unique solution U_ϵ. The whole task is to tend ϵ to 0 under more regularity on the solution. We have tried to adopt this approach, but we faced difficulties because of the nonlinearities in our problem.

3 Long-Time Behavior of Solutions

In this section, we are in a position to prove the second main result of our work. First, we demonstrate that the energy associated with (1.9) is exponentially stable. Moreover, we establish that the solutions decay at an optimal rate.

3.1 Proof of Theorem 1.2

Recall that Theorem 2.2 (see also Proposition 2.3) guarantees the L^2 a priori estimate for the linear system (2.1) whose operator is defined by (2.3)–(2.4). Therefore, the solutions of system (1.9) are globally well-posed. Whereupon, we can treat the exponential stability for this system.

To proceed, consider the following Lyapunov functional

$$V(t) = E(t) - \mu_1 V_1(t) + \mu_2 V_2(t),$$

where $\mu_1, \mu_2 \in \mathbb{R}^+$ will be chosen later. Here, $E(t)$ is the total energy given by (1.8), while

$$V_1(t) = \int_0^L x\eta(t,x)\omega(t,x)\,\mathrm{d}x$$

and

$$V_2(t) = \frac{|\beta|}{2}\tau(t)\int_0^1 (1-\rho)\eta_{xx}^2(t-\tau(t)\rho, L)\,\mathrm{d}\rho.$$

Observe that,

$$(1 - \max\{\mu_1 L, \mu_2\})E(t) \leq V(t) \leq (1 + \max\{\mu_1 L, \mu_2\})E(t), \tag{3.1}$$

by assuming $0 < \mu_1 < 1/L$ and $0 < \mu_2 < 1$.

On the other hand, using system (1.9) and the boundary conditions, we get that

$$\begin{aligned} V_1'(t) &= \int_0^L x\eta_t\omega\,\mathrm{d}x + \int_0^L x\eta\omega_t\,\mathrm{d}x \\ &= -\frac{a_1 L}{2}\begin{pmatrix} \eta_{xx}(t,L) \\ \eta_{xx}(t-\tau(t),L)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix} \alpha^2+1 & \alpha\beta \\ \alpha\beta & \beta^2\end{pmatrix}\begin{pmatrix} \eta_{xx}t,L) \\ \eta_{xx}(t-\tau(t),L)\end{pmatrix} \\ &\quad + \frac{1}{2}\int_0^L \left(\omega^2+\eta^2\right)\mathrm{d}x - \frac{3a}{2}\int_0^L \left(\omega_x^2+\eta_x^2\right)\mathrm{d}x + \frac{5a_1}{2}\int_0^L \left(\omega_{xx}^2+\eta_{xx}^2\right)\mathrm{d}x. \end{aligned} \tag{3.2}$$

In addition, from (2.2) and by integration by parts, we deduce that

$$V_2'(t) = -\frac{|\beta|}{2}\int_0^1 (1-\dot{\tau}(t)\rho)\eta_{xx}^2(t-\tau(t)\rho, L)\mathrm{d}\rho + \frac{|\beta|}{2}\eta_{xx}^2(t,L). \tag{3.3}$$

Thus, from (2.7), (3.2), and (3.3), we have that

$$V'(t) + \lambda V(t) = S_1 + S_2 + S_3, \tag{3.4}$$

where

$$S_1 = \frac{1}{2}\left\langle \Psi_{\mu_1,\mu_2}(\eta_{xx}(t,L),\eta_{xx}(t-\tau(t),L)),(\eta_{xx}(t,L),\eta_{xx}(t-\tau(t),L))\right\rangle,$$

with (recall (1.6))

$$\begin{aligned}\Psi_{\mu_1,\mu_2} &= \Phi_{\alpha,\beta} + \frac{a_1 L\mu_1}{2}\begin{pmatrix}\alpha^2+1 & \alpha\beta\\ \alpha\beta & \beta^2\end{pmatrix} + \frac{|\beta|\mu_2}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\\ S_2 &= -\frac{\mu_1}{2}\int_0^L\left(\omega^2+\eta^2\right)\mathrm{d}x + \frac{3a\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)\mathrm{d}x + \frac{\lambda}{2}\int_0^L\left(\eta^2+\omega^2\right)\mathrm{d}x\\ &\quad + \mu_1\lambda\int_0^L x\eta\omega\,\mathrm{d}x - \frac{5a_1\mu_1}{2}\int_0^L\left(\omega_{xx}^2+\eta_{xx}^2\right)\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}S_3 &= -\mu_2\frac{|\beta|}{2}\int_0^1(1-\dot{\tau}(t)\rho)\eta_{xx}^2(t-\tau(t)\rho,L)\mathrm{d}\rho + \frac{\lambda|\beta|}{2}\tau(t)\int_0^1\eta_{xx}^2(t-\tau(t)\rho,L)\mathrm{d}\rho\\ &\quad + \frac{\mu_2|\beta|\lambda}{2}\tau(t)\int_0^1(1-\rho)\eta_{xx}^2(t-\tau(t)\rho,L)\mathrm{d}\rho,\end{aligned}$$

respectively.

The objective is to show that $V'(t)+\lambda V(t)\le 0$. To do so, let us deal with each term S_i in (3.4), for $i=1,2,3$.
Estimate of S_1: Since the matrix $\Phi_{\alpha,\beta}$ (see (1.6)) is negative definite, it follows from the continuity of the trace and determinant functions that one can choose $\mu_1,\mu_2\in(0,1)$ sufficiently small so that the new matrix Ψ_{μ_1,μ_2} is also negative definite. Thus,

$$S_1\le 0.$$

Estimate of S_2: Observe that using Poincaré inequality, we get

$$\begin{aligned}S_2 &\le \frac{L^2}{2\pi^2}\lambda(1+\mu_1 L)\int_0^L\left(\omega_x^2+\eta_x^2\right)\mathrm{d}x + \frac{3a\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)\mathrm{d}x - \frac{5a_1\mu_1}{2}\int_0^L\left(\omega_{xx}^2+\eta_{xx}^2\right)\mathrm{d}x\\ &\le \left[\frac{L^2}{2\pi^2}\left(\lambda\left(1+\mu_1 L\right)\frac{L^2}{2\pi^2}+3a\mu_1\right)-\frac{5a_1\mu_1}{2}\right]\int_0^L\left(\omega_{xx}^2+\eta_{xx}^2\right)\mathrm{d}x.\end{aligned}$$

Thus,

$$S_2<0,$$

if

$$\lambda<\frac{\mu_1\pi^2\left(5a_1\pi^2-3aL^2\right)}{L^4(1+\mu_1)}.$$

Estimate of S_3: We proceed as in Capistrano-Filho et al. (2024), choosing

$$\lambda<\frac{\mu_2(1-d)}{M(1+\mu_2)},$$

it follows that

$$S_3<0.$$

Therefore, for the estimates above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t)+\lambda V(t)\le 0,$$

and, since $V(t)$ satisfies (3.1), we deduce that

$$E(t)\le\zeta E(0)e^{-\lambda t},\quad \forall t\ge 0,$$

for $\zeta>0$ and $\lambda>0$ fulfilling (1.10) and (1.11), respectively. This achieves the proof of the theorem. □

3.2 Decay Rate: An Optimal Result

We can optimize the value of λ in Theorem 1.2 to obtain the best decay rate for the linear system (1.9) in the following way:

Proposition 3.1. *If the constant μ_1 given in Theorem 1.2 is chosen as follows:*

$$\mu_1 \in \left[0, \frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right), \tag{3.5}$$

then λ has the largest possible value.

Proof. Define the functions f and

$$g : \left[0, \frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right] \to \mathbb{R}$$

by

$$f(\mu_1) = \frac{\mu_1\pi^2\left(5a_1\pi^2 - 3aL^2\right)}{L^4(1+\mu_1 L)},$$

and

$$g(\mu_1) = \frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta| - L(1-d)(a_1^2+\alpha^2)\mu_1}{M\left(2a_1\alpha(1-d) - a_1^2|\beta| - L(1-d)(a_1^2+\alpha^2)\mu_1\right)}(1-d),$$

respectively. On the other hand, let us consider $\lambda(\mu_1) = \min\{f(\mu_1), g(\mu_1)\}$. Thus, we have the following claims.

Claim 2. The function f (respectively, g) is increasing (respectively, decreasing) in the interval

$$\left[0, \frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right).$$

A simple computation shows that

$$f'(\mu_1) > 0, \text{ for all } \mu_1 \geq 0$$

and hence $f'(\mu_1) > 0$ for

$$\mu_1 \in \left[0, \frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right).$$

Furthermore, one can rewrite g as follows:

$$g(\mu_1) = \frac{1-d}{M} - \frac{|\beta|(1-d)^2}{ML(1-d)(a_1^2+\alpha^2)}\left(\frac{1}{\frac{2a_1\alpha(1-d)-a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)} - \mu_1}\right),$$

and thus

$$g'(\mu_1) = -\frac{|\beta|(1-d)^2}{ML(1-d)(a_1^2+\alpha^2)}\left[\frac{1}{\left(\frac{2a_1\alpha(1-d)-a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)} - \mu_1\right)^2}\right] < 0.$$

This ascertains the Claim 2.

Claim 3. There exists only one point μ_1, satisfying (3.5) such that $f(\mu_1) = g(\mu_1)$.

Indeed, since

$$f(0) = 0, \qquad f\left(\frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right) > 0,$$

and

$$g(0) > 0, \qquad g\left(\frac{(2a_1\alpha - |\beta|)(1-d) - a_1^2|\beta|}{L(1-d)(a_1^2+\alpha^2)}\right) = 0,$$

the existence of this point is a direct consequence of the mean value theorem, applied to the function $F = f - g$. The uniqueness follows from the fact that the function $F = f - g$ is increasing in this interval, and Claim 3 holds.

Lastly, thanks to the Claims 2 and 3, the maximum value of the function λ is obtained when μ_1 satisfies (3.5), where $f(\mu_1) = g(\mu_1)$, and the proof of Proposition 3.1 is achieved. □

4 Conclusion

This paper establishes the existence and uniqueness of a solution for a higher-order nonlinear Boussinesq system in a bounded domain, even when a time-dependent delay is present in one of the boundary conditions. Additionally, we prove that solutions to the linearized problem are exponentially stable, both results being obtained under certain conditions related to the system's parameters and the delay. These findings extend the results of the second and third authors in Capistrano-Filho et al. (2024) for a higher-order dispersive system. Further comments on our results are provided below.

(1) It is worth mentioning that the solutions of system (1.4) obtained in Theorem 1.1 are local. Proving the global existence of solutions remains a challenge due to the absence of an a priori L^2 estimate. Specifically, it is challenging to tackle this problem within the energy space for the nonlinear system that includes a delay term.

(2) Observe that the restriction $0 < L < \sqrt{(5a_1/3a)}\pi$ in Theorem 1.2 arises from the Kato smoothing effect, which does not occur in the lower-order Boussinesq system (see, e.g., Capistrano-Filho et al., 2024). This difference is because in system (1.4), we have spatial derivatives of order three and five, both with positive signs. Thus, after performing some integration by parts, the left-hand side of (2.14) contains the H^1-norm with a negative sign and the H^2-norm with a positive sign. To recover the H^2-norm, the Poincaré inequality must be applied, which imposes this restriction on the size of L.

(3) A version of the higher-order Boussinesq system was proposed by Olver (1984, equations (4.7) and (4.8), p. 283) and is given by:

$$\begin{cases} \eta_t + u_x + \frac{1}{6}\beta\left(3\theta^2 - 1\right)u_{xxx} + \frac{1}{120}\beta^2\left(25\theta^4 - 10\theta^2 + 1\right)u_{xxxxx} \\ \quad + \alpha(\eta u)_x + \frac{1}{2}\alpha\beta\left(\theta^2 - 1\right)\left(\eta u_{xx}\right)_x = 0, \\ u_t + \eta_x + \beta\left[\frac{1}{2}\left(1 - \theta^2\right) - \tau\right]\eta_{xxx} + \beta^2\left[\frac{1}{24}\left(\theta^4 - 6\theta^2 + 5\right) + \frac{\tau}{2}\left(\theta^2 - 1\right)\right]\eta_{xxxxx} \\ \quad + \alpha u u_x + \alpha\beta\left[\left(\eta\eta_{xx}\right)_x + \left(2 - \theta^2\right)u_x u_{xx}\right] = 0. \end{cases}$$

Through scaling, we arrive at the following system:

$$\begin{cases} \eta_t + u_x - a u_{xxx} + a_1(\eta u)_x + a_2\left(\eta u_{xx}\right)_x + b u_{xxxxx} = 0, & \text{in } (0,L)\times(0,\infty), \\ u_t + \eta_x - a\eta_{xxx} + a_1 u u_x + a_3\left(\eta\eta_{xx}\right)_x + a_4 u_x u_{xx} + b\eta_{xxxxx} = 0, & \text{in } (0,L)\times(0,\infty), \\ \eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{4.1}$$

where $a > 0, b > 0, a \neq b, a_1 > 0, a_2 < 0, a_3 > 0$ and $a_4 > 0$. System (4.1) was studied in Capistrano-Filho et al. (2019). Using the same boundary conditions as in problem (1.4), we believe that similar results proved in our work can be obtained for system (4.1) without the restriction over L since the sign of the third derivatives in (4.1) is negative instead of positive unlike our case (see system (1.4)).

(4) It is important to point out that system (1.4) is shown to be locally well-posed, and hence we are unable to establish any exponential stability for the nonlinear problem. One interesting research avenue is to show the stability for the nonlinear problem.

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