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JUAN RICARDO MUÑOZ GALEANO

CONTROL AND STABILIZATION OF KDV-KDV AND KP TYPE SYSTEMS

Recife

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CONTROL AND STABILIZATION OF KDV-KDV AND KP TYPE SYSTEMS

Thesis submitted to the Graduate Program in Mathematics of the Federal University of Pernambuco, as a partial requirement for the degree of Doctor of Philosophy in Mathematics.

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JUAN RICARDO MUÑOZ GALEANO

Control and stabilization of KdV-KdV and KP type systems

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"La información es un caos. El conocimiento es el orden espontáneo dentro de ese caos. La libertad es navegar en la onda de esa espontaneidad. Esa es nuestra economía y nuestra guerra." – Mario Mendoza, Buda Blues.

RESUMO

Esta tese apresenta um estudo sobre a estabilização e controle na fronteira de diversos sistemas dispersivos não lineares, incluindo o sistema Boussinesq tipo KdV-KdV, sistema Hirota-Satsuma, a equação de Kadomtsev-Petviashvili (KP) e sua variante de ordem superior, a equação Kawahara-KP (K-KP). Para o sistema Boussinesq tipo KdV-KdV e Hirota-Satsuma, projetamos leis de *feedback* na fronteira que combinam mecanismos de amortecimento e termos de *delay*, demonstrando o decaimento exponencial da energia associada ao sistema, tendo em vista dados iniciais pequenos. Para tal, utilizamos o métodos de Lyapunov e argumentos de ponto fixo. No contexto da equação KP, exploramos o fenômeno do comprimento crítico, derivando desigualdades de observabilidade que levam à controlabilidade de fronteira e estabilização exponencial. Tais resultados dependem do comprimento espacial e são demonstrados usando o Teorema de Paley-Wiener. Por fim, para a equação de K-KP, estabelecemos resultados de estabilidade exponencial local e global através de duas abordagens diferentes, fornecendo constantes ótimas e o tempo mínimo para garantir o decaimento exponencial da energia.

Palavras-chave: Equações dispersivas, dissipação na fronteira, *delay feedback*, estabilidade exponencial, fenômeno de comprimentos críticos.

ABSTRACT

This thesis presents a study on the boundary stabilization and control of several nonlinear dispersive systems, including the Boussinesq KdV-KdV type system, the Hirota-Satsuma system, the Kadomtsev-Petviashvili (KP) equation and its higher-order variant, the Kawahara-KP (K-KP) equation. For the Boussinesq KdV-KdV type system and Hirota-Satsuma system, we design feedback laws at the boundary that combine damping mechanisms and delay terms, demonstrating the exponential decay of the energy associated with the system, given small initial data. For this purpose, we use the Lyapunov method and fixed-point arguments. In the context of the KP equation, we explore the critical length phenomenon, deriving observability inequalities that lead to boundary controllability and exponential stabilization. These results depend on the spatial length and are demonstrated using the Paley-Wiener Theorem. Finally, for the K-KP equation, we establish local and global exponential stability results through two different approaches, providing optimal constants and the minimum time to ensure exponential decay of the energy.

Keywords: Dispersive equations, boundary inputs, delay feedback, exponential stability, critical length phenomenon.

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1 GENERAL INTRODUCTION

The objective of this thesis is to address the controllability and stabilization of dispersive systems that are governed by partial differential equations. Controllability aims to determine if a control input acting in the domain or through the boundary allows us to drive the solution from a given initial state to a desired terminal state. On the other hand, stabilization focuses on studying the behavior of solutions over time and determining if the solutions are asymptotically stable for arbitrarily large values of time. If the answer is positive, then the rate of decay of these solutions will be determined.

The dispersive models studied here are the Boussinesq KdV-KdV type system, the Hirota-Satsuma system, the Kadomtsev-Petviashvili (KP) equation and the Kawahara-Kadomtsev-Petviashvili (K-KP) equation posed on a bounded domain. Before introducing the required concepts and the mathematical description of the problems, we start with a historical review.

1.1 HOW DID THE DISPERSIVE EQUATIONS BEGIN?

In 1834, John Scott Russell, a Scottish naval engineer, observed the Union Canal in Scotland when he witnessed a unique physical phenomenon called a "wave of translation" (RUSSELL, 1844). He observed a wave moving across the canal without changing shape or pace. He stated it precisely:

> *I was observing the motion of a boat that was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it, still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel. Such, in August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.*

This experience caught his curiosity so much that he spent years researching the waves in issue and pushed the scientific community to create a precise mathematical model to characterize this phenomenon precisely. Not satisfied with his findings, he designed a series of tests called *The Hull Construction Wave Line System* to reproduce the wave propagated along the channel in various conditions. His findings revolutionized maritime engineering in the nineteenth century, and he received the Royal Society of Edinburgh's Gold Medal in 1837.

Russell's research disproved several physical theories, including those of Airy (AIRY, 1845) regarding water wave theory, which held that waves could not exist because they would eventually change in shape or speed, and Stokes' theory (STOKES, 1847), which proposed that waves with fixed forms and finite amplitudes could exist, but only in deep water and in periodic form. Nonetheless, Stokes admitted that he knew Russell's theory's incompleteness.

> *It is the opinion of Mr. Russell that the solitary wave is a phenomenon sui generis, in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered.*

The first satisfactory answer was given by Boussinesq (BOUSSINESQ, 1871a) in 1871. The next to get a positive result was Lord Rayleigh (RAYLEIGH, 1876) in 1876, and the last significant result of the 19th century was given by Korteweg and de-Vries (KORTEWEG; VRIES, 1895) in 1895.

Precisely, Boussinesq considered a model of long, incompressible, and rotation-free waves in a shallow channel with a rectangular cross-section neglecting the friction along the boundaries, and he obtained the equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\left[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\right],$$

where (t, x) are the coordinates of a fluid particle at time t, h denotes the amplitude of the wave, H is the height of the water in equilibrium and g is the gravitational constant. Separately, Rayleigh reasoned about the same issue, adding the assumption that there is a stationary wave that vanishes at infinity, and considering only spatial dependence he obtained the description of the behavior of the wave

$$\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 + \frac{3}{H^3}h^2(h - h_0) = 0,$$

with h_0 being the crest of the wave and the other parameters defined as before. Additionally, this equation has an explicit solution given by

$$h(x) = h_0 \operatorname{sech}^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right).$$

In 1876, Rayleigh wrote in his article (RAYLEIGH, 1876):

> *I have lately seen a memoir by Mr. Boussinesq, Comptes Rendus, Vol. LXXII, which contains a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to Boussinesq J.*

In the last result mentioned above, given by Korteweg and de-Vries. They constructed a nonlinear partial differential equation such that the solution describes the phenomenon discovered by Russell. Then, the renowned Korteweg–de Vries (KdV) equation arises,

$$\frac{\partial \eta}{\partial l} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right),$$

in which η denotes the surface elevation above the equilibrium level l, α is a small arbitrary constant related to the motion of the liquid, g is the gravitational constant, and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$, is defined in terms of surface capillary tension T and density ρ. It is possible to simplify the physical constants by the change of variables

$$t \mapsto \frac{1}{2}\sqrt{\frac{g}{l\sigma}}t, \quad x \mapsto -\frac{x}{\sqrt{\sigma}} \quad \text{and} \quad u \mapsto -\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

one obtains the standard KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1.1}$$

which describes the propagation of small amplitude, long wavelength waves on an air-sea interface in a canal of the rectangular cross-section.

After briefly discussing the origins of the dispersive equations, we proceed to introduce the dispersive equations that are involved.

1.2 DISPERSIVE SYSTEMS

Dispersive equations play a crucial role in understanding the behavior and evolution of waves in different physical systems. These equations describe the dispersion phenomenon, which is the dependence of wave propagation on the wavelength or frequency. Roughly speaking, dispersion means that "Waves with different frequencies travel at different velocities," and the dispersion property is related to the linear part. Now, inspired in (LINARES; PONCE, 2015) let's formally define a dispersive partial differential equation. In the one-dimensional context, we consider a linear partial differential equation

$$F\left(\partial_x, \partial_t\right) u(x,t) = 0,$$

where F is a polynomial in the partial derivatives. We look for plane wave solutions of the form $u(x,t) = Ae^{i(kx-\omega t)}$ where A, k, and ω are constants representing the amplitude, the wavenumber, and the frequency, respectively. Hence u will be a solution if and only if

$$F(ik, -i\omega) = 0. \tag{1.2.1}$$

(1.2.1) is called the dispersion relation. This relation characterizes the plane wave motion. In several models, we can write ω as a real function of k, namely, $\omega = \omega(k)$. The phase and group velocities of the waves are defined by

$$c_p(k) = \frac{\omega}{k} \quad \text{and} \quad c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

The waves are called dispersive if the group velocity $c_g = \omega'(k)$ is not constant, i.e., $\omega''(k) \neq 0$.

One of the outstanding examples of a dispersive PDE is the KdV equation (1.1.1). Precisely, the KdV equation is a nonlinear dispersive partial differential equation that models the behavior of long waves in shallow water. It describes the evolution of small-amplitude, long-wavelength waves in a one-dimensional system.

The study on asymptotic models for water waves has been extensively investigated to understand the full water wave system (see (ALVAREZ SAMANIEGO; LANNES, 2008; BONA; COLIN; LANNES, 2005; BONA; LANNES; SAUT, 2008) and references therein) as well arising some systems that model the interaction between the waves as the Boussinesq system introduced by Bona, Chen and Saut (BONA; CHEN; SAUT, 2002) and the interaction of waves coupled on the nonlinearity was introduced by Hirota and Satsuma (HIROTA; SATSUMA, 1981). Furthermore, the two-dimensional description for a wave phenomena that exhibits weak transversality and weak nonlinearity was discovered by Kadomtsev and Petviashvili (KADOMTSEV; PETVIASHVILI, 1970). Next, we specify the dispersive systems involved in this thesis.

1.2.1 The Boussinesq KdV-KdV type system

The Boussinesq system describes the propagation of small amplitude waves on the surface of a water channel. This type of system and its generalizations are highly useful when studying wave propagation in large lakes, oceans, and similar bodies of water.

As early mentioned, Bona, Chen, and Saut (BONA; CHEN; SAUT, 2002) have managed to derive a Boussinesq-type system that describes phenomena of the same nature

$$\begin{cases} \eta_t(x,t) + \omega_x + (\eta\omega)_x + a\omega_{xxx} - b\eta_{xxt} = 0, \\ \omega_t(x,t) + \eta_x + \omega\omega_x + c\eta_{xxx} - d\omega_{xxt} = 0, \end{cases} \tag{1.2.2}$$

which are all first-order approximations of the Euler equations. Here $a, b, c, d \in \mathbb{R}$ fulfills

$$\begin{gathered} a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}\left(1 - \theta^2\right) \geq 0, \\ 0 \leq \theta \leq 1, \quad a + b + c + d = \frac{1}{3}. \end{gathered} \tag{1.2.3}$$

θ specifies which horizontal velocity variables ω represents, $\omega = \omega\theta$ is the nondimensional horizontal velocity in the flow corresponding to the physical velocity at height θh, where h is the undisturbed depth of the liquid. As will appear, the constant in (1.2.3) arise in the form

$$\begin{gathered} a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda, \quad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda), \\ c = \frac{1}{2}\left(1 - \theta^2\right)\mu, \quad d = \frac{1}{2}\left(1 - \theta^2\right)(1-\mu), \end{gathered} \tag{1.2.4}$$

the real parameters λ and μ and do not have a direct physical interpretation like θ does. It is worth noting that (1.2.3) is a consequence of (1.2.4), but the importance of λ and μ is clarified in the modeling of the derived Boussinesq systems (see (BONA; CHEN; SAUT, 2002, Section 2)). Precisely, taking $\theta^2 = \frac{2}{3}$, $\lambda = \mu = 1$, the coupled Boussinesq KdV-KdV system appears as

$$\begin{cases} \eta_t + \omega_x + \frac{1}{6}\omega_{xxx} + (\eta\omega)_x = 0, \\ \omega_t + \eta_x + \frac{1}{6}\eta_{xxx} + \omega\omega_x = 0. \end{cases} \tag{1.2.5}$$

By using the scaling $x \mapsto \frac{x}{\sqrt{6}}$ and $t \mapsto \frac{t}{\sqrt{6}}$ we get the model investigated in this thesis

$$\begin{cases} \eta_t + \omega_x + \omega_{xxx} + (\eta\omega)_x = 0, \\ \omega_t + \eta_x + \eta_{xxx} + \omega\omega_x = 0. \end{cases} \tag{1.2.6}$$

1.2.2 The Hirota-Satsuma system

The Hirota-Satsuma system is a set of nonlinear partial differential equations that arise in the context of soliton theory and integrable systems. Introduced by Hirota and Satsuma (HIROTA; SATSUMA, 1981), this system is a significant model in mathematical physics, particularly in the study of wave propagation and interactions.

The Hirota-Satsuma system consists of two Korteweg-de Vries (KdV) equations coupled on the nonlinear terms, which describe the evolution of two interacting wave fields. It is expressed as:

$$\begin{cases} u_t - a(uu_x + u_{xxx}) = 2bvv_x, \\ v_t + v_{xxx} + 3uv_x = 0, \end{cases} \tag{1.2.7}$$

where a and b are arbitrary constants. Notice that in (1.2.7) the term $2bvv_x$ acts as a force term on the KdV wave system with the linear dispersion relation $\omega(k) = ak^3$. Particularly, in the absence of the effect of v, (1.2.7) can be reduced to the usual KdV equation (1.1.1).

1.2.3 The Kadomtsev-Petviashvili type equation

The KdV equation (1.1.1), which describes a wide class of one-dimensional nonlinear waves in media with weak dispersion, includes special solutions known as solitons. These solitons play an important role in describing the behavior and properties of wave evolution problems. While these waves are typically stable formations, their stability can be affected when strict one-dimensionality is violated. This perturbation was established by Kadomtsev and Petviashvili (KADOMTSEV; PETVIASHVILI, 1970) in 1970 through the consideration of a plane wave $u = \exp(-i\omega t + ikr)$ of small amplitude and with a small wavelength along the x-axis.

The resulting wave phenomena exhibit weak transversality and weak nonlinearity, modeled by the equation

$$u_t + \alpha u_{xxx} + uu_x + \gamma\partial_x^{-1}u_{yy} = 0 \tag{1.2.8}$$

where α, γ are constants. Must be highlighted that the introduced disturbance of the soliton leads to elastic oscillations with weak damping.

In 1993, Karpman (KARPMAN, 1993) introduce a higher order dispersive system

$$u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x + \gamma\partial_x^{-1}u_{yy} = 0. \tag{1.2.9}$$

The fifth-order term may lead to significant qualitative effects on the stability of the transversal packed waves and in the nonlinear structures.

Summarizing, in this thesis, we deal first with dispersive systems; The Boussinesq and the Hirota-Satsuma systems typically deal with one-dimensional coupled wave phenomena, while the Kadomtsev-Petviashvili (KP) equation extends the study to two-dimensional wave propagation with additional complexities.

These previously mentioned systems capture both nonlinear and dispersive effects, making them essential for understanding wave dynamics in different contexts such as hydrodynamics, geophysics, plasma physics, nonlinear optics, etc. These models collectively enhance the understanding of wave dynamics in various physical systems, bridging theoretical insights and practical applications. It is also noteworthy that these three systems are integral to the study of wave dynamics. They are derived from or related to the Korteweg-de Vries (KdV) equation and its generalizations, each addressing different aspects of wave interactions and propagation in various contexts.

1.3 DELAY DIFFERENTIAL EQUATIONS

Delay differential equations (DDEs) naturally arise in various scientific and engineering contexts. The principal reason is that a system's current state not only depends on its present state but also its history. Therefore, the time-delay aspect is intrinsic to many real-world phenomena, such as population dynamics, wave systems, and signal processing. In these phenomena, the system's response is influenced by past states due to inherent delays in feedback mechanisms.

Introducing delays into differential equations leads to a richer and more complex dynamical behavior. These mechanisms can induce stability changes, oscillations, or even chaotic dynamics. Consequently, this makes the analysis and control of such systems both challenging and crucial for applications. Summarizing, delays can significantly impact system stability and performance.

Now, we introduce the simplest mathematical context that motivates the interest to study the delay differential equations. We consider the ODE control system

$$\begin{cases} \dot{y} = f(y,u), \\ y(0) = y_0, \end{cases}$$

where $y \in \mathbb{R}^n$ denotes the state and $u \in \mathbb{R}^m$ denotes the control. Suppose that $f(0,0) = 0$. By employing simple computations we can obtain that there exists a feedback law[1]

$$y \in \mathbb{R}^n \mapsto u(y) \in \mathbb{R}^m,$$

and constants $C, r > 0$ such that, for every solution of the closed-loop system $\dot{y} = f(y, u(y))$ with $|y(0)| \leq r$, we have

$$|y(t)| \leq Ce^{-t}|y(0)|, \quad \forall t \geq 0.$$

[1] For a general context see: Appendix A.2.3

However, if we consider a feedback mechanism that collects information in a past-time the behavior may be affected. More specifically, it is natural to wonder if the solutions of the ODE control system

$$\dot{y} = f(y,u) + k(t)y(t-\tau)$$

decay exponentially.

The next example shows how the delay acts in the asymptotic behavior,

Example 1.3.1. Let us consider the simplest example of a DDE

$$\begin{cases} y'(t) = -y(t-\tau), & t \geq 0, \\ y(t) = 1, & t \in [-\tau, 0]. \end{cases}$$

Here $\tau > 0$ denotes a delay. To solve this equation, we employ the *method of steps*. Note that the solution $y(t)$ for $t \in [(n-1)\tau, n\tau]$, $n \in \mathbb{N}$, can be determined in the following way. For $t \in [0,\tau]$, we have that $t - \tau \in [-\tau, 0]$. Hence,

$$y'(t) = -y(t-\tau) = -1 \implies y(t) = y(0) + \int_0^t (-1)\,ds = 1 - t,$$

for $t \in [0,\tau]$. Then, if we look for $t \in [\tau, 2\tau]$ we get

$$y'(t) = -y(t-\tau) = -[1-(t-\tau)] \implies y(t) = 1 - t + \frac{1}{2}(t-\tau)^2.$$

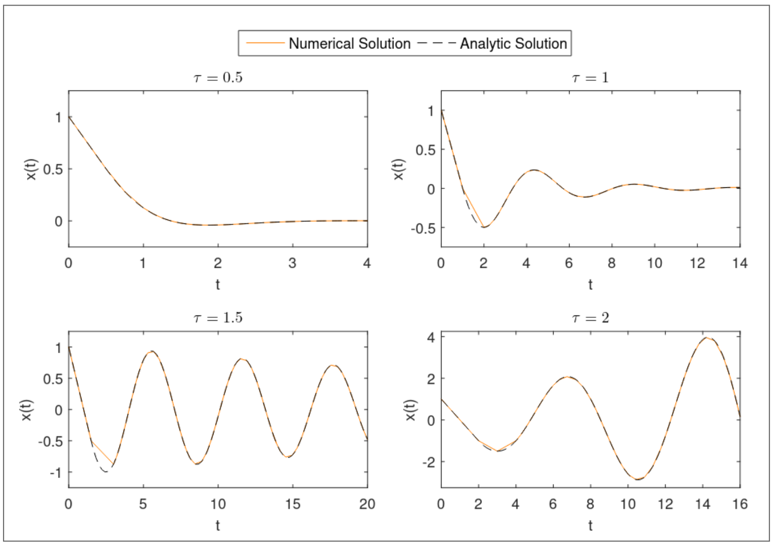
Inductively, follows that, for every $n \in \mathbb{N}$

$$y(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{(t-(k-1)\tau)^k}{k!}, \quad t \in [(n-1)\tau, n\tau].$$

It is noteworthy that $y(t)$ has a unique representation.

A direct use of the package DDE23 in Matlab gives some plots above the behavior of the solution. Observe that figure 1 suggests that a large amount of information in the past affected the exponential decay of the solutions.

Figure 1 – Some solutions of the DDE



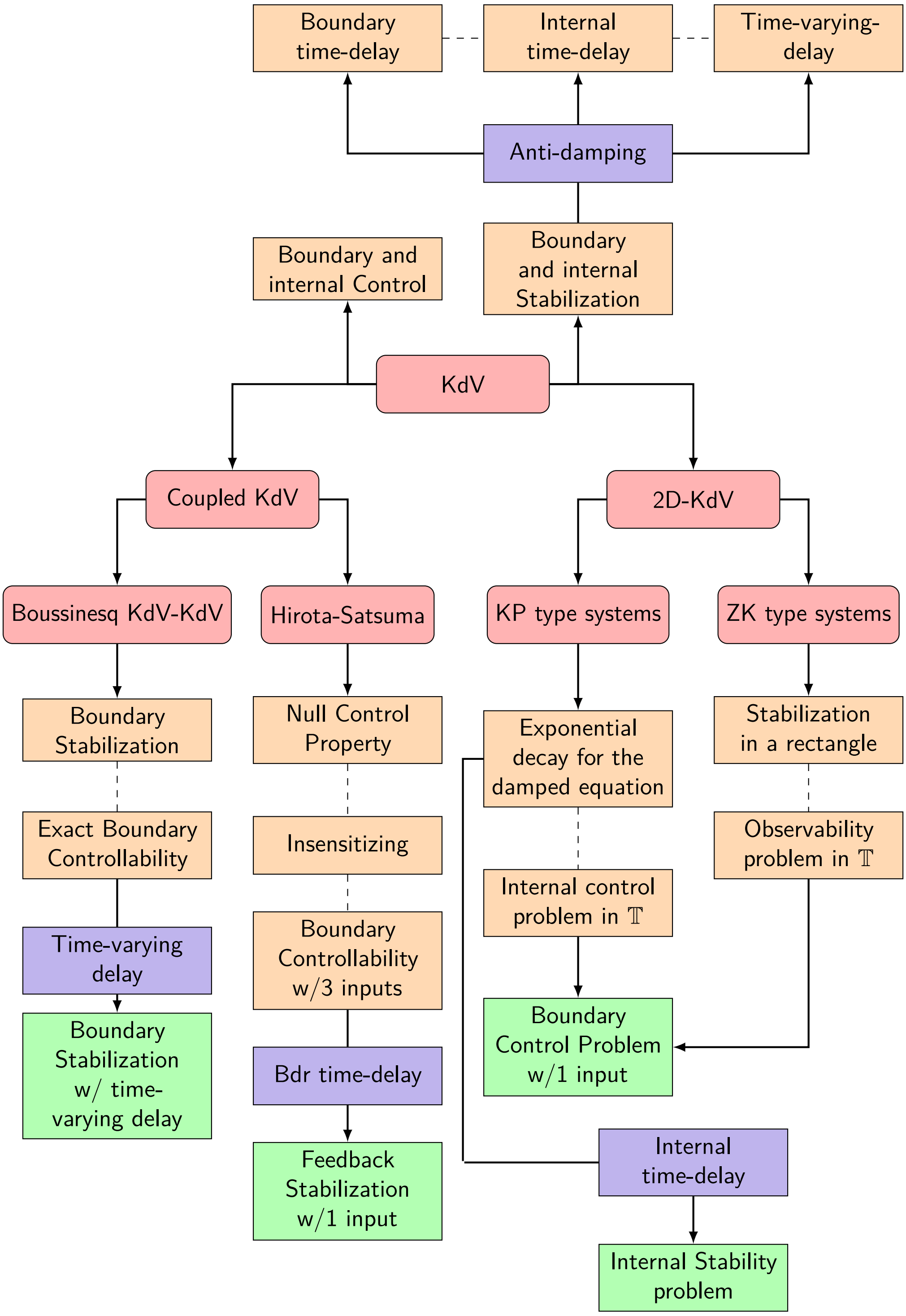
Source: (TERPSTRA, (2016))

1.4 SETTING UP THE PROBLEMS AND MAIN RESULTS

Understanding the behavior of systems and how to manipulate them is essential for numerous applications. These concepts ensure that dynamic systems, represented by mathematical models, behave predictably and remain stable under various conditions. Effective control and stabilization strategies are crucial in industrial and physical applications where achieving a goal with efficiency and performance is paramount. This physical phenomenon can be 'translated' by the differential equations language and consequently studied in a mathematical framework.

Now, we have created a summary diagram 2 that encompasses relevant issues addressed by various authors in this thesis. It is important to note that while other systems such as the Wave equation, KdV-Burgers, Kawahara, Gear–Grimshaw, Kuramoto-Sivashinsky, etc. have also been thoroughly analyzed, not all references are included in the diagram to keep it a reasonable size. We have chosen to start with the KdV due to its relevance in studying dispersive equations and the significant results that have provided a fundamental basis for other developments over the years.

Figure 2 – Diagram of relevant results



Source: Own elaboration

Therefore, this thesis will study two sets of problems. In the first part, we analyze the behavior of the solutions for a coupled KdV-type system under the effect of boundary feedback mechanisms. The different structures on the coupled terms allow us to address different issues *via* appropriate estimates intrinsic to each system. In the second part, we study a two-dimensional generalization of the KdV, the Kadomtsev-Petviashvili equation, here we are looking for the exponential stabilization property under the internal closed-loop feedback mechanism and the exact boundary controllability property. The dimension two problem is challenging and some classical one-dimensional estimates do not work directly.

1.4.1 Boundary stabilization for coupled KdV-KdV type systems with time-delay

The first result (CAPISTRANO-FILHO et al., 2024), joint with Roberto de A. Capistrano–Filho, Boumediène Chentouf, and Victor H. Gonzalez Martinez, addresses the boundary stabilization problem of the Boussinesq KdV-KdV type system. More precisely, we consider the Boussinesq system of KdV-KdV type posed on a bounded domain $[0, L]$

$$\begin{cases} \eta_t(t,x) + \omega_x(t,x) + \omega_{xxx}(t,x) + (\eta(t,x)\omega(t,x))_x = 0, \\ \omega_t(t,x) + \eta_x(t,x) + \eta_{xxx}(t,x) + \omega(t,x)\omega_x(t,x) = 0, \end{cases} \tag{1.4.1}$$

with the following set of boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0, & t \in \mathbb{R}^+, \\ \omega(t,0) = \omega(t,L) = \omega_x(t,L) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{1.4.2}$$

We first notice that the *global Kato smoothing effect* does not hold for the set of boundary conditions (1.4.2) and consequently, the well-posedness issue of employing classical methods remains open. Additionally, under the above boundary conditions, a simple integration by parts yields

$$\frac{d}{dt}E_0(t) = -\int_0^L (\eta(t,x)\omega(t,x))_x \eta(t,x)\, dx,$$

where

$$E_0(t) = \frac{1}{2}\int_0^L (\eta^2(t,x) + \omega^2(t,x))\, dx$$

is the total energy associated with (1.4.1) and (1.4.2). This indicates that *we do not have any control over the energy in the sense that its time derivative does not have a fixed sign*. Therefore, the next natural question arises

Question $\mathcal{A}$: *Is there a suitable set of boundary conditions so that the Kato smoothing effect can be revealed?*

Question $\mathcal{B}$: *Is there a feedback control law that permits the control of the nonlinear term presented in the derivative of the energy associated with the closed-loop system? Moreover, is this desired feedback law strong enough in the presence of a time-dependent delay?*

Question $\mathcal{C}$: *If the answer to these previous questions is yes, does $E_0(t) \to 0$ as $t \to \infty$? If this is the case, can we give an explicit decay rate?*

In this spirit, we design a boundary feedback law and demonstrate the well-posedness of the problem with time-varying delay feedback and smallness restrictions. Ergo, we study the behavior of the system

$$\begin{cases} \eta_t(t,x) + \omega_x(t,x) + \omega_{xxx}(t,x) + (\eta(t,x)\omega(t,x))_x = 0, & \mathbb{R}^+ \times (0,L), \\ \omega_t(t,x) + \eta_x(t,x) + \eta_{xxx}(t,x) + \omega(t,x)\omega_x(t,x) = 0, & \mathbb{R}^+ \times (0,L), \\ \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \omega(t,0) = \omega(t,L) = 0, & t \in \mathbb{R}^+, \\ \omega_x(t,L) = -\alpha\eta_x(t,L) + \beta\eta_x(t-\tau(t),L), & t > 0, \\ \eta_x(t-\tau(0),L) = z_0(t-\tau(0)) \in L^2(0,1), & 0 < t < \tau(0), \\ (\eta(0,x),\omega(0,x)) = (\eta_0(x),\omega_0(x)) \in X_0, & x \in (0,L). \end{cases} \tag{1.4.3}$$

where M and $d < 1$ are positive constants such that the time-dependent function $\tau(t)$ satisfies

$$\begin{cases} 0 < \tau(0) \leq \tau(t) \leq M, \quad \dot{\tau}(t) \leq d < 1, & \forall t \geq 0, \\ \tau \in W^{2,\infty}([0,T]), & T > 0, \end{cases} \tag{1.4.4}$$

and the feedback gains α and β must satisfy the following constraint

$$(2\alpha - |\beta|)(1-d) > |\beta|, \quad \text{for } 0 \leq d < 1. \tag{1.4.5}$$

Then, for total energy associated defined by

$$E(t) = \frac{1}{2}\int_0^L (\eta^2(t,x) + \omega^2(t,x))\,dx + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho,L)\,d\rho. \tag{1.4.6}$$

We obtain an exponential decay of the energy of the linearized Boussinesq KdV-KdV type system. Summarized in the next Theorem

Theorem 1.4.1. *Let $0 < L < \sqrt{3}\pi$. Suppose that (1.4.4) and (1.4.5) are satisfied. Then, for two positive constants μ_1 and μ_2 with $\mu_1 L < 1$, there exist*

$$\zeta = \frac{1 + \max\{\mu_1 L, \mu_2\}}{1 - \max\{\mu_1 L, \mu_2\}}, \tag{1.4.7}$$

and

$$\lambda \leq \min\left\{\frac{\mu_1(3\pi^2 - L^2)}{L^2(1+\mu_1)}, \frac{\mu_2(1-d)}{M(1+\mu_2)}\right\} \tag{1.4.8}$$

such that the energy $E(t)$ given by (1.4.6) associated to the linearized system of (1.4.3) around the origin satisfies

$$E(t) \leq \zeta E(0)e^{-\lambda t}, \quad \text{for all } t \geq 0.$$

The second main result (GONZALEZ MARTINEZ; MUÑOZ, 2024), joint with Victor H. Gonzalez Martinez, is inspired by the impetus to obtain the exponential stability property for a nonlinear coupled KdV type system. Then, we consider the Hirota-Satsuma system, posed on the bounded domain $(0, L)$ with $L > 0$ and $t > 0$,

$$\begin{cases} u_t - \frac{1}{2}u_{xxx} - 3uu_x - 3vv_x = 0 & x \in (0, L),\ t > 0 \\ v_t + v_{xxx} + 3uv_x = 0 & x \in (0, L),\ t > 0. \end{cases} \tag{1.4.9}$$

Physically, in the Hirota-Satsuma system, the interaction between two waves with different speeds focuses on how these waves influence each other with asymmetric dynamics, giving a different framework than that of the KdV-KdV system studied in the previous chapter. Here, we design a feedback mechanism acting on the boundary

$$\begin{cases} u(0) = u(L) = v(0) = v(L) = u_x(0) = 0, & t > 0, \\ v_x(L) = \alpha u_x(L) + \beta u_x(t-h, L), & t > 0, \end{cases} \tag{1.4.10}$$

that involves the parameters α and β that will be related to the feedback gains given from the damping and anti-damping mechanism as the constant time delay that will be denoted by h. Moreover, the interaction between the feedback gains α and β must satisfy the following constraint

$$0 < \alpha^2 + \frac{3}{2}\beta < \frac{1}{2}. \tag{1.4.11}$$

Finally, the initial state for the equations is given by

$$\begin{cases} u(0, x) = u_0(x), v(0, x) = v_0(x) \in L^2(0, L) \\ u_x(t-h, L) = z_0(t-h, L) \in L^2(0, 1). \end{cases} \tag{1.4.12}$$

Then, we can define the total energy associated with the Hirota-Satsuma system (1.4.9)-(1.4.10) as

$$E(t) = \frac{1}{2}\int_0^L u^2(t, x) + v^2(t, x)\, dx + \frac{\beta}{2}h\int_0^1 u_x^2(t - h\rho, L)\, d\rho. \tag{1.4.13}$$

Formally, some integrations by parts allow us to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq \frac{1}{2}\begin{pmatrix} u_x(t, L) \\ u_x(t-h, L) \end{pmatrix}^T \Phi_{\alpha,\beta}\begin{pmatrix} u_x(t, L) \\ u_x(t-h, L) \end{pmatrix} \tag{1.4.14}$$

where

$$\Phi_{\alpha,\beta} = \begin{pmatrix} \alpha^2 - \frac{1}{2} + \beta & \alpha\beta \\ \alpha\beta & \beta^2 - \beta \end{pmatrix}. \tag{1.4.15}$$

is a negative definite matrix. Therefore, from (1.4.14), we obtain that the total energy $E(t)$ associated with the Hirota-Satsuma system is a non-increasing function. Then the natural question arises:

Does $E(t) \to 0$ as $t \to \infty$? If this is the case, can we give an explicit decay rate?

Then, we obtain an affirmative answer read as

Theorem 1.4.2. *Let $L > 0$ and α, β such that (1.4.11) yields. Then, there exists*

$$0 < r < \frac{3}{16L^{\frac{3}{2}}} \tag{1.4.16}$$

such that for every initial data $(u_0, v_0, z_0) \in H$ with $\|(u_0, v_0, z_0)\|_H \leq r$, the energy $E(t)$ defined in (1.4.13) of the Hirota-Satsuma system (1.4.9)-(1.4.10) decays exponentially. More precisely, given μ_1, μ_2 positive constants small enough, then there exists $\kappa = 1 + \max\{\mu_1 L, \mu_2\}$ and

$$\lambda \leq \min\left\{\left[3 - 16L^{\frac{3}{2}}r\right]\frac{\pi^2\mu_1}{2L^2(1+L\mu_1)}, \frac{\mu_2}{h(1+\mu_2)}\right\} \tag{1.4.17}$$

such that $E(t) \leq \kappa E(0)e^{-\lambda t}$, for all $t \geq 0$.

1.4.2 Observability and time-delay stabilization for KP type systems

Now, we look for a two-dimensional generalization of the KdV, called the KP equation. This result (CAPISTRANO-FILHO; GALLEGO; MUÑOZ, 2024) joint with Roberto de A. Capistrano–Filho and Fernando A. Gallego Restrepo investigates two observability inequalities for the Kadomtsev-Petviashvili equation, a two-dimensional generalization of the Korteweg-de Vries equation. In general, the observability inequality is essential to establish the control properties employing the Hilbert Uniqueness Method. Precisely, with these two boundary observations in hand, we deduce the exact controllability and exponential stabilization of the linearized KP-II equation using boundary inputs.

Precisely, we consider the linearized KP-II within a rectangular domain $\Omega := (0, L) \times (0, L)$, $L > 0$

$$u_t + u_x + u_{xxx} + \partial_x^{-1}(u_{yy}) = 0, \quad (x, y) \in \Omega, \ t \in (0, T). \tag{1.4.18}$$

with initial data $u(x, y, 0) = u_0(x, y)$ with $(x, y) \in \Omega$ and boundary conditions

$$\begin{cases} u(0, y, t) = u(L, y, t) = 0, & y \in (0, L), \ t \in (0, T) \\ u(x, 0, t) = u(x, L, t) = 0, & x \in (0, L), \ t \in (0, T) \\ u_x(L, y, t) = h(y, t), & y \in (0, L), \ t \in (0, T) \end{cases} \tag{1.4.19}$$

The study of controllability typically involves finding appropriate control functions that act on the system, and the choice of these controls can depend on the specific structure of the equation, therefore the next question related to the exact control arises:

Question A: *Given an initial state u_0 and a final state u_1 in a certain space, can one find an appropriate boundary control input so that the equation (1.4.18)-(1.4.19) admits a solution u which equals u_0 at time $t = 0$ and u_1 at time $t = T$?*

Moreover, due to the connection between controllability and stabilization, the total energy associated with (1.4.18) is given by

$$E(t) = \frac{1}{2}\int_{\Omega} u^2(x,y,t)\,dx\,dy\,dt. \tag{1.4.20}$$

Then, by associating the control input with feedback, the natural issue appears.

Question B: *Is it possible to choose the control $h(y,t)$ as a feedback damping mechanism such that $E(t) \to 0$ as $t \to +\infty$? If this is the case, can we give the decay rate?*

We provide an exact controllability property for the KP-II equation.

Theorem 1.4.3. *Let $L \in (0,+\infty) \setminus \mathcal{R}$ and $T > 0$. Then the KP-II equation (1.4.18)-(1.4.19) is exactly controllable in time T, that is, for any $u_0, u_T \in L^2(\Omega)$, there exists $h(y,t) \in L^2((0,T)\times(0,L))$ such that the mild solution u of the KP-II equation (1.4.18)-(1.4.19) satisfies $u(\cdot,\cdot,T) = u_T(x,y)$.*

where $\mathcal{R}$ is the *critical set of non controllable lengths* given by

$$\mathcal{R} := \left\{ \frac{\pi}{2n}\sqrt{(3m_1+2m_2+m_3)(m_1+2m_2-m_3)(m_1-2m_2-m_3)(m_1+2m_2+3m_3)} : \right. \\ \left. n, m_1, m_2, m_3 \in \mathbb{Z}^+, \quad m_1 < m_2 < m_3, \quad m_1 + 2m_2 < m_3 \right\}. \tag{1.4.21}$$

Moreover, by choosing a suitable feedback damping mechanism as $h(y,t) = -\alpha u_x(0,y,t)$ with the constraint $0 < |\alpha| \leq 1$, we get the (1.4.18) with a feedback damping mechanism

$$\begin{cases} u(0,y,t) = u(L,y,t) = 0, \quad u_x(L,y,t) = -\alpha u_x(0,y,t), & y \in (0,L),\ t \in (0,T), \\ u(x,0,t) = u(x,L,t) = 0, & x \in (0,L),\ t \in (0,T). \end{cases} \tag{1.4.22}$$

Therefore, taking into account the boundary conditions (1.4.22) we get that

$$\frac{\mathrm{d}}{\mathrm{dt}}E(t) = -\frac{1-\alpha^2}{2}\int_0^L u_x^2(0,y)\,dy - \frac{1}{2}\int_0^L \left(\partial_x^{-1}u_y(0,y)\right)^2 dy \leq 0. \tag{1.4.23}$$

This allows us to obtain exponential stabilization results, the second main result of this section.

Theorem 1.4.4 (Uniform exponential stabilization)**.** *Let $L \in (0,+\infty) \setminus \mathcal{R}$. Then, for any initial data $u_0 \in L^2(\Omega)$ the energy $E(t)$, given by (1.4.20), associated with KP-II system (1.4.24)-(1.4.22) decays exponentially.*

In the last main Theorem of this thesis (CAPISTRANO-FILHO; GONZALEZ MARTINEZ; MUÑOZ, 2023), joint with Roberto de A. Capistrano–Filho and Victor H. Gonzalez Martinez, we present results of stabilization for the higher order of the Kadomtsev-Petviashvili equation. Precisely, we prove with two different approaches that under the presence of a damping mechanism and an internal delay term (anti-damping) the solutions of the Kawahara-Kadomtsev-Petviashvili equation are locally and globally exponentially stable. The main novelty is the optimal constant, as well as the minimal time, that ensures that the energy associated with this system goes to zero exponentially.

Motivated by (CHENTOUF, 2022; CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024; MOURA; NASCIMENTO; SANTOS, 2022; GOMES; PANTHEE, 2011) we will analyze the qualitative properties of the initial-boundary value problem for the K-KP-II equation posed on a bounded domain $\Omega = (0,L)\times(0,L) \subset \mathbb{R}^2$ with localized damping and delay terms

$$\begin{cases} \partial_t u(x,y,t) + \alpha\partial_x^3 u(x,y,t) + \beta\partial_x^5 u(x,y,t) & \\ +\gamma\partial_x^{-1}\partial_y^2 u(x,y,t) + \dfrac{1}{2}\partial_x(u^2(x,y,t)) & (x,y)\in\Omega,\ t>0. \\ +a(x,y)u(x,y,t) + b(x,y)u(x,y,t-h) = 0, & \\ u(0,y,t) = u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\ \partial_x u(L,y,t) = \partial_x u(0,y,t) = \partial_x^2 u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\ u(x,L,t) = u(x,0,t) = 0, & x\in(0,L),\ t\in(0,T), \\ u(x,y,0) = u_0(x,y), \quad u(x,y,t) = z_0(x,y,t), & (x,y)\in\Omega,\ t\in(-h,0). \end{cases} \tag{1.4.24}$$

Here $h > 0$ is the time delay, $\alpha > 0$, $\gamma > 0$ and $\beta < 0$ are real constants. For our purpose, let us consider the following assumption.

Assumption 1. *The real functions $a(x,y)$ and $b(x,y)$ are nonnegative belonging to $L^\infty(\Omega)$. Moreover, $a(x,y) \geq a_0 > 0$ is almost everywhere in a nonempty open subset $\omega \subset \Omega$.*

Our propose here is to present, for the first time, the K-KP-II system not with only a damping mechanism $a(x,y)u$, which plays the role of a feedback-damping mechanism (see e.g. (MOURA; NASCIMENTO; SANTOS, 2022)), but also with an anti-damping, that is, some feedback such that our system does not have decreasing energy. In this context, we would like to prove that the energy associated with the solutions of the system (1.4.24)

$$\begin{aligned} E_u(t) =&\frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy \\ &+\frac{h}{2}\int_0^L\int_0^L\int_0^1 b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy. \end{aligned} \tag{1.4.25}$$

decays exponentially. Precisely, we want to answer the following question:

Does $E_u(t)\to 0$ as $t\to\infty$? If this is the case, can we give the decay rate?

The main result obtained in this problem ensures that without a restrictive assumption on the length L of the domain and with the weight of the delayed feedback small enough the energy (1.4.25) associated with the solution of the system (1.4.24) are locally stable.

Theorem 1.4.5 (Optimal local stabilization)**.** *Assume that the functions $a(x,y), b(x,y)$ satisfy the conditions given in Assumption 1. Let $L>0$, $\xi>1$, $0<\mu<1$ and T_0 given by*

$$T_0=\frac{1}{2\theta}\ln\left(\frac{2\xi\kappa}{\mu}\right)+1, \tag{1.4.26}$$

with $\theta=\frac{3\alpha\eta}{(1+2\eta L)L^2}$, $\kappa=1+\max\left\{2\eta L,\frac{\sigma}{\xi}\right\}$ and $\eta\in\left(0,\frac{\xi-1}{2L(1+2\xi)}\right)$ satisfying

$$\frac{2\alpha\eta}{(2+2\eta L)L^2}=\frac{\sigma}{2h(\xi+\sigma)}$$

where $\sigma=\xi-1-2L\eta(1+2\xi)$. Let $T_{\min}>0$ given by

$$T_{\min}:=-\frac{1}{\nu}\ln\left(\frac{\mu}{2}\right)+\left(\frac{2\|b\|_\infty}{\nu}+1\right)T_0,\ \textit{with}\ \nu=\frac{1}{T_0}\ln\left(\frac{1}{(\mu+\varepsilon)}\right).$$

Then, there exists $\delta>0$, $r>0$, $C>0$ and γ, depending on $T_{\min},\xi,L,h$, such that if $\|b\|_\infty\le\delta$, then for every $(u_0,z_0)\in\mathcal{H}=L^2(\Omega)\times L^2(\Omega\times(0,1))$ satisfying $\|(u_0,z_0)\|_{\mathcal{H}}\le r$, the energy of the system (1.4.24) satisfies

$$E_u(t)\le Ce^{-\gamma t}E_u(0),\ \textit{for all}\ t>T_{\min}.$$

Part I

Boundary stabilization for coupled KdV type systems with time-delay

2 ON THE BOUNDARY STABILIZATION OF THE KDV-KDV SYSTEM WITH TIME-DEPENDENT DELAY

R. A. Capistrano-Filho, B. Chentouf, V. H. Gonzalez Martinez, and J. R. Muñoz, Nonlinear Analysis: Real World Applications, 79:104122 (2024).

2.1 INTRODUCTION

2.1.1 Boussinesq system model

The Boussinesq system is a set of partial differential equations (PDEs) that describe the behavior of waves in fluids with small amplitude and long-wavelength disturbances. It was first introduced by the French mathematician Joseph Boussinesq in the 19th century as a way to model waves in shallow water (BOUSSINESQ, 1871b). Since then, the system has been used to study a wide range of physical phenomena, including ocean currents, atmospheric circulation, and heat transfer in fluids. The Boussinesq system is also an important tool in the study of fluid dynamics and has applications in a variety of fields, including meteorology, oceanography, and engineering.

Recently, Bona *et al.* in (BONA; CHEN; SAUT, 2002; BONA; CHEN; SAUT, 2004) developed a four-parameter family of Boussinesq systems to describe the motion of small-amplitude long waves on the surface of an ideal fluid under gravity and in situations where the motion is sensibly two-dimensional. They specifically investigated a family of systems of the form

$$\begin{cases}\eta_t(t,x)+\omega_x(t,x)+a\omega_{xxx}(t,x)-b\eta_{xxt}(t,x)+(\eta(t,x)\omega(t,x))_x=0,\\ \omega_t(t,x)+\eta_x(t,x)+c\eta_{xxx}(t,x)-d\omega_{xxt}(t,x)+\omega(t,x)\omega_x(t,x)=0,\end{cases}\tag{2.1.1}$$

which are all Euler equation approximations of the same order. Here η represents the elevation of the equilibrium point and $\omega=\omega_\theta$ is the horizontal velocity in the flow at height $\theta\ell$, where $\theta\in[0,1]$ and ℓ is the undisturbed depth of the fluid. The parameters a,b,c,d, that one might choose in a given modeling situation, are required to fulfill the relations $a+b=\frac{1}{2}\left(\theta^2-\frac{1}{3}\right)$ and $c+d=\frac{1}{2}(1-\theta^2)\geq 0$.

When $b=d=0$ and making a scaling argument, we obtain the Boussinesq system of KdV-KdV type

$$\begin{cases}\eta_t(t,x)+\omega_x(t,x)+\omega_{xxx}(t,x)+(\eta(t,x)\omega(t,x))_x=0,\\ \omega_t(t,x)+\eta_x(t,x)+\eta_{xxx}(t,x)+\omega(t,x)\omega_x(t,x)=0,\end{cases}\tag{2.1.2}$$

which is shown to admit global solutions on $\mathbb{R}$ and also has good control properties such as stabilization, and controllability, in periodic framework $\mathbb{T}$[1]. Nonetheless, stabilization properties for the Boussinesq KdV-KdV system on a bounded domain of $\mathbb{R}$ is a challenging problem due to the coupling of the nonlinear and dispersive nature of the PDEs. In this spirit, a few works indicate that appropriate boundary feedback controls provide good stabilization results to the system (2.1.2) on a bounded domain $\mathbb{R}$ (see, for instance, (CAPISTRANO-FILHO; CERPA; GALLEGO, 2023; CAPISTRANO-FILHO; GALLEGO, 2018; CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019; PAZOTO; ROSIER, 2008)). To be more precise, in (PAZOTO; ROSIER, 2008), a set of boundary controls is needed so that the solutions of the system (2.1.2) issuing from small data globally exist and the corresponding energy exponentially decays. Indeed, (2.1.2) is coupled with the following boundary conditions:

$$\begin{cases} \omega(t,0) = \omega_{xx}(t,0) = 0, & t > 0, \\ \omega_x(t,0) = a_0\eta_x(t,0), & t > 0, \\ \omega_x(t,L) = -a_1\eta_x(t,L), & t > 0, \\ \omega_x(t,L) = \eta_x(t,L), \;\; \omega_{xx}(t,L) = -\eta_{xx}(t,L), & t > 0, \end{cases}$$

where $a_0 \geq 0$, whereas $a_1 > 0$. Later, two boundary controls are designed via the backstepping method to obtain a local rapid exponential stabilization result for the solutions to (2.1.2) (CAPISTRANO-FILHO; GALLEGO, 2018). In turn, the main concern in (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019) is the exact controllability of (2.1.2). Specifically, a control of Neumann type is proposed to reach a local exact controllability property as well as the exponential stability of the system. Lastly, the linear variant of (2.1.2) is considered and a single linear boundary control is designed to obtain the rapid stabilization of the solutions (CAPISTRANO-FILHO; CERPA; GALLEGO, 2023).

2.1.2 Problem setting

First, let us consider the KdV-KdV equation (2.1.2) but in a bounded domain $[0,L]$ and with the following set of boundary conditions

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = 0, & t \in \mathbb{R}^+, \\ \omega(t,0) = \omega(t,L) = \omega_x(t,L) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{2.1.3}$$

[1] See (BONA; CHEN; SAUT, 2004) for the real-line case and (CAPISTRANO-FILHO; GOMES, 2023; MICU et al., 2009) for details in the periodic framework.

As mentioned before, note that considering the system described above, two important facts need to be mentioned:

• We first noticed that the *global Kato smoothing effect* does not hold for the set of boundary conditions (2.1.3). This makes impossible the task of showing the well-posedness findings by employing classical methods, such as semigroup theory, and hence the well-posedness problem of this system remains open.

• The second issue is related to the system's energy (2.1.2) and (2.1.3). Under the above boundary conditions, a simple integration by parts yields

$$\frac{d}{dt}E_0(t) = -\int_0^L (\eta(t,x)\omega(t,x))_x \eta(t,x)\,dx,$$

where

$$E_0(t) = \frac{1}{2}\int_0^L (\eta^2(t,x) + \omega^2(t,x))\,dx$$

is the total energy associated with (2.1.2) and (2.1.3). This indicates that *we do not have any control over the energy in the sense that its time derivative does not have a fixed sign.*

Therefore, due to the restriction presented in these two points, the following questions naturally arise:

Question $\mathcal{A}$: *Is there a suitable set of boundary conditions so that the Kato smoothing effect can be revealed?*

Question $\mathcal{B}$: *Is there a feedback control law that permits the control of the nonlinear term presented in the derivative of the energy associated with the closed-loop system? Moreover, is this desired feedback law strong enough in the presence of a time-dependent delay?*

Question $\mathcal{C}$: *If the answer to these previous questions is yes, does* $E_0(t) \to 0$ *as* $t \to \infty$*? If this is the case, can we give an explicit decay rate?*

Our motivation in this work is to give answers to these questions. In this spirit, and to deal with the Boussinesq system of KdV-KdV type (2.1.2), let us consider the set of boundary conditions:

$$\begin{cases} \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \omega(t,0) = \omega(t,L) = 0, & t > 0, \\ \omega_x(t,L) = -\alpha\eta_x(t,L) + \beta\eta_x(t-\tau(t),L), & t > 0, \end{cases} \tag{2.1.4}$$

where $\tau(t)$ is the time-varying delay, while α and β are feedback gains.

Remark 2.1.1. The following remarks are now in order.

i. Note that our new set of boundary conditions contains a damping mechanism $\alpha\eta_x(t, L)$ as well as the time-varying delayed feedback $\beta\eta_x(t - \tau(t), L)$.

ii. The damping mechanism will guarantee the Kato smoothing effect, which is paramount to proving the well-posedness of the system under consideration in this chapter.

iii. The time-varying delay feedback, together with the damping mechanism, permits to drive the energy to 0, as t goes to ∞, giving the stabilization of the system (2.1.2) and (2.1.4), with a precise decay rate.

iv. We point out that our main result, given in the next subsection, ensures the exponential stability of the linearized system associated with (2.1.2)–(2.1.4) employing $\tau(t)$ as a time-varying delay. However, due to the lack of a priori L^2-estimate, it is hard to extend the result to the nonlinear system (2.1.2)–(2.1.4). We instruct the reader to see the discussion about this point in Section 2.4.

It is also noteworthy that the time-delay phenomenon is practically unavoidable because of miscellaneous reasons. Indeed, it often occurs in numerous areas such as biology, mechanics, and engineering due to the dynamics of the actuators and sensors. Having said that, there is in literature a predominant opinion that time delay has intrinsically a disadvantage on the performance of practical systems (see for instance the first papers that treated this subject in the PDEs framework (DATKO, 1988; DATKO, 1997; DATKO; LAGNESE; POLIS, 1986)).

This gives rise to a monumental endeavor in attempting to nullify any negative impact of the presence of a delay on a system. The authors in (NICAISE; PIGNOTTI, 2006; XU; YUNG; LI, 2006) show that the solutions to the wave equation remain stable provided that the delayed term is small, otherwise the stability property is lost. This outcome is extended in (FRIDMAN; NICAISE; VALEIN, 2010) to a general class of second-order evolution equations with unbounded time-dependent delayed control. Similar results are also obtained for numerous systems with time-dependent delay (see for instance (NICAISE; PIGNOTTI, 2008; NICAISE; PIGNOTTI; VALEIN, 2011; NICAISE; VALEIN; FRIDMAN, 2009) and the references therein).

Note also that in the context of dispersive equations, time-delayed feedback is a challenging problem as it can lead to instability or oscillatory behavior in numerous instances. Some recent articles - not exhaustive - already addressed the stabilization problem of dispersive systems with delay. We can cite, for example, (BAUDOUIN; CRÉPEAU; VALEIN, 2019), (CHENTOUF, 2021) and (CAPISTRANO-FILHO et al., 2023) for KdV, KS, and Kawahara equations, where time-delay boundary controls are considered. Furthermore, if the time delay occurs in the equation, the authors in (VALEIN, 2022), (CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024; CHENTOUF, 2022), and (CAPISTRANO-FILHO; GONZALEZ MARTINEZ; MUÑOZ, 2023) showed stabilization results for the KdV, fifth-order KdV, and Kawahara-Kadomtsev-Petviashvili equations, respectively. Finally, we point out that using the time-varying delay, the authors in (PARADA; TIMIMOUN; VALEIN, 2023) obtained stabilization outcomes for the KdV equation. To our best knowledge, this is the only work that considers a coupled dispersive system with a time-dependent delay and we believe that the techniques presented here can be adapted to other systems.

2.1.3 Main results and chapter's outline

To our knowledge, due to the previous restrictions, there is no result combining the damping mechanism and the boundary time-varying delay to guarantee stabilization results for the linearized KdV-KdV system associated with (2.1.2)–(2.1.4). To state the main result and provide answers to the questions previously mentioned, we assume that there exist two positive constants M and $d < 1$ such that the time-dependent function $\tau(t)$ satisfies the following standard conditions:

$$\begin{cases} 0 < \tau(0) \leq \tau(t) \leq M, \quad \dot{\tau}(t) \leq d < 1, & \forall t \geq 0, \\ \tau \in W^{2,\infty}([0,T]), & T > 0. \end{cases} \tag{2.1.5}$$

Furthermore, the feedback gains α and β must satisfy the following constraint

$$(2\alpha - |\beta|)(1-d) > |\beta|, \quad \text{for } 0 \leq d < 1. \tag{2.1.6}$$

or equivalently,

$$\alpha > \frac{|\beta|}{2}\left(\frac{2-d}{1-d}\right), \quad \text{for } 0 \leq d < 1.$$

Next, let $X_0 := L^2(0,L) \times L^2(0,L)$, $H := X_0 \times L^2(0,1)$ and consider the space

$$\mathcal{B} := C([0,T], X_0) \cap L^2(0,T,[H^1(0,L)]^2),$$

whose norm is

$$\|(\eta,\omega)\|_{\mathcal{B}} = \sup_{t\in[0,T]} \|(\eta(t),\omega(t))\|_{X_0} + \|(\eta_x,\omega_x)\|_{L^2(0,T;X_0)}.$$

Whereupon, we are interested in the behavior of the solutions of the system

$$\begin{cases} \eta_t(t,x) + \omega_x(t,x) + \omega_{xxx}(t,x) + (\eta(t,x)\omega(t,x))_x = 0, & \mathbb{R}^+ \times (0,L), \\ \omega_t(t,x) + \eta_x(t,x) + \eta_{xxx}(t,x) + \omega(t,x)\omega_x(t,x) = 0, & \mathbb{R}^+ \times (0,L), \\ \eta(t,0) = \eta(t,L) = \eta_x(t,0) = \omega(t,0) = \omega(t,L) = 0, & t \in \mathbb{R}^+, \\ \omega_x(t,L) = -\alpha\eta_x(t,L) + \beta\eta_x(t-\tau(t),L), & t > 0, \\ \eta_x(t-\tau(0),L) = z_0(t-\tau(0)) \in L^2(0,1), & 0 < t < \tau(0), \\ (\eta(0,x),\omega(0,x)) = (\eta_0(x),\omega_0(x)) \in X_0, & x \in (0,L). \end{cases} \tag{2.1.7}$$

It is noteworthy that the total energy associated with the system (2.1.7) will be defined in H by

$$E(t) = \frac{1}{2}\int_0^L (\eta^2(t,x) + \omega^2(t,x))\,dx + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)\,d\rho. \tag{2.1.8}$$

Thereafter, the principal result ensures that the energy $E(t)$ decays exponentially despite the presence of the delay. An estimate of the decay rate is also provided. This answers each question that we tabled previously.

Theorem 2.1.2. *Let $0 < L < \sqrt{3}\pi$. Suppose that (2.1.5) and (2.1.6) are satisfied. Then, for two positive constants μ_1 and μ_2 with $\mu_1 L < 1$, there exist*

$$\zeta = \frac{1 + \max\{\mu_1 L, \mu_2\}}{1 - \max\{\mu_1 L, \mu_2\}}, \tag{2.1.9}$$

and

$$\lambda \leq \min\left\{\frac{\mu_1(3\pi^2 - L^2)}{L^2(1+\mu_1)}, \frac{\mu_2(1-d)}{M(1+\mu_2)}\right\} \tag{2.1.10}$$

such that the energy $E(t)$ given by (2.1.8) associated to the linearized system of (2.1.7) around the origin satisfies

$$E(t) \leq \zeta E(0)e^{-\lambda t}, \qquad \text{for all } t \geq 0.$$

This outcome brings a new contribution to the stability of the KdV-KdV system with a delay term since in (CAPISTRANO-FILHO; CERPA; GALLEGO, 2023; CAPISTRANO-FILHO; GALLEGO, 2018; CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019; PAZOTO; ROSIER, 2008) no delay was considered. Moreover, unlike these papers, the spectral analysis of the linearized system cannot be conducted due to the time dependency of the delay. In turn, this prevents us from getting the set of critical lengths. The approach used in the current work is direct as it is based on the Lyapunov method.

We end this section by providing an outline of this chapter, which consists of four parts including the Introduction. Section 2.2 discusses the existence of local solutions for the nonlinear Boussinesq KdV-KdV system (2.1.7). Section 2.3 is devoted to proving the stabilization result, Theorem 2.1.2, for the linearized system associated with (2.1.7). Additionally, we have shown that the decay rate λ of Theorem 2.1.2 can be optimized. Finally, in Section 2.4, we will provide some concluding remarks and discuss open problems related to the stabilization of the nonlinear Boussinesq KdV-KdV system (2.1.7).

2.2 WELL-POSEDNESS THEORY

2.2.1 Linear problem

Consider the following linear Cauchy problem

$$\begin{aligned} &\frac{d}{dt}U(t) = A(t)U(t), \\ &U(0) = U_0, \quad t > 0, \end{aligned} \tag{2.2.1}$$

where $A(t)\colon D(A(t)) \subset H \to H$ is densely defined. If $D(A(t))$ is independent of time t, i.e., $D(A(t)) = D(A(0))$, for $t > 0$. The next theorem ensures the existence and uniqueness of the Cauchy problem (2.2.1).

Theorem 2.2.1 ((KATO, 1970)). *Assume that:*

1. $\mathcal{Z} = D(A(0))$ *is a dense subset of* H *and* $D(A(t)) = D(A(0))$*, for all* $t > 0$*,*
2. $A(t)$ *generates a strongly continuous semigroup on* H*. Moreover, the family* $\{A(t)\colon t \in [0,T]\}$ *is stable with stability constants* $C,\ m$ *independent of* t*.*
3. $\partial_t A(t)$ *belongs to* $L^\infty_*([0,T], B(\mathcal{Z}, H))$*, the space of equivalent classes of essentially bounded, strongly measure functions from* $[0,T]$ *into the set* $B(\mathcal{Z}, H)$ *of bounded operators from* $\mathcal{Z}$ *into* H*.*

Then, problem (2.2.1) *has a unique solution* $U \in C([0,T], \mathcal{Z}) \cap C^1([0,T], H)$ *for any initial data in* $\mathcal{Z}$*.*

The task ahead is to apply the above result to ensure the existence of solutions for the linear system associated with (2.1.7). To do that, consider the following linearized system associated with (2.1.7), that is, consider the equation without $\omega(t,x)\omega_x(t,x)$ and $(\eta(t,x)\omega(t,x))_x$.

Following the ideas introduced by in (XU; YUNG; LI, 2006; NICAISE; PIGNOTTI, 2006; NICAISE; VALEIN; FRIDMAN, 2009), let us define the auxiliary variable

$$z(t,\rho) = \eta_x(t-\tau(t)\rho, L),$$

which satisfies the transport equation:

$$\begin{cases} \tau(t)z_t(t,\rho) + (1-\dot{\tau}(t)\rho)z_\rho(t,\rho) = 0, & t>0, \rho\in(0,1), \\ z(t,0) = \eta_x(t,L),\ z(0,\rho) = z_0(-\tau(0)\rho) & t>0,\ \rho\in(0,1). \end{cases} \tag{2.2.2}$$

Now, the space H will be equipped with the inner product

$$\langle(\eta,\omega,z),(\tilde{\eta},\tilde{\omega},\tilde{z})\rangle_t = \langle(\eta,\omega),(\tilde{\eta},\tilde{\omega})\rangle_{X_0} + |\beta|\tau(t)\langle z,\tilde{z}\rangle_{L^2(0,1)}, \tag{2.2.3}$$

for any $(\eta,\omega;z),(\tilde{\eta},\tilde{\omega};\tilde{z})\in H$.

Now, we pick up $U=(\eta,\omega;z)^T$ and consider the time-dependent operator

$$A(t)\colon D(A(t))\subset H\to H$$

given by

$$A(t)(\eta,\omega,z) := \left(-\omega_x-\omega_{xxx}, -\eta_x-\eta_{xxx}, \frac{\dot{\tau}(t)\rho-1}{\tau(t)}z_\rho\right), \tag{2.2.4}$$

with domain defined by

$$D(A(t)) = \left\{ \begin{array}{l|l} (\eta,\omega)\in\left[H^3(0,L)\cap H_0^1(0,L)\right]^2, & \eta_x(0)=0, z(0)=\eta_x(L), \\ z\in H^1(0,1), & \omega_x(L) = -\alpha\eta_x(L)+\beta z(1) \end{array} \right\}. \tag{2.2.5}$$

Whereupon, we rewrite (2.2.2)-(2.2.5) as an abstract Cauchy problem (2.2.1). Moreover, note that $D(A(t))$ is independent of time t since $D(A(t)) = D(A(0))$.

Subsequently, consider the triplet $\{A,H,\mathcal{Z}\}$, with $A=\{A(t)\colon t\in[0,T]\}$ for some $T>0$ fixed and $\mathcal{Z}=D(A(0))$. Now, we can prove a well-posedness result of (2.2.1) related to $\{A,H,\mathcal{Z}\}$.

Theorem 2.2.2. *Assume that α and β are real constants such that (2.1.6) holds. Taking $U_0\in H$, there exists a unique solution $U\in C([0,+\infty),H)$ to (2.2.1). Moreover, if $U_0\in D(A(0))$, then $U\in C([0,+\infty),D(A(0)))\cap C^1([0,+\infty),H)$.*

Proof. The result will be proved classically (see, for instance, (NICAISE; VALEIN; FRIDMAN, 2009)). First, it is not difficult to see that $\mathcal{Z}=D(A(0))$ is a dense subset of H and $D(A(t))=D(A(0))$, for all $t>0$. Therefore, the condition (1) of Theorem 2.2.1 holds.

For the requirement (2) of Theorem 2.2.1, first note that (2.1.5) implies that the norms $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent and does not depend on the choice of t. In fact, for all $t \geq 0$ and $(\eta, \omega; z) \in H$

$$(1 + |\beta|\tau_0)\|(\eta, \omega; z)\|_H^2 \leq \|(\eta, \omega; z)\|_t^2 \leq (1 + |\beta|M)\|(\eta, \omega; z)\|_H^2.$$

Moreover, recalling that $\dot{\tau}$ is bounded and using the mean value theorem follows that, for all $t, s \in [0, T]$

$$\frac{\|U\|_t}{\|U\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}$$

where $U = (\eta, \omega; z) \in H$ and $c > 0$. Indeed, for all $t, s \in [0, T]$,

$$\begin{aligned}\|U\|_t^2 - \|U\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right) \int_0^L \eta^2(t, x) + \omega^2(t, x)\, dx \\ + |\beta| \left(\tau(t) - \tau(s) e^{\frac{c}{\tau_0}|t-s|}\right) \int_0^1 z^2(t, \rho)\, d\rho.\end{aligned}$$

Observe that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. We claim that $\tau(t) - \tau(s) e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. By the mean value Theorem, we get

$$\tau(t) = \tau(s) + \dot{\tau}(a)(t - s), \qquad \text{where } a \in (s, t),$$

consequently

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\dot{\tau}(a)|}{\tau(s)}|t - s|.$$

Since $\dot{\tau}$ is bounded, follows that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0}|t - s| \leq e^{\frac{c}{\tau_0}|t-s|}.$$

and the claim holds.

With this equivalence, we observe that integrating by parts and using the boundary conditions, we have that

$$\begin{aligned}\langle A(t)U, U\rangle_t &= \eta_x(L)\omega_x(L) + \frac{|\beta|}{2}(\dot{\tau}(t) - 1)\eta_x^2(t - \tau(t), L) + \frac{|\beta|}{2}\eta_x^2(L) - \frac{|\beta|}{2}\dot{\tau}(t) \int_0^1 z^2\, d\rho \\ &= \eta_x(L)(-\alpha\eta_x(L) + \beta\eta_x(t - \tau(t), L)) + \frac{|\beta|}{2}(\dot{\tau}(t) - 1)\eta_x^2(t - \tau(t), L) \\ &\quad + \frac{|\beta|}{2}\eta_x^2(L) - \frac{|\beta|}{2}\dot{\tau}(t) \int_0^1 z^2\, d\rho \\ &= \frac{1}{2} \begin{pmatrix} \eta_x(L) \\ \eta_x(t - \tau(t), L) \end{pmatrix}^T \begin{pmatrix} -2\alpha + |\beta| & \beta \\ \beta & |\beta|(\dot{\tau}(t) - 1) \end{pmatrix} \begin{pmatrix} \eta_x(L) \\ \eta_x(t - \tau(t), L) \end{pmatrix} \\ &\quad - \frac{|\beta|}{2}\dot{\tau}(t) \int_0^1 z^2\, d\rho.\end{aligned}$$

Therefore,

$$\langle A(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq \frac{1}{2}\left(\eta_x(L), \eta_x(t-\tau(t), L)\right)\Phi_{\alpha,\beta}\left(\eta_x(L), \eta_x(t-\tau(t), L)\right)^T$$

where

$$\kappa(t) = \frac{(\dot{\tau}(t)^2+1)^{\frac{1}{2}}}{2\tau(t)} \quad \text{and} \quad \Phi_{\alpha,\beta} = \begin{pmatrix} -2\alpha + |\beta| & \beta \\ \beta & |\beta|(d-1) \end{pmatrix}.$$

Invoking (2.1.6), we deduce that $\Phi_{\alpha,\beta}$ is a negative definite matrix and consequently we get

$$\langle A(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0.$$

Thereby, $\tilde{A}(t) = A(t) - \kappa(t)I$ is dissipative.

On the other hand, we claim the following:

Claim 2.2.3. *For all $t \in [0, T]$, the operator $A(t)$ is maximal, or equivalently, we have that $\lambda I - A(t)$ is surjective, for some $\lambda > 0$.*

In fact, let us fix $t \in [0, T]$. Given $(f_1, f_2, h)^T \in H$, we look for $U = (\eta, \omega, z)^T \in D(A(t))$ solution of

$$(\lambda I - A(t))U = (f_1, f_2, h) \iff \begin{cases} \lambda\eta + \omega_x + \omega_{xxx} = f_1, \\ \lambda\omega + \eta_x + \eta_{xxx} = f_2, \\ \lambda z + \left(\dfrac{1-\dot{\tau}(t)\rho}{\tau(t)}\right) z_\rho = h, \\ \eta(0) = \eta(L) = \omega(0) = \omega(L) = \eta_x(0) = 0, \\ \omega_x(L) = -\alpha\eta_x(L) + \beta z(1), z(0) = \eta_x(L). \end{cases} \tag{2.2.6}$$

A straightforward computation gives that z has the explicit form

$$z(\rho) = \begin{cases} \eta_x(L)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\displaystyle\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)\,d\sigma, & \text{if } \dot{\tau}(t) = 0, \\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\rho)}\left[\eta_x(L) + \displaystyle\int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}\,d\sigma\right], & \text{if } \dot{\tau}(t) \neq 0. \end{cases}$$

In particular, $z(1) = \eta_x(L)g_0(t) + g_h(t)$, where

$$g_0(t) = \begin{cases} e^{-\lambda\tau(t)}, & \text{if } \dot{\tau}(t) = 0, \\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}, & \text{if } \dot{\tau}(t) \neq 0, \end{cases}$$

and

$$g_h(t) = \begin{cases} \tau(t)e^{-\lambda\tau(t)}\displaystyle\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma, & \text{if } \dot{\tau}(t) = 0, \\ e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}\displaystyle\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}d\sigma, & \text{if } \dot{\tau}(t) \neq 0. \end{cases}$$

This, together with (2.2.6), leads to claim that η and ω should satisfy

$$\begin{cases} \lambda\eta + \omega_x + \omega_{xxx} = f_1, \\ \lambda\omega + \eta_x + \eta_{xxx} = f_2, \end{cases} \tag{2.2.7}$$

with boundary conditions

$$\begin{cases} \eta(0) = \eta(L) = \omega(0) = \omega(L) = \eta_x(0) = 0, \\ \omega_x(L) = (-\alpha + \beta g_0(t))\eta_x(L) + \beta g_h(t). \end{cases} \tag{2.2.8}$$

Pick $\psi(x,t) = \dfrac{x(x-L)}{L}\beta g_h(t) \in C^\infty([0,L])$ and let $\hat{\omega} := \omega - \psi$. Then, the system (2.2.7) can be rewritten as follows:

$$\begin{cases} \lambda\eta + \omega_x + \omega_{xxx} = f_1 - \psi_x =: \tilde{f}_1, \\ \lambda\omega + \eta_x + \eta_{xxx} = f_2 - \lambda\psi =: \tilde{f}_2, \end{cases} \tag{2.2.9}$$

and must be coupled with (2.2.8). Here, let us mention that for the sake of presentation clarity, we still use ω after translation. One can check that $0 < g_0(t) < 1$. Indeed, if $\dot{\tau}(t) = 0$, then we clearly have $0 < g_0(t) < 1$. In turn, if $\dot{\tau}(t) \neq 0$, then we have two cases to consider, namely $0 < \dot{\tau}(t) < 1$ and $\dot{\tau}(t) < 0$. In the first case, we have $\ln(1-\dot{\tau}(t)) < \ln(1) = 0$ and $\lambda\tau(t)/\dot{\tau}(t) > 0$, which implies that $0 < g_0(t) = e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))} < e^0 = 1$. In the second case, we have $\ln(1-\dot{\tau}(t)) > \ln(1) = 0$ and $\lambda\tau(t)/\dot{\tau}(t) < 0$, which ensures that $0 < g_0(t) = e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))} < e^0 = 1$. We infer from this discussion that $-\tilde{\alpha} := -\alpha + \beta g_0(t) < 0$, thanks to (2.1.6). Thereby, our Claim 2.2.3 is reduced to proving that $\lambda I - \hat{A}$ is surjective, where $\hat{A}$ is given by

$$\hat{A}(\eta,\omega) = (-\omega_x - \omega_{xxx}, -\eta_x - \eta_{xxx}),$$

while its dense domain is

$$D(\hat{A}) := \left\{(\eta,\omega) \in \left[H^3(0,L) \cap H_0^1(0,L)\right]^2 : \eta_x(0) = 0,\ \omega_x(L) = -\tilde{\alpha}\eta_x(L)\right\} \subset X_0.$$

Thanks to (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019, Proposition 4.1), the operators $\hat{A}$ and $\hat{A}^*$ are dissipative, and the desired result follows by Lummer-Phillips Theorem (see, for example, (PAZY, 1983)). This shows the Claim 2.2.3. Consequently, $\tilde{A}(t)$ generates a strongly semigroup on H and $\tilde{A} = \{\tilde{A}(t), t \in [0,T]\}$ is a stable family of generators in H with a stability constant independent of t, and hence the condition (2) of Theorem 2.2.1 is satisfied.

Finally, due to the fact that $\tau \in W^{2,\infty}([0,T])$ for all $T > 0$, we have

$$\dot{\kappa}(t) = \frac{\ddot{\tau}(t)\dot{\tau}(t)}{2\tau(t)\left(\dot{\tau}(t)^2+1\right)^{1/2}} - \frac{\dot{\tau}(t)\left(\dot{\tau}(t)^2+1\right)^{1/2}}{2\tau(t)^2}$$

is bounded on $[0,T]$ for all $T>0$. Moreover,

$$\frac{d}{dt}A(t)U=\left(0,0,\frac{\ddot{\tau}(t)\tau(t)\rho-\dot{\tau}(t)(\dot{\tau}(t)\rho-1)}{\tau(t)^2}z_\rho\right),$$

while the coefficient of z_ρ is bounded on $[0,T]$ and the regularity (3) of Theorem 2.2.1 is fulfilled.

As a consequence, all the assumptions of Theorem 2.2.1 are verified. Therefore, for $U_0\in D(A(0))$, the Cauchy problem

$$\tilde{U}_t(t)=\tilde{A}(t)\tilde{U}(t),\ \tilde{U}(0)=U_0,\quad t>0,$$

has a unique solution $\tilde{U}\in C([0,\infty),H)$ and $\tilde{U}\in C([0,\infty),D(A(0)))\cap C^1([0,\infty),H)$, and consequently the solution of (2.2.1) is $U(t)=e^{\int_0^t\kappa(s)ds}\tilde{U}(t)$. By some computations,

$$\begin{aligned}U_t(t)&=\kappa(t)e^{\int_0^t\kappa(s)ds}\tilde{U}(t)+e^{\int_0^t\kappa(s)ds}\tilde{U}_t(t)\\&=\kappa(t)e^{\int_0^t\kappa(s)ds}\tilde{U}(t)+e^{\int_0^t\kappa(s)ds}\tilde{A}(t)\tilde{U}(t)\\&=e^{\int_0^t\kappa(s)ds}(\kappa(t)\tilde{U}(t)+\tilde{A}(t)\tilde{U}(t))\\&=e^{\int_0^t\kappa(s)ds}A(t)\tilde{U}(t)=A(t)e^{\int_0^t\kappa(s)ds}\tilde{U}(t)\\&=A(t)U(t),\end{aligned}$$

we conclude the proof. □

The next proposition states that the energy (2.1.8) is decreasing along the solutions of (2.2.1). The proof is straightforward and hence omitted.

Proposition 2.2.4. *Suppose α and β are real constants such that (2.1.6) holds. Then, for any mild solution of (2.2.1) the energy $E(t)$ defined by (2.1.8) is non-increasing and*

$$\frac{d}{dt}E(t)=\frac{1}{2}\begin{pmatrix}\eta_x(L)\\\eta_x(t-\tau(t),L)\end{pmatrix}^T\Phi_{\alpha,\beta}\begin{pmatrix}\eta_x(L)\\\eta_x(t-\tau(t),L)\end{pmatrix}. \tag{2.2.10}$$

We end this section by giving *a priori* estimates and the Kato smoothing effect which are essential to obtain the well-posedness of the system (2.1.7). Here, we consider $(S_t(s))_{s\geq0}$ to be the semigroup of contractions associated with the operator $A(t)$.

Proposition 2.2.5. *Let α and β are real constants such that (2.1.6) holds. Then, the map*

$$(\eta_0,\omega_0;z_0)\in H\mapsto(\eta,\omega;z)\in\mathcal{B}\times C(0,T;L^2(0,1))$$

is well-defined, continuous, and fulfills

$$\|(\eta,\omega)\|_{X_0}^2 + |\beta|\|z\|_{L^2(0,1)}^2 \leq \|(\eta_0,\omega_0)\|_{X_0}^2 + |\beta|\|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2, \tag{2.2.11}$$

Furthermore, for every $(\eta_0,\omega_0,z_0)\in H$*, we have that*

$$\|\eta_x(\cdot,L)\|_{L^2(0,T)}^2 + \|z(\cdot,1)\|_{L^2(0,T)}^2 \leq \|(\eta_0,\omega_0)\|_{X_0}^2 + \|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2. \tag{2.2.12}$$

Moreover, the Kato smoothing effect is verified

$$\int_0^T\int_0^L \left(\eta_x^2+\omega_x^2\right)\,dx\,dt \leq C(L,T,\alpha)\left(\|(\eta_0,\omega_0)\|_{X_0}^2 + \|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2\right). \tag{2.2.13}$$

Finally, for the initial data, we have the following estimates

$$\begin{aligned}\|(\eta_0,\omega_0)\|_{X_0}^2 \leq& \frac{1}{T}\|(\eta,\omega)\|_{L^2(0,T;X_0)}^2\\ &+(2\alpha+|\beta|)\|\eta_x(\cdot,L)\|_{L^2(0,T)}^2 + |\beta|\|z(\cdot,1)\|_{L^2(0,1)}^2\end{aligned} \tag{2.2.14}$$

and

$$\|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2 \leq C_1(d,M)\left(\|z(T,\cdot)\|_{L^2(0,1)} + \|z(\cdot,1)\|_{L^2(0,T)}^2\right). \tag{2.2.15}$$

Proof. From (2.2.10) and using that $\Phi_{\alpha,\beta}$ is a symmetric negative definite matrix we obtain that $E'(t)+\eta_x^2(t,L)+z^2(t,1)\leq 0$. Integrating in $[0,s]$, for $0\leq s\leq T$, we get

$$E(s)+\int_0^s \eta_x^2(t,L)\,dt+\int_0^s z^2(t,1)\,dt \leq E(0), \tag{2.2.16}$$

and (2.2.11) is obtained. Taking $s=T$ and since $E(t)$ is a non-increasing function (see Proposition 2.2.4), the estimate (2.2.12) holds. Now, multiplying the first equation of the linearized system associated with (2.1.7) by $x\omega$ and the second one by $x\eta$, adding the results, then integrating by parts in $(0,L)\times(0,T)$ and using (2.2.12), we obtain

$$\begin{aligned}\frac{3}{2}\int_0^T\int_0^L \eta_x^2+\omega_x^2\,dx\,dt \leq& (L+T)\|(\eta_0,\omega_0)\|_{X_0}^2\\ &+\left(\alpha^2+\frac{1}{2}\right)L\left(\|\eta_x(\cdot,L)\|_{L^2(0,T)}^2+\|z(\cdot,1)\|_{L^2(0,T)}^2\right)\\ \leq& C(L,T,\alpha)\left(\|(\eta_0,\omega_0)\|_{X_0}^2+\|z_0(-\tau(0)\cdot)\|_{L^2(0,1)}^2\right),\end{aligned} \tag{2.2.17}$$

where $C(L,T,\alpha):=\max\left\{1,L+T,\left(\alpha^2+\frac{1}{2}\right)L\right\}$, showing (2.2.13). Secondly, we multiply the first equation of the linearized system associated with (2.1.7) by $(T-t)\eta$, while the second one is multiplied by $(T-t)\omega$. Then, adding the results yields

$$\begin{aligned}\frac{T}{2}\|(\eta_0,\omega_0)\|_{X_0}^2 \leq& \frac{1}{2}\|(\eta,\omega)\|_{L^2(0,T,X_0)}^2 + T\left(\alpha+\frac{|\beta|}{2}\right)\int_0^T \eta_x^2(t,L)\,dt\\ &+T\frac{|\beta|}{2}\int_0^T z^2(t,1)\,dt,\end{aligned}$$

where we have used Young's inequality, verifying (2.2.14). Finally, multiplying $(2.2.2)_1$ by z and integrating by parts in $(0,T)\times(0,1)$,

$$\tau_0\int_0^1 z_0^2(-\tau(0)\rho)\,d\rho \leq \int_0^T (1-\dot{\tau}(t))z^2(t,1)\,dt + \tau(T)\int_0^1 z^2(T,\rho)\,d\rho,$$

giving (2.2.15). □

The next result ensures the existence of solutions to the KdV-KdV system with source terms.

Theorem 2.2.6. *Suppose that (2.1.6) and (2.1.5) holds. Let $U_0=(\eta_0,\omega_0,z_0)\in H$ and the source terms $(f_1,f_2)\in L^1(0,T;X_0)$. Then there exists a unique solution $U=(\eta,\omega,z)\in C([0,T],H)$ to*

$$\begin{cases}\eta_t(t,x)+\omega_x(t,x)+\omega_{xxx}(t,x)=f_1, & t>0, x\in(0,L),\\ \omega_t(t,x)+\eta_x(t,x)+\eta_{xxx}(t,x)=f_2, & t>0, x\in(0,L),\end{cases} \tag{2.2.18}$$

with boundary conditions as in (2.1.7). Moreover, for $T>0$, the following estimates hold

$$\begin{cases}\|(\eta,\omega;z)\|_{C([0,T],H)}\leq C(\|(\eta_0,\omega_0,z_0)\|_H+\|(f,g)\|_{L^1(0,T,X_0)}),\\ \|(\eta_x(\cdot,L),z(\cdot,1))\|^2_{[L^2(0,T)]^2}\leq C(\|(\eta_0,\omega_0,z_0)\|^2_H+\|(f,g)\|^2_{L^1(0,T,X_0)}),\\ \|(\eta,\omega)\|_{L^2([0,T],X_1)}\leq C(\|(\eta_0,\omega_0,z_0)\|_H+\|(f,g)\|_{L^1(0,T,X_0)}),\end{cases} \tag{2.2.19}$$

for some constant $C>0$.

Proof. Analogously to the proof of Proposition 2.2.5, it suffices to use (2.2.10) and take into account that $\Phi_{\alpha,\beta}$ is a symmetric negative definite matrix. This implies that there exists $C>0$ such that

$$E'(t)+\eta_x^2(t,L)+z^2(t,1)\leq C\left\langle(\eta,\omega),\ (f_1,f_2)\right\rangle_{X_0}.$$

Integrating the previous inequality on $[0,s]$ for $0\leq s\leq T$, we get

$$E(s)+\int_0^s \eta_x^2(t,L)\,dt+\int_0^s z^2(t,1)\,dt\leq C\left(\int_0^s\left\langle(\eta,\omega),(f_1,f_2)\right\rangle_{X_0}+E(0)\right). \tag{2.2.20}$$

From Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\|(\eta(s,\cdot),\omega(s,\cdot);z(s,\cdot)\|^2_H \leq & C\Big(\|(\eta_0,\omega_0;z_0)\|^2_H\\ &+\|(f_1,f_2)|_{L^1(0,T;X_0)}\|(\eta,\omega)\|_{C([0,T],X_0)}\Big),\end{aligned}$$

and consequently, taking the sup-norm for $s \in [0,T]$ and applying Young's inequality, the estimate (2.2.19)$_1$ is obtained. Additionally, if we consider $s = T$ in (2.2.20), the estimate for the traces (2.2.19)$_2$ is guaranteed. Finally, by using the same Morawetz multipliers as in Proposition 2.2.5, we have

$$\begin{aligned}\int_0^T \int_0^L x\left(f_1\omega(t,x) + f_2\eta(t,x)\right)dxdt \leq& L\|(\eta,\omega,z)\|_{C([0,T],H)}\|(f_1,f_2)\|_{L^1(0,T,X_0)}\\ \leq& C\left(\|(\eta_0,\omega_0;z_0)\|_H^2 + \|(f_1,f_2)\|^2_{L^1(0,T,X_0)}\right),\end{aligned}$$

proving (2.2.19)$_3$. □

2.2.2 Nonlinear problem

Using the theory of local well-posedness of nonlinear systems in (KATO, 1975), it amounts to proving that the map $\Gamma\colon \mathcal{B} \to \mathcal{B}$ has a unique fixed-point in some closed ball $B(0,R) \subset \mathcal{B}$ where $\Gamma(\tilde{\eta},\tilde{\omega}) = (\eta,\omega)$ and (η,ω) are the solution of the system (2.1.7). The next result ensures that the nonlinear terms can be considered as a source term of the linear equation (2.2.18). The proof can be found in (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019).

Proposition 2.2.7. *Let $(\eta,\omega) \in L^2(0,T,[H^1(0,L)]^2)$, so $(\eta\omega)_x$, $\omega\omega_x \in L^1(0,T,X_0)$ and $(\eta,\omega) \in \mathcal{B} \mapsto ((\eta\omega)_x,(\omega\omega_x)) \in L^1(0,T,X_0)$ is continuous. In addition, the following estimate holds,*

$$\begin{aligned}\int_0^T \|((\eta_1\omega_1)_x - (\eta_2\omega_2)_x, \omega_1\omega_{1,x} - \omega_2\omega_{2,x})\|_{X_0}\, dt \leq& KT^{\frac{1}{4}}\left(\|(\eta_1,\omega_1)\|_{\mathcal{B}} + \|(\eta_2,\omega_2)\|_{\mathcal{B}}\right)\\ &\times \|(\eta_1-\eta_2,\omega_1-\omega_2)\|_{\mathcal{B}}\end{aligned}$$

for a constant $K > 0$.

Finally, we are in a position to present the existence of local solutions to (2.1.7).

Theorem 2.2.8. *Let $L, T > 0$ and consider α and β real constants such that (2.1.6) is satisfied. For each initial data $(\eta_0,\omega_0;z_0) \in H$ sufficiently small, $\Gamma\colon \mathcal{B} \to \mathcal{B}$ defined by $\Gamma(\tilde{\eta},\tilde{\omega}) = (\eta,\omega)$ is a contraction. Moreover, there exists a unique solution $(\eta,\omega) \in B(0,R) \subset \mathcal{B}$ of the Boussinesq KdV-KdV nonlinear system (2.1.7).*

Proof. It follows from Theorem 2.2.6 that the map Γ is well defined. Using Proposition 2.2.7 and the *a priori estimates* we obtain

$$\|\Gamma(\tilde{\eta},\tilde{\omega})\|_{\mathcal{B}} = \|(\eta,\omega)\|_{\mathcal{B}} \leq C\left(\|(\eta_0,\omega_0,z_0)\|_H + \|(\tilde{\eta},\tilde{\omega})\|^2_{\mathcal{B}}\right)$$

and

$$\|\Gamma(\tilde{\eta}_1,\tilde{\omega}_1)-\Gamma(\tilde{\eta}_2,\tilde{\omega}_2)\|_{\mathcal{B}} \leq KT^{\frac{1}{4}}\left(\|(\tilde{\eta}_1,\tilde{\omega}_1)\|_{\mathcal{B}}+\|(\tilde{\eta}_2,\tilde{\omega}_2)\|_{\mathcal{B}}\right)\|(\tilde{\eta}_1-\tilde{\eta}_2,\tilde{\omega}_1-\tilde{\omega}_2)\|_{\mathcal{B}}.$$

Now, we restrict Γ to the closed ball $\{(\tilde{\eta},\tilde{\omega})\in\mathcal{B} : \|(\tilde{\eta},\tilde{\omega})\|_{\mathcal{B}}\leq R\}$, with $R>0$ to be determined later. Then, $\|\Gamma(\tilde{\eta},\tilde{\omega})\|_{\mathcal{B}}\leq C\left(\|(\eta_0,\omega_0,z_0)\|_H+R^2\right)$ and

$$\|\Gamma(\tilde{\eta}_1,\tilde{\omega}_1)-\Gamma(\tilde{\eta}_2,\tilde{\omega}_2)\|_{\mathcal{B}} \leq 2RKT^{\frac{1}{4}}\|(\tilde{\eta}_1-\tilde{\eta}_2,\tilde{\omega}_1-\tilde{\omega}_2)\|_{\mathcal{B}}.$$

Next, we pick $R=2C\|(\eta_0,\omega_0,z_0)\|_H$ and $T>0$ such that $2KT^{\frac{1}{4}}R<1$, with $C<2KT^{\frac{1}{4}}$. This leads to claim that

$$\|\Gamma(\tilde{\eta},\tilde{\omega})\|_{\mathcal{B}}\leq R$$

and

$$\|\Gamma(\tilde{\eta}_1,\tilde{\omega}_1)-\Gamma(\tilde{\eta}_2,\tilde{\omega}_2)\|_{\mathcal{B}} < C_1\|(\tilde{\eta}_1-\tilde{\eta}_2,\tilde{\omega}_1-\tilde{\omega}_2)\|_{\mathcal{B}},$$

with $C_1<1$. Lastly, the result is an immediate consequence of the Banach fixed point theorem. □

Remark 2.2.9. We point out that the solutions of the system (2.1.7) obtained in Theorem 2.2.8 are only local. Due to a lack of a priori L^2-estimate, the issue of the global existence of solutions is difficult to address in the energy space for the nonlinear system with a delay term.

2.3 LINEAR STABILIZATION RESULT

Since the L^2 a priori estimate is valid for the linear system, the solutions of the linearized system associated with (2.1.7) are globally well-posed. Therefore, we are ready to prove the main result of this work.

2.3.1 Proof of Theorem 2.1.2

Consider the following Lyapunov functional

$$V(t)=E(t)+\mu_1V_1(t)+\mu_2V_2(t),$$

where $\mu_1,\mu_2\in\mathbb{R}^+$ will be chosen later. Here, $E(t)$ is the total energy given by (2.1.8), while

$$V_1(t)=\frac{1}{2}\int_0^L x\eta(t,x)\omega(t,x)\,dx$$

and

$$V_2(t) = \frac{|\beta|}{2}\tau(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)\,d\rho.$$

Observe that,

$$(1-\max\{\mu_1 L, \mu_2\})E(t) \le V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t) \le (1+\max\{2\mu_1 L, \mu_2\})E(t).$$

The Young's inequality yields that

$$\left|\mu_1 \int_0^L x\eta\omega\,dx\right| \le \mu_1 L \int_0^L |\eta\omega|\;dx \le \frac{\mu_1 L}{2}\int_0^L \left(\eta^2+\omega^2\right)dx. \tag{2.3.1}$$

Moreover,

$$\begin{aligned}
&\left|\mu_1 \int_0^L x\eta\omega\,dx \;+\; \mu_2\cdot\frac{|\beta|}{2}\tau(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)\,d\rho\right| \\
&\qquad \le\; \left|\mu_1 \int_0^L x\eta\omega\,dx\right| + \left|\mu_2\cdot\frac{|\beta|}{2}\tau(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)\,d\rho\right| \\
&\qquad \le\; \frac{\mu_1 L}{2}\int_0^L \left(\eta^2+\omega^2\right)dx + \mu_2\cdot\frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)\,d\rho \\
&\qquad \le\; \max\{\mu_1 L, \mu_2\}\left(\frac{1}{2}\int_0^L \left(\eta^2+\omega^2\right)dx + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)\,d\rho.\right) \\
&\qquad =\; \max\{\mu_1 L, \mu_2\}E(t),
\end{aligned} \tag{2.3.2}$$

and, consequently,

$$(1-\max\{\mu_1 L, \mu_2\})E(t) \le V(t) \le (1+\max\{\mu_1 L, \mu_2\})E(t), \tag{2.3.3}$$

since $\mu_1 L < 1$ by hypothesis.

To obtain the derivative of V_1, we have

$$V_1'(t) = \frac{d}{dt}\left(\int_0^L x\eta\omega dx\right) = \int_0^L x\eta_t\omega dx + \int_0^L x\eta\omega_t dx = I_1 + I_2.$$

Let us analyze each term. For I_1, using the boundary conditions, we get that

$$\begin{aligned}
\int_0^L x\eta_t\omega dx =& -\int_0^L x\omega_x\omega dx - \int_0^L x\omega_{xxx}\omega dx \\
=& \frac{1}{2}\int_0^L \omega^2 dx - \int_0^L \omega_x^2 dx + \int_0^L \frac{x}{2}(\omega_x^2)_x dx \\
=& \frac{1}{2}\int_0^L \omega^2 dx - \int_0^L \omega_x^2 dx + \left(\frac{x}{2}(\omega_x^2)\right)\Big|_0^L - \frac{1}{2}\int_0^L \omega_x^2 dx \\
=& \frac{1}{2}\int_0^L \omega^2 dx - \int_0^L \omega_x^2 dx + \frac{L}{2}(-\alpha\eta_x(L) + \beta\eta_x(t-\tau(t), L))^2 - \frac{1}{2}\int_0^L \omega_x^2 dx.
\end{aligned}$$

Therefore,

$$\int_0^L x\eta_t\omega dx = \frac{1}{2}\int_0^L \omega^2 dx - \frac{3}{2}\int_0^L \omega_x^2 dx + \frac{L}{2}(-\alpha\eta_x(L) + \beta\eta_x(t-\tau(t),L))^2. \tag{2.3.4}$$

For I_2, thanks to the boundary conditions, we have that

$$\begin{aligned}\int_0^L x\eta\omega_t dx =& -\int_0^L x\eta\eta_x dx - \int_0^L x\eta\eta_{xxx} dx\\ =&\frac{1}{2}\int_0^L \eta^2 dx - \int_0^L \eta_x^2 dx + \left(\frac{x}{2}\eta_x^2\right)\Big|_0^L - \frac{1}{2}\int_0^L \eta_x^2 dx\\ =&\frac{1}{2}\int_0^L \eta^2 dx - \int_0^L \eta_x^2 dx - \frac{1}{2}\int_0^L \eta_x^2 dx + \frac{L}{2}\eta_x^2(L)\\ =&\frac{1}{2}\int_0^L \eta^2 dx - \frac{3}{2}\int_0^L \eta_x^2 dx - \int_0^L x\eta\omega\omega_x dx + \frac{L}{2}\eta_x^2(L).\end{aligned} \tag{2.3.5}$$

Adding the identities (2.3.4) and (2.3.5) we obtain the following identity

$$\begin{aligned}V_1'(t) =&\frac{1}{2}\int_0^L \omega^2 dx - \frac{3}{2}\int_0^L \omega_x^2 dx + \frac{L}{2}(-\alpha\eta_x(L) + \beta\eta_x(t-\tau(t),L))^2\\ &+ \frac{1}{2}\int_0^L \eta^2 dx - \frac{3}{2}\int_0^L \eta_x^2 dx + \frac{L}{2}\eta_x^2(L).\end{aligned}$$

Hence,

$$\begin{aligned}V_1'(t) =&\frac{L}{2}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}^T \begin{pmatrix}\alpha^2+1 & -\alpha\beta\\ -\alpha\beta & \beta^2\end{pmatrix}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}\\ &+ \frac{1}{2}\int_0^L \left(\omega^2+\eta^2\right) dx - \frac{3}{2}\int_0^L \left(\omega_x^2+\eta_x^2\right) dx.\end{aligned}$$

Let

$$V_2(t) = \frac{|\beta|}{2}\tau(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho.$$

Remembering that

$$-\tau(t)\partial_t\eta_x(t-\tau(t)\rho,L) = (1-\dot{\tau}(t)\rho)\partial_\rho\eta_x(t-\tau(t)\rho,L)$$

we have, by integration by parts, that

$$
\begin{aligned}
V_2'(t) = & \frac{|\beta|}{2}\dot{\tau}(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho \\
& + |\beta|\tau(t)\int_0^1 (1-\rho)\eta_x(t-\tau(t-\tau(t)\rho, L)\partial_t\eta_x(t-\tau(t)\rho, L)d\rho \\
= & \frac{|\beta|}{2}\dot{\tau}(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho \\
& + |\beta|\int_0^1 (\rho-1)(1-\dot{\tau}(t)\rho)\eta_x(t-\tau(t)\rho, L)\partial_\rho\eta_x(t-\tau(t)\rho, L)d\rho \\
= & \frac{|\beta|}{2}\dot{\tau}(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho \\
& + \frac{|\beta|}{2}\int_0^1 (\rho-1)(1-\dot{\tau}(t)\rho)\left(\eta_x^2(t-\tau(t)\rho, L)\right)_\rho d\rho \\
= & \frac{|\beta|}{2}\dot{\tau}(t)\int_0^1 (1-\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho \\
& + \frac{|\beta|}{2}\int_0^1 \left[(1-\rho)(1-\dot{\tau}(t)\rho)\right]_\rho \eta_x^2(t-\tau(t)\rho, L)d\rho \\
& + \frac{|\beta|}{2}\left[(\rho-1)(1-\dot{\tau}(t)\rho)\eta_x^2(t-\tau(t)\rho, L)\right]_{\rho=0}^{\rho=1} \\
= & -\frac{|\beta|}{2}\int_0^1 (1-\dot{\tau}(t)\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho + \frac{|\beta|}{2}\eta_x^2(t, L),
\end{aligned}
$$

that is,

$$
V_2'(t) = -\frac{|\beta|}{2}\int_0^1 (1-\dot{\tau}(t)\rho)\eta_x^2(t-\tau(t)\rho, L)d\rho + \frac{|\beta|}{2}\eta_x^2(t, L). \tag{2.3.6}
$$

Since the energy of our problem is given by

$$
E(t) = \frac{1}{2}\int_0^L \left(\eta^2 + \omega^2\right) dx + \frac{|\beta|}{2}\tau(t)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)d\rho,
$$

yields that

$$
E'(t) = \frac{1}{2}\begin{pmatrix} \eta_x(t, L) \\ \eta_x(t-\tau(t), L) \end{pmatrix}^T \Phi_{\alpha,\beta} \begin{pmatrix} \eta_x(t, L) \\ \eta_x(t-\tau(t), L) \end{pmatrix},
$$

with

$$
\Phi_{\alpha,\beta} = \begin{pmatrix} -2\alpha + |\beta| & \beta \\ \beta & |\beta|(d-1) \end{pmatrix}.
$$

Let

$$
V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t).
$$

Then,

$$
\begin{aligned}
V'(t)+\lambda V(t)=&E'(t)+\mu_1V_1'(t)+\mu_2V_2'(t)+\lambda E(t)+\lambda\mu_1V_1(t)+\lambda\mu_2V_2(t)\\
=&\frac{1}{2}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}^T\Phi_{\alpha,\beta}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}\\
&+\frac{\mu_1L}{2}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}^T\begin{pmatrix}\alpha^2+1 & -\alpha\beta\\ -\alpha\beta & \beta^2\end{pmatrix}\begin{pmatrix}\eta_x(t,L)\\ \eta_x(t-\tau(t),L)\end{pmatrix}\\
&+\frac{\mu_1}{2}\int_0^L\left(\omega^2+\eta^2\right)dx-\frac{3\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)dx\\
&-\mu_2\frac{|\beta|}{2}\int_0^1(1-\dot\tau(t)\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho+\frac{\mu_2|\beta|}{2}\eta_x^2(t,L)\\
&+\frac{\lambda}{2}\int_0^L\left(\eta^2+\omega^2\right)dx+\frac{\lambda|\beta|}{2}\tau(t)\int_0^1\eta_x^2(t-\tau(t)\rho,L)d\rho\\
&+\mu_1\lambda\int_0^Lx\eta\omega dx+\frac{\mu_2|\beta|\lambda}{2}\tau(t)\int_0^1(1-\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V'(t)+\lambda V(t)=&\frac{1}{2}\left\langle\Psi_{\mu_1,\mu_2}(\eta_x(t,L),\eta_x(t-\tau(t),L)),(\eta_x(t,L),\eta_x(t-\tau(t),L))\right\rangle\\
&+\frac{\mu_1}{2}\int_0^L\left(\omega^2+\eta^2\right)dx-\frac{3\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)dx\\
&+\frac{\lambda}{2}\int_0^L\left(\eta^2+\omega^2\right)dx+\mu_1\lambda\int_0^Lx\eta\omega dx\\
&-\mu_2\frac{|\beta|}{2}\int_0^1(1-\dot\tau(t)\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho+\frac{\lambda|\beta|}{2}\tau(t)\int_0^1\eta_x^2(t-\tau(t)\rho,L)d\rho\\
&+\frac{\mu_2|\beta|\lambda}{2}\tau(t)\int_0^1(1-\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho\\
=&M+S_1+S_2.
\end{aligned}
$$

Here, the terms Ψ_{μ_1,μ_2}, M, S_1 and S_2 are given by

$$
\Psi_{\mu_1,\mu_2}=\Phi_{\alpha,\beta}+L\mu_1\begin{pmatrix}\alpha^2+1 & -\alpha\beta\\ -\alpha\beta & \beta^2\end{pmatrix}+|\beta|\mu_2\begin{pmatrix}1&0\\0&0\end{pmatrix}, \tag{2.3.7}
$$

$$
\begin{aligned}
M=&\frac{1}{2}\left\langle\Psi_{\mu_1,\mu_2}(\eta_x(t,L),\eta_x(t-\tau(t),L)),(\eta_x(t,L),\eta_x(t-\tau(t),L))\right\rangle,\\
S_1=&\frac{\mu_1}{2}\int_0^L\left(\omega^2+\eta^2\right)dx-\frac{3\mu_1}{2}\int_0^L\left(\omega_x^2+\eta_x^2\right)dx+\frac{\lambda}{2}\int_0^L\left(\eta^2+\omega^2\right)dx+\mu_1\lambda\int_0^Lx\eta\omega dx,
\end{aligned}
$$

and

$$
\begin{aligned}
S_2=&-\mu_2\frac{|\beta|}{2}\int_0^1(1-\dot\tau(t)\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho+\frac{\lambda|\beta|}{2}\tau(t)\int_0^1\eta_x^2(t-\tau(t)\rho,L)d\rho\\
&+\frac{\mu_2|\beta|\lambda}{2}\tau(t)\int_0^1(1-\rho)\eta_x^2(t-\tau(t)\rho,L)d\rho,
\end{aligned}
$$

respectively.

Now we need to prove that $V'(t)+\lambda V(t)\leq 0$. To do that, let us analyze each term above.

Estimate for M: From the properties of $\Phi_{\alpha,\beta}$ and the continuity of the trace and determinant functions, we can ensure that Ψ_{μ_1,μ_2} is negative definite. Thus,

$$M \leq 0.$$

Estimate for S_1: Observe that using Poincaré inequality, we get that

$$\begin{aligned} S_1 \leq & \frac{1}{2}\left(\lambda(1+\mu_1 L)+\mu_1\right)\int_0^L \left(\omega^2+\eta^2\right)dx - \frac{3\mu_1}{2}\int_0^L \left(\omega_x^2+\eta_x^2\right)dx \\ \leq & \left[\frac{L^2}{2\pi^2}\left(\lambda(1+\mu_1 L)+\mu_1\right) - \frac{3\mu_1}{2}\right]\int_0^L \left(\omega_x^2+\eta_x^2\right)dx. \end{aligned}$$

Thus,

$$S_1 < 0,$$

if

$$\lambda < \frac{\mu_1(3\pi^2 - L^2)}{L^2(1+\mu_1)}.$$

Estimate for S_2: Note that

$$\begin{aligned} S_2 \leq & -\frac{\mu_2|\beta|}{2}(1-d)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)d\rho + \frac{\lambda|\beta|M}{2}\int_0^1 \eta_x^2(t-\tau(t)\rho, L)d\rho \\ & + \frac{\lambda\mu_2|\beta|M}{2}\int_0^1 \eta_x^2(t-\tau(t)\rho, L)d\rho \\ \leq & \frac{|\beta|}{2}\left(\lambda M + \lambda\mu_2 M - \mu_2(1-d)\right)\int_0^1 \eta_x^2(t-\tau(t)\rho, L)d\rho. \end{aligned}$$

Then, choosing

$$\lambda < \frac{\mu_2(1-d)}{M(1+\mu_2)}$$

we have that

$$\frac{|\beta|}{2}\left(\lambda M + \lambda\mu_2 M - \mu_2(1-d)\right) < 0.$$

Therefore, for $\zeta > 0$ and $\lambda > 0$ fulfilling (2.1.9) and (2.1.10), respectively, we have

$$\frac{d}{dt}V(t) + \lambda V(t) \leq 0 \iff E(t) \leq \zeta E(0)e^{-\lambda t}, \quad \forall t \geq 0,$$

since $V(t)$ satisfies (2.3.3). This achieves the proof of the theorem. □

2.3.2 Optimization of the decay rate

We can optimize the value of λ in Theorem 2.1.2 to obtain the best decay rate for the linear system associated with (2.1.7) in the following way:

Proposition 2.3.1. *Choosing the constant μ_1 as follows*

$$\mu_1 \in \left[0, \frac{(2\alpha - |\beta|)(1-d) - |\beta|}{L(1-d)(1+\alpha^2)}\right), \tag{2.3.8}$$

we claim that λ has the largest possible value.

Proof. Define the functions f and $g : \left[0, \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}\right] \to \mathbb{R}$ by

$$f(\mu_1) = \frac{\mu_1 (3\pi^2 - L^2)}{L^2(1+\mu_1 L)},$$

and

$$g(\mu_1) = \frac{(2\alpha - |\beta|)(1-d) - |\beta| - L(1-d)(1+\alpha^2)\mu_1}{M(2\alpha(1-d) - |\beta| - L(1-d)(1+\alpha^2)\mu_1)}(1-d).$$

Then, let $\lambda(\mu_1) = \min\{f(\mu_1), g(\mu_1)\}$ we have the following claims.

Claim 2.3.2. *The function f is increasing in the interval $\left[0, \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}\right)$ while the function g is decreasing in the same interval.*

In fact, note that if

$$f(\mu_1) = \frac{(3\pi^2 - L^2)}{L^3}\left(1 - \frac{1}{1+\mu_1 L}\right) \implies f'(\mu_1) = \frac{(3\pi^2 - L^2)}{L^2(1+\mu_1 L)^2} > 0.$$

In particular, $f'(\mu_1) > 0$ for $\mu_1 \in \left[0, \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}\right)$. Analogously, as

$$g(\mu_1) = \frac{1-d}{M} - \frac{|\beta|(1-d)^2}{ML(1-d)(1+\alpha^2)}\left(\frac{1}{\frac{2\alpha(1-d)-|\beta|}{L(1-d)(1+\alpha^2)} - \mu_1}\right),$$

so

$$g'(\mu_1) = -\frac{|\beta|(1-d)^2}{ML(1-d)(1+\alpha^2)}\left[\frac{1}{\left(\frac{2\alpha(1-d)-|\beta|}{L(1-d)(1+\alpha^2)} - \mu_1\right)^2}\right] < 0,$$

since

$$\mu_1 < \frac{(2\alpha - |\beta|)(1-d) - |\beta|}{L(1-d)(1+\alpha^2)} < \frac{2\alpha(1-d) - |\beta|}{L(1-d)(1+\alpha^2)},$$

showing the claim 2.3.2.

Claim 2.3.3. *There exists only one point satisfying (2.3.8) such that $f(\mu_1) = g(\mu_1)$.*

Indeed, to show the existence of this point, it is sufficient to note that $f(0) = 0$,

$$f\left(\frac{(2\alpha - |\beta|)(1-d) - |\beta|}{L(1-d)(1+\alpha^2)}\right) = \frac{(3\pi^2 - L^2)}{2L^3}\left(1 - \frac{(1-d)(1+\alpha^2)}{(1-d)(1+\alpha^2) + (2\alpha - |\beta|)(1-d) - |\beta|}\right) > 0$$

and

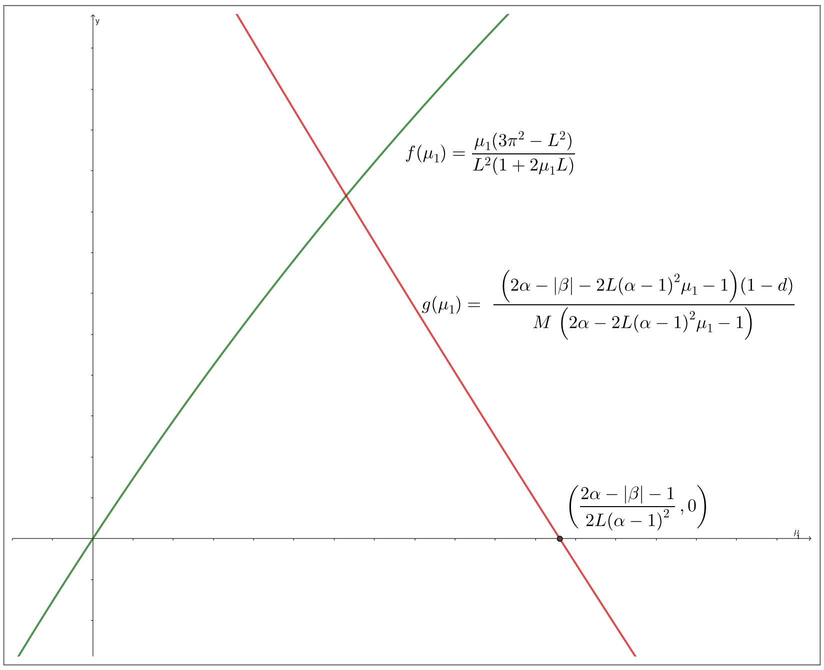
$$g(0) = \frac{1-d}{M}\left(1 - \frac{|\beta|(1-d)}{2\alpha(1-d) - |\beta|}\right) > 0, \quad g\left(\frac{(2\alpha - |\beta|)(1-d) - |\beta|}{L(1-d)(1+\alpha^2)}\right) = 0.$$

The uniqueness follows from the fact that f is increasing while g is decreasing in this interval and claim 2.3.3 holds.

Finally, taking into account the claims 2.3.2 and 2.3.3, the maximum value of the function λ must be reached at the point μ_1 satisfying (2.3.8), where $f(\mu_1) = g(\mu_1)$, and the Proposition 2.3.1 is achieved. □

We can illustrate, this in Figure 3 below, the situation of the previous proposition taking, for instance, $L = 5$, $d = \frac{1}{2}$, $\alpha = 1$, $\beta = \frac{1}{2}$ and $M = 3$, when $\lambda(\mu_1) = \min\{f(\mu_1), g(\mu_1)\}$:

Figure 3 – Ilustration of Proposition 2.3.1



Source: Own elaboration

2.4 CONCLUDING DISCUSSION

This chapter was concerned with the local well-posedness for the system (2.1.7) and stabilization of the energy associated with the linearized KdV-KdV system posed on a bounded domain. We proved the local well-posedness result by considering a linear combination of the damping mechanism and a time-varying delay term. Moreover, since we have the global solution associated with the linearized system, the energy method is used to show the exponential stabilization outcome for the linearized system.

2.4.1 Further comments

The following remarks are worth mentioning.

1. The well-posedness finding is not proved directly. The main issue is due to the time-varying delay term that makes the associated operator for the system time-dependent. Therefore, we invoked the ideas introduced by Kato (KATO, 1970) to solve an abstract Cauchy problem of the "hyperbolic" type.

2. In (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019), the authors showed the stabilization result when $\beta = 0$. In this case, using the classical compactness-uniqueness argument, they found a restrictive condition on the spatial length, that is, the stabilization follows if only if
$$L \notin \mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2} \,:\, k,\, l \in \mathbb{N}^* \right\}.$$
Additionally, in (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019), the decay rate could not be characterized. In turn, due to the presence of the time-varying delay term in our problem, the restriction on the spatial length is $L \in (0, \sqrt{3}\pi)$, which seems reasonable. Last but not least, the decay rate of the energy is explicitly provided contrary to (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019). However, the drawback of our result is that it is only true for the linearized system.

3. It is noteworthy that the strategy used in (PAZOTO; ROSIER, 2008), and more recently in (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019) ensures the global solution of the nonlinear system (2.1.7) **without delay**. However, such a strategy can not be applied when a time-dependent delay occurs. This is because in this case, the system is non-autonomous.

In addition to that, this strategy fails to provide the desired result (global existence of solutions) for the nonlinear system even if a constant delay $\tau(t) = h$ is considered. The reason is our operator A, defined by (2.2.4), has a transport part with nonhomogeneous boundary conditions given by the equation (2.2.2) and hence we can not expect to control the solution of the transport part in the space $H^{1/3}(0,1)$ in terms of the $L^2(0,1)$ norm of the initial data. Thus, for the full system (2.1.7) with a constant delay $\tau(t) = h$, another approach needs to be applied. We discuss it in the last subsection of the work.

4. Naturally, it would be interesting to make a comparison between the KdV-KdV and the KdV models. Two important facts appear:

   - The Lyapunov approach provides a direct way to deal with the nonlinear system KdV equation, as shown in (PARADA; TIMIMOUN; VALEIN, 2023). In this work, stability results for the KdV equation with time-varying delay are established using the same techniques. In comparison to our work, two KdV equations are coupled by the nonlinearities; thus the complexity of the problem suggests choosing a different Lyapunov functional and deals only with the linearized system.
   - Another interesting comparison is about the energy decay rate associated with the KdV and KdV-KdV models, at least for the linear problem. In both cases, the explicit decay rate is shown.

5. A calculation shows that taking μ_1 and μ_2 in Theorem 2.1.2 such that

$$\mu_1 < \min\left\{\frac{2\alpha-|\beta|}{L(1+\alpha^2)}, \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}\right\} = \frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}$$

and

$$\begin{aligned}\mu_2 = \min\Bigg\{&\frac{(2\alpha-|\beta|)-L(1+\alpha^2)\mu_1}{|\beta|},\\ &\frac{(2\alpha-\beta)(1-d)-|\beta|-L(1-d)(1+\alpha^2)\mu_1}{|\beta|(1-d)}\Bigg\}\\ =&\frac{(2\alpha-\beta)(1-d)-|\beta|-L(1-d)(1+\alpha^2)\mu_1}{|\beta|(1-d)},\end{aligned}$$

implies that Ψ_{μ_1,μ_2}, given by (2.3.7), is negative definite provide that $|\alpha| < 1$.

In fact, recall

$$\begin{aligned}\Psi_{\mu_1,\mu_2} &= \begin{pmatrix} -2\alpha+|\beta|+\mu_1 L(1+\alpha^2)+\mu_2|\beta| & \beta(1-L\mu_1\alpha) \\ \beta(1-L\mu_1\alpha) & |\beta|(d-1)+L\mu_1\beta^2 \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.\end{aligned}$$

In order to Ψ_{μ_1,μ_2} be negative definite, the term a_{11} must be negative,

$$-2\alpha+|\beta|+L\mu_1(1+\alpha^2)+|\beta|\mu_2<0 \Longleftrightarrow \mu_2<\frac{(2\alpha-|\beta|)-L(1+\alpha^2)\mu_1}{|\beta|}$$

with

$$2\alpha-|\beta|-L(1+\alpha^2)\mu_1>0,$$

which implies that μ_1 must satisfy

$$\mu_1<\frac{2\alpha-|\beta|}{L(1+\alpha^2)}.$$

Moreover, we need that

$$\det\Psi_{\mu_1,\mu_2}=\begin{vmatrix} -2\alpha+|\beta|+L\mu_1(\alpha^2+1)+|\beta|\mu_2 & \beta(1-L\mu_1\alpha) \\ \beta(1-L\mu_1\alpha) & |\beta|(d-1)+L\mu_1\beta^2 \end{vmatrix}>0.$$

Note that,

$$\begin{aligned}\det\Psi_{\mu_1,\mu_2} =&|\beta|\left[(L\mu_1)^2|\beta|+L\mu_1(1+\mu_2)|\beta|^2-L\mu_1(\alpha^2+1)(1-d)\right.\\ &\left.-\left((-2\alpha+|\beta|)(1-d)+|\beta|\mu_2(1-d)+|\beta|\right)\right].\end{aligned}$$

Since

$$(L\mu_1)^2|\beta|+L\mu_1|\beta|^2(1+\mu_2)>0,$$

in order to the determinant of Ψ_{μ_1,μ_2} be positive, we only need

$$-L\mu_1(1-d)(1+\alpha^2)-((-2\alpha+|\beta|)(1-d)+|\beta|\mu_2(1-d)+|\beta|)=0$$

that is,

$$-L\mu_1(1-d)(1+\alpha^2)+(2\alpha-|\beta|)(1-d)-|\beta|\mu_2(1-d)-|\beta|=0.$$

Thus, we have

$$\mu_2=\frac{(2\alpha-|\beta|)(1-d)-|\beta|-L(1-d)(1+\alpha^2)\mu_1}{|\beta|(1-d)}$$

with

$$\mu_1<\frac{(2\alpha-|\beta|)(1-d)-|\beta|}{L(1-d)(1+\alpha^2)}.$$

6. From Theorem 2.1.2 and item (4), it follows that when $L < \sqrt{3}\pi$ and by taking $\mu_1, \mu_2 > 0$ so that $\mu_1 L < 1$ and

$$\mu_1 < \frac{(2\alpha - |\beta|)(1-d) - |\beta|}{L(1-d)(1+\alpha^2)},$$
$$\mu_2 = \frac{(2\alpha - |\beta|)(1-d) - |\beta| - L(1-d)(1+\alpha^2)\mu_1}{|\beta|(1-d)},$$

we reach that $E(t) \leq \zeta E(0)e^{-\lambda t}$, for all $t \geq 0$ where

$$\lambda \leq \min\left\{\frac{\mu_1(3\pi^2 - L^2)}{L^2(1+\mu_1)}, \frac{\mu_2(1-d)}{M(1+\mu_2)}\right\} \quad \text{and} \quad \zeta = \frac{1+\max\{\mu_1 L, \mu_2\}}{1-\max\{\mu_1 L, \mu_2\}}.$$

3 EXPONENTIAL STABILIZATION FOR THE HIROTA-SATSUMA SYSTEM BY BOUNDARY TIME-DELAY INPUT

V. H. Gonzalez Martinez, and J. R. Muñoz, arXiv:2408.15705 [math.AP], (Submitted).

3.1 INTRODUCTION

The Hirota-Satsuma serves as a model for understanding certain types of nonlinear wave interactions and phenomena that arise in the propagation of nonlinear waves in shallow water or the behavior of waves in stratified fluids, in various physical systems and their properties provide insights into the behavior of nonlinear waves in different physical systems.

In 1981, Hirota and Satsuma (HIROTA; SATSUMA, 1981) introduced a system that includes two real functions depending on the time and the space denoted by, $u = u(t, x)$ and $v = v(t, x)$, modeling the interactions of two long waves with different dispersion relations. The system is given by the equations:

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) - 2bvu_x = 0, & x \in \mathbb{R},\ t \geq 0, \\ v_t + v_{xxx} + 3uvx = 0, & x \in \mathbb{R},\ t \geq 0. \end{cases}$$

The asymptotic behavior of dispersive systems, described by partial differential equations (PDEs), has been a significant research focus recently. The main goal has been to develop control mechanisms, such as feedback and boundary controls, to stabilize these systems by ensuring energy decay or mitigating disturbances. Significant progress has been made in stabilizing systems on bounded domains, like KdV, Kawahara, and Boussinesq-type systems, often achieving exponential stabilization through damping mechanisms.

Physically, the Hirota-Satsuma system specifically models two waves with different speeds, while the KdV-KdV system typically involves two waves with symmetric interaction. It is noteworthy that the Boussinesq KdV-KdV type system with boundary time-dependent delay was studied by the authors in (CAPISTRANO-FILHO et al., 2024) obtaining the exponential decay. However, due to the lack of regularity of this system, only the linearized version admits this property. Inspired by this issue, our main objective is to obtain the global well-posedness and then, describe the asymptotic behavior with a boundary time-delay feedback for a KdV-type system that includes the coupled nonlinear terms.

Additionally, the study of PDEs with time delays has gained attention due to their relevance in various fields, such as biology and engineering. Time delays, caused by factors like measurement lag or computation time, can both destabilize a system and improve its performance, depending on their implementation (See (ARARUNA; CAPISTRANO-FILHO; DORONIN, 2012; GALLEGO, 2018; CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024; CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019; DATKO, 1988; DATKO; LAGNESE; POLIS, 1986; NICAISE; PIGNOTTI, 2006; PARADA; TIMIMOUN; VALEIN, 2023; PAZOTO; ROSIER, 2008; ROSIER, 1997; VALEIN, 2022) and therein).

Among the main novelties presented here, the method employed allows us to prove the exponential stability and give an explicit decay rate of the Hirota-Satsuma system. Moreover, the main result does not involve any restriction on the size of the spatial interval, it is obtained for the nonlinear system and the techniques present here can be adapted for a time-varying delayed system as in (CAPISTRANO-FILHO et al., 2024).

3.1.1 Problem Setting and main results

Let us describe the problem which we are interested in studying. Consider the bounded domain $(0, L)$ with $L > 0$ and $t > 0$. Then, the Hirota-Satsuma system is given by

$$\begin{cases} u_t - \frac{1}{2}u_{xxx} - 3uu_x - 3vv_x = 0 & x \in (0, L),\ t > 0, \\ v_t + v_{xxx} + 3uv_x = 0 & x \in (0, L),\ t > 0, \\ u(t, 0) = u(t, L) = v(t, 0) = v(t, L) = u_x(t, 0) = 0, & t > 0, \\ v_x(t, L) = \alpha u_x(t, L) + \beta u_x(t - h, L), & t > 0, \\ u(0, x) = u_0(x) \in L^2(0, L), & x \in (0, L) \\ v(0, x) = v_0(x) \in L^2(0, L), & x \in (0, L) \\ u_x(t - h, L) = z_0(t - h, L) \in L^2(0, 1), & t \in (0, 1). \end{cases} \tag{3.1.1}$$

where u_0, v_0 denotes the initial data and z_0 denotes the delayed data. Furthermore, the system (3.1.1) involves the parameters α and β that will be related to the feedback gains given from the damping anti-damping mechanism as the constant time delay that will be denoted by h. Moreover, the interaction between the feedback gains α and β must satisfy the following constraint

$$0 < \alpha^2 + \frac{3}{2}\beta < \frac{1}{2}. \tag{3.1.2}$$

Then, we can define the total energy associated with the Hirota-Satsuma system (3.1.1) as

$$E(t) = \frac{1}{2}\int_0^L u^2(t, x) + v^2(t, x)\, dx + \frac{\beta}{2}h\int_0^1 u_x^2(t - h\rho, L)\, d\rho. \tag{3.1.3}$$

Formally, some integrations by parts allow us to deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq \frac{1}{2}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L) \end{pmatrix}^T \Phi_{\alpha,\beta} \begin{pmatrix} u_x(t,L) \\ u_x(t-h,L) \end{pmatrix} \tag{3.1.4}$$

where

$$\Phi_{\alpha,\beta} = \begin{pmatrix} \alpha^2 - \frac{1}{2} + \beta & \alpha\beta \\ \alpha\beta & \beta^2 - \beta \end{pmatrix}$$

is a negative definite matrix. Indeed, the first entry satisfies

$$\frac{1}{2} > \alpha^2 + \frac{3}{2}\beta > \alpha^2 + \beta \implies \alpha^2 + \beta - \frac{1}{2} < 0.$$

By using (3.1.2) we get $\frac{1}{2} + \beta^2 > \alpha^2 + \frac{3}{2}\beta$, then the determinant is such that

$$\begin{aligned} \frac{1}{\beta}\det\Phi &= \left(\alpha^2 - \frac{1}{2} + \beta\right)(\beta - 1) - \alpha^2\beta \\ &= -\alpha^2 - \frac{3}{2}\beta + \beta^2 + \frac{1}{2} > 0. \end{aligned}$$

Therefore, from (3.1.4), we obtain that the total energy $E(t)$ associated with the Hirota-Satsuma system is a non-increasing function. Then the natural question arises:

Does $E(t) \to 0$ as $t \to \infty$? If this is the case, can we give an explicit decay rate?

Before presenting our positive answer to this question and the main result of this work let us define the functional spaces that will be used throughout the analysis, $X_0 := [L^2(0,L)]^2$ and $H := X_0 \times L^2(0,1)$ and consider

$$\mathcal{B} := C([0,T], X_0) \cap L^2(0,T,[H^1(0,L)]^2)$$

with the associated norm $\|(u,v)\|_{\mathcal{B}} = \sup_{t\in[0,T]} \|(u(t),v(t)\|_{X_0} + \|(u_x,v_x)\|_{L^2(0,T,X_0)}$.

Then, we ensure that the total energy associated with the Hirota-Satsuma system (3.1.1) decays exponentially, that is,

Theorem 3.1.1. *Let $L > 0$ and α, β such that (3.1.2) yields. Then, there exists $0 < r < 3/16L^{\frac{3}{2}}$ such that for every initial data $(u_0, v_0, z_0) \in H$ with $\|(u_0,v_0,z_0)\|_H \leq r$, the energy $E(t)$ defined in (3.1.3) of the Hirota-Satsuma system (3.1.1) decays exponentially. More precisely, given μ_1, μ_2 positive constants small enough, then there exists $\kappa = 1 + \max\{\mu_1 L, \mu_2\}$ and*

$$\lambda \leq \min\left\{\frac{\pi^2\mu_1(3 - 16L^{\frac{3}{2}}r)}{2L^2(1+L\mu_1)}, \frac{\mu_2}{h(1+\mu_2)}\right\}$$

such that $E(t) \leq \kappa E(0)e^{-\lambda t}$, for all $t \geq 0$.

We end this section by providing an outline of this chapter: Section 3.2 is devoted to the proofs of the well-posedness, first dealing with the linear system and then by the Kato smoothing addressing the global well-posedness. Next, in Section 3.3 we use Lyapunov's approach and obtain the exponential stability of the solutions of the Hirota-Satsuma system issued from small initial data.

3.2 WELL-POSEDNESS

In this section, we address the well-posedness of the Hirota-Satsuma system, that is, we obtain the existence of solutions, that in conjunction with some *a priori* estimates and the Kato smoothing effect, allow us to deal with the nonlinear systems and obtain a prove the global well-posedness of the solutions.

3.2.1 Linear problem

First, we consider the linearization of (3.1.1) around the origin

$$\begin{cases} u_t - \frac{1}{2}u_{xxx} = 0 & x \in (0,L),\ t > 0 \\ v_t + v_{xxx} = 0 & x \in (0,L),\ t > 0 \\ u(t,0) = u(t,L) = v(t,0) = v(t,L) = u_x(t,0) = 0, & t > 0, \\ v_x(t,L) = \alpha u_x(t,L) + \beta u_x(t-h,L), & t > 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x) \in L^2(0,L) \\ u_x(t-h,L) = z_0(t-h,L) \in L^2(0,1). \end{cases} \tag{3.2.1}$$

Now, following the idea introduced in (NICAISE; PIGNOTTI, 2006), let us introduce the change of variables $z(t,\rho) = u_x(t-h\rho, L)$ with $\rho \in (0,1)$ that satisfies the transport equation

$$\begin{cases} hz_t(t,\rho) + z_\rho(t,\rho) = 0, & \rho \in (0,1),\ t > 0 \\ z(t,0) = u_x(t,L), z(0,\rho) = z_0(-h\rho), & \rho \in (0,1),\ t > 0 \end{cases} \tag{3.2.2}$$

and consider H equipped with the inner product[1]

$$\langle (u,v,z), (\overline{u},\overline{v},\overline{z}) \rangle = \langle (u,v), (\overline{u},\overline{v}) \rangle_{X_0} + \beta h \langle z, \overline{z} \rangle_{L^2(0,1)}$$

[1] This new inner product is equivalent to the usual inner product on H

for any $(u,v,z),(\overline{u},\overline{v},\overline{z})\in H$. Now, pick up $U=(u,v,z)$ and recast (3.2.1)-(3.2.2) as a Cauchy abstract problem

$$\frac{\mathrm{d}}{\mathrm{d}t}U = AU, \quad U(0)=U_0, \quad t>0,$$

where $A\colon D(A)\subset H\to H$ is the operator given by

$$A(u,v,z) := \Big(\frac{1}{2}u_{xxx}, -v_{xxx}, -\frac{1}{h}z_\rho\Big)$$

with a densely defined domain

$$D(A) := \left\{ \begin{array}{r|l} (u,v)\in[H^3(0,L)\cap H_0^1(0,L)]^2, & u_x(0)=0,\ z(0)=u_x(L), \\ z\in H^1(0,1) & v_x(L)=\alpha u_x(L)+\beta z(1) \end{array} \right\} \subset H$$

Proposition 3.2.1. *Suppose that (3.1.2) yields. Then A generates a continuous semigroup of contractions $(S(t))_{t\geq 0}$ in H.*

Proof. Clearly, A is densely defined and closed, so we are done if we prove that A and its adjoint A^* are both dissipative in H. It is readily seen that $A^*\colon D(A^*)\subset H\to H$ is given by

$$A^*(\eta,\omega,\theta) := \Big(-\frac{1}{2}\eta_{xxx}, \omega_{xxx}, \frac{1}{h}\theta_\rho\Big) \tag{3.2.3}$$

with domain

$$D(A^*) := \left\{ \begin{array}{r|l} (\eta,\omega)\in[H^3(0,L)\cap H_0^1(0,L)]^2, & \omega_x(0)=0,\ \theta(1)=\omega_x(L), \\ \theta\in H^1(0,1) & \eta_x(L)=2\alpha\omega_x(L)+2\beta\theta(0) \end{array} \right\} \subset H. \tag{3.2.4}$$

Let $(u,v,z)\in D(A)$, then performing some integrations by parts holds that

$$\langle A(u,v,z);(u,v,z)\rangle_H \leq \frac{1}{2}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}^T \Phi_{\alpha,\beta}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}^T \leq 0. \tag{3.2.5}$$

Where $\Phi_{\alpha,\beta}$ is the negative definite matrix given by (3.1.1). On the other hand, let $(\eta,\omega,\theta)\in D(A^*)$, then

$$\langle A^*(\eta,\omega,\theta);(\eta,\omega,\theta)\rangle_H \leq \frac{1}{2}\begin{pmatrix} \omega_x(L) \\ \theta(0)\end{pmatrix}^T \tilde{\Phi}_{\alpha,\beta}\begin{pmatrix} \omega_x(L) \\ \theta(0)\end{pmatrix} \tag{3.2.6}$$

where

$$\tilde{\Phi}_{\alpha,\beta} = \begin{pmatrix} 2\alpha^2+\beta-1 & \alpha\beta \\ \alpha\beta & 2\beta^2-\beta \end{pmatrix}. \tag{3.2.7}$$

It is not difficult to verify that under the assumption (3.1.2), $\tilde{\Phi}_{\alpha,\beta}$ is negative definite and consequently

$$\langle A^*(\eta,\omega,\theta);(\eta,\omega,\theta)\rangle_H \leq \frac{1}{2}\begin{pmatrix}\omega_x(L)\\ \theta(0)\end{pmatrix}^T \tilde{\Phi}\begin{pmatrix}\omega_x(L)\\ \theta(0)\end{pmatrix} \leq 0. \tag{3.2.8}$$

Summarizing , A and A^* are dissipative, from (PAZY, 1983, Corollary 4.4, page 15) the result yields. □

Notice that, $E'(t) = \langle A(u,v,z);(u,v,z)\rangle_H$, then we can establish the next proposition to state that the energy (3.1.3) is decreasing along the solutions of (3.2.1).

Proposition 3.2.2. *Suppose that α and β are real constants such that (3.1.2) holds. Then for any mild solution of (3.2.1) the energy $E(t)$ defined by (3.1.3) is non-increasing and*

$$E'(t) \leq \frac{1}{2}\begin{pmatrix}u_x(t,L)\\ u_x(t-h,L)\end{pmatrix}^T \Phi_{\alpha,\beta}\begin{pmatrix}u_x(t,L)\\ u_x(t-h,L)\end{pmatrix}$$

The following proposition (whose proof is analogous to Proposition 2.4 in (CAPISTRANO-FILHO et al., 2024)) provides useful estimates for the mild solutions of (3.2.1). The first ones are standard energy estimates, while the last one reveals a Kato smoothing effect.

Proposition 3.2.3. *Let α and β are real constant such that (3.1.2) holds. Then, the map*

$$(u_0,v_0,z_0)\in H \mapsto (u,v,z)\in \mathcal{B}\times C(0,T;L^2(0,1))$$

is well defined, continuous and fulfills

$$\|(u,v)\|^2_{X_0} + \beta\|z\|^2_{L^2(0,1)} \leq \|(u_0,v_0)\|^2_{X_0} + \beta\|z_0(-h\cdot)\|^2_{L^2(0,1)}. \tag{3.2.9}$$

Furthermore, for every $(u_0,v_0,z_0)\in H$, we have that

$$\|u_x(\cdot,L)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)} \leq \|(u_0,v_0)\|^2_{X_0} + \|z_0(-h\cdot)\|^2_{L^2(0,1)} \tag{3.2.10}$$

Moreover, the Kato smoothing effect is verified, that is,

$$\int_0^T\int_0^L u_x^2 + v_x^2\,dx\,dt \leq C(L,\alpha,\beta)\left(\|(u_0,v_0)\|^2_{X_0} + \|z_0(-h\cdot)\|^2_{L^2(0,1)}\right). \tag{3.2.11}$$

Proof. The proof of estimates (3.2.9)-(3.2.10) is analogous to Proposition 2.4 in (CAPISTRANO-FILHO et al., 2024). Now, we use Morawetz multipliers technique. Multiplying $(3.2.1)_1$ by $(L-x)u$

and (3.2.1)$_2$ by xv adding the results and integrating by parts follows that

$$
\begin{aligned}
0 = & \frac{1}{2}\int_0^L (L-x)\left[u^2(T,x) - u_0^2(x)\right]\,dx + \frac{1}{2}\int_0^L x\left[v^2(T,x) - v_0^2(x)\right]\,dx \\
& \frac{3}{4}\int_0^T\int_0^L u_x^2\,dx\,dt + \frac{3}{2}\int_0^T\int_0^L v_x^2\,dx\,dt - \frac{L}{2}\int_0^T \left[\alpha u_x(t,L) + \beta z(1)\right]^2\,dt
\end{aligned}
$$

This implies, by using (3.2.9) and (3.2.10)

$$
\begin{aligned}
\frac{3}{4}\int_0^T\int_0^L u_x^2\,dx\,dt + \frac{3}{2}\int_0^T\int_0^L v_x^2\,dx\,dt = & \frac{1}{2}\int_0^L (L-x)\left[u_0^2 - u^2(T,x)\right]\,dx \\
& + \frac{1}{2}\int_0^L x\left[v_0^2 - v^2(T,x)\right]\,dx + \frac{L}{2}\int_0^T \left[\alpha u_x(t,L) + \beta z(1)\right]^2\,dt \\
\leq & \frac{L}{2}\|(u_0,v_0)\|_{X_0}^2 - \frac{L}{2}\|(u(T,x),v(T,x))\|_{X_0}^2 \\
& + (\alpha^2+\beta^2)L\left[\int_0^T u_x^2(t,L)\,dt + \int_0^T z^2(1)\,dt\right] \\
\leq & L\|(u_0,v_0)\|_{X_0}^2 + (\alpha^2+\beta^2)L\left(\|u_x(\cdot,L)\|_{L^2(0,T)}^2 + \|z(\cdot,1)\|_{L^2(0,T)}^2\right) \\
\leq & C(L,\alpha,\beta)\left(\|(u_0,v_0)\|_{X_0}^2 + \|z_0(-h\cdot)\|_{L^2(0,1)}^2\right)
\end{aligned}
$$

Consequently (3.2.11) is verified with $C(L,\alpha,\beta) = \frac{4}{3}L(1+\alpha^2+\beta^2)$. □

Remark 3.2.4. The regularity of the Hirota-Satsuma system differs from that of the KdV-KdV system due to its asymmetric structure. In the KdV-KdV system, the symmetric coupling allows for the use of symmetric Morawetz multipliers to achieve regularizing effects like the Kato smoothing effect. However, the asymmetric interaction in the Hirota-Satsuma system, particularly due to its nonlinear and linear coupling terms, makes symmetric multipliers insufficient for obtaining the same regularity results, requiring a different multipliers techniques for handling the smoothing effect.

3.2.2 Nonlinear problem

Here, we aim to obtain the well-posedness for the Hirota-Satsuma system (3.1.1), we decompose the procedure in two steps. We start by turning our attention to consider the linear system (3.2.1) with source terms $f_1, f_2 \in L^1(0,T,X_0)$,

$$\begin{cases} u_t - \frac{1}{2}u_{xxx} = f_1 & x \in (0,L),\ t>0 \\ v_t + v_{xxx} = f_2 & x \in (0,L),\ t>0 \\ u(t,0) = u(t,L) = v(t,0) = v(t,L) = u_x(t,0) = 0, & t>0, \\ v_x(t,L) = \alpha u_x(t,L) + \beta u_x(t-h,L), & t>0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x) \in L^2(0,L) \\ u_x(t-h,L) = z_0(t-h,L) \in L^2(0,1). \end{cases} \tag{3.2.12}$$

By Kato's smoothing, we can ensure that the system is well-posed. More precisely, we have the following result:

Theorem 3.2.5. *Assume that (3.1.2) holds. Let $U_0 = (u_0, v_0, z_0) \in H$ and the source terms $f_1, f_2 \in L^1(0,T,X_0)$. So, there exists a unique solution $U = (u,v,z) \in C([0,T],H)$ to (3.2.12). Moreover, for $T>0$, there exists $C>0$ such that the following estimates hold*

$$\begin{aligned} \|(u,v,z)\|_{C([0,T],H)} &\le C\left(\|(u_0,v_0,z_0)\|_H + \|(f_1,f_2)\|_{L^1(0,T,X_0)}\right), \\ \|(u_x(\cdot,L), z(\cdot,1))\|^2_{L^2(0,T)} &\le C\left(\|(u_0,v_0,z_0)\|^2_H + \|(f_1,f_2)\|^2_{L^1(0,T,X_0)}\right), \\ \|(u,v)\|_{L^2(0,T,[H^1(0,L)]^2)} &\le C\left(\|(u_0,v_0,z_0)\|_H + \|(f_1,f_2)\|_{L^1(0,T,X_0)}\right). \end{aligned}$$

Proof. We can proceed as Theorem 2.5 in (CAPISTRANO-FILHO et al., 2024). □

In the second step, we can address the well-posedness of the nonlinear system (3.1.1) by associating the source terms (f_1, f_2) with the nonlinear terms $(uu_x + vv_x, uv_x)$. Essentially, we need to prove that the map $\Gamma\colon \mathcal{B} \to \mathcal{B}$ has a unique fixed-point in some closed ball $B(0,R) \subset \mathcal{B}$. This map is defined by $\Gamma(\tilde{u},\tilde{v}) = (u,v)$, and (u,v) are the solution of the system (3.1.1). First, in the next Proposition we guarantee that the nonlinear terms can be considered a source term of the linear equation (3.2.12).

Proposition 3.2.6. *Let $(u,v) \in L^2(0,T,[H^1(0,L)]^2)$, so uv_x, $uu_x \in L^1(0,T,X_0)$ and $(u,v) \in \mathcal{B} \mapsto (uu_x + vv_x, uv_x) \in L^1(0,T,X_0)$ is continuous. In addition, the following estimate holds,*

$$\begin{aligned} \int_0^T &\|(u_1u_{1,x} + v_1v_{1,x} - (u_2u_{2,x} + v_2v_{2,x}), u_1v_{1,x} - u_2v_{2,x})\|_{X_0}\, dt \\ &\le K\left(\|(u_1,v_1)\|_{\mathcal{B}} + \|(u_2,v_2)\|_{\mathcal{B}}\right)\|(u_1-u_2, v_1-v_2)\|_{\mathcal{B}} \end{aligned} \tag{3.2.13}$$

for a constant $K>0$.

Proof. By the Sobolev immersion $H^1(0,L) \hookrightarrow L^\infty(0,L)$, follows that

$$\|uv_x\|_{L^2(0,L)} \leq \|u\|_{L^\infty(0,L)}\|v_x\|_{L^2(0,L)} \leq K\|u\|_{H^1(0,L)}\|v\|_{H^1(0,L)}.$$

Consequently, there exists a constant $K > 0$ such that

$$\begin{aligned} &\|(u_1u_{1,x} + v_1v_{1,x} - (u_2u_{2,x} + v_2v_{2,x}), u_1v_{1,x} - u_2v_{2,x})\|_{X_0} \\ &\qquad \leq K\left(\|(u_1,v_1)\|_{[H^1(0,L)]^2} + \|(u_2,v_2)\|_{[H^1(0,L)]^2}\right)\|(u_1-u_2, v_1-v_2)\|_{[H^1(0,L)]^2}. \end{aligned}$$

Then, by integrating on $[0,T]$ and using the Cauchy-Schwarz inequality, (3.2.13) holds. □

Finally, we are in a position to present the existence of solutions to the Hirota-Satsuma System (3.1.1).

Theorem 3.2.7. *Let $L, T > 0$ and consider α and β real constants such that (3.1.2) is satisfied. For each initial data $(u_0, v_0; z_0) \in H$ sufficiently small, $\Gamma\colon \mathcal{B} \to \mathcal{B}$ defined by $\Gamma(\tilde{u},\tilde{v}) = (u,v)$ is a contraction. Moreover, there exists a unique solution $(u,v) \in B(0,R) \subset \mathcal{B}$ of the Hirota-Satsuma system (3.1.1).*

Proof. It follows from Theorem 3.2.5 that the map Γ is well defined. Using Proposition 3.2.6 and the *a priori estimates* (3.2.5) we obtain that

$$\|\Gamma(\tilde{u},\tilde{v})\|_{\mathcal{B}} = \|(u,v)\|_{\mathcal{B}} \leq C\left(\|(u_0,v_0,z_0)\|_H + \|(\tilde{u},\tilde{v})\|^2_{\mathcal{B}}\right),$$

and

$$\|\Gamma(\tilde{u}_1,\tilde{v}_1) - \Gamma(\tilde{u}_2,\tilde{v}_2)\|_{\mathcal{B}} \leq K\left(\|(\tilde{u}_1,\tilde{v}_1)\|_{\mathcal{B}} + \|(\tilde{u}_2,\tilde{v}_2)\|_{\mathcal{B}}\right)\|(\tilde{u}_1-\tilde{u}_2, \tilde{v}_1-\tilde{v}_2)\|_{\mathcal{B}}.$$

Now, we restrict Γ to the closed ball $\{(\tilde{u},\tilde{v}) \in \mathcal{B} : \|(\tilde{u},\tilde{v})\|_{\mathcal{B}} \leq R\}$, with $R > 0$ to be determined later. Then, $\|\Gamma(\tilde{u},\tilde{v})\|_{\mathcal{B}} \leq C\left(\|(u_0,v_0,z_0)\|_H + R^2\right)$ and

$$\|\Gamma(\tilde{u}_1,\tilde{v}_1) - \Gamma(\tilde{u}_2,\tilde{v}_2)\|_{\mathcal{B}} \leq 2RK\|(\tilde{u}_1-\tilde{u}_2, \tilde{v}_1-\tilde{v}_2)\|_{\mathcal{B}}.$$

Next, we pick $R = 2C\|(u_0,v_0,z_0)\|_H$ such that $2KR < 1$, with $C < 2K$. This leads to claim that

$$\|\Gamma(\tilde{u},\tilde{v})\|_{\mathcal{B}} \leq R$$

and

$$\|\Gamma(\tilde{u}_1,\tilde{v}_1) - \Gamma(\tilde{u}_2,\tilde{v}_2)\|_{\mathcal{B}} < C_1\|(\tilde{u}_1-\tilde{u}_2, \tilde{v}_1-\tilde{v}_2)\|_{\mathcal{B}},$$

with $C_1 < 1$. Finally, the result yields as consequence of the Banach fixed point theorem. □

Remark 3.2.8. In contrast to what we have for the KdV-KdV system (see Remark 2 in (CAPISTRANO-FILHO et al., 2024)), note that the solutions of the Hirota Satsuma system (3.1.1) obtained in Theorem 3.2.7 are global thanks to the Proposition 3.2.2 that lies, essentially, in the fact that the nonlinearities preserve the nonincreasing character of the energy $E(t)$ for the nonlinear system (3.1.1).

3.3 BOUNDARY EXPONENTIAL STABILIZATION

By constructing an appropriate perturbation to the Energy, we can systematically analyze the stability properties of the Hirota-Satsuma system. This section will delve into applying Lyapunov's approach, achieving the desired boundary exponential stabilization for the Hirota-Satsuma system.

Proof of Theorem 3.1.1. Let us introduce the Lyapunov functional $V(t)$ defined as

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t)$$

where $\mu_1, \mu_2 \in \mathbb{R}^+$ will be chosen later, $E(t)$ is the total energy given by (3.1.3),

$$V_1(t) = \frac{1}{2}\int_0^L (L-x)u^2(t,x) + xv^2(t,x)\,dx \text{ and } V_2(t) = \frac{\beta h}{2}\int_0^1 (1-\rho)u_x^2(t-h\rho,L)\,d\rho.$$

Notice that $E(t)$ and $V(t)$ are equivalent in the sense,

$$E(t) \leq V(t) \leq (1+\max\{\mu_1 L, \mu_2\})\,E(t).$$

To obtain the exponential decay, we are going to estimate $V'(t) + \lambda V(t)$.

Using equation (3.1.1) and performing integration by parts we obtain

$$V_1'(t) = -\frac{3}{4}\int_0^L u_x^2\,dx - \frac{3}{2}\int_0^L v_x^2\,dx + \frac{L}{2}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}^T \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2\end{pmatrix}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}$$
$$+\int_0^L u^3\,dx + 3\int_0^L (L-2x)uvv_x\,dx.$$

On the other hand, observe that

$$V_2'(t) = -\frac{\beta}{2}\int_0^1 u_x^2(t-h\rho,L)\,d\rho + \frac{1}{2}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}^T \begin{pmatrix} \beta & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} u_x(t,L) \\ u_x(t-h,L)\end{pmatrix}$$

Gathering all these results, follows that

$$
\begin{aligned}
V'(t)+\lambda V(t) \leq & \frac{1}{2}\left\langle \Psi_{\mu_1,\mu_2}(u_x(t,L),u_x(t-h\rho,L),(u_x(t,L),u_x(t-h\rho,L)\right\rangle \\
& -\frac{3}{4}\mu_1\int_0^L\left(u_x^2+v_x^2\right)dx+\frac{\lambda}{2}(1+L\mu_1)\int_0^L\left(u^2+v^2\right)dx \\
& +\int_0^L \mu_1 u^3\,dx+3\int_0^L \mu_1(L-2x)uvv_x\,dx \\
& +\frac{\beta}{2}(\lambda h+\lambda h\mu_2-\mu_2)\int_0^1 u_x^2(t-h\rho,L)\,d\rho.
\end{aligned}
$$

Here,

$$
\Psi_{\mu_1,\mu_2}=\Phi_{\alpha,\beta}+L\mu_1\begin{pmatrix}\alpha^2 & \alpha\beta\\ \alpha\beta & \beta^2\end{pmatrix}+\mu_2\begin{pmatrix}\beta & 0\\ 0 & 0\end{pmatrix}. \tag{3.3.1}
$$

Due to the continuity of the trace and the determinant we can choose μ_1,μ_2 small enough (see Remark 3.3.1) such that Ψ_{μ_1,μ_2} is definite negative and consequently

$$
\left\langle \Psi_{\mu_1,\mu_2}(u_x(t,L),u_x(t-h\rho,L),(u_x(t,L),u_x(t-h\rho,L)\right\rangle \leq 0. \tag{3.3.2}
$$

Then, by (3.3.2) and employing Poincaré's inequality holds that

$$
\begin{aligned}
V'(t)+\lambda V(t) \leq & \left[\frac{\lambda L^2}{2\pi^2}(1+L\mu_1)-\frac{3}{4}\mu_1\right]\int_0^L\left(u_x^2+v_x^2\right)dx+\int_0^L \mu_1 u^3\,dx \\
& +3\int_0^L \mu_1(L-2x)uvv_x\,dx+\frac{\beta}{2}(\lambda h+\lambda h\mu_2-\mu_2)\int_0^1 u_x^2(t-h\rho,L)\,d\rho.
\end{aligned}
$$

Let us deal with the nonlinear terms, by the Sobolev embedding $H_0^1(0,L)\hookrightarrow L^\infty(0,L)$ and the generalized Hölder's inequality we obtain

$$
\begin{aligned}
\int_0^L u^3\,dx \leq & \|u(t,\cdot)\|^2_{L^\infty(0,L)}\int_0^L |u|\,dx \leq L^{\frac{3}{2}}r\|u_x(t,\cdot)\|^2_{L^2(0,L)} \\
3\int_0^L (L-2x)uvv_x\,dx \leq & 3L\|u(t,\cdot)\|_{L^2(0,L)}\|v(t,\cdot)\|_{L^\infty(0,L)}\|v_x(t,\cdot)\|_{L^2(0,L)} \\
\leq & 3L^{\frac{3}{2}}r\|v_x(t,\cdot)\|^2_{L^2(0,L)}
\end{aligned}
$$

Therefore, taking the constants λ and r as in the statement of Theorem 3.1.1, we obtain

$$
\begin{aligned}
V'(t)+\lambda V(t) \leq & \left[\frac{\lambda L^2}{2\pi^2}(1+L\mu_1)-\frac{3}{4}\mu_1+4L^{\frac{3}{2}}\mu_1 r\right]\int_0^L\left(u_x^2+v_x^2\right)dx \\
& +\frac{\beta}{2}(\lambda h+\lambda h\mu_2-\mu_2)\int_0^1 u_x^2(t-h\rho,L)\,d\rho \leq 0.
\end{aligned}
$$

Consequently, by using Gronwall's inequality the result yields. □

Remark 3.3.1. Taking μ_1 and μ_2 in Theorem 3.1.1 satisfying

$$
\mu_1<\min\left\{\frac{1-2\beta-2\alpha^2}{2L\alpha^2},\frac{1-2\alpha^2-3\beta}{L(2\alpha^2+\beta)}\right\}
$$

and

$$\mu_2 < \min\left\{\frac{1-2\beta-2(1+L\mu_1)\alpha^2}{2\beta}, \frac{1-2(1+L\mu_1)\alpha^2-(1+L\mu_1)\beta-2\beta}{2\beta}\right\}$$

the matrix Ψ_{μ_1,μ_2}, given by (3.3.1), is negative definite. In fact, recall

$$\Psi_{\mu_1,\mu_2} = \begin{pmatrix} \alpha^2 - \frac{1}{2} + \beta + L\mu_1\alpha^2 + \mu_2\beta & (1+L\mu_1)\alpha\beta \\ (1+L\mu_1)\alpha\beta & \beta^2 - \beta + L\mu_1\beta^2 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.$$

In order to Ψ_{μ_1,μ_2} be negative definite, $a_{1,1}$ must be negative, that is

$$\alpha^2 - \frac{1}{2} + \beta + L\mu_1\alpha^2 + \mu_2\beta < 0 \Longleftrightarrow \mu_2 < \frac{1-2\beta-2(1+L\mu_1)\alpha^2}{2\beta}$$

since

$$1-2\beta-2\alpha^2 > 2L\mu_1\alpha^2 \Longleftrightarrow 0 < \mu_1 < \frac{1-2\beta-2\alpha^2}{2L\alpha^2}.$$

Additionally,is mandatory that $\det\Psi_{\mu_1,\mu_2} > 0$. Observe that

$$\begin{aligned}\det\Psi_{\mu_1,\mu_2} &= \beta\left[\frac{1}{2} + (1+\mu_2)(1+L\mu_1)\beta^2 - (1+\mu_2)\beta - (1+L\mu_1)\alpha^2 - \frac{1}{2}(1+L\mu_1)\beta\right] \\ &\geq \beta\left[\frac{1}{2} - (1+\mu_2)\beta - (1+L\mu_1)\alpha^2 - \frac{1}{2}(1+L\mu_1)\beta\right].\end{aligned}$$

This implies that $\det\Psi_{\mu_1,\mu_2} > 0$, if and only if

$$\frac{1}{2}-(1+\mu_2)\beta-(1+L\mu_1)\alpha^2-\frac{1}{2}(1+L\mu_1)\beta > 0 \Longleftrightarrow \mu_2 < \frac{1-2(1+L\mu_1)\alpha^2-(1+L\mu_1)\beta-2\beta}{2\beta}$$

with

$$0 < -2(1+L\mu_1)\alpha^2-(1+L\mu_1)\beta-2\beta \Longleftrightarrow \mu_1 < \frac{1-2\alpha^2-3\beta}{L(2\alpha^2+\beta)}$$

Part II

Observability and stabilization for KP type systems

4 CRITICAL LENGTHS FOR THE LINEAR KADOMTSEV-PETVIASHVILI EQUATION

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4.1 INTRODUCTION

The renowned Korteweg de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$

was initially formulated by Boussinesq (BOUSSINESQ, 1877) and Korteweg-de Vries (KORTEWEG; VRIES, 1895) to describe the movement of water waves within a canal. Specifically, it is widely recognized as a mathematical representation of the one-way movement of small amplitude long waves in nonlinear dispersive systems.

If we consider two spatial dimensions, wave phenomena exhibiting weak transversality and weak nonlinearity can be modeled by the Kadomtsev-Petviashvili (KP) equation

$$(u_t + uu_x + au_{xxx})_x + bu_{yy} = 0 \tag{4.1.1}$$

introduced by Kadomtsev and Petviashvili (KADOMTSEV; PETVIASHVILI, 1970) where $u = u(x, y, t)$ and a, b are constants. Note that, by scaling transformations, the coefficients in equation (4.1.1) can be set to $a = 1$, $b^2 = 1$. We consider the rewritten and scaled version of equation (4.1.1) for the sequel

$$u_t + uu_x + u_{xxx} + b\partial_x^{-1}u_{yy} = 0, \quad b = \pm 1.$$

If $b = -1$, we will refer to it as KP-I. On the other hand, if $b = 1$, we will refer to it as KP-II and distinguish between the focusing and defocusing cases, respectively.

4.1.1 Problem setting

Numerous aspects of the KP equation have been thoroughly examined, such as its well-posedness, stability of solitary waves, its integrability, etc (see (IÓRIO; NUNES, 1998; ISAZA; MEJIA, 2001; LEVANDOSKY, 2000; LEVANDOSKY; SEPÚLVEDA; VILLAGRÁN, 2008; PANTHEE, 2005; TAKAOKA, 2000) and therein). However, the controllability problem needs to be studied better.

The controllability problem involves determining whether it is possible to "manipulate" the solution at specific points in space and time to transition the system from an initial to a final state. Moreover, controllability and the asymptotic behavior of solutions are closely related; selecting an appropriate control mechanism can achieve exponential stability of the solutions. In this context, controllability and stabilization problems for one-dimensional dispersive equations and systems, such as KdV, Kawahara, and Boussinesq equations, posed on various domains have been successfully addressed using different techniques (e.g (ROSIER, 1997; GLASS; GUERRERO, 2009; GALLEGO, 2018; CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019) and references therein). Specifically, the internal controllability of the KP-II equation on a periodic domain was studied in (RIVAS; SUN, 2020), and the asymptotic behavior of solutions for the KP-II and K-KP-II equations with internal feedback mechanisms on bounded domains was examined in (GOMES; PANTHEE, 2011; MOURA; NASCIMENTO; SANTOS, 2022), respectively. Additionally, the exponential stabilization of the K-KP-II equation with internal damping and time-delayed feedback was investigated in (CAPISTRANO-FILHO; GONZALEZ MARTINEZ; MUÑOZ, 2023).

An important fact needs to be spotlighted: Due to the regularity of the solutions for the KP-II equation and the dimension of the domain, the Sobolev embeddings employed in the one-dimensional cases cannot be used to address the well-posedness for the nonlinear case using the fixed point argument. Therefore, we deal with the linearized version of the KP-II.

Let us consider the linearized KP-II within a rectangular domain $\Omega := (0, L) \times (0, L)$, $L > 0$

$$u_t + u_x + u_{xxx} + \partial_x^{-1}(u_{yy}) = 0, \quad (x, y) \in \Omega, \ t \in (0, T). \tag{4.1.2}$$

with initial data $u(x, y, 0) = u_0(x, y)$ with $(x, y) \in \Omega$ and boundary conditions

$$\begin{cases} u(0, y, t) = u(L, y, t) = 0, \quad u_x(L, y, t) = h(y, t), & y \in (0, L), \ t \in (0, T) \\ u(x, 0, t) = u(x, L, t) = 0, & x \in (0, L), \ t \in (0, T) \end{cases} \tag{4.1.3}$$

Here, the operator ∂_x^{-1} is defined by[1]

$$\partial_x^{-1}\varphi(x, y, t) = \psi(x, y, t) \quad \text{such that} \quad \psi(L, y, t) = 0 \quad \text{and} \quad \psi_x(x, y, t) = \varphi(x, y, t).$$

The study of controllability typically involves finding appropriate control functions that act on the system, and the choice of these controls can depend on the specific structure of the equation, therefore the next question related to the exact control arises:

[1] It can be shown that the definition of operator ∂_x^{-1} is equivalent to $\partial_x^{-1}u(x, y, t) = -\int_x^L u(s, y, t)\, ds$.

Problem A: *Given an initial state u_0 and a final state u_1 in a certain space, can one find an appropriate boundary control input so that the equation (4.1.2)-(4.1.3) admits a solution u which equals u_0 at time $t = 0$ and u_1 at time $t = T$?*

Additionally, the total energy associated to (4.1.2) is given by

$$E(t) = \frac{1}{2}\int_{\Omega} u^2(x, y, t)\, dx\, dy\, dt. \tag{4.1.4}$$

A feedback process is one in which the system's state determines the way the control has to be exerted at any time. Therefore, the notions of control and stability are extremely related and play an important role in applications. In this framework, we say that the control is given by a feedback law, then the natural issue appears.

Problem B: *Is it possible to choose the control $h(y, t)$ as a feedback damping mechanism such that $E(t) \to 0$ as $t \to +\infty$? If this is the case, can we give the decay rate?*

To our knowledge, the boundary control properties of the KP-II equation posed on a rectangle are completely open. This chapter aims to investigate boundary observability and, consequently, obtain exact boundary controllability and the asymptotic behavior of the solutions with one control.

4.1.2 Notations and main results

Let us introduce the functional space required for our analysis before presenting answers to our questions. Given $\Omega \subset \mathbb{R}^2$, let us define $X^k(\Omega)$ to be the Sobolev space

$$X^k(\Omega) := \left\{ \varphi \in H^k(\Omega) \colon \begin{array}{r} \partial_x^{-1}\varphi(x, y) = \psi(x, y) \in H^k(\Omega), \\ \psi(L, y) = 0,\ \partial_x \psi(x, y) = \varphi(x, y) \end{array} \right\}$$

endowed with the norm $\|\varphi\|^2_{X^k(\Omega)} = \|\varphi\|^2_{H^k(\Omega)} + \|\partial_x^{-1}\varphi\|^2_{H^k(\Omega)}$. Let us define the normed space $H_x^k(\Omega)$ as

$$H_x^k(\Omega) := \left\{ \varphi \colon \partial_x^j \varphi \in L^2(\Omega),\ \text{for } 0 \le j \le k \right\}$$

with the norm $\|\varphi\|^2_{H_x^k(\Omega)} = \sum_{j=0}^k \|\partial_x^j \varphi\|^2_{L^2(\Omega)}$. Similarly, we consider the space

$$X_x^k(\Omega) := \left\{ \varphi \in H_x^k(\Omega) \colon \begin{array}{r} \partial_x^{-1}\varphi(x, y) = \psi(x, y) \in H_x^k(\Omega), \\ \psi(L, y) = 0,\ \partial_x \psi(x, y) = \varphi(x, y) \end{array} \right\} \tag{4.1.5}$$

with norm $\|\varphi\|^2_{X_x^k(\Omega)} = \|\varphi\|^2_{H_x^k(\Omega)} + \|\partial_x^{-1}\varphi\|^2_{H_x^k(\Omega)}$. Finally, $H_{x0}^k(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ in $H_x^k(\Omega)$.

For $T>0$ let us introduce the following sets

$$\mathcal{B}_{X^k} := C\left([0,T];L^2(\Omega)\right) \cap L^2(0,T;X^k_{x0}(\Omega))$$

and

$$\mathcal{B}_{H^k} := C\left([0,T];L^2(\Omega)\right) \cap L^2(0,T;H^k_{x0}(\Omega))$$

endowed with its natural norms

$$\|u\|_{\mathcal{B}_{X^k}} := \max_{t\in[0,T]} \|u(\cdot,\cdot,t)\|_{L^2(\Omega)} + \left(\int_0^T \|u(\cdot,\cdot,t)\|^2_{X^k_{x0}(\Omega)}\,dt\right)^{\frac{1}{2}}$$

and

$$\|u\|_{\mathcal{B}_{H^k}} := \max_{t\in[0,T]} \|u(\cdot,\cdot,t)\|_{L^2(\Omega)} + \left(\int_0^T \|u(\cdot,\cdot,t)\|^2_{H^k_{x0}(\Omega)}\,dt\right)^{\frac{1}{2}},$$

respectively. The next result will be repeatedly referenced in this chapter, and it is known as the anisotropic Gagliardo-Nirenberg inequality.

Theorem 4.1.1 ((BESOV; IL'IN; NIKOL'SKII, 1978, Theorem 15.7)). *Let β and $\alpha^{(j)}$, for $j=1,\dots,N$, denote n- dimensional multi-indices with non-negative-integer-valued components. Suppose that $1<p^{(j)}<\infty$, $1<q<\infty$, $0<\mu_j<1$ with*

$$\sum_{j=1}^{N}\mu_j = 1, \quad \frac{1}{q} \le \sum_{j=1}^{N}\frac{\mu_j}{p^{(j)}}, \quad \beta - \frac{1}{q} = \sum_{j=1}^{N}\mu_j\left(\alpha^{(j)} - \frac{1}{p^{(j)}}\right).$$

Then, for $f(x)\in C_0^\infty(\mathbb{R}^n)$, $\left\|D^\beta f\right\|_q \le C\prod_{j=1}^N \left\|D^{\alpha^{(j)}} f\right\|^{\mu_j}_{p^{(j)}}$, where for non-negative multi-index $\beta=(\beta_1,\dots,\beta_N)$ we denote D^β by $D^\beta = D^{\beta_1}_{x_1}\dots D^{\beta_n}_{x_n}$ and $D^{\beta_i}_{x_i} = \frac{\partial^{\beta_i}}{\partial x_i^{k_i}}$.

In real-world applications, achieving control or stabilization with a minimal number of inputs is often desirable. In our case, we impose the boundary conditions (4.1.3) with only one input acting along the right side of the rectangle Ω, enabling us to achieve exact boundary control and stabilization. However, due to the presence of the drift term $\partial_x u$, the spectral properties of the operator will be affected. While high frequencies remain asymptotically preserved, low frequencies may undergo significant changes, potentially leading to certain eigenfunctions becoming uncontrollable for specific values of L. A length L is termed *critical* when the equation loses controllability. Precisely, this phenomenon was first noticed in (ROSIER, 1997) for the KdV equation and states that the set of critical lengths for the linear control system, namely

$$\begin{cases} u_t + u_{xxx} + u_x = 0, & (x,t)\in(0,L)\times(0,T), \\ u(0,t)=u(L,t)=0,\ u_x(L,t)=h(t), & t\in(0,T), \end{cases} \tag{4.1.6}$$

is characterized by

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2}; \quad k, l \in \mathbb{Z}^+ \right\}. \tag{4.1.7}$$

Some steps ultimately helped determine the critical lengths for the system (4.1.6). The first reduction led to a unique continuation property for the adjoint system. Next, a spectral problem was formulated with an additional condition, that is,

$$\begin{cases} \lambda u(x) + u'(x) + u'''(x) = 0, \\ u(0) = u(L) = u'(0) = u'(L). \end{cases} \tag{4.1.8}$$

The existence of a nontrivial solution of (4.1.8), if the length L is *critical*, is stated in the next lemma.

Lemma 4.1.2 ((ROSIER, 1997, Lemma 3.5)). *Let $L \in (0, +\infty)$. Then exists $\lambda \in \mathbb{C}$ and a nontrivial solution $u \in H^3(0, L)$ of (4.1.8) if and only if $L \in \mathcal{N}$.*

By following those ideas, consider the set

$$\mathcal{R} := \left\{ \frac{\pi}{4n} \sqrt{(3m_1 + 2m_2 + m_3)(m_1 - 2m_2 - m_3)(m_1 + 2m_2 - m_3)(m_1 + 2m_2 + 3m_3)} : \\ n, m_1, m_2, m_3 \in \mathbb{N}, \text{ with } |m_1 - m_3| > 2m_2 > 0 \right\}. \tag{4.1.9}$$

The first main result gives an exact controllability property for the KP-II equation.

Theorem 4.1.3. *Let $L \in (0, +\infty) \setminus \mathcal{R}$ and $T > 0$. Then the KP-II equation (4.1.2)-(4.1.3) is exactly controllable in time T, that is, for any $u_0, u_T \in L^2(\Omega)$, there exists $h(y,t) \in L^2((0,T) \times (0,L))$ such that the mild solution u of the KP-II equation (4.1.2)-(4.1.3) satisfies $u(\cdot, \cdot, T) = u_T(x, y)$.*

Next, by choosing a suitable feedback damping mechanism as $h(y,t) = -\alpha u_x(0, y, t)$ with the constraint $0 < |\alpha| \leq 1$, we get the (4.1.2) with a feedback damping mechanism

$$\begin{cases} u(0,y,t) = u(L,y,t) = 0, \quad u_x(L,y,t) = -\alpha u_x(0,y,t), & y \in (0,L), \ t \in (0,T), \\ u(x,0,t) = u(x,L,t) = 0, & x \in (0,L), \ t \in (0,T). \end{cases} \tag{4.1.10}$$

Therefore, taking into account the boundary conditions (4.1.10) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = -\frac{1 - \alpha^2}{2} \int_0^L u_x^2(0, y)\, dy - \frac{1}{2} \int_0^L \left(\partial_x^{-1} u_y(0, y) \right)^2 dy \leq 0. \tag{4.1.11}$$

This allows us to obtain exponential stabilization results, the second main result of the work.

Theorem 4.1.4 (Uniform exponential stabilization). *Let $L \in (0,+\infty) \setminus \mathcal{R}$. Then, for any initial data $u_0 \in L^2(\Omega)$ the energy $E(t)$, given by (4.1.4), associated with KP-II system (4.1.2)-(4.1.10) decays exponentially.*

Finally, to give a detailed description of the asymptotic behavior of the solutions and avoid the critical set phenomena, we employ Lyapunov's approach to obtain an explicit decay rate for the stabilization problem. The result is the following one.

Theorem 4.1.5 (Explicit decay rate). *Let $0 < L < \sqrt{3}$ and suppose that*

$$\nu < \frac{\gamma(3-L^2)}{(1+L\gamma)L^2}, \quad \rho = 1 + L\gamma, \tag{4.1.12}$$

where γ is a positive constant such that

$$0 < \gamma < \frac{1-\alpha^2}{L\alpha^2}. \tag{4.1.13}$$

Then, for any initial data $u_0 \in L^2(\Omega)$ the energy given by (4.1.4), associated with KP-II system (4.1.2)-(4.1.10), decays exponentially. More precisely, there exist two positive constants ν and ρ such that

$$E(t) \leq \rho E(0) e^{-\nu t}, \quad \forall t > 0.$$

The proofs of the Theorems 4.1.3 and 4.1.4 are a consequence of the compactness-uniqueness argument due to J.-L. Lions (LIONS, 1988b). The main idea is to prove an observability inequality that reduces our problem to prove a unique continuation property. This unique continuation is achieved thanks to the good properties of the spectral problem associated with both problems.

Note that, differently of Theorem 4.1.4, Theorem 4.1.5 does not ensure the critical length phenomenon, however, a restrictive assumption on the length L is necessary to control the energy of the system. This happens because the Lyapunov method does not involve the spectrum of the operator associated with the stabilization problem. We recommend that the reader consult the Section 4.4 for more details. We point out that the main novelty of our work is to give an explicit characterization of the set (4.1.9) for a dispersive type system after several years. Many authors tried in the last decades to give a new set of critical lengths for the dispersive system, we can cite, for example, (CAPISTRANO-FILHO et al., 2023; CAPISTRANO-FILHO; GALLEGO; PAZOTO, 2019; CERPA; RIVAS; ZHANG, 2013; GLASS; GUERRERO, 2010) and therein. After the pioneering work (ROSIER, 1997), only the following articles were able to characterize the explicit critical length in the context of the bounded domain for KdV-type equation (CAPISTRANO-FILHO; PAZOTO; ROSIER, 2016; CAPISTRANO-FILHO; GALLEGO; PAZOTO,

2017; CAPISTRANO-FILHO; PAZOTO; ROSIER, 2019; CAICEDO; CAPISTRANO-FILHO; ZHANG, 2017) and (DORONIN; LARKIN, 2015) for the 2D Zakharov-Kuznetsov equation posed on bounded rectangles and on a strip. So, few works were able to present a new and complete picture of the critical length set for dispersive type systems in one or two-dimensional cases.

Remark 4.1.6. We finish our introduction with a few remarks.

i. Note that if we consider $m_1 = m_2 = 1$, $m_3 = 4$ and $n = 15$ in (4.1.9), we have that $L = \frac{\sqrt{3}\pi}{4}$ is a explicit critical length. Moreover, taking $m_1 = 4k - 1, m_2 = m_3 = 1, n^2 = 3k^2 - 3$, where $k \in \mathbb{Z}^+$, with $k > 1$, yields that $L = 4k\pi$.

ii. By choosing $m_1 = m_2 = 1$, $m_3 = 7$ and $n = 12$ in (4.1.9), we get $L = 2\pi$. Therefore, a noncontrollable solution on the x-axis for the KP-II equation (4.1.2)-(4.1.3) is given by $u(x, y, t) = \cos(x) - 1$.

iii. From the definition of $\mathcal{N}$ (see (4.1.7)) and denoting

$$\mathcal{N}^\star = \{2k\pi : k \in \mathbb{Z}^+\},$$

we obtain $\mathcal{N}^\star \subset \mathcal{N}$. Thus, under the assumptions $m_1 = m_2 = k$, $m_3 = 7m_1 = 7k$, in (4.1.9), we have

$$\mathcal{R}^\star = \left\{ \frac{24k^2\pi}{n} : k, n \in \mathbb{Z}^+ \right\} \subset \mathcal{R},$$

where $\mathcal{R}$. So in this case, $\mathcal{N}^\star \subseteq \mathcal{R}^\star$, if $n = 12^3 p$ and $k = 12p$, with $p \in \mathbb{Z}^+$.

iv. Observe that Theorems 4.1.3 and 4.1.4 ensures that the observability inequalities 4.3.1 and 4.3.16 holds iff $L \notin \mathcal{R}$, or rigorously speaking, the controllability (and stabilization) holds iff $L \notin \mathcal{R}$.

v. Removing the drift term u_x in the equation (4.1.2), Theorems 4.1.3 and 4.1.4 holds for any $L > 0$, that is, no restriction on the length L appears.

4.1.3 Chapter's outline

This chapter is organized into four sections, including the introduction. In Section 4.2, we address the well-posedness problem associated with the KP-II system. Section 4.3 presents two types of observability inequalities, which allow us to explicitly characterize the critical length described by the set (4.1.9). These observability inequalities are crucial for establishing

Theorems 4.1.3 and 4.1.4, which are proved in Section 4.4. In the same section, we use Lyapunov's method to prove Theorem 4.1.5, helping to relax the critical length condition for the stabilization problem.

4.2 WELL-POSEDNESS FOR THE KP-II

4.2.1 Homogeneous system

Let us start with the well-posedness of the following problem

$$\begin{cases} u_t + u_x + u_{xxx} + \partial_x^{-1}(u_{yy}) = 0, & (x,y) \in \Omega,\ t \in (0,T), \\ u(0,y,t) = u(L,y,t) = u_x(L,y,t) = 0, & y \in (0,L),\ t \in (0,T) \\ u(x,0,t) = u(x,L,t) = 0, & x \in (0,L),\ t \in (0,T), \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{4.2.1}$$

Associated with this system, denote the operator $A\colon D(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined by $Au = -u_x - u_{xxx} - \partial_x^{-1} u_{yy}$,with dense domain given by

$$D(A) := \left\{ u \in H_x^3(\Omega) \cap X^2(\Omega) \,\middle|\, u(0,y) = u(L,y) = u(x,0) = u(x,L) = u_x(L,y) = 0 \right\}. \tag{4.2.2}$$

We present some useful results to establish the linear system's well-posedness.

Lemma 4.2.1. *The operator A is closed and the adjoint $A^*\colon D(A^*) \subset L^2(\Omega) \to L^2(\Omega)$ is given by*[2]

$$A^*\theta = \theta_x + \theta_{xxx} + \left(\partial_x^{-1}\right)^* \theta_{yy} \tag{4.2.3}$$

with dense domain

$$D(A^*) := \left\{ \theta \in H_x^3(\Omega) \cap X^2(\Omega) \middle| \theta(0,y) = \theta(L,y) = \theta(x,0) = \theta(x,L) = \theta_x(0,y) = 0 \right\}. \tag{4.2.4}$$

Proof. Let $u \in D(A)$ and $\theta \in D(A^*)$. We consider two functions, f, g such that $u_y(x,y) = f_x(x,y)$ and $\theta_y(x,y) = g_x(x,y)$, with $f(L,y) = 0$ and $g(0,y) = 0$. Firstly, we estimate the

2 Observe that for the adjoint problem, the operator $\left(\partial_x^{-1}\right)^*$ is defined as $\left(\partial_x^{-1}\right)^* \varphi(x,y) = \psi(x,y)$ such that $\psi(0,y) = 0$ and $\psi_x(x,y) = \varphi(x,y)$. For this case, the definition is equivalent to $\left(\partial_x^{-1}\right)^* \varphi(x,y,t) = \int_0^x \varphi(s,y,t)\, ds$.

product of the θ by the nonlocal term $\partial_x^{-1}u_{yy}$, so we have by integration by parts and the traces properties that

$$
\begin{aligned}
-\int_\Omega \theta(x,y)\partial_x^{-1}u_{yy}(x,y)\,dx\,dy &= -\int_\Omega \theta(x,y)f_y(x,y)\,dx\,dy \\
&= \int_\Omega u(x,y)\left(\partial_x^{-1}\right)^*\theta_{yy}(x,y)\,dx\,dy.
\end{aligned}
$$

Consequently, we can estimate the duality product $\langle Au,\theta\rangle_{L^2(\Omega)}$ as follows

$$
\begin{aligned}
\langle Au,\theta\rangle_{L^2(\Omega)} &= \int_\Omega u(x,y)\left(\theta_x(x,y)\,dx\,dy + \theta_{xxx}(x,y)\,dx\,dy + \partial_x^{-1}\theta_{yy}(x,y)\right)\,dx\,dy \\
&= \langle u, A^*\theta\rangle_{L^2(\Omega)}.
\end{aligned}
$$

Finally, note that $A^{**}=A$, then A is closed. □

Proposition 4.2.2. *The operator A is the infinitesimal generator of a C_0-semigroup in $L^2(\Omega)$.*

Proof. Let $u\in D(A)$ and call $f_x(x,y)=u_y(x,y)$ with $f(L,y)=0$. From (4.2.2) it follows, thanks to the integration of parts, that

$$
\langle Au,u\rangle_L^2(\Omega) = \frac{1}{2}\int_0^L [u_x^2(L,y)-u_x^2(0,y)]\,dy + \frac{1}{2}\int_0^L [f^2(L,y)-f^2(0,y)]\,dy.
$$

Therefore, we have that

$$
\langle Au,u\rangle_{L^2(\Omega)} = -\frac{1}{2}\int_0^L u_x^2(0,y)dy - \frac{1}{2}\int_0^L (\partial_x^{-1}u_y(0,y))^2\,dy \le 0. \tag{4.2.5}
$$

Similarly, let $\theta\in D(A^*)$ and $g_x(x,y)=\theta_y(x,y)$ with $g(0,y)=0$. Then

$$
\langle \theta, A^*\theta\rangle_{L^2(\Omega)} = -\frac{1}{2}\int_0^L [\theta_x^2(L,y)-\theta_x^2(0,y)]\,dy - \frac{1}{2}\int_\Omega (g^2(x,y))_x\,dx\,dy
$$

From (4.2.4), it yields that

$$
\langle \theta, A^*\theta\rangle_{L^2(\Omega)} = -\frac{1}{2}\int_0^L \theta_x^2(L,y)\,dy - \frac{1}{2}\int_0^L \left((\partial_x^{-1})^*\theta_y(L,y)\right)^2\,dy \le 0.
$$

Thus, from (PAZY, 1983, Corollary 4.4, page 15), the proposition holds. □

Now, we can state the following theorem regarding the existence of solutions to the Cauchy problem.

Theorem 4.2.3. *Then, for each initial data $u_0\in L^2(\Omega)$ there exists a unique mild solution $u\in C([0,\infty),L^2(\Omega))$ for the system (4.2.1). Moreover, if the initial data $u_0\in D(A)$ the solution u belongs to $C([0,\infty);D(A))\cap C^1([0,\infty);L^2(\Omega))$.*

Proof. From Proposition 4.2.2, it follows that A generates a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ in $L^2(\Omega)$ (see (PAZY, 1983, Corollary 1.4.4)). $\square$

The next proposition provides valuable estimates for the mild solutions of the equation (4.2.1), including the energy estimate, Kato smoothing effect, and the existence of the traces.

Proposition 4.2.4. *Let $u_0 \in L^2(\Omega)$, then the map $u_0 \in L^2(\Omega) \mapsto u \in \mathcal{B}_{H^1}$ is well-defined, continuous and satisfies*

$$\|u(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}, \quad \forall t \in [0,T]$$

and

$$\|u\|^2_{L^2(0,T;H^1_x(\Omega))} \leq C(T,L)\|u_0\|^2_{L^2(\Omega)}, \tag{4.2.6}$$

for some positive constant $C := C(T,L) = \frac{2(2T+L)}{3} > 0$. Moreover, if the initial data $u_0 \in L^2(\Omega)$, we get the following trace estimate

$$\|u_x(0,\cdot,\cdot)\|^2_{L^2((0,T)\times(0,L))} \leq \|u_0\|^2_{L^2(\Omega)} \tag{4.2.7}$$

and the estimate

$$\begin{aligned}\int_\Omega u_0^2(x,y)\,dx\,dy \leq \frac{1}{T}\int_0^T\int_\Omega u^2\,dx\,dy\,dt + \int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt \\ + \int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt.\end{aligned} \tag{4.2.8}$$

Proof. Observe that (4.2.5) implies that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\Omega u^2\,dx\,dy &= \int_\Omega u(t)\left(-u_x(t) - u_{xxx}(t) - \partial_x^{-1}u_{yy}(t)\right)\,dx\,dy \\ &= \langle Au(t), u(t)\rangle_{L^2(\Omega)} \leq 0.\end{aligned}$$

Integrating in $[0,s]$, for $0 \leq s \leq T$ we get

$$\|u\|_{C([0,T];L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}.$$

To see that $u \in L^2(0,T;H^1_x(\Omega))$ we multiply the equation (4.2.1) by xu. Integrating by parts in $\Omega \times (0,T)$, we obtain that

$$\begin{aligned}&\frac{3}{2}\int_0^T\int_\Omega u_x^2\,dx\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega \left(\partial_x^{-1}u_y\right)^2\,dx\,dy\,dt \\ &\quad = \frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt + \frac{1}{2}\int_\Omega xu_0^2(x,y)\,dx\,dy - \frac{1}{2}\int_\Omega xu^2(x,y,T)\,dx\,dy.\end{aligned}$$

Then,

$$\begin{aligned}\frac{3}{2}\|u_x\|^2_{L^2(0,T;L^2(\Omega))} &\leq \frac{3}{2}\int_0^T\int_\Omega u_x^2\,dx\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega \left(\partial_x^{-1}u_y\right)^2\,dx\,dy\,dt\\ &= \frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt + \frac{1}{2}\int_\Omega xu_0^2(x,y)\,dx\,dy - \frac{1}{2}\int_\Omega xu^2(x,y,T)\,dx\,dy\\ &\leq \left(\frac{T}{2}+L\right)\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore,

$$\|u\|^2_{L^2(0,T;H^1_x(\Omega))} = \int_0^T \|u(\cdot,\cdot,t)\|^2_{L^2(\Omega)}\,dt + \int_0^T \|u_x(\cdot,\cdot,t)\|^2_{L^2(\Omega)}\,dt \leq \frac{2\,(2T+L)}{3}\|u_0\|^2_{L^2(\Omega)},$$

giving (4.2.6). On the other hand, for the trace estimate, we multiply the equation (4.2.1) by u and integrate by parts in $\Omega\times(0,T)$,

$$\begin{aligned}&\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt + \int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt\\ &= \int_\Omega u_0^2(x,y)\,dx\,dy - \int_\Omega u^2(x,y,T)\,dx\,dy.\end{aligned}$$

This implies

$$\begin{aligned}\|u_x(0,\cdot,\cdot)\|^2_{L^2((0,T)\times(0,L))} &\leq \|u_x(0,\cdot,\cdot)\|^2_{L^2(0,T)\times(0,L)} + \left\|\partial_x^{-1}u_y(0,\cdot,\cdot)\right\|^2_{L^2(0,T)\times(0,L)}\\ &\leq \|u_0\|^2_{L^2(\Omega)},\end{aligned}$$

and (4.2.7) holds.

Finally, multiplying the equation (4.2.1) by $(T-t)u$ and integrating by parts in $\Omega\times(0,T)$ yields that

$$\begin{aligned}\int_\Omega u_0^2(x,y)\,dx\,dy \leq \frac{1}{T}\int_0^T\int_\Omega u^2\,dx\,dy\,dt &+ \int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt\\ &+ \int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt,\end{aligned}$$

and the result is proven. □

4.2.2 Adjoint system

We consider the following time-backward homogeneous problem

$$\begin{cases}\theta_t + \theta_x + \theta_{xxx} + \left(\partial_x^{-1}\right)^*\theta_{yy} = 0, & (x,y)\in\Omega,\ t\in(0,T),\\ \theta(0,y,t) = \theta(L,y,t) = \theta_x(0,y,t) = 0, & y\in(0,L),\ t\in(0,T),\\ \theta(x,0,t) = \theta(x,L,t) = 0, & x\in(0,L),\ t\in(0,T),\\ \theta(x,y,T) = \theta_T(x,y), & (x,y)\in\Omega.\end{cases} \tag{4.2.9}$$

Note that, by the variable change $t \mapsto T - t$ the problem (4.2.9) can be formulated as

$$\begin{cases} \theta_t - \theta_x - \theta_{xxx} - \left(\partial_x^{-1}\right)^* \theta_{yy} = 0, & (x,y) \in \Omega,\ t \in (0,T), \\ \theta(0,y,t) = \theta(L,y,t) = \theta_x(0,y,t) = 0, & y \in (0,L),\ t \in (0,T), \\ \theta(x,0,t) = \theta(x,L,t) = 0, & x \in (0,L),\ t \in (0,T), \\ \theta(x,y,0) = \theta_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{4.2.10}$$

That is equivalent to abstract Cauchy problem $\theta_t = A^*\theta$ with initial data $\theta(x,y,0) = \theta_0(x,y)$, where A^* is defined by (4.2.3) with dense domain (4.2.4), by using Lemma 4.2.1 we can establish the well-posedness for the problem (4.2.10) and consequently the well-posedness for (4.2.9) yields. Precisely we get the following result.

Theorem 4.2.5. *Then, for each initial data $\theta_0 \in L^2(\Omega)$ there exists a unique mild solution $\theta \in C\left([0,\infty), L^2(\Omega)\right)$ for the problem (4.2.10). Moreover, if the initial data $\theta_0 \in D(A^*)$ the solutions are classical such that $\theta \in C\left([0,\infty), D(A^*)\right) \cap C^1\left([0,\infty), L^2(\Omega)\right)$.*

4.2.3 KP-II equation with feedback

Let us first pay attention to the existence and regularity issues for the solutions for (4.1.2) with a feedback mechanism given by (4.1.10). Gathering this information, we get the system

$$\begin{cases} u_t + u_x + u_{xxx} + \partial_x^{-1}(u_{yy}) = 0, & (x,y) \in \Omega,\ t \in (0,T), \\ u(0,y,t) = u(L,y,t) = 0, & y \in (0,L),\ t \in (0,T) \\ u(x,0,t) = u(x,L,t) = 0, \quad u_x(L,y,t) = -\alpha u_x(0,y,t), & y \in (0,L),\ t \in (0,T) \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{4.2.11}$$

We introduce the operator $\tilde{A} : D(\tilde{A}) \subset L^2(\Omega) \to L^2(\Omega)$ defined by $\tilde{A}u := -u_x - u_{xxx} - \partial_x^{-1} u_{yy}$ on the dense domain given by

$$D(\tilde{A}) := \left\{ u \in H_x^3(\Omega) \cap X^2(\Omega) \,\middle|\, \begin{array}{l} u(0,y) = u(L,y) = u(x,0) = u(x,L) = 0 \\ u_x(L,y) = -\alpha u_x(0,y),\ 0 < |\alpha| \leq 1 \end{array} \right\}.$$

By analogous arguments as in Lemma 4.2.1 we get that $\tilde{A}$ is closed and the adjoint $\tilde{A}^* : D(\tilde{A}^*) \subset L^2(\Omega) \to L^2(\Omega)$ by $\tilde{A}^* v = v_x + v_{xxx} + \left(\partial_x^{-1}\right)^* v_{yy}$ with dense domain

$$D(\tilde{A}^*) := \left\{ v \in H_x^3(\Omega) \cap X^2(\Omega) \,\middle|\, \begin{array}{l} v(0,y) = v(L,y) = v(x,0) = v(x,L) = 0 \\ v_x(0,y) = \alpha v_x(L,y),\ 0 < |\alpha| \leq 1 \end{array} \right\}.$$

Proposition 4.2.6. *The operator $\tilde{A}$ is the infinitesimal generator of a C_0-semigroup in $L^2(\Omega)$.*

Proof. Similarly as the proof of the Lemma 4.2.2, we consider $u \in D(\tilde{A})$ then

$$\left\langle \tilde{A}u, u \right\rangle_{L^2(\Omega)} = -\frac{(1-\alpha^2)}{2} \int_0^L u_x^2(0,y)\, dy - \frac{1}{2} \int_0^L (\partial_x^{-1} u_y(0,y))^2\, dy \leq 0.$$

Similarly, let $v \in D(\tilde{A}^*)$ therefore

$$\left\langle v, \tilde{A}^* v \right\rangle_{L^2(\Omega)} = -\frac{(1-\alpha^2)}{2} \int_0^L v_x^2(L,y)\, dy - \frac{1}{2} \int_0^L \left((\partial_x^{-1})^* v_y(L,y) \right)^2\, dy \leq 0.$$

Thus, from (PAZY, 1983, Corollary 4.4, page 15), the proposition holds. □

Therefore, we are in a position to present the subsequent theorem concerning the existence of solutions for the Cauchy abstract problem:

Theorem 4.2.7. *For the initial data $u_0 \in L^2(\Omega)$, there exists a unique solution $u \in C\left([0,\infty), L^2(\Omega)\right)$ for the KP-II equation (4.2.11). Moreover, if the initial data $u_0 \in D(\tilde{A})$ the solutions are classical such that $u \in C\left([0,\infty), D(\tilde{A})\right) \cap C^1\left([0,\infty), \mathcal{H}\right).$*

Proof. From Proposition 4.2.6, it follows that $\tilde{A}$ generates a strongly continuous semigroup of contractions $(\tilde{S}(t))_{t\geq 0}$ in $L^2(\Omega)$ (see (PAZY, 1983, Corollary 1.4.4)). □

Proposition 4.2.8. *For any mild solution of (4.2.11), the energy $E(t)$ is non-increasing and there exists a constant $C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq -C \left[\int_0^L u_x^2(0,y,t)\, dy + \int_0^L (\partial_x^{-1} u_y(0,y,t))^2\, dy \right] \tag{4.2.12}$$

where $C = C(\alpha, \gamma)$ is given by $C = \min\left\{ \frac{1-\alpha^2}{2}, \frac{1}{2} \right\}$.

Proof. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \int_\Omega u(t) \left(-u_x(t) - u_{xxx}(t) - \partial_x^{-1} u_{yy}(t) \right) dxdy = \langle Au(t), u(t) \rangle_{L^2(\Omega)}.$$

From the proof of Proposition 4.2.6, we conclude (4.2.12). □

The following estimates for the mild solution $u(\cdot) = \tilde{S}(\cdot)(u_0)$ of the KP-II equation provide useful information for the regularity and the asymptotic behavior. The first ones are standard energy estimates and the last one reveals that a slightly different Kato smoothing effect holds,

Proposition 4.2.9. *Let $u_0 \in L^2(\Omega)$ and the mild solution given by $u(\cdot) = \tilde{S}(\cdot)(u_0)$. Then,*

$$\begin{aligned}\frac{1}{2}\int_\Omega u_0^2(x,y)\,dx\,dy - \frac{1}{2}\int_\Omega u^2(x,y,T)\,dx\,dy =& \frac{1-\alpha^2}{2}\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt \\ &+ \frac{1}{2}\int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt,\end{aligned} \tag{4.2.13}$$

and

$$\begin{aligned}\frac{T}{2}\int_\Omega u_0^2(x,y)\,dx\,dy =& \frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt + \frac{1-\alpha^2}{2}\int_0^T\int_0^L (T-t)u_x^2(0,y,t)\,dy\,dt \\ &+ \frac{1}{2}\int_0^T\int_0^L (T-t)\left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt,\end{aligned}$$

for any $T > 0$. Moreover, we have

$$\|u\|^2_{L^2(0,T;H^1_x(\Omega))} \leq C(T,L,\alpha)\|u_0\|^2_{L^2(\Omega)},$$

where $C := C(T,L,\alpha) = \left(\frac{4T}{3} + \frac{2L}{3(1-\alpha^2)}\right) > 0$.

Proof. Following the same arguments that in Proposition (4.2.4) and taking into account the boundary conditions of the KP-II equation (4.2.11) we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_0^2(x,y)\,dx\,dy - \frac{1}{2}\int_\Omega u^2(x,y,T)\,dx\,dy \\ &\qquad = \frac{1-\alpha^2}{2}\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt + \frac{1}{2}\int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt,\end{aligned}$$

$$\begin{aligned}&\frac{T}{2}\int_\Omega u_0^2(x,y)\,dx\,dy = \frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt \\ &\quad + \frac{1-\alpha^2}{2}\int_0^T\int_0^L (T-t)u_x^2(0,y,t)\,dy\,dt + \frac{1}{2}\int_0^T\int_0^L (T-t)\left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt,\end{aligned}$$

and

$$\begin{aligned}&\frac{3}{2}\int_0^T\int_\Omega u_x^2\,dx\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega \left(\partial_x^{-1}u_y\right)^2\,dx\,dy\,dt = \frac{1}{2}\int_\Omega x u_0^2(x,y)\,dx\,dy \\ &- \frac{1}{2}\int_\Omega u^2(x,y,T)\,dx\,dy + \frac{L\alpha^2}{2}\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt + \frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt.\end{aligned} \tag{4.2.14}$$

Note that, from (4.2.13),

$$\begin{aligned}(1-\alpha^2)\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt \leq& (1-\alpha^2)\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt \\ &+ \int_0^T\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\,dt \\ \leq& 4E(0).\end{aligned}$$

Therefore (4.2.14) implies

$$\begin{aligned}\frac{3}{2}\int_0^T\int_\Omega u_x^2\,dx\,dy\,dt &\le \frac{1}{2}\int_\Omega xu_0^2(x,y)\,dx\,dy-\frac{1}{2}\int_\Omega u^2(x,y,T)\,dx\,dy\\ &\quad+\frac{L\alpha^2}{2}\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt+\frac{1}{2}\int_0^T\int_\Omega u^2\,dx\,dy\,dt\\ &\le \frac{2L+T(1-\alpha^2)}{2(1-\alpha^2)}\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus,

$$\|u\|^2_{L^2(0,T;H^1_x(\Omega))}=\int_0^T\|u(\cdot,\cdot,t)\|^2_{H^1_x(\Omega)}\,dt\le\left(T+\frac{2L+T(1-\alpha^2)}{3(1-\alpha^2)}\right)\|u_0\|^2_{L^2(\Omega)}.$$

Summarizing,

$$\|u\|^2_{L^2(0,T;H^1_x(\Omega))}\le\left(\frac{4T}{3}+\frac{2L}{3(1-\alpha^2)}\right)\|u_0\|^2_{L^2(\Omega)},$$

showing the result. □

4.2.4 Nonhomegenous system

Let us consider problem (4.1.2) with the nonhomogeneous boundary condition (4.1.3) and initial data $u_0(x,y)$. Given the lack of appropriate hidden regularity results that would allow us to achieve optimal existence and uniqueness outcomes within the framework of the usual Sobolev spaces, the transposition (or duality) method yields the existence of weak solutions.

As usual, we begin with a formal computation. Fix $\theta_0\in D(A^*)$ arbitrarily, where $D(A^*)$ is given by (4.2.4) and multiply (4.1.2)-(4.1.3) by the solution of (4.2.10). We obtain for every fixed $S\in[0,T]$ the equality

$$0=\int_\Omega u(x,y,s)\theta(x,y,s)\,dx\,dy-\int_\Omega u_0(x,y)\theta_0(x,y)\,dx\,dy-\int_0^S\int_0^L h(y,t)\theta_x(L,y,t)\,dy\,dt.$$

Putting

$$L_S(u_0)=-\langle u_0,\theta_0\rangle_{D(A^*)',D(A^*)}-\int_0^S\int_0^L h(y,t)\theta_x(L,y,t)\,dy\,dt.$$

we may rewrite this identity as

$$L_S(u_0)=\langle u(S),\theta(S)\rangle_{D(A^*)',D(A^*)},\quad\forall\theta_0\in D(A^*).\tag{4.2.15}$$

This leads to the following definition.

Definition 4.2.10. Given $T>0$, $u_0\in D(A^*)'$ and $h(y,t)\in L^2((0,T)\times(0,L))$, we say that u is solution (by transposition) of (4.1.2)-(4.1.3) if $u\in C([0,T],D(A^*)')$ and if (4.2.15) is satisfied for all $S\in\mathbb{R}$.

Now, we can establish the well-posedness for the nonhomogeneous problem, which is classic and we will omit the proof.

Proposition 4.2.11. *Let $T > 0$, $u_0 \in D(A^*)'$ and $h(y,t) \in L^2((0,T) \times (0,L))$. Then there exists a unique solution $u \in C([0,T], D(A^*)')$ of (4.1.2)-(4.1.3) such that*

$$\|u\|_{L^\infty(0,T;D(A^*)')} \leq C \left(\|u_0\|_{D(A^*)'} + \|h\|_{L^2((0,T)\times(0,L))} \right).$$

4.3 BOUNDARY OBSERVATIONS

4.3.1 Observation for controllability

It is well known in control theory (KOMORNIK; LORETI, 2005; LIONS, 1988b) that the exact controllability of a system is equivalent to proving an observability inequality. Taking into account, the following observability will ensure the controllability of the KP-II system (4.1.2)-(4.1.3).

Theorem 4.3.1. *Let $L \in (0, +\infty) \setminus \mathcal{R}$ and $T > 0$. Then exists $C(T, L) > 0$ such that $\theta_T(x,y) \in L^2(\Omega)$ satisfies*

$$\|\theta_T\|^2_{L^2(\Omega)} \leq C \int_0^T \int_0^L |\theta_x(L,y,t)|^2 \, dy \, dt, \tag{4.3.1}$$

where θ is the solution of (4.2.9) with initial data $\theta_T(x,y)$.

Proof. To prove this, first, making the change of variables $t \mapsto T - t$, $x \mapsto L - x$, transform (4.2.9) into the homogeneous problem (4.2.1). Then, (4.3.1) is equivalent to

$$\|u_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_0^L |u_x(0,y,t)|^2 \, dy \, dt. \tag{4.3.2}$$

To establish the observability inequality (4.3.2) we proceed in several steps.

First step. By contradiction, assume that (4.3.2) does not hold. Then, there exists a sequence $(u_0^n)_n \subset L^2(\Omega)$ such that

$$1 = \|u_0^n\|^2_{L^2(\Omega)} > n \int_0^T \int_0^L (u_x^n(0,y,t))^2 \, dy \, dt, \tag{4.3.3}$$

for all $n \in \mathbb{N}$. Observe that (4.3.3) is equivalent to the next two assertions: $\|u_0^n\|^2_{L^2(\Omega)} = 1$, and

$$\int_0^T \int_0^L (u_x^n(0,y,t))^2 \, dy \, dt \longrightarrow 0, \quad \text{in } L^2((0,T) \times (0,L)),$$

where u^n is the solution of (4.2.1) with initial data u_0^n for each $n \in \mathbb{N}$. From (4.2.6) and (4.3.3) we get that $(u^n)_n$ is bounded in $L^2(0,T;H_x^1(\Omega))$. Then, the sequence defined by

$$u_t^n = -u_x^n - u_{xxx}^n - \partial_x^{-1} u_{yy}^n$$

is bounded in $L^2(0,T;H^{-2}(\Omega))$. Since $u^n(\cdot,\cdot,t) \in H_x^1(\Omega)$, thus $u_x^n(\cdot,\cdot,t) \in L^2(\Omega) \subset H^{-2}(\Omega)$. Moreover, thanks to the fact that $u_x^n(\cdot,\cdot,t) \in L^2(\Omega)$ we get that $u_{xxx}^n(\cdot,\cdot,t) \in H^{-2}(\Omega)$.

Claim 1. Note that

$$\partial_x^{-1} u_{yy}^n \in L^2(0,T;H^{-2}(\Omega)).$$

Indeed, we can recognize $L^2(\Omega)$ as a pivot space once we have $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-2}(\Omega)$. Observe that, as $u^n(\cdot,\cdot,t) \in H_x^1(\Omega)$ implies that $u_x^n(\cdot,\cdot,t) \in L^2(\Omega)$. Now, defining a function f^n such that $u_y^n = f_x^n$ with $f^n(L,y,t) = 0$, by using the anisotropic Gagliardo-Niremberg inequality (Theorem 4.1.1), we obtain

$$\left\|\partial_x^{-1} u_{yy}^n(\cdot,\cdot,t)\right\|_{L^2(\Omega)} \leq C \left\|f_x^n(\cdot,\cdot,t)\right\|_{L^2(\Omega)} = C \left\|u_y^n(\cdot,\cdot,t)\right\|_{L^2(\Omega)} \leq C^2 \left\|u_x^n(\cdot,\cdot,t)\right\|_{L^2(\Omega)} < \infty,$$

that is, $\partial_x^{-1} u_{yy}^n(\cdot,\cdot,t) \in L^2(\Omega)$. After that by employing a duality product and the Cauchy-Schwarz inequality follows that

$$\left|\left\langle \partial_x^{-1} u_{yy}^n, \xi \right\rangle_{H^{-2}(\Omega),H_0^2(\Omega)}\right| \leq C\|\xi\|_{L^2(\Omega)} \|u_y^n(\cdot,\cdot,t)\|_{L^2(\Omega)} \leq C^2 \|\xi\|_{L^2(\Omega)} \|f_x^n(\cdot,\cdot,t)\|_{L^2(\Omega)}.$$

Note that, $(u^n)_n$ bounded in $L^2(0,T;H_x^1(\Omega))$ implies, in particular, that $(u_x^n)_n$ is bounded in $L^2(0,T;L^2(\Omega))$, so

$$\left\|\partial_x^{-1} u_{yy}^n\right\|_{L^2(0,T;H^{-2}(\Omega))}^2 \leq C^2 \int_0^T \|u_x^n(\cdot,\cdot,t)\|_{L^2(\Omega)}^2 \, dt \leq C^2 \|u_x\|_{L^2(0,T;H_x^1(\Omega))}^2,$$

showing claim 1.

It is important to emphasize that $H_x^1(\Omega) \hookrightarrow L^2(\Omega)$, and since

$$H_x^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-2}(\Omega),$$

we can apply the Aubin-Lions lemma to obtain that $(u^n)_n$ is relatively compact in $L^2(0,T;L^2(\Omega))$, that is, exists a subsequence, still denoted $(u^n)_n$, such that

$$u^n \longrightarrow u \text{ in } L^2(0,T;L^2(\Omega))$$

and

$$\|u\|_{L^2(0,T;L^2(\Omega))} = 1.$$

Claim 2. We have that

$$\partial_x^{-1}u_{yy}^n \longrightarrow \partial_x^{-1}u_{yy} \quad \text{in} \quad L^2(0,T;H^{-2}(\Omega)).$$

In fact, from the definition of (4.1.5) we have that $\partial_x^{-1}u^n = \varphi^n$ where $\varphi_x^n = u^n$, $u^n(\cdot,\cdot,t) \in H_x^1(\Omega)$ and $\varphi^n(\cdot,\cdot,t) \in H_x^1(\Omega)$. Since $\partial_x^{-1}u_{yy}^n = \varphi_{yy}^n$ follows that

$$\begin{aligned}\left\|\partial_x^{-1}u_{yy}^n(\cdot,\cdot,t) - \partial_x^{-1}u_{yy}(\cdot,\cdot,t)\right\|_{H^{-2}(\Omega)} &\leq C\|\varphi^n(\cdot,\cdot,t) - \varphi(\cdot,\cdot,t)\|_{L^2(\Omega)} \\ &\leq CL^2\|\varphi_x^n(\cdot,\cdot,t) - \varphi_x(\cdot,\cdot,t)\|_{L^2(\Omega)} \\ &= CL^2\|u^n(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^2(\Omega)} \longrightarrow 0,\end{aligned}$$

showing claim 2.

From (4.2.8) and (4.3.3) we get that $(u_0^n)_n$ is a Cauchy sequence in $L^2(\Omega)$ and then, at least for a subsequence converges to some $u_0 \in L^2(\Omega)$ such that

$$\|u_0\|_{L^2(\Omega)} = 1. \tag{4.3.4}$$

Recall that by the semigroup solution representation $u = S(\cdot)u_0$, and from (4.3.3) follows that

$$0 = \liminf_{n\to+\infty}\left[\int_0^T\int_0^L (u_x^n(0,y,t))^2\,dy\,dt\right] \geq \int_0^T\int_0^L (u_x(0,y,t))^2\,dy\,dt$$

Then, $u_x(0,\cdot,\cdot) = 0$ a.e. $(0,L)\times(0,T)$. Hence, taking into account the previous claims, u is a solution for

$$\begin{cases} u_t + u_x + u_{xxx} + \partial_x^{-1}u_{yy} = 0, \\ u(0,y) = u(L,y) = u(x,0) = u(x,L) = u_x(L,y) = u_x(0,y) = 0, \\ u(x,y,0) = u_0(x,y), \end{cases}$$

satisfying (4.3.4).

Second step. We reduce the proof of the observability inequality into a spectral problem as done in (ROSIER, 1997) for the KdV equation. The result is the following one.

Lemma 4.3.2. *For any $L \in (0,+\infty)\setminus\mathcal{R}$ and $T>0$, let N_T denote the space of all the initial states, $u_0 \in L^2(\Omega)$ for which the solution $u(t) = S(t)u_0$ of (4.2.1) satisfies $u_x(0,y) = 0$. Then $N_T = \{0\}$.*

Proof. Using the arguments as those given in (ROSIER, 1997, Lemma 3.4), follows that if $N_T \neq \emptyset$, the map $u_0 \in N_T \mapsto \tilde{A}(N_T) \subset \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of

N_T) has (at least) one eigenvalue; hence, there exist $\lambda \in \mathbb{C}$ and $u_0 \in H^3_x(\Omega) \cap H^2_y(\Omega)$ such that

$$\begin{cases} \lambda u_0(x,y) + u_{x,0}(x,y) + u_{xxx,0}(x,y) + \partial_x^{-1} u_{yy,0}(x,y) = 0, \\ u_0(0,y) = u_0(L,y) = u_0(x,0) = u_0(x,L) = u_{x,0}(L,y) = u_{x,0}(0,y) = 0, \end{cases} \tag{4.3.5}$$

with $(x,y) \in \Omega$. To conclude the proof of the lemma, we will prove this does not hold. □

Third step. To obtain the contradiction, it remains to prove that a duple (λ, u_0) as above does not exist. Precisely, this step is to show the no existence of a nontrivial solution for the spectral problem (4.3.5).

Lemma 4.3.3. *Let $L \in (0, +\infty)$.Consider the following assertion:*

$$(\mathcal{F}) \quad \exists \lambda \in \mathbb{C}, \exists u_0 \in H^3_x(\Omega) \cap H^2_y(\Omega) \backslash \{0\} \textit{ fulfilling } (4.3.5).$$

Then, $(\mathcal{F})$ holds if and only if $L \in \mathcal{R}$.

Proof. To simplify the notation, let us denote $u(x,y) = u_0(x,y)$. Then (4.3.5) can be rewritten as

$$\begin{cases} \lambda u + u_x + u_{xxx} + \partial_x^{-1} u_{yy} = 0, \\ u(0,y) = u(L,y) = u_x(0,y) = u_x(L,y) = u(x,0) = u(x,L) = 0. \end{cases}$$

Separating variables[3] as $u(x,y) = p(x) \cdot q(y)$ we infer

$$\lambda p(x)q(y) + p'(x)q(y) + p'''(x)q(y) + \partial_x^{-1}\left(p(x)q''(y)\right) = 0,$$

that is,

$$q(y)\left(\lambda p(x) + p'(x) + p'''(x)\right) + q''(y)\partial_x^{-1} p(x) = 0$$

equivalently, if there exists a constant $\xi \in \mathbb{R}$ such that

$$\frac{q''(y)}{q(y)} = \frac{-(\lambda p(x) + p'(x) + p'''(x))}{\partial_x^{-1} p(x)} = \xi.$$

Therefore, we get one ODE for each variable

$$\begin{cases} \lambda p(x) + p'(x) + p'''(x) + \xi \partial_x^{-1} p(x) = 0 \\ p(0) = p(L) = p'(0) = p'(L) = 0 \end{cases} \tag{4.3.6}$$

[3] The existence of a solution using the separation of variables is ensured by (KOORNWINDER, 1980, Lemma 3.5).

and

$$\begin{cases} q''(y) - \xi q(y) = 0 \\ q(0) = q(L) = 0. \end{cases} \tag{4.3.7}$$

Notice that there are three different possibilities for the general solution of (4.3.7):

1. If $\xi = 0$,

$$q(y) = C_1 + C_2 y,$$

and the boundary conditions implies that $C_1 = C_2 = 0$.

2. If ξ is a positive real number, say $\xi = a^2 > 0$, then

$$q(y) = C_1 e^{-ay} + C_2 e^{ay},$$

and again, the boundary conditions imply that $C_1 = C_2 = 0$.

3. Finally, if $\xi = -a^2 < 0$, then

$$q(y) = C_1 \sin(ay) + C_2 \cos(ay).$$

From the boundary conditions we get

$$\begin{cases} q(0) = C_2 = 0, \\ q(L) = C_1 \sin(aL) + C_2 \cos(aL) = 0, \end{cases}$$

from which C_1 is arbitrary, $C_2 = 0$ and

$$a_n = \frac{n\pi}{L}, \quad n \in \mathbb{N}.$$

Thus, only in the last case, we find a non-trivial solution $q_n(y) = C_1 \sin\left(\frac{n\pi}{L} y\right)$. Consequently (4.3.6) transforms in

$$\begin{cases} \lambda p(x) + p'(x) + p'''(x) - \left(\frac{n\pi}{L}\right)^2 \partial_x^{-1} p(x) = 0 \\ p(0) = p(L) = p'(0) = p'(L) = 0. \end{cases} \tag{4.3.8}$$

Now, assume that there exists $p \in H^3(0, L) \setminus \{0\}$ solution of (4.3.8) and denote by

$$\hat{p}(k) = \int_0^L e^{-ixk} p(x) dx.$$

Then, multiplying (4.3.8) by e^{-ixk}, integrating by parts over $(0, L)$ and using the boundary conditions we obtain

$$\left(-\left(\frac{n\pi}{L}\right)^2 \frac{1}{(ik)} + \lambda + (ik) + (ik)^3\right) \hat{p}(k) = \left(\frac{n\pi}{L}\right)^2 \frac{1}{(ik)} \partial_x^{-1} p(0) + p''(0) - e^{iLk} p''(L).$$

Indeed, we calculate the Fourier transform for the inverse term

$$\widehat{\partial_x^{-1}p}(k) = \int_0^L e^{-ixk}\partial_x^{-1}p(x)\,dx = -\frac{1}{ik}\int_0^L \left[e^{-ixk}\right]_x \partial_x^{-1}p(x)\,dx = \frac{1}{ik}\hat{p}(k) + \frac{1}{ik}\partial_x^{-1}p(0)$$

and the Fourier transform to the third term

$$\widehat{p'''}(k) = \int_0^L e^{-ixk}p'''(x)\,dx = (ik)^3\hat{p}(k) + e^{-iLk}p''(L) - p''(0).$$

Due to the boundary conditions, the Fourier transform of p' and p'' have traces equal to zero. Consequently,

$$\left(\lambda + (ik) + (ik)^3 - \left(\frac{n\pi}{L}\right)^2 \frac{1}{(ik)}\right)\hat{p}(k) = \left(\frac{n\pi}{L}\right)^2 \frac{1}{(ik)}\partial_x^{-1}p(0) + p''(0) - e^{-iLk}p''(L),$$

hence, multiplying by (ik) we get

$$\left(\lambda(ik) + (ik)^2 + (ik)^4 - \left(\frac{n\pi}{L}\right)^2\right)\hat{p}(k) = \left(\frac{n\pi}{L}\right)^2 \partial_x^{-1}p(0) + (ik)p''(0) - (ik)e^{-iLk}p''(L),$$

by rearranging the equation above setting $\lambda = i\sigma \in \mathbb{C}$, follows that

$$\hat{p}(k) = \frac{\left(\frac{n\pi}{L}\right)^2 \partial_x^{-1}p(0) + (ik)p''(0) - (ik)e^{-iLk}p''(L)}{k^4 - k^2 - \sigma k - \left(\frac{n\pi}{L}\right)^2}.$$

Using the Paley-Wiener theorem, we can characterize the Fourier transform by showing the existence of σ and $\alpha_0, \alpha_1, \alpha_2$ such that the application

$$f(k) := \frac{\left(\frac{n\pi}{L}\right)^2 \alpha_0 + (ik)\alpha_1 - (ik)e^{-iLk}\alpha_2}{k^4 - k^2 - \sigma k - \left(\frac{n\pi}{L}\right)^2} = \frac{R(k)}{Q(k)} \tag{4.3.9}$$

satisfies

(i) f is an entire function in $\mathbb{C}$.

(ii) $\displaystyle\int_0^\infty |f(k)|^2\left(1+|k|^2\right)^2\,dk < \infty$.

(iii) Exists some positive constants C, N such that for all $k \in \mathbb{C}$,

$$|f(k)| \le C(1+|k|)^N e^{L|\mathrm{Im}\,k|}.$$

Now, we focus on analyzing the roots of R and Q defined above. $R(k)$ must be entire if their roots are also roots of $Q(k)$. Let $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ be the roots of $Q(k)$. We have to consider two cases:

- **C**ase 1: Suppose that $\alpha_0 = 0$.

If it is the case, the related trace value $\partial_x^{-1}p(0)=0$ and (4.3.9) turns in

$$f(k)=\frac{(ik)\alpha_1-(ik)e^{-iLk}\alpha_2}{k^4-k^2-\sigma k-\left(\frac{n\pi}{L}\right)^2}=\frac{R_1(k)}{Q(k)}. \tag{4.3.10}$$

Note that $Q(0)=-\left(\frac{n\pi}{L}\right)^2\neq 0$ and consequently $k=0$ cannot be root of $R_1(k)$ and $Q(k)$ simultaneously. Since the roots of $R_1(k)$ are simple, unless $\alpha_1=\alpha_2=0$, (i) holds, then (ii) and (iii) are satisfied. It follows that our problem is equivalent to the existence of complex numbers $\sigma,\kappa_0\in\mathbb{C}$ and $m_1,m_2,m_3\in\mathbb{N}^*$such that, if we set

$$\kappa_1:=\kappa_0+m_1\frac{2\pi}{L},\quad \kappa_2:=\kappa_1+m_2\frac{2\pi}{L}\quad\text{and}\quad \kappa_3:=\kappa_2+m_3\frac{2\pi}{L}$$

we have

$$Q(k)=(k-\kappa_0)(k-\kappa_1)(k-\kappa_2)(k-\kappa_3).$$

By using the Girard–Newton formula, we get

$$\begin{cases} 0 & =\sum_{j=0}^3\kappa_j,\\ -1 & =\kappa_0\sum_{j=1}^3\kappa_j+\kappa_1\sum_{j=2}^3\kappa_j+\kappa_2\kappa_3,\\ \sigma & =\kappa_0\kappa_1\sum_{j=2}^3\kappa_j+\kappa_2\kappa_3\sum_{j=0}^1\kappa_j,\\ -\left(\frac{n\pi}{L}\right)^2 & =\prod_{j=0}^3\kappa_j. \end{cases} \tag{4.3.11}$$

Observe that, κ_2,κ_3 can be rewritten in terms of κ_0 as

$$\kappa_2=\kappa_0+(m_1+m_2)\frac{2\pi}{L}\quad\text{and}\quad \kappa_3=\kappa_0+(m_1+m_2+m_3)\frac{2\pi}{L}.$$

From the first relation in (4.3.11), it follows that

$$\kappa_0=\frac{-\pi(3m_1+2m_2+m_3)}{2L}.$$

Using the third relation in (4.3.11), we get

$$\sigma=2\left(\frac{\pi}{L}\right)^3(m_1+2m_2+m_3)(m_3^2-m_1^2)$$

Finally, thanks to the fourth relation of (4.3.11), we obtain

$$\begin{aligned} L&=\frac{\pi}{4n}\sqrt{(3m_1+2m_2+m_3)\,(m_1-2m_2-m_3)\,(m_1+2m_2-m_3)\,(m_1+2m_2+3m_3)}\\ &=\frac{\pi}{4n}\sqrt{(3m_1+2m_2+m_3)\,((m_1-m_3)^2-4m_2^2)\,(m_1+2m_2+3m_3)} \end{aligned}$$

provided that

$$|m_1-m_3|>2m_2>0.$$

- **C**ase 2: Suppose that $\alpha_0 \neq 0$.

For this case notice that

$$R(k) = \frac{1}{2}\left(R(k) + \overline{R(\bar{k})}\right) + \frac{1}{2}\left(R(k) - \overline{R(\bar{k})}\right),$$

where

$$\overline{R(\bar{k})} = \left(\frac{n\pi}{L}\right)^2 \alpha_0 - (ik)\alpha_1 + (ik)e^{iLk}\alpha_2.$$

Therefore,

$$R(k) + \overline{R(\bar{k})} = 2\left(\frac{n\pi}{L}\right)^2 \alpha_0 + (ik)\alpha_2\left(e^{iLk} - e^{-iLk}\right) = 2\left(\frac{n\pi}{L}\right)^2 \alpha_0 - 2k\alpha_2 \sin(Lk) \quad (4.3.12)$$

and

$$R(k) - \overline{R(\bar{k})} = 2(ik)\alpha_1 - (ik)\alpha_2\left(e^{iLk} + e^{-iLk}\right) = 2(ik)\alpha_1 - 2(ik)\alpha_2\cos(Lk). \quad (4.3.13)$$

Here, we used that

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Rearranging (4.3.12) and (4.3.13) and gathering the expressions, we get

$$\begin{cases} \left(\frac{n\pi}{L}\right)^2 \alpha_0 - k\alpha_2\sin(Lk) = 0, \\ \alpha_1 k - k\alpha_2\cos(Lk) = 0, \end{cases} \iff \begin{cases} k\alpha_2\sin(Lk) = \left(\frac{n\pi}{L}\right)^2 \alpha_0, \\ k\alpha_2\cos(Lk) = \alpha_1 k, \end{cases}$$

or, equivalently

$$\begin{cases} k^2\alpha_2^2\sin^2(Lk) = \left(\frac{n\pi}{L}\right)^4 \alpha_0^2, \\ k^2\alpha_2^2\cos^2(Lk) = \alpha_1^2 k^2. \end{cases} \quad (4.3.14)$$

Adding the results in (4.3.14), follows that

$$\alpha_2^2 k^2\left(\sin^2(Lk) + \cos^2(Lk)\right) = \alpha_2^2 k^2 = \left(\frac{n\pi}{L}\right)^4 \alpha_0^2 + \alpha_1^2 k^2$$

that is,

$$k^2(\alpha_2^2 - \alpha_1^2) = \alpha_0^2\left(\frac{n\pi}{L}\right)^4 \neq 0. \quad (4.3.15)$$

From (4.3.15), it follows that

$$\kappa_j^2 = \frac{\alpha_0^2}{(\alpha_2^2 - \alpha_1^2)}\left(\frac{n\pi}{L}\right)^4, \quad \forall j = 0, 1, 2, 3.$$

Thus, if $\alpha_2^2 - \alpha_1^2 > 0$, we have that

$$\kappa_j = \pm\sqrt{\frac{\alpha_0^2}{(\alpha_2^2 - \alpha_1^2)}\left(\frac{n\pi}{L}\right)^2},$$

and if $\alpha_2^2 - \alpha_1^2 < 0$, it follows that

$$\kappa_j = \pm i\sqrt{\frac{\alpha_0^2}{(\alpha_2^2 - \alpha_1^2)}\left(\frac{n\pi}{L}\right)^2}.$$

In any case, without loss of generality, we deduce that

$$Q(k) = (k - \kappa_0)^2(k - \kappa_2)^2.$$

Note that $Q(k)$ (see (4.3.10)) cannot have two roots of order two. Indeed, note that

$$\begin{aligned} Q(k) &= k^4 - k^2 - \sigma k - \left(\frac{n\pi}{L}\right)^2 \\ &= (k - \kappa_0)^2(k - \kappa_2)^2 \\ &= k^4 - 2(\kappa_0 + \kappa_2)k^3 + (\kappa_2^2 + 4\kappa_0\kappa_2 + \kappa_0^2)k^2 - 2(\kappa_0\kappa_2^2 + \kappa_0^2\kappa_2)k + \kappa_0^2\kappa_2^2. \end{aligned}$$

Therefore, κ_0 and κ_2 must satisfy

$$\begin{cases} \kappa_0 + \kappa_2 = 0; \\ \kappa_2^2 + 4\kappa_0\kappa_2 + \kappa_2^2 = -1; \\ \kappa_0^2\kappa_2^2 = -\left(\frac{n\pi}{L}\right)^2. \end{cases}$$

The first relation implies that $\kappa_0 = -\kappa_2$, then from the second relation we get $\kappa_0^2 = \frac{1}{4}$. With this in hand, from the third relation we obtain that $\frac{1}{16} = -\left(\frac{n\pi}{L}\right)^2$ giving a contradiction. Hence, $(\mathcal{F})$ holds if and only if $L \in \mathcal{R}$. This completes the proof of the Lemma 4.3.3 and, consequently, the proof of Lemma 4.3.2. □

With these lemmas in hand the proof of Theorem 4.3.1 is achieved. □

4.3.2 Observation for stabilization

Due to the structure of the energy for the KP-II with one boundary feedback, we have a slightly different observability inequality that will ensure the stabilization of the KP-II system (4.1.2)-(4.1.10).

Theorem 4.3.4. *Let $L \in (0,+\infty) \setminus \mathcal{R}$ and $T > 0$. Then exists $C(T,L) > 0$ such that $u_0(x,y) \in L^2(\Omega)$ satisfies*

$$\|u_0\|_{L^2(\Omega)} \le C \left(\int_0^T \int_0^L u_x^2(0,y,t)\,dy\,dt + \int_0^T \int_0^L (\partial_x^{-1} u_y(0,y,t))^2\,dy\,dt \right), \tag{4.3.16}$$

where u is the solution of (4.1.2)-(4.1.10) with initial data $u_0(x,y)$.

Proof. Essentially the proof of this result is similar as done in the previous subsection. Perhaps it is important to highlight that in the proof of (4.3.16) we get an additional condition to get the contradiction but the reasoning is the same. Recalling the spectral problem

$$\begin{cases} \lambda u + u_x + u_{xxx} + \partial_x^{-1} u_{yy} = 0 \\ u(0,y) = u(L,y) = u_x(0,y) = u_x(L,y) = u(x,0) = u(x,L) = \partial_x^{-1} u(0,y) = 0. \end{cases}$$

Separating variables as $u(x,y) = p(x) \cdot q(y)$ we infer

$$\lambda p(x)q(y) + p'(x)q(y) + p'''(x)q(y) + \partial_x^{-1} \left(p(x)q''(y)\right) = 0,$$

that is,

$$q(y)\left(\lambda p(x) + p'(x) + p'''(x)\right) + q''(y)\partial_x^{-1} p(x) = 0$$

equivalently, exists a constant $\xi \in \mathbb{R}$

$$\frac{q''(y)}{q(y)} = \frac{-(\lambda p(x) + p'(x) + p'''(x))}{\partial_x^{-1} p(x)} = \xi.$$

Therefore, we get one ODE for each variable

$$\begin{cases} \lambda p(x) + p'(x) + p'''(x) + \xi \partial_x^{-1} p(x) = 0 \\ p(0) = p(L) = p'(0) = p'(L) = \partial_x^{-1} p(0) = 0. \end{cases} \tag{4.3.17}$$

and

$$\begin{cases} q''(y) - \xi q(y) = 0, \\ q(0) = q(L) = 0. \end{cases} \tag{4.3.18}$$

By (4.3.18), q has non-trivial solutions $q_n(y) = C_1 \sin\left(\frac{n\pi}{L} y\right)$, where $\xi = -\left(\frac{n\pi}{L}\right)^2$ and $C_1 \in \mathbb{R}$. Consequently (4.3.17) transforms in

$$\begin{cases} \lambda p(x) + p'(x) + p'''(x) - \left(\frac{n\pi}{L}\right)^2 \partial_x^{-1} p(x) = 0, \\ p(0) = p(L) = p'(0) = p'(L) = \partial_x^{-1} p(0) = 0. \end{cases} \tag{4.3.19}$$

Now, by the same reasoning as in the proof of the Theorem 4.3.1, denoting by $\hat{p}(k) = \int_0^L e^{-ixk}p(x)\,dx$, multiplying (4.3.19) by e^{-ixk}, integrating by parts over $(0, L)$ and using the boundary conditions we obtain

$$-\left(\frac{n\pi}{L}\right)^2 \frac{1}{(ik)}\hat{p}(k) + \lambda\hat{p}(k) + (ik)\hat{p}(k) + (ik)^3\hat{p}(k) = p''(0) - e^{-iLk}p''(L).$$

Consequently, after rearranging and setting $\lambda = i\sigma$,

$$\hat{p}(k) = (ik)\frac{p''(0) - e^{-iLk}p''(L)}{k^4 - k^2 - \sigma k - \left(\frac{n\pi}{L}\right)^2}.$$

Using the Paley-Wiener theorem, we can characterize the Fourier transform by showing the existence of σ and α_1, α_2 such that the application

$$f(k) := (ik)\frac{(\alpha_1 - e^{-iLk}\alpha_2)}{k^4 - k^2 - \sigma k - \left(\frac{n\pi}{L}\right)^2} \tag{4.3.20}$$

satisfies the same properties of the function f in the Lemma 4.3.3. Observe that (4.3.20) is equal to (4.3.10) and therefore (4.3.16) holds. □

4.4 CONTROL RESULTS OF KP-II EQUATION

4.4.1 Boundary controllability

To prove the control result let us follow the H.U.M. developed by J.-L. Lions (LIONS, 1988b). Next, we define the exact controllability property.

Definition 4.4.1. Let $T > 0$. We say that the system (4.1.2)-(4.1.3) is exactly controllable in time T if for any initial and final data $u_0, u_T \in L^2(\Omega)$, there exists $h \in L^2((0,T)\times(0,L))$ such that $u(\cdot,\cdot,T) = u_T(\cdot,\cdot)$.

Now we use the adjoint system to give an equivalent condition for the Definition 4.4.1. The condition is the following one.

Lemma 4.4.2. *Let $u_0, u_T \in L^2(\Omega)$. Then there exists $h \in L^2((0,T)\times(0,L))$, such that the solution u of (4.1.2)-(4.1.3) satisfies $u(x,y,T) = u_T(x,y)$ if and only if*

$$\int_\Omega \theta_T(x,y)u(x,y,T)\,dx\,dy = \int_0^T\int_0^L h(y,t)\theta_x(L,y,t)\,dy\,dt + \int_\Omega \theta(x,y,0)u_0(x,y)\,dx\,dy, \tag{4.4.1}$$

for any $\theta_T(x,y) \in L^2(\Omega)$ and θ being the solution of the adjoint problem (4.2.9).

Proof of Theorem 4.1.3. Notice that, the relation (4.4.1) may be seen as an optimality condition for the critical points of the functional $\Lambda(\theta_T) : L^2(\Omega) \to \mathbb{R}$, defined by

$$\Lambda(\theta_T) = \frac{1}{2}\,\|\theta_x(L,y,t)\|^2_{L^2((0,T)\times(0,L))} - \int_\Omega \theta_T u(x,y,T)\,dx\,dy, \tag{4.4.2}$$

where θ is the solution of (4.2.9). A control may be obtained from the solution of the homogeneous system (4.2.9) with the initial data minimizing the functional Λ. Hence, the controllability is reduced to a minimization problem. To guarantee that Λ defined by (4.4.2) has a unique minimizer we use the next fundamental result in the calculus of variations.

Theorem 4.4.3 (See (BREZIS, 1983))**.** *Let H be a reflexive Banach space, K a closed convex subset of H and $\Lambda : K \to \mathbb{R}$ a function with the following properties:*

1. *Λ is convex and lower semi-continuous;*

2. *If K is unbounded then Λ is coercive. Then Λ attains its minimum in K, i. e. there exists $x_0 \in K$ such that*

$$\Lambda\left(x_0\right) = \min_{x\in K} \varphi(x).$$

Note that Λ defined in (4.4.2) is continuous and convex. The existence of a minimum is ensured if we prove that Λ is also coercive, which is obtained with the *observability inequality* given by (4.3.1). Thus,

$$\Lambda(\theta_T) = \frac{1}{2}\|\theta_x(L,\cdot,\cdot)\|^2_{L^2((0,T)\times(0,L))} - \int_\Omega \theta_T u(x,y,T)\,dx\,dy \geq \frac{C^{-1}}{2}\|\theta_T\|_{L^2(\Omega)}.$$

Therefore, the KP-II equation (4.1.2)-(4.1.3) is exactly controllable. □

4.4.2 Boundary stabilization

We aim to show that the KP-II equation (4.1.2)-(4.1.3) can be stabilized by selecting an appropriate feedback-damping control law. Specifically, by choosing $h(y,t) = -\alpha u_x(0,y,t)$ with $0 < |\alpha| \leq 1$, we can establish our second main result. Recall that the energy of the KP-II equation, defined by (4.1.4), is a nonincreasing function due to (4.1.11).

Proof of the Theorem 4.1.4. Through the utilization of energy dissipation, specifically (4.2.12), in conjunction with the *observability inequality* (4.3.16), we obtain the exponential stabilization. Indeed, notice that, integrating (4.2.12) over $[0,T]$, it follows that

$$E(T) - E(0) \leq -C\left[\int_0^T\int_0^L u_x^2(0,y,t)\,dy\,dt + \int_0^T\int_0^L (\partial_x^{-1}u_y(0,y,t))^2\,dy\,dt\right].$$

Thus, we have that

$$E(T) - E(0) \leq -CE(0).$$

Since the energy is dissipative, it follows that $E(T) \leq E(0)$, thus

$$E(T) - E(0) \leq -CE(T),$$

which implies that

$$E(T) \leq \delta E(0), \text{ where } \delta = \frac{C}{1+C} < 1.$$

Now, applying the same argument on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots$, yields that

$$E(mT) \leq \delta E((m-1)T) \leq \cdots \leq \delta^m E(0).$$

Thus, we have

$$E(mT) \leq e^{-\mu_0 mT} E(0) \quad \text{with} \quad \mu_0 = \frac{1}{T} \ln\left(1 + \frac{1}{C}\right) > 0.$$

For an arbitrary $t > 0$, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the nonincreasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\mu_0(m-1)T} E(0) \leq \frac{1}{\delta} e^{-\mu_0 t} E(0),$$

showing uniform exponential stability. □

4.4.3 An explicit decay rate

Finally, by using Lyapunov's approach we can give an explicit decay rate for the solutions of (4.1.2)-(4.1.3).

Proof of Theorem 4.1.5. Firstly, let us consider the following Lyapunov's functional $V(t) = E(t) + \gamma V_1(t)$, where γ is a constant to be fixed later, $E(t)$ is given by (4.1.4) and

$$V_1(t) = \frac{1}{2} \int_\Omega x u^2(x,y,t)\, dx\, dy.$$

Note that $E(t)$ and $V(t)$ are equivalent in the following sense $E(t) \leq V(t) \leq (1 + \gamma L)E(t)$ and observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} V_1(t) = &-\frac{3}{2} \int_\Omega u_x^2(x,y,t)\, dx\, dy + \frac{1}{2} \int_\Omega u^2(x,y,t)\, dx\, dy \\ &+ \frac{L\alpha^2}{2} \int_0^L u_x^2(0,y,t)\, dy - \frac{1}{2} \int_\Omega \left(\partial_x^{-1} u_y(x,y,t)\right)^2\, dx\, dy.\end{aligned}$$

Recalling that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\frac{(1-\alpha^2)}{2}\int_0^L u_x^2(0,y,t)\,dy - \frac{1}{2}\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy$$

and gathering the results holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t) = &\frac{1}{2}\left[L\alpha^2\gamma+\alpha^2-1\right]\int_0^L u_x^2(0,y,t)\,dy - \frac{1}{2}\int_0^L (\partial_x^{-1}u_y(0,y,t))^2\,dy\\ &-\frac{3}{2}\gamma\int_\Omega u_x^2(x,y,t)\,dx\,dy + \frac{1}{2}\gamma\int_\Omega u^2(x,y,t)\,dx\,dy\,dt - \frac{1}{2}\gamma\int_\Omega (\partial_x^{-1}u_y(x,y,t))^2\,dx\,dy.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t)+2\nu V(t) = &\frac{1}{2}\left[L\alpha^2\gamma+\alpha^2-1\right]\int_0^L u_x^2(0,y,t)\,dy - \frac{1}{2}\int_0^L (\partial_x^{-1}u_y(0,y,t))^2\,dy\\ &-\frac{3}{2}\gamma\int_\Omega u_x^2(x,y,t)\,dx\,dy + \frac{1}{2}\gamma\int_\Omega u^2(x,y,t)\,dx\,dy\,dt\\ &-\frac{1}{2}\gamma\int_\Omega (\partial_x^{-1}u_y(x,y,t))^2\,dx\,dy\\ &+\frac{\nu}{2}\int_\Omega u^2(x,y,t)\,dx\,dy + \frac{\nu\gamma}{2}\int_\Omega xu^2(x,y,t)\,dx\,dy\\ \le &\frac{1}{2}\left[L\alpha^2\gamma+\alpha^2-1\right]\int_0^L u_x^2(0,y,t)\,dy - \frac{3}{2}\gamma\int_\Omega u_x^2(x,y,t)\,dx\,dy\\ &+\frac{\nu(1+L\gamma)+\gamma}{2}\int_\Omega u^2(x,y,t)\,dx\,dy.\end{aligned}$$

Consequently, applying the Poincaré inequality yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t)+\nu V(t) \le &\frac{1}{2}\left[L\alpha^2\gamma+\alpha^2-1\right]\int_0^L u_x^2(0,y,t)\,dy\\ &+\frac{1}{2}\left[(\nu(1+L\gamma)+\gamma)L^2-3\gamma\right]\int_\Omega u_x^2(x,y,t)\,dx\,dy.\end{aligned}$$

Follows from (4.1.12) and (4.1.13) that $\frac{\mathrm{d}}{\mathrm{d}t}V(t)+\nu V(t)\le 0$, then by the Gronwall's inequality and the equivalence between $E(t)$ and $V(t)$ the proof is complete. □

4.5 FINAL REMARKS

This work dealt with the KP-II equation posed on a rectangle, a bi-dimensional generalization of the KdV equation. Under certain hypotheses of the spatial length L, that is, $L\in(0,+\infty)\backslash\mathcal{R}$, with $\mathcal{R}$ defined by

$$\begin{aligned}\mathcal{R} := \Bigg\{&\frac{\pi}{4n}\sqrt{(3m_1+2m_2+m_3)(m_1-2m_2-m_3)(m_1+2m_2-m_3)(m_1+2m_2+3m_3)}:\\ &n,m_1,m_2,m_3\in\mathbb{N},\ \text{with}\ \ |m_1-m_3|>2m_2>0\Bigg\},\end{aligned}$$

the boundary controllability and stabilization are achieved by using the compactness-uniqueness method, which reduces the problem to show a unique continuation property. This property is a consequence of a spectral problem associated with the KP-II operator. Additionally, to relax the condition over L, we employed Lyapunov's approach to finding an explicit decay rate for the solutions of the feedback-closed KP-II system.

5 STABILIZATION OF THE KAWAHARA-KADOMTSEV-PETVIASHVILI EQUATION WITH TIME-DELAYED FEEDBACK

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5.1 INTRODUCTION

In the last years, properties of the asymptotic models for water waves have been extensively studied to understand the full water wave system[1]. As we know, we can formulate the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form. Some physical conditions give us the so-called long waves or shallow water waves. For example, in one spatial dimensional case, the so-called Kawahara equation which is an equation derived by Hasimoto and Kawahara in (HASIMOTO, 1970; KAWAHARA, 1972) that takes the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45}u_{xxxxx} = 0. \tag{5.1.1}$$

If we look for two spatial dimensional, wave phenomena that exhibit weak transversality and weak nonlinearity are modeled by the Kadomtsev-Petviashvili (KP) equation

$$u_t + \alpha u_{xxx} + \gamma \partial_x^{-1} u_{yy} + uu_x = 0, \tag{5.1.2}$$

where $u = u(x, y, t)$ and α, β, γ are constants it was introduced by Kadomtsev and Petviashvili (see (KADOMTSEV; PETVIASHVILI, 1970)) in 1970. In 1993, Karpman included the higher-order dispersion in (5.1.2) leads to a fifth-order generalization of the KP equation (KARPMAN, 1993)

$$u_t + \alpha u_{xxx} + \beta u_{xxxxx} + \gamma \partial_x^{-1} u_{yy} + uu_x = 0, \tag{5.1.3}$$

which will be called the Kawahara-Kadomtsev-Petviashvili equation (K-KP). Note that, by scaling transformations on the variables x, t, and u, the coefficients in equation (5.1.3) can be set to $\alpha > 0, \beta < 0, \gamma^2 = 1$. For the sequel, we consider this scaled form of the equation:

$$u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} + \gamma \partial_x^{-1} u_{yy} = 0, \quad \gamma = \pm 1. \tag{5.1.4}$$

When $\gamma = -1$ we will refer to the case as K-KP-I and for $\gamma = 1$ as K-KP II, respectively. This is motivated in analogy with the usual terminology for the KP equation, which distinguishes the

[1] See for instance (BONA; LANNES; SAUT, 2008; LANNES, 2013) and references therein, for a rigorous justification of various asymptotic models for surface and internal waves.

two cases for the sign of the ratio of the highest derivative terms in x and y, that is, focusing and defocusing cases, respectively.

It is important to point out that there are several physical applications in modeling long water waves in a shallow water regime with a strong dispersion represented by systems (5.1.1)–(5.1.4). We can cite at least two of them, the first one is to describe both the wave speed and the wave amplitude (HARAGUS, 1996), and the second one is modeling plasma waves with strong dispersion (KAWAHARA, 1972).

5.1.1 Problem setting

There is an important advance in control theory to understand how the damping mechanism acts in the energy of systems governed by a partial differential equation. In particular, exponential stability for dispersive equations related to water waves posed on bounded domains has been intensively studied. For example, it is well known that the KdV equation (MENZALA; VASCONCELLOS; ZUAZUA, 2002), Boussinesq system of KdV-KdV type (PAZOTO; ROSIER, 2008), Kawahara equation (ARARUNA; CAPISTRANO-FILHO; DORONIN, 2012) and others are exponentially stable using the Compactness-Uniqueness developed by J.L. Lions (LIONS, 1988a). Other results as obtained in (CAPISTRANO-FILHO; CERPA; GALLEGO, 2023) and in (CAPISTRANO-FILHO; GALLEGO, 2018) are obtained using Urquiza's and Backstepping approach. All these results use damping mechanisms in the equation or the boundary as a control.

Recently, in (CHENTOUF, 2022; CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024), the authors obtained exponential decay for a fifth-order KdV type equation via the Compactness-Uniqueness argument and Lyapunov approach. Additionally to that, in (GOMES; PANTHEE, 2011) and (MOURA; NASCIMENTO; SANTOS, 2022), exponential decay for the KP-II and K-KP-II were shown[2]. In both works, the authors can prove regularity and well-posedness for these equations and show that the energy associated with this equation decays exponentially in the presence of a damping term acting in the equation.

As we can see in these articles there is interest in the mathematical context in the study of the asymptotic behavior of the solution of the equation (5.1.4). Additionally, as pointed out, the model under consideration in this chapter has importance in the context of the dispersive equation as well as, physical motivation. So, motivated by (CHENTOUF, 2022; CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024; MOURA; NASCIMENTO; SANTOS, 2022; GOMES; PANTHEE,

[2] See also the reference therein for stabilization of KP-II and K-KP-II.

2011) we will analyze the qualitative properties of the initial-boundary value problem for the K-KP-II equation posed on a bounded domain $\Omega = (0, L) \times (0, L) \subset \mathbb{R}^2$ with localized damping and delay terms

$$\begin{cases} \partial_t u(x,y,t) + \alpha\partial_x^3 u(x,y,t) + \beta\partial_x^5 u(x,y,t) & \\ + \gamma\partial_x^{-1}\partial_y^2 u(x,y,t) + \frac{1}{2}\partial_x(u^2(x,y,t)) & (x,y) \in \Omega,\ t > 0. \\ + a(x,y)u(x,y,t) + b(x,y)u(x,y,t-h) = 0, & \\ u(0,y,t) = u(L,y,t) = 0, & y \in (0,L),\ t \in (0,T), \\ \partial_x u(L,y,t) = \partial_x u(0,y,t) = \partial_x^2 u(L,y,t) = 0, & y \in (0,L),\ t \in (0,T), \\ u(x,L,t) = u(x,0,t) = 0, & x \in (0,L),\ t \in (0,T), \\ u(x,y,0) = u_0(x,y), \quad u(x,y,t) = z_0(x,y,t), & (x,y) \in \Omega,\ t \in (-h,0). \end{cases} \tag{5.1.5}$$

Here $h > 0$ is the time delay, $\alpha > 0$, $\gamma > 0$ and $\beta < 0$ are real constants. Additionally, define the operator $\partial_x^{-1} := \partial_x^{-1}\varphi(x,y,t) = \psi(x,y,t)$ such that $\psi(L,y,t) = 0$ and $\partial_x\psi(x,y,t) = \varphi(x,y,t)$[3] and, for our purpose, let us consider the following assumption.

Assumption 2. *The real functions $a\,(x,y)$ and $b\,(x,y)$ are nonnegative belonging to $L^\infty(\Omega)$. Moreover, $a(x,y) \geq a_0 > 0$ is almost everywhere in a nonempty open subset $\omega \subset \Omega$.*

Our propose here is to present, for the first time, the K-KP-II system not with only a damping mechanism $a(x,y)u$, which plays the role of a feedback-damping mechanism (see e.g. (MOURA; NASCIMENTO; SANTOS, 2022)), but also with an anti-damping, that is, some feedback such that our system does not have decreasing energy. In this context, we would like to prove that the energy associated with the solutions of the system (5.1.5)

$$\begin{aligned} E_u(t) =& \frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy \\ &+ \frac{h}{2}\int_0^L\int_0^L\int_0^1 b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy. \end{aligned} \tag{5.1.6}$$

decays exponentially. Precisely, we want to answer the following question:

Does $E_u(t) \to 0$ as $t \to \infty$? If this is the case, can we give the decay rate?

3 It can be shown that the definition of operator ∂_x^{-1} is equivalent to $\partial_x^{-1}u(x,y,t) = \int_x^L u(s,y,t)\,ds$.

5.1.2 Notation and main results

Before presenting answers to this question, let us introduce the functional space that will be necessary for our analysis. Given $\Omega \subset \mathbb{R}^2$ let us define $X^k(\Omega)$ to be the Sobolev space

$$X^k(\Omega) := \left\{ \begin{array}{l} \varphi \in H^k(\Omega)\colon \partial_x^{-1}\varphi(x,y) = \psi(x,y) \in H^k(\Omega) \text{ such that,} \\ \psi(L,y) = 0 \text{ and } \partial_x \psi(x,y) = \varphi(x,y) \end{array} \right\} \tag{5.1.7}$$

endowed with the norm $\|\varphi\|^2_{X^k(\Omega)} = \|\varphi\|^2_{H^k(\Omega)} + \|\partial_x^{-1}\varphi\|^2_{H^k(\Omega)}$. We also define the normed space $H^k_x(\Omega)$,

$$H^k_x(\Omega) := \left\{\varphi\colon \partial^j_x \varphi \in L^2(\Omega), \text{ for } 0 \leq j \leq k\right\} \tag{5.1.8}$$

with the norm $\|\varphi\|^2_{H^k_x(\Omega)} = \sum_{j=0}^k \|\partial^j_x \varphi\|^2_{L^2(\Omega)}$ and the space

$$X^k_x(\Omega) := \left\{ \begin{array}{l} \varphi \in H^k_x(\Omega)\colon \partial_x^{-1}\varphi(x,y) = \psi(x,y) \in H^k_x(\Omega) \text{ such that} \\ \psi(L,y) = 0 \text{ and } \partial_x \psi(x,y) = \varphi(x,y) \end{array} \right\} \tag{5.1.9}$$

with $\|\varphi\|^2_{X^k_x(\Omega)} = \|\varphi\|^2_{H^k_x(\Omega)} + \|\partial_x^{-1}\varphi\|^2_{H^k_x(\Omega)}$. Finally, $H^k_{x0}(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ in $H^k_x(\Omega)$.

The next result will be used repeatedly throughout the chapter:

Theorem 5.1.1 ((BESOV; IL'IN; NIKOL'SKII, 1978, Theorem 15.7)). *Let β and $\alpha^{(j)}$, for $j = 1, \ldots, N$, denote n- dimensional multi-indices with non-negative-integer-valued components. Suppose that $1 < p^{(j)} < \infty$, $1 < q < \infty$, $0 < \mu_j < 1$ with*

$$\sum_{j=1}^N \mu_j = 1, \quad \frac{1}{q} \leq \sum_{j=1}^N \frac{\mu_j}{p^{(j)}}, \quad \text{and} \quad \beta - \frac{1}{q} = \sum_{j=1}^N \mu_j \left(\alpha^{(j)} - \frac{1}{p^{(j)}}\right).$$

Then, for $f(x) \in C_0^\infty(\mathbb{R}^n)$,

$$\left\|D^\beta f\right\|_q \leq C \prod_{j=1}^N \left\|D^{\alpha^{(j)}} f\right\|^{\mu_j}_{p^{(j)}}.$$

Where, for non-negative multi-index $\beta = (\beta_1, \ldots, \beta_N)$ we denote D^β by $D^\beta = D^{\beta_1}_{x_1} \ldots D^{\beta_n}_{x_n}$ and $D^{\beta_i}_{x_i} = \frac{\partial^{\beta_i}}{\partial x_i^{k_i}}$

The first result of the chapter ensures that without a restrictive assumption on the length L of the domain and with the weight of the delayed feedback small enough the energy (5.1.6) associated with the solution of the system (5.1.5) are locally stable.

Theorem 5.1.2 (Optimal local stabilization)**.** *Assume that the functions $a(x,y)$, $b(x,y)$ satisfy the conditions given in Assumption 2. Let $L>0$, $\xi>1$, $0<\mu<1$ and T_0 given by*

$$T_0=\frac{1}{2\theta}\ln\left(\frac{2\xi\kappa}{\mu}\right)+1, \tag{5.1.10}$$

with $\theta=\frac{3\alpha\eta}{(1+2\eta L)L^2}$, $\kappa=1+\max\left\{2\eta L,\frac{\sigma}{\xi}\right\}$ and $\eta\in\left(0,\frac{\xi-1}{2L(1+2\xi)}\right)$ satisfying

$$\frac{2\alpha\eta}{(2+2\eta L)L^2}=\frac{\sigma}{2h(\xi+\sigma)}$$

where $\sigma=\xi-1-2L\eta(1+2\xi)$. Let $T_{\min}>0$ given by

$$T_{\min}:=-\frac{1}{\nu}\ln\left(\frac{\mu}{2}\right)+\left(\frac{2\|b\|_\infty}{\nu}+1\right)T_0,\ \textit{with}\ \nu=\frac{1}{T_0}\ln\left(\frac{1}{(\mu+\varepsilon)}\right).$$

Then, there exists $\delta>0$, $r>0$, $C>0$ and γ, depending on $T_{\min},\xi,L,h$, such that if $\|b\|_\infty\leq\delta$, then for every $(u_0,z_0)\in\mathcal{H}=L^2(\Omega)\times L^2(\Omega\times(0,1))$ satisfying $\|(u_0,z_0)\|_{\mathcal{H}}\leq r$, the energy of the system (5.1.5) satisfies

$$E_u(t)\leq Ce^{-\gamma t}E_u(0),\ \textit{for all}\ t>T_{\min}.$$

Now on, following the ideas in (CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024), we obtain some stability properties about the next system, called μ_i–system. Note that if we choose $a(x,y)=\mu_1 a(x,y)$ and $b(x,y)=\mu_2 a(x,y)$ in (5.1.5), where μ_1 and μ_2 are real constants we obtain the system

$$\begin{cases}\partial_t u(x,y,t)+\alpha\partial_x^3 u(x,y,t)+\beta\partial_x^5 u(x,y,t) & \\ +\gamma\partial_x^{-1}\partial_y^2 u(x,y,t)+\frac{1}{2}\partial_x(u^2(x,y,t)) & (x,y,t)\in\Omega\times\mathbb{R}^+\\ +a(x,y)\left(\mu_1 u(x,y,t)+\mu_2 u(x,y,t-h)\right)=0, & \\ u(0,y,t)=u(L,y,t)=0, & y\in(0,L),\ t\in(0,T),\\ \partial_x u(L,y,t)=\partial_x u(0,y,t)=\partial_x^2 u(L,y,t)=0, & y\in(0,L),\ t\in(0,T),\\ u(x,L,t)=u(x,0,t)=0, & x\in(0,L),\ t\in(0,T),\\ u(x,y,0)=u_0(x,y),\quad u(x,y,t)=z_0(x,y,t), & (x,y)\in\Omega,\ t\in(-h,0).\end{cases} \tag{5.1.11}$$

Here, $\mu_1>\mu_2$ are positive real numbers and $a(x,y)$ satisfies Assumption 2. We define the total energy associated to (5.1.11)

$$\begin{aligned}E_u(t)=&\frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy\\ &+\frac{\xi}{2}\int_0^L\int_0^L\int_0^1 a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy,\end{aligned} \tag{5.1.12}$$

where $\xi > 0$ satisfies

$$h\mu_2 < \xi < h(2\mu_1 - \mu_2). \tag{5.1.13}$$

Note that the derivative of the energy (5.1.12) satisfies

$$\begin{aligned}\frac{d}{dt}E_u(t) \leq &- C\left(\int_0^L \partial_x^2 u(0,y,t)^2\,dy + \int_0^L \left(\partial_x^{-1}\partial_y u(0,y,t)\right)^2\,dy\right.\\ &\left.+\int_0^L\int_0^L a(x,y)u^2(x,y,t-h)\,dx\,dy\right)\end{aligned} \tag{5.1.14}$$

for $C := C(\mu_1,\mu_2,\xi,h) \geq 0$. This indicates that the function $a(x,y)$ plays the role of a feedback-damping mechanism, at least for the linearized system. Therefore, for the system (5.1.11) we split the behavior of the solutions into two parts. Employing Lyapunov's method, it can be deduced that the energy $E_u(t)$ goes exponentially to zero as $t \to \infty$, however, the initial data needs to be sufficiently small in this case. Precisely, the second local result can be read as follows:

Theorem 5.1.3 (Local stabilization). *Let $L > 0$. Assume that $a(x,y) \in L^\infty(\Omega)$ is a non-negative function, that relation (5.1.13) holds and $\beta < -\frac{1}{30}$. Then, there exists*

$$0 < r < \frac{\sqrt[4]{216\alpha^3}}{CL^{\frac{5}{2}}}$$

such that for every $(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H}$ satisfying $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq r$, the energy defined in (5.1.12) decays exponentially. More precisely, there exists two positive constants θ and κ such that $E_u(t) \leq \kappa E_u(0)e^{-2\theta t}$ for all $t > 0$. Here,

$$\theta < \min\left\{\frac{\eta}{(1+2\eta L)L^2}\left[3\alpha - \frac{1}{2}C^{\frac{4}{3}}r^{\frac{4}{3}}L^{\frac{10}{3}}\right], \frac{\xi\sigma}{2h(\xi+\sigma\xi)}\right\}, \qquad \kappa = 1 + \max\{2\eta L, \sigma\}$$

and η and σ are positive constants such that

$$\begin{aligned}&\sigma < \frac{2h}{\xi}\left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2h}\right)\\ &\eta < \min\left\{\frac{1}{2L\mu_2}\left[\frac{\xi}{h} - \mu_2\right], \frac{1}{2L\mu_1 + L\mu_2}\left[\mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2h}(1+\sigma)\right]\right\}.\end{aligned}$$

The last result of the chapter is still related to the system (5.1.11), removing the hypothesis of the initial data being small. To do that, we use the compactness-uniqueness argument due to J.-L. Lions (LIONS, 1988b), which reduces our problem to prove an *observability inequality* for the nonlinear system (5.1.11) and removes the hypotheses that the initial data are small enough.

Theorem 5.1.4 (Global stabilization). *Let $a \in L^\infty(\Omega)$ satisfies Assumption 2. Suppose that $\mu_1 > \mu_2$ satisfies (5.1.13). Let $R > 0$, then there exists $C = C(R) > 0$ and $\nu = \nu(R) > 0$ such that E_u, defined in (5.1.12) decays exponentially as t tends to infinity, when $\|(u_0, z_0)\|_{\mathcal{H}} \leq R$.*

5.1.3 Novelty and outline

We finish the introduction by highlighting some facts about our problem in comparison with the works previously mentioned, as well as, the organization of this chapter.

a. Observe that the absence of drift term u_x, in comparison with Kawahara equation in (CHENTOUF, 2022; CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024), leads to get stabilization results without restriction in the length of the spatial domain. This term is not important in our analysis, the term only plays an important role in the problems where the control (damping or delay) is acting in the boundary condition[4].

b. As stated earlier, we introduce an anti-damping together with the damping mechanism to show that the energy of the system (5.1.5) decays exponentially. Compared with the known result (MOURA; NASCIMENTO; SANTOS, 2022), the novelty of this chapter is twofold:

   1. Our work gives the precise decay rate, see Theorems 5.1.2 and 5.1.3.
   2. Lyapunov's method shows an optimal decay rate in terms of θ in Theorem 5.1.2. Observe that the value of θ can be optimized as a function of η, that is, we can choose
$$\eta \in \left(0, \frac{\xi - 1}{2L(1+2\xi)}\right) \tag{5.1.15}$$
such that the value of θ is the largest possible, which implies that the decay rate θ thus obtained is the best one. This can be seen defining the functions $f, g : \left[0, \frac{\xi-1}{2L(1+2\xi)}\right] \longrightarrow \mathbb{R}$ by
$$f(\eta) = \frac{3\alpha\eta}{L^2(1+2\eta L)}, \quad g(\eta) = \frac{\xi - 1 - 2L\eta(1+2\xi)}{2h(2\xi - 1 - 2\eta L(1+2\xi))},$$
and considering $\gamma(\eta) = \min\{f(\eta), g(\eta)\}$. So, the function f is increasing in the interval $\left[0, \frac{\xi-1}{2L(1+2\xi)}\right)$ while the function g is decreasing in this same interval. In fact, note that
$$f(\eta) = \frac{3\alpha}{2L^3}\left(1 - \frac{1}{1+2\eta L}\right)$$
and
$$g(\eta) = \frac{1}{2h} - \left(\frac{\xi}{4hL(1+2\xi)}\right)\left(\frac{1}{\frac{\xi}{2L(1+2\xi)} + \frac{\xi-1}{2L(1+2\xi)} - \eta}\right).$$

[4] For details about this situation the authors suggest reference (CAPISTRANO-FILHO et al., 2023).

If $-\frac{1}{2L} < \eta$, then

$$f'(\eta) = \frac{3\alpha}{2L^3} \frac{2L}{(1+2L\eta)^2} > 0.$$

In particular, $f'(\eta) > 0$ when

$$\eta \in \left[0, \frac{\xi - 1}{2L(1+2\xi)}\right).$$

Analogously,

$$g'(\eta) = -\left(\frac{\xi}{4hL(1+2\xi)}\right) \frac{1}{\left(\frac{\xi}{2L(1+2\xi)} + \frac{\xi-1}{2L(1+2\xi)} - \eta\right)^2} < 0,$$

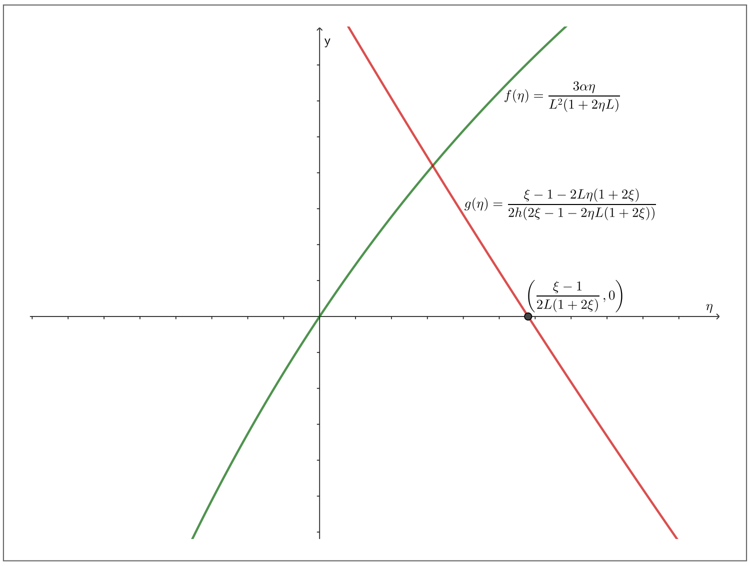
since $\xi > 1$ and $\eta < \frac{\xi-1}{2L(1+2\xi)}$, showing our claim. Now, we claim that there exists only one point satisfying (5.1.15) such that $f(\eta) = g(\eta)$. To show the existence of this point, it is sufficient to note that $f(0) = 0$, $g\left(\frac{\xi-1}{2L(1+2\xi)}\right) = 0$ and

$$f\left(\frac{\xi - 1}{2L(1+2\xi)}\right) = \frac{3\alpha}{2L^3}\left(\frac{3\xi - 1}{3\xi}\right) > 0, \quad g(0) = \frac{1}{2h}\left(1 - \frac{\xi}{2\xi - 1}\right) > 0.$$

The uniqueness follows from the fact that f is increasing while g is decreasing in this interval.

c. Taking into account the above information about f and g, the maximum value of the function must be reached at the point η satisfying (5.1.15), where $f(\eta) = g(\eta)$. The figure 4 below shows, in a simple case, what was said earlier to the functions f and g when we consider some values, for example, $L = 1$, $\xi = 2.3$, $\alpha = 0.5$ and $h = 1.5$:

Figure 4 – Maximum of $\gamma(\eta) = \min\{f(\eta), g(\eta)\}$.



Source: Own elaboration

d. Still concerning the Theorem 5.1.2, observe that we do not need to localize the solution of the transport equation in a small subset of $(0, L)$ as in (VALEIN, 2022, Section 4). Moreover, we emphasize that we can take $a = 0$ in Theorem 5.1.2. Finally, it is important to mention that we do not know if the time $T_{\min}$ is optimal.

e. Aiming to present optimal decay results, note that for the nonlinear system, we obtain one stabilization result with no restriction in the length of the spatial domain but carries a restriction in one parameter of the system, see Theorem 5.1.3. Once again, it is possible to waive one of the conditions (either the restriction on L or a restriction in one parameter of the system). Observe that, using Theorem 5.1.1 like as (5.3.11) below, we have

$$\begin{aligned} \int_0^L \int_0^L u^3(x,y,t)\,dx\,dy &\leq cL \left\| u_{xx} \right\|_{L^2(\Omega)}^{\frac{1}{2}} \left\| u \right\|_{L^2(\Omega)}^{\frac{5}{2}} \\ &\leq \frac{1}{4}(CL)^4 \left\| u_{xx} \right\|_{L^2(\Omega)}^2 + \frac{3}{4} r^{\frac{4}{3}} \left\| u \right\|_{L^2(\Omega)}^2 . \end{aligned} \tag{5.1.16}$$

This estimate allows obtaining, with an analogous argument, another result for exponential stability without restriction in the parameter β but with restriction in the length L of the domain. Thus, in Theorem 5.1.3, we can remove the hypothesis over β, however, a hypothesis over L is necessary. The result is the following:

Theorem 5.1.5 (Local stabilization-*bis*)**.** *Let $0 < L < \sqrt[4]{\frac{-30\beta}{C}}$. Assume that $a(\cdot,\cdot) \in L^\infty(\Omega)$ is a non-negative function and that the relation (5.1.13) holds. Then, there exists $0 < r < \frac{\sqrt[4]{216\alpha^3}}{CL^{\frac{5}{2}}}$ such that for every $(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H}$ satisfying $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq r$, the energy defined in (5.1.12) decays exponentially. More precisely, there exists two positive constants θ and κ such that $E_u(t) \leq \kappa E_u(0) e^{-2\theta t}$ for all $t > 0$, where θ, κ, η and σ are positive constants defined as in Theorem 5.1.3.*

f. The results obtained here can be easily adapted for the KP-II system (5.1.2) with or without the drift term u_x, extending the results of (MOURA; NASCIMENTO; SANTOS, 2022) and (GOMES; PANTHEE, 2011).

The chapter is organized as follows:

- Section 5.2 is dedicated to showing the well-posedness of the system.
- Section 5.3 is devoted to proving the first, and optimal, local stability result, that is, Theorem 5.1.2.

- In Section 5.4 we are able to prove the exponential stability, Theorem 5.1.3, for the energy associated with the μ_i–system (5.1.11).
- Additionally, to extend the local property to the global one, in Section 5.4 we give the proof of Theorem 5.1.4.

5.2 μ_i-SYSTEM: WELL-POSEDNESS

In this section, we deal with the study of the μ_i-system (5.1.11) that is essential to obtain results for (5.1.5). Since the results are classical, we just give the main results and the idea of the proofs.

5.2.1 Linear system

Here, we use semigroup theory to obtain well-posedness results for the linear system associated with (5.1.11). To do that, consider $z(x,y,\rho,t) = u(x,y,t-\rho h)$, for $(x,y) \in \Omega$, $\rho \in (0,1)$ and $t > 0$. Then $z(x,y,\rho,t)$ satisfies the transport equation

$$\begin{cases} h\partial_t z(x,y,\rho,t) + \partial_\rho z(x,y,\rho,t) = 0, & (x,y) \in \Omega,\ \rho \in (0,1),\ t > 0, \\ z(x,y,0,t) = u(x,y,t), & (x,y) \in \Omega,\ t > 0, \\ z(x,y,\rho,0) = z_0(x,y,\rho,-\rho h), & (x,y) \in \Omega,\ \rho \in (0,1). \end{cases} \tag{5.2.1}$$

Let $\mathcal{H} = L^2(\Omega) \times L^2(\Omega \times (0,1))$ a Hilbert space equipped with the inner product

$$\begin{aligned} \langle (u,z)\,(v,w) \rangle_{\mathcal{H}} = & \int_0^L \int_0^L u(x,y)v(x,y)\,dx\,dy \\ & + \xi \left\| a \right\|_\infty \int_0^L \int_0^L \int_0^1 z(x,y,\rho)w(x,y,\rho)\,d\rho\,dx\,dy, \end{aligned}$$

with ξ satisfies (5.1.13). To study the well-posedness in the Hadamard sense, we need to rewrite the linear system associated with (5.1.11) as an abstract problem. Let $U(t) = (u(\cdot,\cdot,t), z(\cdot,\cdot,\cdot,t))$ and denote $z(1) := z(x,y,1,t)$. From the linear system associated with

(5.1.11) and (5.2.1) we get the next system

$$
\begin{cases}
\partial_t u(x,y,t) + \alpha\partial_x^3 u(x,y,t) + \beta\partial_x^5 u(x,y,t) & \\
+\gamma\partial_x^{-1}\partial_y^2 u(x,y,t) + \frac{1}{2}\partial_x(u^2(x,y,t)) & (x,y,t)\in\Omega\times\mathbb{R}^+ \\
+a(x,y)\left(\mu_1 u(x,y,t)+\mu_2 z(1)\right) = 0 & \\
u(0,y,t) = u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\
\partial_x u(L,y,t) = \partial_x u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\
\partial_x u(0,y,t) = \partial_x^2 u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\
u(x,L,t) = u(x,0,t) = 0, & x\in(0,L),\ t\in(0,T), \\
u(x,y,0) = u_0(x,y), & (x,y)\in\Omega \\
h\partial_t z(x,y,\rho,t) + \partial_\rho z(x,y,\rho,t) = 0, & (x,y)\in\Omega,\ \rho\in(0,1),\ t>0, \\
z(x,y,0,t) = u(x,y,t), & (x,y)\in\Omega,\ t>0, \\
z(x,y,\rho,0) = z_0(x,y,\rho,-\rho h), & (x,y)\in\Omega,\ \rho\in(0,1).
\end{cases}
\tag{5.2.2}
$$

which is equivalent to

$$
\begin{cases}
\frac{d}{dt}U(t) = AU(t), \\
U(0) = (u_0(x,y), z_0(x,y,-\rho h))
\end{cases}
\tag{5.2.3}
$$

where $A\colon D(A)\subset\mathcal{H}\to\mathcal{H}$ is defined by

$$
A(u,z) = \left(-\alpha\partial_x^3 u - \beta\partial_x^5 u - \gamma\partial_x^{-1}\partial_y^2 u - a(x,y)(\mu_1 u + \mu_2 z(1)); -h^{-1}\partial_\rho z\right) \tag{5.2.4}
$$

with the dense domain given by

$$
D(A) := \left\{
\begin{array}{l|l}
(u,z)\in\mathcal{H}: & u(0,y) = u(L,y) = u(x,0) = u(x,L) = 0, \\
u\in H_x^5(\Omega)\cap X^2(\Omega), & \partial_x u(L,y) = \partial_x u(0,y) = \partial_x^2 u(L,y) = 0, \\
\partial_\rho z\in L^2(\Omega\times(0,1)), & z(x,y,0) = u(x,y)
\end{array}
\right\}.
$$

The next result is classical and can be omitted.

Lemma 5.2.1. *The operator A is closed and the adjoint $A^*\colon D(A^*)\subset\mathcal{H}\to\mathcal{H}$ is given by*

$$
A^*(u,z) = \left(\alpha\partial_x^3 u + \beta\partial_x^5 u + \gamma\partial_x^{-1}\partial_y^2 u - a(x,y)\mu_1 u + \frac{\xi\left\|a\right\|_\infty}{h}z(\cdot,\cdot,0); h^{-1}\partial_\rho z\right)
$$

with dense domain

$$
D(A^*) := \left\{
\begin{array}{l|l}
(u,z)\in\mathcal{H}: & u(0,y) = u(L,y) = u(x,0) = u(x,L) = 0, \\
u\in H_x^5(\Omega)\cap X^2(\Omega), & \partial_x u(L,y) = \partial_x u_x(0,y) = \partial_x^2 u(0,y) = 0, \\
\partial_\rho z\in L^2(\Omega\times(0,1)), & z(x,y,1) = -\frac{a(x,y)h\mu_2}{\xi\left\|a\right\|_\infty}u(x,y)
\end{array}
\right\}.
$$

Proposition 5.2.2. *Assume that $a \in L^\infty(\Omega)$ is a nonnegative function and (5.1.13) is satisfied. Then A is the infinitesimal generator of a C_0-semigroup in $\mathcal{H}$.*

Proof. Let $U = (u, z) \in D(A)$, then

$$\langle AU, U\rangle_{\mathcal{H}} \leq \frac{\xi \left\| a \right\|_\infty}{2h} \int_0^L \int_0^L u^2(x, y)\, dx\, dy. \tag{5.2.5}$$

Hence, for $\lambda = \frac{\xi\|a\|_\infty}{2h}$ we have $\langle (A - \lambda I)U, U\rangle_{\mathcal{H}} \leq 0$ (resp. $\langle (A^* - \lambda I)U, U\rangle_{\mathcal{H}} \leq 0$, for $U = (u, z) \in D(A^*)$). Since $A - \lambda I$ is a densely defined closed linear operator, and both $A - \lambda I$ and $(A - \lambda I)^*$ are dissipative, A generate an infinitesimal C_0-semigroup on $\mathcal{H}$. □

The next theorem establishes the existence of solutions for the abstract Cauchy problem (5.2.3). This result is a consequence of the previous proposition.

Theorem 5.2.3. *Assume that $a \in L^\infty(\Omega)$ and (5.1.13) is satisfied. Then, for each initial data $U_0 \in \mathcal{H}$ there exists a unique mild solution $U \in C\left([0, \infty), \mathcal{H}\right)$ for the system (5.2.3). Moreover, if the initial data $U_0 \in D(A)$ the solutions are classical such that $U \in C\left([0, \infty), D(A)\right) \cap C^1\left([0, \infty), \mathcal{H}\right)$.*

Next results are devoted to showing *a priori* and regularity estimates for the solutions of (5.2.3).

Proposition 5.2.4. *Let $a \in L^\infty(\Omega)$ be a nonnegative function and consider that (5.1.13) holds. Then, for any mild solution of (5.2.3) the energy E_u, defined by (5.1.12), is non-increasing and there exists a constant $C > 0$ such that*

$$\begin{aligned}\frac{d}{dt}E_u(t) \leq & -C\left(\int_0^L \partial_x^2 u(0, y, t)^2\, dy + \int_0^L (\partial_x^{-1}\partial_y u(0, y, t))^2\, dy\right.\\ & \left. + \int_0^L \int_0^L a(x, y)u^2\, dx\, dy + \int_0^L \int_0^L a(x, y)u^2(x, y, t - h)\, dx\, dy\right)\end{aligned} \tag{5.2.6}$$

where $C = C(\beta, \gamma, \xi, h, \mu_1, \mu_2)$ is given by

$$C = \min\left\{-\frac{\beta}{2}, \frac{\gamma}{2}, \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2h}, -\frac{\mu_2}{h} + \frac{\xi}{2h}\right\}.$$

Proof. First, multiply $(5.2.2)_1$ by $u(x, y, t)$ and integrate by parts in $L^2(\Omega)$. After that, multiply $(5.2.2)_5$ by $z(x, y, \rho, t)$ and integrate by parts in $L^2(\Omega \times (0, 1))$. Finally, adding the results, and the proposition follows. □

To use the contraction principle and to obtain the Kato smoothing effect, for $T > 0$, we introduce the following sets:

$$\begin{aligned}\mathcal{B}_X =& C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T, X^2_{x0}(\Omega)\right),\\ \mathcal{B}_H =& C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T, H^2_{x0}(\Omega)\right)\end{aligned}$$

endowed with its natural norms

$$\begin{aligned}\|y\|_{\mathcal{B}_X} &= \max_{t\in[0,T]} \|y(\cdot,\cdot,t)\|_{L^2(\Omega)} + \left(\int_0^T \|y(\cdot,\cdot,t)\|^2_{X^2_{x0}(\Omega)}\, dt\right)^{\frac{1}{2}},\\ \|y\|_{\mathcal{B}_H} &= \max_{t\in[0,T]} \|y(\cdot,\cdot,t)\|_{L^2(\Omega)} + \left(\int_0^T \|y(\cdot,\cdot,t)\|^2_{H^2_{x0}(\Omega)}\, dt\right)^{\frac{1}{2}}.\end{aligned}$$

Here, $X^2_{x0}(\Omega)$ denotes the space

$$X^k_{x0}(\Omega) := \left\{\begin{array}{l}\varphi \in H^k_{x0}(\Omega)\colon \partial_x^{-1}\varphi(x,y) = \psi(x,y) \in H^k_{x0}(\Omega) \text{ with}\\ \psi(L,y) = 0 \text{ and } \partial_x\psi(x,y) = \varphi(x,y).\end{array}\right\} \tag{5.2.7}$$

Proposition 5.2.5. *Let $a \in L^\infty(\Omega)$ be a nonnegative function. Then, the map*

$$(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H} \mapsto (u,z) \in \mathcal{B}_X \times C\left([0,T], L^2\left(\Omega\times(0,1)\right)\right)$$

is continuous and for $(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H}$, the following estimates are satisfied

$$\begin{aligned}&\frac{1}{2}\int_0^L\int_0^L u^2(x,y)\,dx\,dy + \frac{\xi}{2}\int_0^L\int_0^L\int_0^1 a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy\\ &\le \frac{1}{2}\int_0^L\int_0^L u_0^2\,dx\,dy + \frac{\xi}{2}\int_0^L\int_0^L\int_0^1 a(x,y)z_0^2(x,y,-\rho h)\,d\rho\,dx\,dy,\end{aligned} \tag{5.2.8}$$

$$\begin{aligned}\frac{3\alpha}{2}\int_0^T\int_0^L\int_0^L \partial_x u(x,y,t)^2\,dx\,dy\,dt - \frac{5\beta}{2}\int_0^T\int_0^L\int_0^L \partial_x^2 u(x,y,t)^2\,dx\,dy\,dt&\\ \le \mathcal{C}(a,\mu_1,\mu_2,L)(1+T)\,\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}&\end{aligned} \tag{5.2.9}$$

and

$$\begin{aligned}&\|u_0\|^2_{L^2(\Omega)} \le \frac{1}{T}\int_0^T\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy\,dt - \beta\int_0^T\int_0^L \partial_x^2 u(0,y,t)^2\,dy\,dt\\ &+ \gamma\int_0^T\int_0^L \left(\partial_x^{-1}\partial_y u(0,y,t)\right)^2 dy\,dt + \int_0^T\int_0^L\int_0^L a(x,y)\mu_2 u^2(x,y,t-h)\,dx\,dy\,dt\\ &+ (2\mu_1+\mu_2)\int_0^T\int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy\,dt.\end{aligned} \tag{5.2.10}$$

Proof. The proof is classical and uses the Morawetz multipliers. Precisely, first, (5.2.8) follows from (5.2.6). To get the other two inequalities for $(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H}$, multiplying $(5.2.2)_5$ by $z(x,\rho,t)$ and $(5.2.2)_1$ by $xu(x,y,t)$ and integrating by parts in $\Omega\times(0,T)$, (5.2.9) holds. Finally, multiplying $(5.2.2)_1$ by $(T-t)u(x,y,t)$ and integrating by parts in $\Omega\times(0,T)$ we obtain (5.2.10). □

5.2.2 Linear system with source term

We will study the system (5.2.2), with a source term $f(x,y,t)$ on the right-hand side. The next result ensures the well-posedness of this system.

Proposition 5.2.6. *Assume that $a(x,y)\in L^\infty(\Omega)$ is a nonnegative function and that (5.1.13) is satisfied. For any $(u_0, z_0(\cdot,\cdot,-h(\cdot)))\in\mathcal{H}$ and $f\in L^1\left(0,T,L^2(\Omega)\right)$, there exists a unique mild solution for (5.2.2) with the source term $f(x,y,t)$ on the right-hand side in the class*

$$(u, u(\cdot,\cdot,t-h(\cdot)))\in\mathcal{B}_X\times C\left([0,T],L^2(\Omega\times(0,1))\right).$$

Moreover, we have

$$\|(u,z)\|_{C([0,T],\mathcal{H})}\leq e^{\frac{\xi\|a\|_\infty}{2h}T}\left(\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}+\|f\|_{L^1(0,T,L^2(\Omega))}\right) \tag{5.2.11}$$

and

$$\delta\|u\|^2_{L^2(0,T,H^2_x(\Omega))}\leq\mathcal{C}\left(\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|^2_{\mathcal{H}}+\|f\|^2_{L^1(0,T,L^2(\Omega))}\right) \tag{5.2.12}$$

where

$$\mathcal{C}=\mathcal{C}\left(a,\mu_1,\mu_2,L,T,h\right)=\frac{3L}{2}+L\|a\|_\infty(\mu_1+\mu_2)+\delta\left(1+T+e^{\frac{\xi\|a\|_\infty}{h}T}\right)$$

and $\delta=\min\{1,3\alpha/2,-5\beta/2\}$.

Proof. Note that A is an infinitesimal generator of a C_0-semigroup $(e^{tA})_{t\geq0}$ satisfying $\left\|e^{tA}\right\|_{\mathcal{L}(\mathcal{H})}\leq e^{\frac{\xi\|a\|_\infty}{2h}t}$ and the system can be rewritten as a first order system with source term $(f(\cdot,\cdot,t),0)$, showing the well-posed in $C([0,T],\mathcal{H})$. Finally, observe that the right-hand side is not homogeneous, since

$$\left|\int_0^T\int_\Omega xf(x,y,t)u(x,y,t)\,dx\,dy\,dt\right|\leq\frac{L}{2}\|u\|^2_{C([0,T];L^2(0,L))}+\frac{L}{2}\|f\|^2_{L^1(0,T,L^2(\Omega))},$$

showing the result. □

5.2.3 Nonlinear system: Global results.

In this last part, we consider the nonlinear term uu_x as a source term.

Proposition 5.2.7. *If $u\in\mathcal{B}_X$ then $uu_x\in L^1(0,T;L^2(\Omega))$ and the map $u\in\mathcal{B}_X\mapsto u\partial_x u\in L^1(0,T;L^2(\Omega))$ is continuous. In particular, exists $K>0$, such that, for all $u,v\in\mathcal{B}_X$ we have*

$$\|u\partial_x u-v\partial_x v\|_{L^1(0,T,L^2(\Omega))}\leq K\left(\|u\|_{\mathcal{B}_X}+\|v\|_{\mathcal{B}_X}\right)\|u-v\|_{\mathcal{B}_X}. \tag{5.2.13}$$

Proof. The Hölder inequality and the Sobolev embedding $H^2_{x0}(\Omega) \hookrightarrow L^\infty(\Omega)$ gives us

$$\|u\partial_x u - v\partial_x v\|_{L^1(0,T,L^2(\Omega))} \leq C_1 \cdot C \cdot T^{\frac{1}{4}} \left(\|u\|_{\mathcal{B}_H} + \|v\|_{\mathcal{B}_H}\right) \|u - v\|_{\mathcal{B}_H}, \tag{5.2.14}$$

for $u, v \in \mathcal{B}_X$. Note that, $u \in \mathcal{B}_X$ implies that $u(\cdot,\cdot,t) \in H^2_{x0}(\Omega)$ and consequently $u(\cdot,\cdot,t) \in H^1_{x0}(\Omega)$ and $u_x(\cdot,\cdot,t) \in H^1_{x0}(\Omega)$. Here, using the definition of the operator ∂_x^{-1} and the Poincaré's inequality we obtain,

$$\left\|\partial_x^{-1}(u\partial_x u)\right\|_{L^1(0,T,L^2(\Omega))} \leq L^2 \|u\partial_x u\|_{L^1(0,T,L^2(\Omega))} \tag{5.2.15}$$

So, from (5.2.14), with $v = 0$, and (5.2.15) we get $u\partial_x u \in L^1(0,T,L^2(\Omega))$ and the proof is complete. □

We prove the global well-posedness of the K-KP-II with a delay term.

Proposition 5.2.8. *Let $L > 0$, $a(x,y) \in L^\infty(\Omega)$ be a nonnegative function and that (5.1.13) holds. Then, for all initial data $(u_0, z_0(\cdot,\cdot,-h(\cdot)) \in \mathcal{H}$, there exists a unique $u \in \mathcal{B}_X$ solution of (5.1.11). Moreover, there exist constants $\mathcal{C} > 0$ and $\delta \in (0,1]$ such that*

$$\delta \|u\|^2_{L^2(0,T,H^2_x(\Omega))} \leq \mathcal{C}\left(\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|^2_{\mathcal{H}} + \|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|^{\frac{10}{3}}_{\mathcal{H}}\right). \tag{5.2.16}$$

Proof. To obtain the global existence of solutions we show the local existence and use the *a priori* estimate below, which is proved using the multipliers method and Gronwall's inequality:

$$\|(u(\cdot,\cdot,t), u(\cdot,\cdot,t-h))\|^2_{\mathcal{H}} \leq e^{\frac{\xi\|a\|_\infty}{h}t} \|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|^2_{\mathcal{H}} \tag{5.2.17}$$

With the previous inequality in hands, the local existence and uniqueness of solutions of (5.1.11) holds. Precisely, pick $(u_0, z_0(\cdot,\cdot,-h(\cdot))) \in \mathcal{H}$ and $u \in \mathcal{B}_X$, consider the map $\Phi\colon \mathcal{B}_X \to \mathcal{B}_X$ defined by $\Phi(u) = \tilde{u}$, where $\tilde{u}$ is solution of (5.1.11) with the source term $f = -u\partial_x u$. Then, $u \in \mathcal{B}_X$ is the solution for (5.1.11) if and only if u is a fixed point of Φ. To show this, we need to prove that Φ is a contraction.

If $T < 1$ from (5.2.11), (5.2.12) and Proposition 5.2.7 we get

$$\begin{aligned}\|\Phi u\|_{\mathcal{B}_X} \leq& \sqrt{\delta^{-1}\mathcal{C}}\left(1 + \sqrt{T} + e^{\frac{\xi\|a\|_\infty}{2h}T}\right) \|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \\ &+ \sqrt{\delta^{-1}\mathcal{C}} \cdot C_1 \cdot C\left(2T^{\frac{1}{4}} + T^{\frac{1}{4}} e^{\frac{\xi\|a\|_\infty}{2h}T}\right) \|u\|^2_{\mathcal{B}_X}\end{aligned}$$

and

$$\|\Phi u - \Phi v\|_{\mathcal{B}_X} \leq S\left(1 + \sqrt{T} + e^{\frac{\xi\|a\|_\infty}{2h}T}\right) T^{\frac{1}{4}} \left(\|u\|_{\mathcal{B}_X} + \|v\|_{\mathcal{B}_X}\right) \|u - v\|_{\mathcal{B}_X},$$

where $S = \sqrt{\delta^{-1}\mathcal{C}} \cdot C_1 \cdot C$. Now, consider the application Φ restricted to the closed ball

$$\left\{ u \in \mathcal{B} \colon \ \|u\|_{\mathcal{B}_X} \leq R \right\},$$

with $R > 0$ such that $R = 4\sqrt{\delta^{-1}\mathcal{C}}\, \|(u_0, z_0(\cdot,\cdot,-h(\cdot))\|_{\mathcal{H}}$ and $T > 0$ satisfying

$$T < 1, \quad e^{\frac{\xi \|a\|_\infty}{2h} T} < 2, \quad 2T^{\frac{1}{4}} + T^{\frac{1}{4}} e^{\frac{\xi\|a\|_\infty}{2h} T} < \frac{1}{2\sqrt{\delta^{-1}\mathcal{C}} \cdot C_1 \cdot C_2 R}$$

holds that Φ is a contraction. From Banach's fixed point theorem, application Φ has a unique fixed point. □

5.3 THE DAMPING-DELAYED SYSTEM: OPTIMAL LOCAL RESULT

This section deals with the behavior of the solutions associated with (5.1.5). The first result ensures local stability considering the perturbed system. After that, we are in a position to prove the first main result of this chapter, Theorem 5.1.2.

5.3.1 Preliminaries

We are interested in analyzing the well-posedness of (5.1.5) with total energy associated defined by (5.1.6) that satisfies

$$\begin{aligned} \frac{d}{dt}E_u(t) \leq& \int_0^L\int_0^L b(x,y)u^2\,dx\,dy + \frac{\beta}{2}\int_0^L u_{xx}^2(0,y,t)\,dy \\ &- \frac{\gamma}{2}\int_0^L \left(\partial_x^{-1}u_y(0,y,t)\right)^2 dy - \int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy. \end{aligned} \tag{5.3.1}$$

This implies that the energy is not decreasing, in general, since the term $b(x,y) \geq 0$. So, we consider the following perturbation system

$$\begin{cases} \partial_t u(x,y,t) + \alpha\partial_x^3 u(x,y,t) + \beta\partial_x^5 u(x,y,t) & \\ +\gamma\partial_x^{-1}\partial_y^2 u(x,y,t) + a(x,y)u(x,y,t) & (x,y)\in\Omega,\ t>0. \\ +b(x,y)(\xi u(x,y,t) + u(x,y,t-h)) = f, & \\ u(0,y,t) = u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\ \partial_x u(L,y,t) = \partial_x u(0,y,t) = \partial_x^2 u(L,y,t) = 0, & y\in(0,L),\ t\in(0,T), \\ u(x,L,t) = u(x,0,t) = 0, & x\in(0,L),\ t\in(0,T), \\ u(x,y,0) = u_0(x,y), \quad u(x,y,t) = z_0(x,y,t), & (x,y)\in\Omega,\ t\in(-h,0), \end{cases} \tag{5.3.2}$$

with $f=-\frac{1}{2}\partial_x(u^2(x,y,t)$, which is "close" to (5.1.5), where ξ a positive constant, and now the following energy associated with the perturbed system

$$\begin{aligned}E_u(t)=&\frac{1}{2}\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy\\&+\frac{\xi h}{2}\int_0^L\int_0^L\int_0^1 b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy,\end{aligned}\tag{5.3.3}$$

is decreasing. In fact, note that

$$\begin{aligned}&\frac{d}{dt}E_u(t)\leq\frac{\beta}{2}\int_0^L u_{xx}^2(0,y,t)\,dy-\frac{\gamma}{2}\int_0^L\left(\partial_x^{-1}u_y(0,y,t)\right)^2\,dy\\&-\int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy+\frac{1}{2}\int_0^L\int_0^L(b(x,y)-\xi b(x,y))u^2(x,y,t)\,dx\,dy\\&+\frac{1}{2}\int_0^L\int_0^L(b(x,y)-\xi b(x,y))u^2(x,y,t-h)\,dx\,dy\leq 0,\end{aligned}$$

for $\xi>1$. Note that the system (5.3.2) can be written as a first-order system

$$\begin{cases}\dfrac{\partial}{\partial t}U(t)=AU(t),\\ U(0)=(u_0(x,y),z_0(x,y,-\rho h))\,.\end{cases}\tag{5.3.4}$$

Here $A=A_0+B$ with domain $D(A)=D(A_0)$, A_0 is defined by

$$A_0(u,z)=\left((-\alpha\partial_x^3-\beta\partial_x^5-\gamma\partial_x^{-1}\partial_y^2-a(x,y))u-b(x,y)(\xi u+z(\cdot,\cdot,1)),-h^{-1}\partial_\rho z\right)$$

and the bounded operator B is defined by $B(u,z)=(\xi b(x,y)u,0)$, for all $(u,z)\in\mathcal{H}$. Observe that the system (5.3.4) has a classical solution (see Proposition 5.2.2).

Consider $(e^{A_0t})_{t\geq0}$ the C_0–semigroup associated with A_0. First, let us prove the exponential stability of the system (5.3.2), with $f=0$, by using Lyapunov's approach. To do that, let us consider the following Lyapunov functional

$$V(t)=E_u(t)+\eta V_1(t)+\sigma V_2(t),$$

where η and σ are suitable constants to be fixed later, $E_u(t)$ is the energy defined by (5.3.3), $V_1(t)$ is giving by

$$V_1(t)=\int_0^L\int_0^L xu^2(x,y,t)\,dx\,dy\tag{5.3.5}$$

and $V_2(t)$ is defined by

$$V_2(t)=\frac{h}{2}\int_0^L\int_0^L\int_0^1(1-\rho)b(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy.\tag{5.3.6}$$

Note that $E_u(t)$ and $V(t)$ are equivalent in the following sense

$$E(t)\leq V(t)\leq\left(1+\max\left\{2\eta L,\frac{\sigma}{\xi}\right\}\right)E(t)\tag{5.3.7}$$

Then, we have the next results for exponential stability to the system (5.3.2) with $f=0$.

Proposition 5.3.1. *Let $L>0$. Assume that $a(x,y)$ and $b(x,y)$ belonging to $L^\infty(\Omega)$ are nonnegative functions, $b(x,y)\geq b_0>0$ in ω and $\xi>1$. Then for every $(u_0,z_0(\cdot,\cdot,-h(\cdot)))\in\mathcal{H}$ the energy defined in (5.3.3) decays exponentially. More precisely, there exists two positive constants θ and κ such that $E_u(t)\leq\kappa E_u(0)e^{-2\theta t}$ for all $t>0$. Here,*

$$\theta<\min\left\{\frac{3\alpha\eta}{(1+2\eta L)L^2},\frac{\sigma}{2h(\xi+\sigma)}\right\},\qquad \kappa=1+\max\left\{2\eta L,\frac{\sigma}{\xi}\right\}$$

and η and σ are positive constants such that $\sigma=\xi-1-2L\eta(1+2\xi)$ and $\eta<\frac{\xi-1}{2L(1+2\xi)}$.

Proof. Consider $(u_0,z_0(\cdot,\cdot,-h(\cdot)))\in D(A_0)$. Let u solution of the linear system associated with (5.3.2). Differentiating (5.3.5) and using $(5.3.2)_1$, we obtain

$$\begin{aligned}\frac{d}{dt}V_1(t)=&-3\alpha\int_0^L\int_0^L u_x^2(x,y,t)\,dx\,dy+5\beta\int_0^L\int_0^L u_{xx}^2(x,y,t)\,dx\,dy\\&-\gamma\int_0^L\int_0^L\left(\partial_x^{-1}u_y\right)^2\,dx\,dy-2\int_0^L\int_0^L xa(x,y)u^2(x,y,t)\,dx\,dy\\&-2\int_0^L\int_0^L x\xi b(x,y)u^2(x,y,t)\,dx\,dy\\&-2\int_0^L\int_0^L xb(x,y)u(x,y,t)u(x,y,t-h)\,dx\,dy.\end{aligned}$$

Therefore, for $\theta>0$, η and σ chosen as in the statement of proposition we have $\frac{d}{dt}V(t)+2\theta V(t)\leq 0$, which is equivalent to

$$E_u(t)\leq\left(1+\max\left\{2\eta L,\frac{\sigma}{\xi}\right\}\right)e^{-2\theta t}E(0),\quad\forall t>0,$$

thanks to (5.3.7). □

The next result shows that the energy (5.1.6) associated with the system (5.3.2) with appropriate source term f decays exponentially.

Proposition 5.3.2. *Consider $a(\cdot,\cdot),b(\cdot,\cdot)\in L^\infty(\Omega)$ nonnegative functions, $b(x,y)\geq b_0>0$ in ω and $\xi>1$. So, there exists $\delta>0$ such that if $\|\beta\|\leq\delta$ then, for every initial data $(u_0,z_0(\cdot,\cdot,-h(\cdot))\in\mathcal{H}$ the energy of the system $E_u(t)$, defined in (5.1.5) is exponentially stable.*

Proof. Consider a function v satisfying the system (5.3.2) with $f=0$, initial condition $v(x,y,0)=u_0(x,y)$, and $z^1(1)=u(x,y,t-h)$ where z^1 satisfies

$$\begin{cases}hz_t^1(x,y,\rho,t)+z_\rho^1(x,y,\rho,t)=0, & (x,y)\in\Omega,\ \rho\in(0,1),\ t>0\\ z^1(x,y,0,t)=v(x,y,t), & (x,y)\in\Omega,t>0\\ z^1(x,y,\rho,0)=v(x,y,-\rho h)=z_0(x,y,-\rho h), & (x,y)\in\Omega,\ \rho\in(0,1).\end{cases}\tag{5.3.8}$$

and w satisfying the source system associated with (5.3.2) with $f=\xi b(x,y)v(x,y,t)$, initial condition $w(x,y,0)=0$ and $z^2(1)=u(x,y,t-h)$ where z^2 satisfies

$$\begin{cases} hz_t^2(x,y,\rho,t)+z_\rho^2(x,y,\rho,t)=0, & (x,y)\in\Omega,\ \rho\in(0,1),\ t>1 \\ z^2(x,y,0,t)=w(x,y,t), & (x,y)\in\Omega,\ t>0 \\ z^2(x,y,\rho,0)=0, & (x,y)\in\Omega,\ \rho\in(0,1) \end{cases} \tag{5.3.9}$$

Define $u=v+w$ and $z=z^1+z^2$, then u satisfies the linear system associated with (5.1.5) where $z(1)=u(x,y,t-h)$with z satisfying the equation (5.2.1).

Now, fix $0<\mu<1$ and choose

$$T_0=\frac{1}{2\theta}\ln\left(\frac{2\xi\kappa}{\mu}\right)+1 \implies \kappa e^{-2\theta T_0}<\frac{\mu}{2\xi},$$

where η,σ,θ and κ are given in the Proposition 5.3.1. As $E_v(0)\le\xi E_u(0)$, follows that

$$E_v(T_0)\le\kappa e^{-2\theta T_0}E_v(0)\le\frac{\mu}{2\xi}E_v(0)\le\frac{\mu}{2}E_u(0).$$

Observe that

$$E_u(T_0)\le 2E_v(T_0)+\|(w(\cdot,\cdot,T_0),w(\cdot,\cdot,T_0-h(\cdot)))\|_{\mathcal{H}}.$$

Since A generates a C_0 semi-group we have that

$$\begin{aligned}\|(w(\cdot,\cdot,T_0),w(\cdot,\cdot,T_0-h(\cdot)))\|_{\mathcal{H}} &\le \int_0^{T_0}e^{\frac{1+3\xi}{2}(T_0-s)}\left(\int_0^L|\xi b(x,y)v|^2\,dx\right)^{\frac{1}{2}}ds\\ &\le\sqrt{2\kappa}\xi\|b\|_\infty E_v(0)^{\frac{1}{2}}\int_0^{T_0}e^{\frac{1+3\xi}{2}(T_0-s)}e^{-\theta s}\,ds\\ &\le 2\xi^2\|b\|_\infty^2 e^{(3\xi+1)T_0}\kappa E_v(0),\end{aligned}$$

thanks to the fact that

$$\int_0^{T_0}e^{\frac{1+3\xi}{2}(T_0-s)}e^{-\theta s}\,ds=\frac{e^{\frac{1+3\xi}{2}T_0}-e^{-\theta T_0}}{\frac{1+3\xi}{2}+\theta}\quad\text{and}\quad\frac{1+3\xi}{2}+\theta>2.$$

For $\varepsilon>0$ such that $0<\mu+\varepsilon<1$ and $\|b\|_\infty\le\min\left\{\frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon^3\kappa}e^{\frac{1+3\xi}{2}\left(\frac{1}{2\theta}\ln\left(\frac{2\xi\kappa}{\mu}\right)+2\right)}},1\right\}$, we obtain that,

$$E_u(T_0)\le\mu E_u(0)+2\xi^3\|b\|_\infty^2e^{(1+3\xi)T_0}\kappa E_u(0)<(\mu+\varepsilon)E_u(0).$$

Finally, considering a boot-strap and induction arguments, for T_0 defined by (5.1.10), we can construct another solution that satisfies the linear system associated with (5.3.2) such that the following inequality holds $E_u(mT_0)\le(\mu+\varepsilon)^mE_u(0)$, for all $m\in\mathbb{N}$. Picking $t>T_0$, we note that there exists $m\in\mathbb{N}$ such that $t=mT_0+s$ with $0\le s<T_0$, then

$$E_u(t)\le e^{(2\|b\|_\infty+\nu)T_0}e^{-\nu t}E_u(0),$$

where

$$\nu = \frac{1}{T_0}\ln\left(\frac{1}{\mu+\varepsilon}\right), \tag{5.3.10}$$

showing the result. □

5.3.2 Proof of Theorem 5.1.2

With the previous result in hand, in this section, we are going to prove a local stabilization result with an optimal decay rate. Using the same arguments in Section 5.2.3 we have that (5.1.5) is well-posed. Besides that, we have, by using Gronwall's inequality, that

$$\|(u(\cdot,\cdot,t),u(\cdot,\cdot,t-h(\cdot)))\|_{\mathcal{H}}^2 \leq e^{2\xi\|b\|_\infty t}\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}^2$$

This implies directly that

$$\|u\|_{C([0,T],L^2(\Omega))} \leq e^{\xi\|b\|_\infty T}\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}$$

and

$$\|u\|_{L^2(0,T,L^2(\Omega))} \leq T^{\frac{1}{2}}e^{\xi\|b\|_\infty T}\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}$$

Now, multiplying the system (5.1.5) by $xu(x,y,t)$, integrating by parts in $\Omega\times(0,T)$ we get

$$\begin{aligned}&\frac{3\alpha}{2}\int_0^T\int_0^L\int_0^L u_x^2(x,y,t)\,dx\,dy\,dt-\frac{5\beta}{2}\int_0^T\int_0^L\int_0^L u_{xx}^2(x,y,t)\,dx\,dy\,dt\\&\leq\left(\frac{L}{2}+L\left(\|a\|_\infty+\|b\|_\infty\right)Te^{2\xi\|b\|_\infty T}\right)\|(u_0,z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}}^2\\&+\int_0^T\int_0^L\int_0^L|u(x,y,t)|^3\,dx\,dy\,dt\end{aligned}$$

From

$$\int_0^L\int_0^L u^3(x,y,t)\,dx\,dy \leq \frac{\varepsilon^4}{4}\|u\|_{H_x^2(\Omega)}^2+\frac{3}{4}\left(\frac{CL}{\varepsilon}\right)^{\frac{4}{3}}\|u\|_{L^2(\Omega)}^{\frac{10}{3}}, \tag{5.3.11}$$

and taking $E_u(0)\leq 1$, yields

$$\|u\|_{B_H}^2 \leq \tilde{\mathcal{K}}\left(1+Te^{2\|b\|_\infty T}+Te^{\frac{10}{3}\|b\|_\infty T}+e^{2\|b\|_\infty T}\right)E_u(0),$$

where

$$\tilde{\mathcal{K}} := \frac{1}{\min\{1,3\alpha/2,-5\beta/2\}}\left(\frac{L}{2}+L(\|a\|_\infty+\|b\|_\infty)+\frac{1}{4}\left(\frac{cL}{\tilde{\delta}}\right)^{\frac{4}{3}}\right)$$

Observe that, by definition, $\partial_x^{-1}u(\cdot,\cdot,t)=\varphi(\cdot,\cdot,t)\in H_{x0}^2$ such that $\partial_x\varphi(\cdot,\cdot,t)=u(\cdot,\cdot,t)$. Since $u\in H_{x0}^2$, using Poincaré's inequality, we have that

$$\left\|\partial_x^{-1}u(\cdot,\cdot,t)\right\|_{L^2(\Omega)}=\|\varphi(\cdot,\cdot,t)\|_{L^2(\Omega)}\leq L^2\|\partial_x\varphi(\cdot,\cdot,t)\|_{L^2(\Omega)}=L^2\|u(\cdot,\cdot,t)\|_{L^2(\Omega)}.$$

Therefore,

$$\|u\|_{\mathcal{B}_X}^2 \leq (1+L^2)\tilde{\mathcal{K}}\left(1+Te^{2\|b\|_\infty T}+Te^{\frac{10}{3}\|b\|_\infty T}+e^{2\|b\|_\infty T}\right)E_u(0).$$

Let $(u_0, z_0(\cdot,\cdot,-h(\cdot)))$ be a initial data satisfying $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq r$, where r to be chosen later. The solution u of (5.1.5) can be written as $u = u^1 + u^2$ where u^1 is solution of the linear system associated with (5.1.5) considering the initial data $u^1(x,y,0) = u_0(x,y)$ and $u^1(x,y,t) = z_0(x,y,t)$ and u^2 fulfills the nonlinear system (5.1.5) with initial data $u^2(x,y,0) = 0$ and $u^2(x,y,t) = 0$.

Fix $\mu \in (0,1)$, follows the same ideas introduced by (CAPISTRANO-FILHO; GONZALEZ MARTINEZ, 2024, Appendix A), there exists, $T_1 > 0$ such that

$$e^{(2\|b\|_\infty+\nu)T_0-\nu T_1} < \frac{\eta}{2} \Longleftrightarrow T_1 > -\frac{1}{\nu}\ln\left(\frac{\eta}{2}\right)+\left(\frac{2\|b\|_\infty}{\nu}+1\right)T_0$$

with ν is defined by (5.3.10) satisfying $E_{u^1}(T_1) \leq \frac{\mu}{2}E_{u^1}(0)$. This implies together with (5.3.11) that

$$\begin{aligned}
E_u(T_1) \leq& \mu E_u(0) + \left\|(u^2(\cdot,\cdot,T_1), u^2(\cdot,\cdot,T_1-h(\cdot)))\right\|_{\mathcal{H}}^2\\
\leq& \mu E_u(0) + e^{(1+3\xi)T_1}\|uu_x\|^2_{L^1(0,T_1,L^2(\Omega))}\\
\leq& \mu E_u(0) + e^{(1+3\xi)T_1}C_1^2C_2^2T^{\frac{1}{2}}\|u\|^4_{\mathcal{B}_X}\\
\leq& (\mu+\mathcal{R})E_u(0),
\end{aligned}$$

where

$$\mathcal{R} = e^{(1+3\xi)T_1}C_1^2C_2^2T_1^{\frac{1}{2}}(1+L^2)^2\tilde{\mathcal{K}}^2\left(1+T_1e^{2\|b\|_\infty T_1}+T_1e^{\frac{10}{3}\|b\|_\infty T_1}+e^{2\|b\|_\infty T_1}\right)^2 r.$$

Therefore, given $\varepsilon > 0$ such that $\mu + \varepsilon < 1$, we take $r > 0$ such that

$$r < \frac{\varepsilon}{e^{(1+3\xi)T_1}C_1^2C_2^2T_1^{\frac{1}{2}}(1+L^2)^2\tilde{\mathcal{K}}^2\left(1+T_1e^{2\|b\|_\infty T_1}+T_1e^{\frac{10}{3}\|b\|_\infty T_1}+e^{2\|b\|_\infty T_1}\right)^2}$$

to obtain $E_u(T_1) \leq (\mu+\varepsilon)E_u(0)$, with $\mu+\varepsilon < 1$. Using a prolongation argument, first for the time $2T_1$ and after for mT_1, the result is obtained. □

5.4 μ_i-SYSTEM: STABILITY RESULTS

The main objective of this section is to prove the local and global exponential stability for the solutions of (5.1.11) using two different approaches.

5.4.1 Local stabilization: Proof of Theorem 5.1.3

Consider the Lyapunov's functional $V(t) = E_u(t) + \eta V_1(t) + \sigma V_2(t)$, where $E_u(t)$ is defined by (5.1.12), $V_1(t)$ defined by (5.3.5) and

$$V_2(t) = \frac{\xi}{2}\int_0^L\int_0^L\int_0^1 (1-\rho)a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy. \tag{5.4.1}$$

Using the same argument as in the proof of Proposition 5.3.1 we see that

$$\begin{aligned}\frac{d}{dt}V(t) + 2\theta V(t) \le& \left(\frac{\mu_2}{2} - \frac{\xi}{2h} + \eta L\mu_2\right)\int_0^L\int_0^L a(x,y)u^2(x,y,t-h)\,dx\,dy\\ &+\left(\theta\xi - \frac{\xi}{2h}\sigma + \theta\sigma\xi\right)\int_0^L\int_0^L\int_0^1 a(x,y)u^2(x,y,t-\rho h)\,d\rho\,dx\,dy\\ &+\left(\frac{\mu_2}{2} - \mu_1 + \frac{\xi}{2h} + 2\eta L\mu_1 + \eta L\mu_2 + \frac{\xi}{2h}\sigma\right)\int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy\\ &+(\theta + 2\theta\eta L)\int_0^L\int_0^L u^2(x,y,t)\,dx\,dy - 3\alpha\eta\int_0^L\int_0^L u_x^2(x,y,t)\,dx\,dy\\ &+\frac{2}{3}\eta\int_0^L\int_0^L u^3(x,y,t)\,dx\,dy + 5\beta\eta\int_0^L\int_0^L u_{xx}^2(x,y,t)\,dx\,dy,\end{aligned} \tag{5.4.2}$$

for all $\theta > 0$. Note that, thanks to Theorem 5.1.1 we have

$$\int_0^L\int_0^L u^3(x,y,t)\,dx\,dy \le \frac{1}{4}\left\|u_{xx}\right\|^2_{L^2(\Omega)} + \frac{3}{4}(CL)^{\frac{4}{3}}r^{\frac{4}{3}}\left\|u\right\|^2_{L^2(\Omega)}.$$

Putting this previous inequality in (5.4.2), and using Poincaré's inequality and (5.1.16), we get

$$\begin{aligned}\frac{d}{dt}V(t)+2\theta V(t) \le& \left(5\beta\eta + \frac{1}{6}\eta\right)\int_0^L\int_0^L u_{xx}^2(x,y,t)\,dx\,dy\\ &+\left(\theta(1+2\eta L)L^2 + \frac{1}{2}\eta C^{\frac{4}{3}}r^{\frac{4}{3}}L^{\frac{10}{3}} - 3\alpha\eta\right)\int_0^L\int_0^L u_x^2(x,y,t)\,dx\,dy.\end{aligned}$$

Consequently, taking the previous constant as in the statement of the theorem we have that

$$V'(t) + 2\gamma V(t) \le 0. \tag{5.4.3}$$

Finally, from the following relation $E(t) \le V(t) \le (1 + \max\{2\eta L, \sigma\})\,E(t)$ and (5.4.3), we obtain

$$E(t) \le V(t) \le e^{-2\theta t}V(0) \le (1+\max\{2\eta L,\sigma\})e^{-2\sigma t}E(0), \quad \forall t > 0,$$

and Theorem 5.1.3 is proved. □

5.4.2 Global stabilization: Proof of Theorem 5.1.4

As is classical in control theory, Theorem 5.1.4 is a consequence of the following observability inequality

$$E_u(0) \leq \mathcal{C}\left(\int_0^T\int_0^L \partial_x^2 u(0,y,t)^2\,dy + \int_0^T\int_0^L (\partial_x^{-1}\partial_y u(0,y,t))^2\,dy\,dt + \int_0^T\int_0^L\int_0^L a(x,y)(u^2(x,y,t-h)+u^2(x,y,t)\,dx\,dy\,dt\right) \tag{5.4.4}$$

Observe that using the same ideas of (5.2.10), we get

$$\begin{aligned} T\|u_0\|^2_{L^2(\Omega)} \leq{}& \|u\|^2_{L^2(0,T,L^2(\Omega))} - \beta T\int_0^T\int_0^L \partial_x^2 u(0,y,t)^2\,dy\,dt\\ &+\gamma T\int_0^T\int_0^L \left(\partial_x^{-1}\partial_y u(0,y,t)\right)^2\,dy\,dt\\ &+T(2\mu_1+\mu_2)\int_0^T\int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy\,dt\\ &+T\int_0^T\int_0^L\int_0^L a(x,y)\mu_2 u^2(x,y,t-h)\,dx\,dy\,dt \end{aligned} \tag{5.4.5}$$

Moreover, multiplying $(5.2.2)_5$ by $a(x,y)\xi z(x,y,\rho,s)$, integrating in $\Omega\times(0,1)\times(0,T)$ and taking in account that $z(x,y,\rho,t) = u(x,y,t-\rho h)$ we obtain

$$\begin{aligned} &\int_0^L\int_0^L\int_0^1 a(x,y)z^2(x,\rho,0)\,d\rho\,dx\,dy\\ &\leq \frac{1}{hT}\int_0^T\int_0^L\int_0^L a(x,y)u^2(x,y,t)\,dx\,dy\,dt\\ &+\left(\frac{1}{Th}+\frac{1}{h}\right)\int_0^T\int_0^L\int_0^L a(x,y)u^2(x,y,t-h)\,dx\,dy\,dt \end{aligned} \tag{5.4.6}$$

Gathering (5.4.7) and (5.4.5), we see that to show (5.4.4) is sufficient to prove that for any T and $R>0$, there exists $K := K(R,T) > 0$ such that

$$\begin{aligned} \|u\|^2_{L^2(0,T,L^2(0,L))} \leq{}& K\left(\int_0^T\int_0^L \partial_x^2 u(0,y,t)^2\,dy\right.\\ &+\int_0^T\int_0^L(\partial_x^{-1}\partial_y u(0,y,t))^2\,dy\,dt + \int_0^T\int_0^L\int_0^L a(x,y)u^2\,dx\,dy\,dt\\ &\left.+\int_0^T\int_0^L\int_0^L a(x,y)u^2(x,y,t-h)\,dx\,dy\,dt\right) \end{aligned} \tag{5.4.7}$$

holds for all solutions of (5.1.11) with initial data $\|(u_0, z_0(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq R$.

To prove it, let us argue by contradiction. Suppose that (5.4.7) does not holds, then there exists a sequence $(u^n)_n \subset \mathcal{B}_X$ of solutions of (5.1.11) with initial data $\|(u_0^n, z_0^n(\cdot,\cdot,-h(\cdot)))\|_{\mathcal{H}} \leq R$ such that $\lim_{n\to\infty}\frac{\|u^n\|^2_{L^2(0,T,L^2(\Omega))}}{B(u^n)} = +\infty$ where

$$\begin{aligned} B(u^n) ={}& \int_0^T\int_0^L |\partial_x^2 u^n(0,y,t)|^2\,dy + \int_0^T\int_0^L |(\partial_x^{-1}\partial_y u^n(0,y,t))|^2\,dy\,dt\\ &+\int_0^T\int_0^L\int_0^L a(x,y)\left(|u^n(x,y,t)|^2 + |u^n(x,y,t-h)|^2\right)\,dx\,dy\,dt. \end{aligned}$$

Let $\lambda_n = \|u^n\|_{L^2(0,T,L^2(\Omega))}$ and $v^n(x,y,t) = 1/\lambda_n u^n(x,y,t)$, then v^n satisfies (5.1.11)$_1$ with the following boundary conditions

$$\begin{cases} v^n(0,y,t) = v^n(L,y,t) = 0, & y \in (0,L), t>0, \\ \partial_x v^n(L,y,t) = \partial_x v^n(0,y,t) = \partial_x^2 v^n(L,y,t) = 0, & y \in (0,L), t>0, \\ v^n(x,L,t) = v^n(x,0,t) = 0, & x \in (0,L), t>0 \\ v^n(x,y,0) = \frac{u_0}{\lambda_n}(x,y), \quad v^n(x,y,t) = \frac{z_0}{\lambda_n}(x,y,t), & (x,y) \in \Omega, \ t \in (-h,0) \\ \|v^n\|^2_{L^2(0,T,L^2(\Omega))} = 1 \end{cases} \tag{5.4.8}$$

and $B(v^n) \to 0$ as $n \to \infty$. Therefore, we have from (5.4.5) that

$$\|v^n(\cdot,\cdot,t)\|^2_{L^2(\Omega)} \leq \frac{1}{T}\|v^n\|^2_{L^2(0,T,L^2(\Omega))} + cB(v^n) \tag{5.4.9}$$

which together with (5.4.8)$_6$ and $B(v^n) \to 0$ gives that $(v^n(\cdot,\cdot,0))_n$ is bounded in $L^2(\Omega)$. Additionally to that, the following inequality (see (5.4.6))

$$\int_\Omega \int_0^1 a(x,y)\frac{1}{\lambda_n^2}|z^n(x,\rho,0)|^2 \, d\rho\, dx\, dy \leq \frac{1}{hT}\int_0^T \int_\Omega a(x,y)|v^n(x,y,t)|^2 \, dx\, dy\, dt$$
$$+ \left(\frac{1}{hT} + \frac{1}{h}\right)\int_0^T \int_\Omega a(x,y)|v^n(x,y,t-h)|^2 \, dx\, dy\, dt$$

ensures that $\left(\sqrt{a(x,y)}v^n(\cdot,\cdot,-h(\cdot))\right)_n$ is bounded in $L^2(\Omega\times(0,1))$ and from (5.2.8), $(\lambda_n)_n \subset \mathbb{R}$ is bounded. On the other hand, as a consequence of Proposition 5.2.5 we have that $(v^n)_n$ is bounded in $L^2(0,T,H^2_x(\Omega))$. Now, using Theorem 5.1.1, we get

$$\|v^n v^n_x\|_{L^2(0,T,L^1(\Omega))} \leq C^2\|v^n\|^{\frac{3}{2}}_{L^\infty(0,T,L^2(\Omega))}\|v^n\|_{L^2(0,T,H^2_x(\Omega))}$$

and $(v^n v^n_x)_n$ is bounded in $L^2(0,T,L^1(\Omega))$. Defining $\partial_y v^n = \partial_x \varphi^n$, and using once again Theorem 5.1.1 we have $\left\|\partial_x^{-1}v^n_{yy}\right\|_{L^2(\Omega)} \leq C^2\|v^n_x\|_{L^2(\Omega)} < \infty$. Consequently, using the Cauchy-Schwarz inequality

$$\left|\left\langle \partial_x^{-1}v^n_{yy}, \xi\right\rangle_{H^{-3}(\Omega),H^3_0(\Omega)}\right| \leq \left\|\varphi^n_y\right\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \leq C^2\|v^n_x\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)}.$$

Observe that $(v^n)_n$ bounded in $L^2(0,T;H^2_x(\Omega))$ implies, in particular, $(v^n_x)_n$ is bounded in $L^2(0,T,L^2(\Omega))$, so

$$\left\|\partial_x^{-1}v^n_{yy}\right\|^2_{L^2(0,T;H^{-3}(\Omega))} \leq C\int_0^T \|v^n_{xx}\|_{L^2(\Omega)}\|v^n\|_{L^2(\Omega)}\, dt \leq \frac{C}{2}\|v^n\|_{L^2(0,T,H^2_x(\Omega))},$$

where we used that $H^2_x(\Omega) \subset L^2(\Omega)$.

Thus, the previous analysis ensures that

$$\begin{aligned} v_t^n(x,y,t) =& -\alpha v_{xxx}^n(x,y,t) + \beta v_{xxxxx}^n(x,y,t) + \gamma \partial_x^{-1} v_{yy}^n(x,y,t) \\ &+ \lambda_n v^n(x,y,t) v_x^n(x,y,t) + a(x,y)\left(\mu_1 v^n(x,y,t) + \mu_2 v^n(x,y,t-h)\right), \end{aligned}$$

is bounded in $L^2(0,T,H^{-3}(\Omega))$, which together with a classical compactness results[5], give us the existence of a sequence $(v^n)_n$ relatively compact in $L^2(0,T,L^2(\Omega))$, that is, there exists a subsequence, still denoted $(v^n)_n$,

$$v_n \to v \text{ in } L^2(0,T,L^2(\Omega)) \tag{5.4.10}$$

with $\|v\|_{L^2(0,T,L^2(\Omega))} = 1$.

Finally, from weak lower semicontinuity of convex functional, we obtain

$$v(x,y,t) = 0 \in \omega \times (0,T) \text{ and } \partial_x^2 v(0,y,t) = 0 \text{ in } (0,L)\times(0,T). \tag{5.4.11}$$

Since $(\lambda_n)_n$ is bounded, we can extract a subsequence denoted $(\lambda_n)_n$ which converges to $\lambda \geq 0$.

We claim that $\partial_x^{-1}\partial_y^2 v^n \to \partial_x^{-1}\partial_y^2 v$ in $L^2(0,T,H^{-2}(\Omega))$. In fact, from definition of $\mathcal{B}_X$ we have $\partial_x^{-1} v^n = \varphi^n$ where $\partial_x \varphi^n = v^n$, $v^n(\cdot,\cdot,t) \in H_{x0}^1(\Omega)$ and $\varphi^n(\cdot,\cdot,t) \in H_{x0}^1(\Omega)$. Since $\partial_x^{-1}\partial_y^2 v^n = \partial_y^2 \varphi^n$ we obtain

$$\begin{aligned} &\left\|\partial_x^{-1} v_{yy}^n(\cdot,\cdot,t) - \partial_x^{-1} v_{yy}(\cdot,\cdot,t)\right\|_{H^{-2}(\Omega)} = \left\|\varphi_{yy}^n(\cdot,\cdot,t) - \varphi_{yy}(\cdot,\cdot,t)\right\|_{H^{-2}(\Omega)} \\ &\leq c\left\|\varphi^n(\cdot,\cdot,t) - \varphi(\cdot,\cdot,t)\right\|_{L^2(\Omega)} \leq cL^2\left\|\varphi_x^n(\cdot,\cdot,t) - \varphi_x(\cdot,\cdot,t)\right\|_{L^2(\Omega)} \\ &= cL^2\left\|v^n(\cdot,\cdot,t) - v(\cdot,\cdot,t)\right\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, the desired convergence follows from the previous inequality and convergence (5.4.10).

Therefore, from the above convergences $v(x,y,t)$ satisfies (5.4.11) and (5.1.11) with the following conditions

$$\begin{cases} v(0,y,t) = v(L,y,t) = 0, & y \in (0,L),\ t > 0 \\ \partial_x v(L,y,t) = \partial_x v(0,y,t) = \partial_x^2 v(L,y,t) = 0, & y \in (0,L),\ t > 0 \\ v(x,L,t) = v(x,0,t) = 0, & x \in (0,L),\ t > 0 \\ \|v\|_{L^2(0,T,L^2(\Omega))} = 1. \end{cases} \tag{5.4.12}$$

[5] See (SIMON, 1987).

Thus, for $\lambda = 0$ we obtain $v = 0$, thanks to Holmgren's uniqueness theorem, which is a contradiction with the fact that $\|v\|_{L^2(0,T,L^2(\Omega))} = 1$. Otherwise, if $\lambda > 0$, we can show that $v \in L^2(0, T, H_x^5(\Omega) \cap X^2(\Omega))$ and applying (MOURA; NASCIMENTO; SANTOS, 2022, Theorem 1.2), follows that $u \equiv 0$ in $\Omega \times (0, T)$, achieving Theorem 5.1.4. $\square$

6 COMMENTS ABOUT FUTURE RESEARCH

6.1 OPEN PROBLEMS FOR THE BOUSSINESQ KDV-KDV TYPE SYSTEM

There are some points to be raised.

- **A time-varying delay feedback.** The main difficulty when dealing with the problem (2.1.7) is how to prove the global well-posedness. This is due to the lack of the L^2 a priori estimate. It is worth mentioning that, in this case, the semigroup theory or multipliers method cannot be applied, due to a restriction of "controlling" the solutions of the transport equation in specific norms. We believe that a variation of the approach introduced by Bona *et al.* in (BONA; SUN; ZHANG, 2003) can be adapted. However, this remains a promising research avenue, and the stabilization problem for the nonlinear system (2.1.7) needs to be investigated.

- **Variation of feedback-law.** Considering two internal damping mechanisms and a linear combination of boundary damping and time-varying delay feedback, a similar result of our work can be proved. Due to the restriction of the well-posedness problem, we cannot remove the boundary damping. However, an open problem is to remove one internal damping mechanism and make $\beta = 0$. We believe that the Carleman estimate shown in (BARCENA PETISCO; GUERRERO; PAZOTO, 2022) can be used to investigate all these cases.

- **Optimal decay rate.** Note that the Proposition 2.3.1 gives the optimality of λ for the stabilization problem related to the linear system associated with (2.1.7). In turn, it is still an open problem to obtain an optimal decay rate for both the linear and nonlinear problems without additional conditions for the parameters α and β.

6.2 SOME COMMENTS ABOUT THE HIROTA-SATSUMA SYSTEM

- **Exploration of Saturation Input Feedback Mechanisms** An interesting extension of the current work on the Hirota-Satsuma system involves investigating the effects of alternative feedback mechanisms, such as saturation input feedback. Unlike the time-delayed feedback mechanisms, saturation input introduces nonlinear constraints that can model some limitations in real-world applications. This approach is particularly relevant in systems where the input cannot exceed a certain threshold, such as in mechanical and electrical systems with actuator constraints. The ongoing work focuses on the design of stabilization strategies under these saturation feedback conditions, and preliminary results suggest potential advantages in achieving faster stabilization rates and improved robustness against perturbations. This work is still in progress and aims to provide new insights into designing another closed-loop feedback mechanism for the Hirota-Satsuma system.

- **Internal stabilization via unique continuation property.** One potential area for future research involves exploring ways to stabilize the Hirota-Satsuma system using internal feedback mechanisms instead of relying on boundary feedback. This approach requires using unique continuation properties that allow extending local solution properties to a wider domain. We plan to apply the unique continuation property similar to the one developed by Bhandari (See (BHANDARI, 2024)) to achieve exponential stabilization by leveraging the system's internal structure. This problem opens up new possibilities for exploring complex control strategies contributing to a better understanding of the internal dynamics of the Hirota-Satsuma system and its applications in various physical models.

6.3 PROBLEMS TO BE ADDRESSED ON THE BOUNDARY OBSERVATION FOR KP-II

- **The nonlinear problem.** Observe that due to the lack of regularity, we can not address the nonlinear problem with less regular initial data, that is,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + \partial_x^{-1}u_{yy} = 0, & (x,y)\in\Omega,\ t\in(0,T), \\ u(0,y,t) = u(L,y,t) = 0, \quad u_x(L,y,t) = h(y,t), & y\in(0,L),\ t\in(0,T) \\ u(x,0,t) = u(x,L,t) = 0, & x\in(0,L),\ t\in(0,T), \\ u(x,y,0) = u_0(x,y), & (x,y)\in\Omega. \end{cases} \tag{6.3.1}$$

So, in the nonlinear context, the natural questions are:

($\mathcal{A}$) Is the nonlinear problem (6.3.1) well-posed in the class $C\left([0,T];L^2(\Omega)\right)\cap L^2(0,T;H^1_{x0}(\Omega))$?

($\mathcal{B}$) Is the nonlinear system (6.3.1) controllable for any $L>0$?

($\mathcal{C}$) Is there a feedback lack such that the nonlinear system (6.3.1) is exponentially stable?

- **Controllability in the critical lengths.** The boundary control problem for a spatial domain with a critical length $\mathcal{R}^\star$ for which the linearized control system is not controllable arises. We believe that employing a similar approach as in (CERPA, 2007; CORON; CRÉPEAU, 2004) the nonlinear term gives the local exact controllability around the origin provided that the time of control is large enough.

6.4 COMMENTS ON THE INTERNAL ISSUES FOR THE K-KP-II

- **Internal control via Carleman estimates** An open issue to research for the Kawahara-Kadomtsev-Petviashvili (K-KP) equation will focus on addressing the internal control problem using Carleman estimates. In particular, for proving unique continuation properties and obtaining observability inequalities, which are crucial for internal controllability results. However, applying Carleman estimates to the K-KP equation poses significant challenges due to the involvement of two spatial dimensions and the presence of a nonlocal term. The nonlocal term complicates the derivation of suitable Carleman weights, and the two-dimensional nature of the equation adds further difficulty in constructing appropriate control functions.
- **Moving controls and adaptation strategies.** Another promising avenue for future research involves exploring the use of moving controls for the internal stabilization of the K-KP equation. Moving controls are dynamic control strategies where the control region changes over time, which can be particularly effective in managing systems governed by PDEs with complex dynamics. The main challenge will require a careful analysis on understanding of how control regions can be optimized over time to achieve the controllability property.
- **Stabilization Using Different Feedback Mechanisms.** Additionally, the K-KP-II equation could explore the stabilization using other feedback mechanisms, such as nonlinear or state-dependent feedback. For example, introducing feedback mechanisms that depend nonlinearly on the state or its derivatives could provide more flexibility in shaping the response of the system and enhancing robustness against disturbances. This approach could involve saturation-type feedback or delay-dependent feedback, where the closed-loop feedback mechanism action is a nonlinear function of the system state. These mechanisms can introduce additional complexity in the mathematical analysis, especially

in a multi-dimensional setting with a nonlocal term, as seen in the presented chapter for the K-KP-II equation. However, they may offer significant benefits in terms of stabilization speed and robustness that are better suited to complex, real-world applications where simple linear feedback may not be sufficient.

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APPENDIX A – SOME GENERAL RESULTS

A.1 BASIC THEORY

In this section, we are focused on studying and recalling some notions and tools used throughout this thesis.

A.1.1 Distributions and Sobolev spaces

Throughout this subsection we are inspired in (ADAMS, 1975; MEDEIROS; MIRANDA, 1989; BREZIS, 2011; SCHWARTZ, 1966) and in the references therein.

We refer to a *domain*, denoted by Ω, for a nonempty open set in n-dimensional real space $\mathbb{R}^n$. We will focus on the differentiability and integrability of functions defined on the set Ω. Given $n \in \mathbb{N}$, if $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} \dots x_n^{\alpha^n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Moreover, if $D_j = \frac{\partial}{\partial x_j}$, then

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha^n}$$

denotes a differential operator of order $|\alpha|$. Notice that, $D^{(0,\dots,0)}u = u$.

If α and β are two multi-indices, we say that $\beta \leq \alpha$ provided $\beta_j \leq \alpha_j$ for $1 \leq j \leq n$. Then $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. Moreover, we also denote $\alpha! = \alpha_1! \dots \alpha_n!$ and if $\beta \leq \alpha$,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

With this, for u, v regular enough functions we state the Leibniz rule given by

$$D^\alpha(uv) = \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u(x) D^{\alpha-\beta} v(x).$$

Let $\Omega \subset \mathbb{R}^n$, we denote by $\overline{\Omega}$ the closure of Ω in $\mathbb{R}^n$. Let u a function defined on Ω, we describe the *support* of u to be the set

$$\operatorname{supp}(u) = \overline{\{x \in \Omega \colon u(x) \neq 0\}}.$$

We say that u has *compact support* in Ω if $\operatorname{supp}(u)$ is compact.

For any $m \in \mathbb{N}$, let $C^m(\Omega)$ denote the vector spaces

$$C^m(\Omega) = \{\phi\colon D^\alpha \phi, |\alpha| \leq m \text{ is continuous on } \Omega\}.$$

We denote $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$. The subspaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consists of all those functions in $C(\Omega)$ and $C^\infty(\Omega)$, respectively, that have compact support in Ω.

Example A.1.1. Let $\theta\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\theta(x) = \begin{cases} e^{-x^{-2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

then $\theta \in C^\infty(\mathbb{R})$.

The example above in $\mathbb{R}$ is a motivation to build classical examples of C_0^∞ functions in $\mathbb{R}^n$,

Example A.1.2. Let $\Omega \subset \mathbb{R}^n$ be an open such that $B_1(0) = \{x \in \mathbb{R}^n; \|x\| < 1\}$ is compactly contained in Ω. Let us consider $f : \Omega \to \mathbb{R}$ such that

$$f(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \|x\| < 1, \\ 0, & \|x\| \geq 1 \end{cases}$$

where $x = (x_1, \ldots, x_n)$ and $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$. We have $f \in C^\infty(\Omega)$ and $\mathrm{supp}(f) = \overline{B_1(0)}$ is compact, that is, $f \in C_0^\infty(\Omega)$.

Definition A.1.3. We say that $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$, denoted by $\varphi_n \to \varphi$, if

(i) There exists a compact K of Ω such that $\mathrm{supp}(\varphi) \subset K$ and $\mathrm{supp}(\varphi_n) \subset K,\ \forall n \in \mathbb{N}$;

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in K, for all multi-index α.

By $D(\Omega)$ we represent the space $C_0^\infty(\Omega)$, equipped with the convergence defined above and will be called *space of test functions on* Ω.

We define a *distribution over* Ω, as defined by Schwartz, to any linear form T over $\mathcal{D}(\Omega)$ that is continuous in the sense of convergence defined above, that is, for every sequence $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ that converges to $\varphi \in \mathcal{D}(\Omega)$, then $(\langle T, \varphi_n\rangle)_n \subset \mathbb{K}$ converges to $\langle T, \varphi\rangle \in \mathbb{K}$[1].

[1] Observe that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\langle T, \varphi\rangle$ is the evaluation of T in φ, i.e. $T(\varphi)$

Remark A.1.4. The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the space of (Schwartz) distributions on Ω. $\mathcal{D}'(\Omega)$ is given the weak-star topology as the dual of $\mathcal{D}(\Omega)$, and is a locally convex topological vector space (TVS) with that topology.

The following example of scalar distributions plays a key role in the theory. First, recall that a function u defined a.e on Ω is said to be *locally integrable* on Ω provided $u \in L^1(\omega)$ for every open ω compactly contained in Ω and we write $u \in L^1_{\text{loc}}(\Omega)$.

Example A.1.5. For every $u \in L^1_{loc}(\Omega)$ we can associate a distribution $T_u : \mathcal{D}(\Omega) \to \mathbb{R}$, defined by

$$\langle T_u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx,$$

that will be uniquely determined by u.

Remark A.1.6. Not every distribution $T \in \mathcal{D}'(\Omega)$ is on the form T_u for some $u \in L^1_{\text{loc}}(\Omega)$. Indeed, if $0 \in \Omega$, there can be no locally integrable function δ over Ω such that for every $\varphi \in \mathcal{D}(\Omega)$

$$\int_\Omega \delta(x)\varphi(x)\,dx = \varphi(0).$$

However, the linear functional δ defined on $\mathcal{D}(\Omega)$ by $\langle \delta, \varphi \rangle = \varphi(0)$ can be shown that is continuous, and hence a distribution on Ω. It is called *Dirac distribution*.

Lemma A.1.7 (Du Bois Raymond)**.** *Let $u \in L^1_{loc}(\Omega)$. Then*

$$\int_\Omega u(x)\varphi(x)dx = 0, \ \forall \varphi \in \mathcal{D}(\Omega),$$

if and only if $u = 0$ almost everywhere in Ω.

Let α a multi-index and $\varphi \in \mathcal{D}(\Omega)$, if $u \in C^{|\alpha|}(\Omega)$, then integrating by parts $|\alpha|$ times leads to

$$\int_\Omega (D^\alpha u(x))\,\varphi(x)\,dx = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\varphi(x)\,dx.$$

This motivates the definition of the derivative $D^\alpha T$ of a distribution $T \in \mathcal{D}'(\Omega)$

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|}\langle T, D^\alpha \varphi \rangle, \ \forall \varphi \in \mathcal{D}(\Omega).$$

It is notable that:

- Each distribution T over Ω has derivatives of all orders.

- $D^\alpha T$ is a distribution over Ω, where $T \in \mathcal{D}'(\Omega)$. In fact, it is easily seen that $D^\alpha T$ is linear. Now, we show that it is continuous, consider $(\varphi_n)_n \subset \mathcal{D}(\Omega)$ converging to $\varphi \in \mathcal{D}(\Omega)$. Thus,

$$|\langle D^\alpha T, \varphi_n\rangle - \langle D^\alpha T, \varphi\rangle| \leq |\langle T, D^\alpha \varphi_n - D^\alpha \varphi\rangle| \Rightarrow 0$$

  when $n \to \infty$.

- The map $D^\alpha : \mathcal{D}'(\Omega) \Rightarrow \mathcal{D}'(\Omega)$, such that $T \mapsto D^\alpha T$, is linear and continuous in the sense of convergence defined in $\mathcal{D}'(\Omega)$.

Example A.1.8. Let $u\colon \mathbb{R} \to \mathbb{R}$ the *Heaviside* function defined by

$$u(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Notice that $u \in L^1_{\text{loc}}(\mathbb{R})$ but $u' = \delta \notin L^1_{\text{loc}}(\mathbb{R})$. Indeed,

$$\langle u', \varphi\rangle = -\langle u, \varphi'\rangle = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta, \varphi\rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$.

The example A.1.8 shows that the derivative of a $L^1_{\text{loc}}(\Omega)$ function is not, in general, besides to $L^1_{\text{loc}}(\Omega)$. This motivates the well-recognized definition of Sobolev spaces that will be introduced later. First, for $1 \leq p < \infty$, we denote by $L^p(\Omega)$ the space of (classes of) functions $u\colon \Omega \to \mathbb{R}$ measurable in Ω such that $|u|^p$ is Lebesgue integrable in Ω. This is a Banach space with the norm

$$\|u\|^p_{L^p(\Omega)} = \int_\Omega |u(x)|^p\, dx.$$

When $p = \infty$, $L^\infty(\Omega)$ consists of all essentially bounded functions in Ω equipped with the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)| = \inf\left\{C\colon |v(x)| \leq C \text{ a.e. in } \Omega\right\}.$$

When $p = 2$ we have a Hilbert space $L^2(\Omega)$ with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\, dx,$$

and induced norm

$$\|u\|^2_{L^2(\Omega)} = \int_\Omega |u(x)|^2\, dx.$$

Given an integer $m > 0$, by $W^{m,p}(\Omega),\ 1 \leq p \leq \infty$, represents the Sobolev space of order m, over Ω of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$, for every multi-index α, with $|\alpha| \leq m$. $W^{m,p}(\Omega)$ is a vector space, whatever $1 \leq p < \infty$. Considering the following norm

$$\|u\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p \, dx$$

when $1 \leq p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \operatorname{supp}_{x\in\Omega} |D^\alpha u(x)|$$

when $p = \infty$, then Sobolev spaces $W^{m,p}(\Omega)$ is a Banach space.

When $p = 2$, the space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which equipped with the inner product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) \, dx$$

is a Hilbert space.

Let us denote by $W_0^{m,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ relative to the norm of the space $W^{m,p}(\Omega)$, i.e.

$$\overline{C_0^\infty(\Omega)}^{W^{m,p}(\Omega)} = W_0^{m,p}(\Omega).$$

Whenever Ω is bounded at least in one direction x_i of $\mathbb{R}^n$, the norm of $W_0^{m,p}(\Omega)$ is given by

$$\|u\|^p_{W_0^{m,p}(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \, dx.$$

We denote by $W^{-m,q}(\Omega)$ the topological dual of $W_0^{m,p}(\Omega)$, where $1 \leq p < \infty$ and q is the Hölder conjugated index of p[2]. We write $H^{-m}(\Omega)$ to denote the topological dual of $H_0^m(\Omega)$.

Let X and Y be two normed vector spaces such that $X \subseteq Y$. If the inclusion map $i\colon x \in X \mapsto x \in Y$ is continuous for every $x \in X$, then X is said to be *continuously embedded* in Y and will be denoted $X \hookrightarrow Y$.

Theorem A.1.9 (Sobolev embeddings). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and consider an integer $m \geq 1$ and $1 \leq p < \infty$. Then,*

(i) If $\frac{1}{p} - \frac{m}{n} > 0$, then $W^{m,p} \hookrightarrow L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{p} - \frac{m}{n}$;

(ii) If $\frac{1}{p} - \frac{m}{n} = 0$, then $W^{m,p} \hookrightarrow L^q(\Omega)$, for all $p \leq q < +\infty$;

(iii) If $\frac{1}{p} - \frac{m}{n} < 0$, then $W^{m,p} \hookrightarrow L^\infty(\Omega)$.

[2] q is said to be the Hölder conjugated index of $1 \leq p \leq \infty$ if $\frac{1}{p} + \frac{1}{q} = 1$

Theorem A.1.10 (Rellich-Kondrachov). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and consider $n \geq 2$. Then,*

(i) If $p < n$, then $W^{1,p(\Omega)}$ is compactly embedding in $L^q(\Omega)$, for all $1 \leq q < \frac{np}{n-p}$;

(ii) If $p = n$, then $W^{1,p}(\Omega)$ is compactly embedding in $L^q(\Omega)$, for all $1 \leq q < q$;

(iii) If $p > n$ then $W^{1,p}(\Omega)$ is compactly embedding in $C^0(\overline{\Omega})$.

We will denote by $L^p(0,T;X),\ 1 \leq p < \infty$, the space of Banach of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X^p$ is Lebesgue integrable in $(0,T)$, with the norm

$$\|u(t)\|^p_{L^p(0,T;X)} = \int_0^T \|u(t)\|_X^p \, dt.$$

Furthermore, if $p = \infty$, $L^\infty(0,T;X)$ represents the Banach space of (classes of) functions u, defined in $(0,T)$ with values in X, that are strongly measurable and $\|u(t)\|_X$ has supreme essential finite in $(0,T)$, with the norm

$$\|u(t)\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t \in (0,T)} \|u(t)\|_X.$$

Remark A.1.11. When $p = 2$ and X is a Hilbert space, the space $L^2(0,T;X)$ is a Hilbert space, whose inner product is given by

$$\langle u, v \rangle_{L^2(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_X \, dt.$$

Consider the space $L^p(0,T;X),\ 1 < p < \infty$, with X being Hilbert separable space, then we can associate the topological dual space

$$[L^p(0,T;X)]' \simeq L^q(0,T;X'),$$

where p and q are Hölder conjugated index. When $p = 1$, we will associate

$$[L^1(0,T;X)]' \simeq L^\infty(0,T;X').$$

Given a Banach space X. The vector space of linear and continuous maps of $\mathcal{D}(0,T)$ on X is called the Space of Vector Distributions on $(0,T)$ with values in X and denoted by $\mathcal{D}'(0,T;X)$.

Example A.1.12. Given $u \in L^p(0,T;X), 1 \leq p < \infty$, and $\varphi \in \mathcal{D}(0,T)$ the application $T_u : \mathcal{D}(0,T) \to X$, defined by

$$T_u(\varphi) = \int_0^T u(t)\varphi(t)dt,$$

Bochner's integral on X, is linear and continuous in the sense of convergence of $\mathcal{D}(0,T)$, so a vector distribution. The map $u \mapsto T_u$ is injective, so we can identify u with T_u and, in this sense, we have

$$L^p(0,T;X) \subset \mathcal{D}'(0,T;X).$$

Given $S \in \mathcal{D}'(0,T;X)$, inspired on the previous derivative of distribution, we define the derivative of order m as being the vector distribution over $(0,T)$ with values in X given for

$$\left\langle \frac{\mathrm{d}^m S}{\mathrm{d}t^n}, \varphi \right\rangle = (-1)^m \left\langle S, \frac{\mathrm{d}^n \varphi}{\mathrm{d}t^n} \right\rangle, \text{ for all } \varphi \in \mathcal{D}(0,T).$$

Let us consider the Banach space

$$W^{m,p}(0,T;X) = \left\{ u \in L^p(0,T;X) \colon u^{(j)} \in L^p(0,T,X), j = 1, \ldots, m \right\},$$

where $u^{(j)}$ represents the $j-$th derivative of u in the sense of distributions and the space is endowed with the norm

$$\|u\|^p_{W^{m,p}(0,T;X)} = \sum_{j=0}^{m} \|u^{(j)}\|^p_{L^p(0,T;X)}.$$

When $p = 2$ and X is a Hilbert space, the space $W^{m,2}(0,T;X)$ will be denoted by $H^m(0,T;X)$, which, equipped with the inner product

$$\langle u, v \rangle_{H^m(0,T;X)} = \sum_{j=0}^{m} \langle u^{(j)}, v^{(j)} \rangle_{L^2(0,T;X)},$$

is a Hilbert space. It is denoted by $H_0^m(0,T;X)$ the closure, in $H^m(0,T;X)$, of $\mathcal{D}(0,T;X)$ and by $H^{-m}(0,T;X)$ the topological dual of $H_0^m(0,T;X)$.

A.1.2 Interpolation of Sobolev spaces

Most of the results that we will enunciate in this subsection, as well as their demonstrations, can be found in (LIONS; MAGENES, 1968).

Let X and Y be two separable Hilbert spaces, with continuous and dense embedding, $X \hookrightarrow Y$. Let $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ be the inner products of X and Y, respectively. We will denote by $D(S)$, the set of all functions u defined in X, such that the application $v \mapsto \langle u, v \rangle_X, v \in X$, is continuous in the topology induced by Y. Then, $\langle u, v\rangle_X = \langle Su, v\rangle_Y$ defines S, as an (unbounded) operator on Y with domain $D(S)$, dense in Y. Since S is a self-adjoint and strictly positive operator by using the spectral decomposition of self-adjoint operators, we can define S^θ, $\theta \in \mathbb{R}$. In particular we will use $A = S^{\frac{1}{2}}$. The operator A, is self-adjoint, positive defined on Y , with domain X and

$$\langle u, v\rangle_X = \langle Au, Av\rangle_Y, \quad \text{for all } u, v \in X.$$

Definition A.1.13. Under the previous assumptions, we define the intermediate space

$$[X, Y]_\theta = D(A^{1-\theta}),\ 0 \le \theta \le 1,$$

equipped with the norm

$$\|u\|^2_{[X,Y]_\theta} = \|u\|^2_Y + \|A^{1-\theta}u\|^2_Y.$$

Note that:

(i) $X \hookrightarrow [X, Y]_\theta \hookrightarrow Y$.

(ii) $\|u\|_{[X,Y]_\theta} \le \|u\|^{1-\theta}_X \|u\|^\theta_Y$.

(iii) If $0 < \theta_0 < \theta_1 < 1$, then $[X, Y]_{\theta_0} \hookrightarrow [X, Y]_{\theta_1}$.

(iv) $[[X, Y]_{\theta_0}, [X, Y]_{\theta_1}]_\theta = [X, Y]_{(1-\theta)\theta_0+\theta\theta_1}$.

The following results will be used throughout this thesis.

Theorem A.1.14. *Let $\Omega \subset \mathbb{R}^n$ and $m > 0$. Then,*

$$H^m_0(\Omega) = \{u \in H^m(\Omega); u|_{\partial\Omega} = 0\},$$

and $u|_{\partial\Omega}$ is, by definition, the trace[3] of u on $\partial\Omega$.

Theorem A.1.15. *Let $\Omega \subset \mathbb{R}^n$ and $\theta_1 \ge \theta_2 \ge 0$, $\theta_1, \theta_2 \ne k + \frac{1}{2}$, for any integer k. If $s = (1-\theta)\theta_1 + \theta\theta_2 \ne k + \dfrac{1}{2}$, then*

$$[H^{\theta_1}_0(\Omega), H^{\theta_2}_0(\Omega)] = H^s_0(\Omega)$$

and

$$[H^m_0(\Omega), L^2(\Omega)]_\theta = H^s_0(\Omega), \quad s = (1-\theta)m \ne k + \frac{1}{2}$$

with equivalent norms.

[3] We call of u over $\partial\Omega$ as trace, to a continuous linear map $\gamma_0 \colon H^1(\Omega) \to H^{\frac{1}{2}(\partial\Omega)}$

A.1.3 Classical remarkable results

Now, let us present a series of classical results that will be used throughout this thesis. The results are classical and the proofs will be omitted (see (ADAMS, 1975; BREZIS, 2011) and references therein).

Lemma A.1.16 (Young's Inequality)**.** *Let a and b be positive constants, $1 \leq p, q \leq \infty$, such that p and q are Hölder conjugated index. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, for all $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q.$$

Lemma A.1.17 (Gronwall's Inequality)**.** *Let $u(t)$ be a non-negative differentiable function on $[0, T]$, satisfying*

$$u'(t) \leq f(t)u(t) + g(t)$$

where $f(t)$ and $g(t)$ are integrable functions over $[0, T]$. Then,

$$u(t) \leq e^{\int_0^t f(\tau)d\tau} \left[u(0) + \int_0^t g(s) e^{-\int_0^s f(\tau)d\tau} ds \right], \forall t \in [0, T].$$

Lemma A.1.18 (Cauchy-Schwarz's Inequality)**.** *Let $(E, \langle \cdot, \cdot \rangle)$ be a vector space with an inner product and $\| \cdot \|$ the induced norm of the inner product, then*

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \ \forall x, y \in E.$$

Furthermore, equality holds if and only if x and y are linearly independent.

Lemma A.1.19 (Hölder's Inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, consider $1 \leq p, q \leq \infty$ such that p and q are Hölder conjugated. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} = \int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Lemma A.1.20 (Generalized Hölder's Inequality)**.** *Let $f_j \in L^{p_j}(\Omega)$ for $0 \leq j \leq k$ such that $\frac{1}{p} = \sum_{j=1}^k \frac{1}{p_k} \leq 1$. Then $f_1 \dots f_k \in L^p(\Omega)$ and yields that*

$$\|f_1 \dots f_k\|_{L^p(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \dots \|f_k\|_{L^{p_k}(\Omega)}.$$

Lemma A.1.21 (Poincaré-Friedrichs inequality)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$, then for every $1 \leq p < \infty$ there exists a constant $C = C(\Omega, p) > 0$, such that*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

Remark A.1.22. Poincaré's inequality remains true if Ω has a finite measure and also if Ω has a bounded projection on some axis.

Lemma A.1.23 (Gagliardo–Nirenberg inequality (first form))**.** *Let $I = (0,1)$, $1 \leq q < \infty$ and $1 < r \leq \infty$. Then*

$$\|u\|_{L^\infty(I)} \leq C\|u\|^a_{W^{1,r}(I)}\|u\|^{1-a}_{L^q(I)}, \ \forall u \in W^{1,r}(I)$$

for some constant $C = C(q,r)$, where $0 < a < 1$ is defined by $a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}$.

Lemma A.1.24 (Gagliardo-Nirenberg inequality (Usual applications))**.** *Let $\Omega \subset \mathbb{R}^n$ be a regular bounded open set.*

(i) Let $u \in L^p(\Omega) \cap W^{2,r}(\Omega)$ with $1 \leq p, r \leq \infty$. Then $u \in W^{1,q}(\Omega)$ where $\frac{1}{q} = \frac{1}{2}\left(\frac{1}{p} + \frac{1}{r}\right)$, and

$$\|\nabla u\|^2_{L^q(\Omega)} \leq C\|u\|_{W^{2,r}(\Omega)}\|u\|_{L^p(\Omega)}.$$

(ii) Let $1 \leq q \leq p < \infty$. Then, for $a = 1 - \frac{q}{p}$,

$$\|u\|_{L^p(\Omega)} \leq C\|u\|^{1-a}_{L^q(\Omega)}\|u\|^a_{W^{1,n}(\Omega)}, \quad \forall u \in W^{1,N}(\Omega).$$

(iii) Let $1 \leq q \leq p \leq \infty$ and $r > N$. Then

$$\|u\|_{L^p(\Omega)} \leq C\|u\|^{1-a}_{L^q(\Omega)}\|u\|^a_{W^{1,r}(\Omega)}, \quad \forall u \in W^{1,r}(\Omega),$$

where

$$a = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{N} - \frac{1}{r}}.$$

The following Theorem can be found at (AUBIN, 1963).

Theorem A.1.25 (Aubin-Lions)**.** *Let X_0, X and X_1 be Banach spaces such that $X_0 \subset X \subset X_1$ with X_0 compactly embedded in X and $X \hookrightarrow X_1$. Suppose that $1 < p, q \leq \infty$ and*

$$W = \{u \in L^p([0,T]; X_0) \colon u_t \in L^q([0,T]; X_1)\}.$$

(i) If $p < \infty$ then W is compactly embedded into $L^p([0,T], X)$.

(ii) If $p = \infty$ and $q > 1$ then $W \hookrightarrow C([0,T]; X)$ is compact.

A.1.4 Semigroup theory

The semigroup theory provides a framework for analyzing the time evolution of systems described by PDEs, conducting mainly existence and uniqueness issues through the properties of operators. Consequently, some definitions and results will be presented. The results contained here can be found in (PAZY, 1983). In the sequel, we will denotes by $(X, \|\cdot\|_X)$ a Banach space.

Definition A.1.26. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a *semigroup of a bounded linear operator on* X if

(i) $T(0) = I$, where I is the identity operator on X;

(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;

A semigroup of a bounded linear operator $T(t)$ is *uniformly continuous* if

$$\lim_{t \to 0^+} \|(T(t) - I)x\|_X = 0, \quad \forall x \in X.$$

The linear operator A is defined by

$$D(A) = \left\{ x \in X \colon \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \left. \frac{\mathrm{d}^+ T(t)x}{\mathrm{d}t} \right|_{t=0} \quad \text{for } x \in D(A)$$

is the *infinitesimal generator* of the semigroup $T(t)$, $D(A)$ is the *domain* of A.

Corollary A.1.27. *Let $T(t)$ be a uniformly continuous semigroup of a bounded linear operator. Then*

(i) There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.

(ii) There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.

(iii) The operator A defined in item (b) is the infinitesimal generator of $T(t)$.

(iv) The application $t \mapsto T(t)$ is differentiable in norm and

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = AT(t) = T(t)A.$$

Definition A.1.28. A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup of a bounded linear operator if

$$\lim_{t \to 0^+} T(t)x = x, \quad \forall x \in X.$$

A strongly continuous semigroup of a bounded linear operator on X will be called a *semigroup of class* C_0 or simply a C_0-semigroup.

Theorem A.1.29. *Let $T(t)$ be a C_0 semigroup. There exists constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad 0 \leq t < \infty.$$

Corollary A.1.30. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$ then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.*

A.1.4.1 Two theorems that generate semigroup

Let $T(t)$ be a C_0-semigroup. Look back on that exists constants $\omega \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$. If $\omega = 0$, $T(t)$ is called *uniformly bounded*, and if moreover $M = 1$ it is called a C_0-semigroup of contractions. Here, we focus on characterizing the infinitesimal generators of C_0 semigroups of contractions. Some conditions on the behavior of the resolvent of an operator $A\colon D(A) \subset X \to X$, which are necessary and sufficient for A to be the infinitesimal generator of a C_0-semigroup, are given.

Recall that if A is a linear (not necessarily bounded) operator in X, the *resolvent set* $\rho(A)$ of A is

$$\rho(A) := \{\lambda \in \mathbb{C}\colon \lambda I - A \text{ is invertible }\}.$$

The family $R(\lambda\colon A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the *resolvent* of A.

Theorem A.1.31 (Hille-Yosida)**.** *A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$,$t \geq 0$ if and only if*

(i) A is closed and $\overline{D(A)} = X$;

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\|R(\lambda\colon A)\| \leq \frac{1}{\lambda}.$$

To present another characterization of the infinitesimal generator of a C_0 semigroup of contractions, we need some preliminaries. Let X be a Banach space and let X^* its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{ x^*;\ x^* \in X^* \text{ and } \langle x^*, x \rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \right\}.$$

Notice that the Hahn-Banach theorem implies that that $F(x) \neq \emptyset$ for every $x \in X$.

Definition A.1.32. A linear operator $A : D(A) \subset X \to X$ is *dissipative* if for every $x \in D(A)$ there is a $x^* \in F(X)$ such that $\text{Re}\langle Ax, x^* \rangle \leq 0$.

Now, we establish a useful characterization of dissipative operators,

Theorem A.1.33. *A linear operator A is dissipative if and only if*

$$\|(\lambda I - A)x\| \geq \lambda \|x\|$$

for all $x \in D(A)$ and $\lambda > 0$.

The second result that gives a characterization for C_0 semigroup of contractions is stated as

Theorem A.1.34 (Lumer-Phillips)**.** *Let A be a linear operator with dense domain $D(A)$ in X.*

(i) If A is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(ii) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$,

$$\mathit{Re}\langle Ax, x^* \rangle \leq 0.$$

Finally, as a consequence of the Theorem above, we highlight one of the most useful results for the generation of C_0 semigroup of contractions

Corollary A.1.35. *Let A be a densely defined closed linear operator. If both A and A^* (adjoint of A) are dissipative, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

A.1.4.2 The abstract Cauchy problem

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator. Given $x \in X$, the abstract Cauchy problem for A with initial data x consists of finding a solution $u(t)$ to the initial value problem (I.V.P.)

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t), \quad t > 0, \\ u(0) = x. \end{cases} \tag{A.1.1}$$

Now, let us introduce a notion of a solution to the problem (A.1.1).

Definition A.1.36 (Classical solution). By a *classical solution* of (A.1.1) we mean a function $u : \mathbb{R}^+ \to X$such that $u(t)$ is continuous for all $t \geq 0$, continuously differentiable and $u(t) \in D(A)$ for all $t > 0$ that satisfies (A.1.1).

Remark A.1.37. We want to empathize on two points about the classical solutions:

- Note that since $u(t) \in D(A)$ for $t > 0$ and u is continuous at $t = 0$, (A.1.1) cannot have solution for $x \notin \overline{D(A)}$.
- It is clear that if A is the infinitesimal generator of a C_0 semigroup $T(t)$, the abstract Cauchy problem for A has a solution, namely $u(t) = T(t)x$, for every $x \in D(A)$. Moreover, it is not difficult to show that this is the only solution of (A.1.1).

We turn our attention to the non-homogeneous abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), \quad t > 0, \\ u(0) = x. \end{cases} \tag{A.1.2}$$

where $f : [0, T) \to X$. We suppose that A is the infinitesimal generator of a C_0 semigroup $T(t)$ with corresponding homogeneous equation (A.1.1) has a unique solution for every initial value $x \in D(A)$.

Definition A.1.38 (Classical solution). A function $u : [0, T) \to X$ is a *classical solution* of (A.1.2) on $[0, T)$ if u is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $u(t) \in D(A)$ for $0 < t < T$ and (A.1.2) is satisfied for all $t \in [0, T)$.

Suppose that $u(t)$ is a classical solution of (A.1.2). Then $g(s) = T(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\frac{\mathrm{d}g}{\mathrm{d}s} = -AT(t-s)u(s) + T(t-s)\frac{\mathrm{d}u}{\mathrm{d}s} = T(t-s)f(s).$$

Hence, If $f \in L^1(0,T;X)$ then $S(t-s)f(s)$ is integrable on $[0,t]$ and integrating from 0 to t yields[4]

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds. \tag{A.1.3}$$

Corollary A.1.39. *If $f \in L^1(0,T;X)$ then for every $x \in X$ the initial value problem (A.1.2) has at most one solution. If it has a solution, this is given by (A.1.3)*

For every $f \in L^1(0,T;X)$ the right-hand side of (A.1.3) is a continuous function on $[0,T)$. It is natural to consider it as a generalized solution of (A.1.2) even if it is not differentiable and does not strictly satisfy the equation in the classical sense. Therefore we define,

Definition A.1.40. Let $x \in X$ and $f \in L^1(0,T;X)$. The function $u \in C([0,T];X)$ given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \; 0 \le t \le T,$$

is the mild solution of the non-homogeneous Cauchy problem (A.1.2) on $[0,T]$.

The definition of a mild solution of the abstract Cauchy problem (A.1.2) coincides when $f \equiv 0$ with the definition of $T(t)x$ as the mild solution of the corresponding homogeneous equation. Moreover, not every mild solution of (A.1.2) is indeed a (classical) solution even in the case $f \equiv 0$.

Next, let us present another notion of solution to the abstract Cauchy problem (A.1.2)

Definition A.1.41 (Strong solution)**.** A function u which is differentiable almost everywhere on $[0,T]$ such that $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1([0,T];X)$ is called a *strong solution* of the abstract Cauchy problem (A.1.2) if $u(0) = x$ and

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t),$$

almost everywhere on $[0,T]$.

Notice that if $A = 0$ and $f \in L^1([0,T];X)$, the abstract Cauchy problem (A.1.2) has usually no solution unless $f \in C([0,T];X)$. However, (A.1.2)has always a strong solution given by

$$u(t) = u(0) + \int_0^t f(s)ds.$$

Furthermore, if u is a strong solution of (A.1.2) and $f \in L^1([0,T];X)$, can be showed that u is a mild solution as well.

[4] The representation of solution (A.1.3) is known also *Duhamel's formula*

Finally, we deal with the nonlinear case. Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) = f(t, u(t)), \quad t > t_0, \\ u(t_0) = u_0. \end{cases} \tag{A.1.4}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on a Banach space X and $f\colon [t_0, T] \times X \to X$ is a continuous in t and satisfies the Lipschitz condition[5] on u. By the aforementioned arguments can be established a solution u that satisfies the integral equation

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s) f(s, u(s))\, ds,$$

which means that is a **mild solution**. Consequently, we have the following classical result which assures the existence and uniqueness of these mild solutions

Theorem A.1.42. *Let $f\colon [t_0, T] \times X \to X$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X. If $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X then for every $u_0 \in X$, the abstract Cauchy problem (A.1.4) has a unique mild solution $u \in C([t_0, T]; X)$. Moreover, the mapping $u_0 \mapsto u$ is Lipschitz continuous from X into $C([t_0, T]; X)$.*

Additionally, can be spotlighted some points

- If $u_0, v_0 \in X$ are initial data and u, v are its respective mild solutions of (A.1.4), then

$$\|u(t) - v(t)\|_X \leq M e^{LMt} \|u_0 - v_0\|_X.$$

- If $u_0 \in D(A)$, then u is a strong solution of (A.1.4) on $[t_0, T]$, for $T > t_0$.

A.2 SOME CLASSICAL CONCEPTS ABOUT CONTROL AND STABILIZATION

Here, we present some definitions, tools, as well as techniques, that will be useful throughout this manuscript and are inspired in (LIONS, 1988a; LIONS, 1988b; RUSSELL, 1978; ZUAZUA, 2006; SLOTINE; LI, 1990; CORON, 2007; SLEMROD, 1974).

5 We said that $f\colon [t_0, T] \times X \to X$ satisfies the Lipschitz condition if there exists $L > 0$ such that

$$\|f(\cdot, u) - f(\cdot, v)\|_X \leq L\|u - v\|_X, \ \forall u, v \in X.$$

A.2.1 Control for finite-dimensional linear systems

Some essential concepts of control and stabilization come from finite dimensional systems (ODE) and after generalization in some sense to infinite dimensional systems (PDE). Therefore, let us consider $m, n \in \mathbb{N}^*$, $T > 0$ and the finite-dimensional system

$$\begin{cases} x'(t) = Ax(t) + Bv(t), \quad 0 < t < T, \\ x(0) = x^0, \end{cases} \tag{A.2.1}$$

where $m \leq n$, A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and $x^0 \in \mathbb{R}^n$. The function $x\colon [0,T] \to \mathbb{R}^n$ represents the *state* and $u\colon [0,T] \to \mathbb{R}^m$ are called the *control*. The most desirable goal is, of course, controlling the system using a minimum number of m of controls.

Note that, by the variations of constants formula, if $u \in L^2(0,T;\mathbb{R}^m)$, (A.2.1) has a unique solution $x \in H^1(0,T;\mathbb{R}^n)$ given by

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)\,ds, \quad \forall t \in [0,T]. \tag{A.2.2}$$

Definition A.2.1. We said (A.2.1) is *exactly controllable* in time $T > 0$ if given any initial and final data $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0,T;\mathbb{R}^m)$ such that the solution (A.2.2) of (A.2.1) satisfies $x(T) = x^1$.

- The aim of the control consists in driving the solution from the initial data x^0 to the final one x^1 in time T by acting on the system through the control u.
- It is desirable to make the number of controls m to be as small as possible. However, this may affect the control properties of the system.

By making a variable change, can we consider $x^1 = 0$, this motivates the following definition

Definition A.2.2. We said (A.2.1) is *null controllable* in time $T > 0$ if given any initial and final data $x^0 \in \mathbb{R}^n$ there exists $u \in L^2(0,T;\mathbb{R}^m)$ such that the solution (A.2.2) of (A.2.1) satisfies $x(T) = 0$.

Remark A.2.3. Exact and null controllability are equivalent properties in the case of finite dimensional linear systems. But this is not necessarily the case for nonlinear systems, or, for strongly time-irreversible infinite dimensional systems.

A.2.2 Control as a minimization problem

Let us introduce the homogeneous *adjoint system* of (A.2.1)

$$\begin{cases} -\varphi' = A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{A.2.3}$$

where A^* denotes the adjoint matrix of A. Next, by the adjoint properties, we have a characterization for the exact controllability property,

Lemma A.2.4. *An initial data $x^0 \in \mathbb{R}^n$ of (A.2.1) is driven to zero in time T by using a control $u \in L^2(0,T)$ if and only if*

$$\int_0^T \langle u, B^*\varphi \rangle \, dt + \langle x^0, \varphi(0) \rangle = 0 \tag{A.2.4}$$

for any $\varphi_T \in \mathbb{R}^n$, φ being the solution of the adjoint system (A.2.3)

Moreover, (A.2.4) is an optimality condition for the critical points of the functional $J \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2} \int_0^T |B^*\varphi|^2 \, dt + \langle x^0, \varphi(0) \rangle$$

with φ the solution of the adjoint system (A.2.3) with initial data φ_T at time $t = T$. More precisely,

Lemma A.2.5. *Suppose that J has a minimizer $\hat{\varphi}_T \in \mathbb{R}^n$ and let $\hat{\varphi}$ be the solution of the adjoint system (A.2.3) with initial data $\hat{\varphi}_T$. Then*

$$u = B^*\hat{\varphi}$$

is a control of system (A.2.1) with initial data x^0.

The lemma (A.2.5) gives a variational method to obtain the control[6] as a minimum of the functional J. Remark that J is continuous. Therefore, the existence of a minimum is ensured if J is coercive too, that is,

$$\lim_{|\varphi_T| \to \infty} J(\varphi_T) = \infty. \tag{A.2.5}$$

The coercivity of J, (A.2.5), follows from the next concept named as *observability*,

[6] This is not the unique possible functional allowing to build the control.

Definition A.2.6. We said that (A.2.3) is *observable* in time $T > 0$ if there exists $C > 0$ such that

$$\int_0^T |B^*\varphi|^2 \, dt \geq C|\varphi(0)|^2, \quad \forall \varphi_T \in \mathbb{R}^n, \tag{A.2.6}$$

where φ being the solution of (A.2.3).

Remark A.2.7. The observability inequality (A.2.6) is equivalent to the following assertion: there exists $C > 0$ such that

$$\int_0^T |B^*\varphi|^2 \, dt \geq C|\varphi_T|^2, \quad \forall \varphi_T \in \mathbb{R}^n, \tag{A.2.7}$$

where φ being the solution of (A.2.3).

Finally, the next Theorem ensures that the exact controllability can be reduced to the study of observability.

Theorem A.2.8. *The system* (A.2.1) *is exactly controllable in time* T *if and only if* (A.2.3) *is observable in time* T.

A.2.3 A feedback stabilization problem

In a practical context, the stabilization problem for a system can be defined as finding a mechanism that ensures the system's state remains close to a desired point over time. Controllability is often a prerequisite for stabilization. If a system is not controllable, it may be impossible to design a control input that drives the system to the desired equilibrium state, making stabilization unattainable. Hence, heuristically, we can see stabilization as a controllability problem when the control is exerted at any time.

Here, we suppose that A is a skew-adjoint matrix, that is $A^* = -A$. Additionally, in this case, $\langle Ax, x\rangle = 0$. Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{A.2.8}$$

When the control is not acting, the energy of the solutions of (A.2.8) is conserved, that is, is constant over the time,

$$|x(t)| = |x^0|, \quad \forall t \geq 0.$$

The *stabilization* problem can be stated in the next way. Suppose that (A.2.8) is controllable, then we look for a solution of the system (A.2.8) such that with *feedback* control

$$u(t) = Lx(t) \tag{A.2.9}$$

has a *exponential decay*, that is, there exists $C > 0$ and $\lambda > 0$ such that

$$|x(t)| \leq Ce^{-\lambda t}|x^0| \tag{A.2.10}$$

for any solution. In particular, the control u given by (A.2.9) acts in real-time from the state x. More precisely, we are looking for an operator L such that the solution of the system

$$x' = (A + BL)x$$

has an exponential decay rate. Observe that due to the representation of solutions, the decay can not be faster than exponential.

Theorem A.2.9. *If A is skew-adjoint and the system (A.2.8) is controllable then $L = -B^*$ stabilizes the system, that is, the solution of*

$$\begin{cases} x' = Ax - BB^*x \\ x(0) = x^0 \end{cases} \tag{A.2.11}$$

has an exponential decay.

Remark A.2.10. To prove the Theorem A.2.9 a fundamental estimate is sufficient to obtain the exponential decay, that is, there exists $T > 0$ and $C > 0$ such that

$$\int_0^T |B^*x|^2 \, dt \geq C^{-1}|x(0)|^2, \tag{A.2.12}$$

for any solution x of (A.2.11). Note that (A.2.12) is an observability type inequality and this shows how the controllability and stabilization are related *via* an inequality.

A.2.4 Control and stabilization extended to infinite dimensional systems

All of the concepts and results mentioned above can be generalized (in some sense) to infinite dimensional systems. Let $T > 0$, H and V be real Hilbert spaces and consider the following control system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = Au + Bv, \quad 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{A.2.13}$$

where u denotes the states and $v \in L^2(0,T;V)$ is the control. The operator $A\colon D(A) \to H$ is a linear operator and $B \in \mathcal{L}(V, D(A^*)')$[7], where $D(A^*)'$ denotes the dual space of $D(A^*)$ and A^* is the adjoint of the operator A. Additionally, A^* is associated with the homogeneous adjoint system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = -A^*\varphi, \quad 0 < t < T, \\ \varphi(T) = \varphi_T, \end{cases} \tag{A.2.14}$$

Now, we state the most classical notions of controllability for the abstract system (A.2.13),

Definition A.2.11. The system (A.2.13) is *exactly controllable* in time $T > 0$ if, for every initial and final data $u_0, u_T \in H$, there exists $v \in L^2(0,T;V)$ such that the solution of (A.2.13) satisfies $u(T) = u_T$.

Definition A.2.12. The system (A.2.13) is *null controllable* in time $T > 0$ if, for every initial data $u_0 \in H$, there exists $v \in L^2(0,T;V)$ such that the solution of (A.2.13) satisfies $u(T) = 0$.

Definition A.2.13. The system (A.2.13) is *approximately controllable* in time $T > 0$ if, for every initial and final data $u_0, u_T \in H$, and $\varepsilon > 0$, there exists $v \in L^2(0,T;V)$ such that the solution of (A.2.13) satisfies

$$\|u(T) - u_T\|_H \leq \varepsilon.$$

Similar to the mentioned for finite-dimensional, a control may be obtained from the solution of the homogeneous system (A.2.14) with the initial data minimizing the functional $J\colon H \to \mathbb{R}$ given by

$$J(\varphi) = \frac{1}{2}\int_0^T \langle u, B^*\varphi\rangle_H \, dt + \langle u_0, \varphi(0)\rangle_H - \langle u_T, \varphi_T\rangle_H.$$

Hence, the controllability is reduced to a minimization problem. To guarantee that J has a unique minimizer we use the next fundamental result in the calculus of variations.

Theorem A.2.14 (See (BREZIS, 2011))**.** *Let H be a reflexive Banach space, K a closed convex subset of H and $J : K \to \mathbb{R}$ a function with the following properties:*

(i) J is convex

(ii) J is lower semi-continuous

[7] This functional setting gives the possibility to consider boundary control operators (instead of the stronger one $B \in \mathcal{L}(V, H)$)

(iii) If K is unbounded then J is coercive, i.e.

$$\lim_{\|x\|\to\infty} J(x) = \infty.$$

Then J attains its minimum in K, i. e. there exists $x_0 \in K$ such that

$$J(x_0) = \min_{x\in K} \varphi(x)$$

Note that J is continuous and convex. The existence of a minimum is ensured if J is also coercive, which is obtained with the *observability inequality*

$$\int_0^T \|B^*\varphi\|_H \, dt \geq C\|\varphi(0)\|_H^2, \quad \forall\varphi(0) \in H. \tag{A.2.15}$$

Finally, consider the uncontrolled case, $v \equiv 0$ in (A.2.13). Let $\overline{u}$ be an equilibrium solution, that is, $A\overline{u} = 0$, with $\overline{u} \in D(A)$.

Definition A.2.15. We said that $\overline{u}$ is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \overline{u}\| \leq \delta$, the unique mild solution u of (A.2.13) satisfies

$$\|u(t) - \overline{u}\| < \varepsilon, \quad \forall t \geq 0.$$

Definition A.2.16. We said that $\overline{u}$ is *asymptotically stable* if is stable and there exists $\delta > 0$ such that for all $u_0 \in H$ with $\|u_0 - \overline{u}\| \leq \delta$, the unique mild solution u of (A.2.13) satisfies

$$\lim_{t\to\infty} \|u(t) - \overline{u}\| = 0.$$

Definition A.2.17. We said that $\overline{u}$ is *exponentially stable* if is asymptotically stable and there exists $\lambda > 0$ such that for all $u_0 \in H$ the unique mild solution u of (A.2.13) satisfies

$$\|u(t) - \overline{u}\| < e^{-\lambda t}\|u(t) - \overline{u}\|.$$

The largest constant λ which may be utilized in the exponential stability is called the rate of convergence.

A.2.5 A summary of some feedback mechanisms

Feedback mechanisms are crucial in understanding how systems evolve, and they are widely studied across engineering, physics, biology, and even computation. In general, these mechanisms will influence a system's behavior and will be separated into two types

- **Positive feedback**: Essentially, this mechanism tends to stabilize a system by reducing deviations from a desired state.

- **Negative feedback**: which can amplify deviations, potentially leading to growth or instability.

Essentially, in this thesis, we focus on two damping mechanisms and time delay. A damping mechanism is a form of feedback that can reduce oscillations, while time delay can destabilize a system by introducing phase shifts. Now, we look more precisely at these effects:

First, damping refers to the effect of reducing the amplitude of oscillations in a system over time, often due to energy loss (like friction or resistance). Mathematically, damping is usually represented by a term proportional to one state of the differential equation. For example, in a second-order linear differential equation:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0$$

the damping is associated with the velocity (or first derivative). Here,

- m is the mass,
- c is the damping coefficient,
- k is the stiffness, and
- $x(t)$ is the displacement as a function of time.

The term $c\dot{x}(t)$ represents the damping force. When c is large, the system returns to equilibrium more quickly, reducing oscillations.

On the other hand, time delay in a feedback system refers to a lag between the input and the output. This can cause oscillations, instability, or even chaos in some cases. Mathematically, time delay is often modeled with a delay differential equation (DDE), such as:

$$\dot{x}(t) = f(x(t), x(t - \tau))$$

where:

- $x(t)$ is the state of the system at time t,
- τ is the time delay.

Time delay can introduce complex dynamics because the system's future state depends not just on its current state but also on its past.

A.2.6 The direct method of Lyapunov

Lyapunov's direct method (also called the second method of Lyapunov) allows us to determine the stability of a system without explicitly integrating the differential equation. The method is a generalization of the idea that if there is some "energy" in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need to define exactly what one means by "energy." Let B_ε be a ball of size ε around the origin, $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.

Definition A.2.18 (Locally positive definite functions (lpdf))**.** A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a *locally positive definite function* if for some $\varepsilon > 0$ and some continuous, strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$,

$$V(0,t) = 0 \quad \text{and} \quad V(x,t) \geq \alpha(\|x\|) \quad \forall x \in B_\epsilon, \forall t \geq 0.$$

A locally positive definite function is locally like an energy function. Functions which are globally like energy functions are called positive definite functions:

Definition A.2.19 (Positive definite functions (pdf))**.** A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is a *positive definite function* if it is locally positive definite and, additionally, $\alpha(p) \to \infty$ as $p \to \infty$.

To bound the energy function from above, we define nonincreasing as follows:

Definition A.2.20 (Nonincreasing functions)**.** A continuous function $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is *nonincreasing* if for some $\varepsilon > 0$ and some continuous, strictly increasing function $\beta : \mathbb{R}_+ \to \mathbb{R}$,

$$V(x,t) \leq \beta(\|x\|) \quad \forall x \in B_\epsilon, \forall t \geq 0$$

Based on these definitions, the following theorem enables us to assess the stability of a system by analyzing a suitable energy function. Essentially, the theorem states that if $V(x,t)$ is a locally positive definite function and $\dot{V}(x,t) \leq 0$, we can infer the stability of the equilibrium point. The time derivative of V is evaluated along the system's trajectories

$$\dot{V}\Big|_{\dot{x}=f(x,t)} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f.$$

In what follows, by $\dot{V}$ we will mean $\dot{V}\Big|_{\dot{x}=f(x,t)}$.

Theorem A.2.21 (Basic theorem of Lyapunov)**.** *Let $V(x,t)$ be a non-negative function with derivative $\dot{V}$ along the trajectories of the system.*

(i) If $V(x,t)$ is locally positive definite and $\dot{V}(x,t) \leq 0$ locally in x and for all t, then the origin of the system is locally stable (in the sense of Lyapunov).

(ii) If $V(x,t)$ is locally positive definite and nonincreasing, and $\dot{V}(x,t) \leq 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).

(iii) If $V(x,t)$ is locally positive definite and nonincreasing, and $-\dot{V}(x,t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.

(iv) If $V(x,t)$ is positive definite and nonincreasing, and $-\dot{V}(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.

Theorem A.2.21 gives sufficient conditions for the stability of the origin of a system. It does not, however, give a prescription for determining the Lyapunov function $V(x,t)$. Since the theorem only gives sufficient conditions, the search for a Lyapunov function establishing the stability of an equilibrium point could be arduous. The utility of this theorem is limited by the lack of a computable technique for generating Lyapunov functions.

Theorem A.2.22. *$x^* = 0$ is an exponentially stable equilibrium point of $\dot{x} = f(x,t)$ if and only if there exists an $\varepsilon > 0$ and a function $V(x,t)$ which satisfies*

$$\begin{cases} \alpha_1 \|x\|^2 \leq V(x,t) \leq \alpha_2 \|x\|^2 \\ \dot{V}\Big|_{\dot{x}=f(x,t)} \leq -\alpha_3 \|x\|^2 \\ \left\| \frac{\partial V}{\partial x}(x,t) \right\| \leq \alpha_4 \|x\| \end{cases}$$

for some positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\|x\| \leq \varepsilon$.

A.2.7 Lyapunov theory

Given a control system, the first and most important question about its various properties is whether it is stable, because, in the words of Slotine and Li (see (SLOTINE; LI, 1990)), *an unstable control system is typically useless and potentially dangerous.* Here, we will present the concept of a stable system according to the Russian mathematician Alexandr Mikhailovich

Lyapunov (1857-1917), presenting some definitions and the *direct method* introduced by him at the end of the 19th century in the work *The General Problem of Motion Stability*, which includes two methods for stability analysis (the so-called linearization method and the direct method) and which was first published in 1892.

The direct method determines the stability properties of a nonlinear system by constructing an "energy-like" scalar function for the system and examining the time variation of the function. This function is known as the Lyapunov function, as we will see later.

Definition A.2.23. A nonlinear dynamic system can usually be represented by a set of nonlinear differential equations in the form

$$u'(t) = g(u, t) \tag{A.2.16}$$

where g is a nonlinear vector function and u is the state vector in $\mathbb{R}^n$.

A particular value of the state vector is also called a point because it corresponds to a point in the state space. The number n is called the order of the system. A solution $u(t)$ of the equations (A.2.16) generally corresponds to a state-space curve for t ranging from zero to infinity. This curve is often called a state trajectory or a system trajectory.

Note that although the equation (A.2.16) does not explicitly contain the control input as a variable, it is directly applicable to feedback control systems. The reason is that the equation (A.2.16) can represent the closed-loop dynamics of a feedback control system, with the control input being a function of state u and time t, disappearing in the loop dynamics closed. Specifically, if $u' = g(u, v, t)$ where $v = h(u, t)$ is a selected control law, then the closed-loop dynamics is $u' = g[u, h(u, t), t]$ which can be rewritten in the form (A.2.16). The equation (A.2.16) can also represent dynamic systems where no control signal is involved.

Definition A.2.24. The nonlinear system (A.2.16) is said to be autonomous if g does not explicitly depend on time, so

$$u' = g(u). \tag{A.2.17}$$

Otherwise, the system is called non-autonomous.

The fundamental difference between autonomous and non-autonomous systems resides in the fact that the trajectory of the autonomous system is independent of the initial time, whereas that of a non-autonomous system generally is not.

Definition A.2.25. A state u^* is an equilibrium point of the system (A.2.16) if $u(t_0) = u^*$ implies

$$g(u,t) = 0, \ \forall t \geq t_0.$$

In other words, $u(t) = u^*, \forall t \geq t_0$.

To simplify the notation and analysis of the stability of the system (A.2.16) at a specific equilibrium point, we can assume that such equilibrium point is the origin, since introducing a new variable $y = u - u^*$ and replacing $u = y + u^*$ in the equations of the system (A.2.16) a new set of equations in the variable y is obtained

$$y' = g(y + u^*, t) = h(y,t). \tag{A.2.18}$$

It can be easily verified that there is a one-to-one correspondence between the solutions of (A.2.16) and those of (A.2.18) and that, in addition, $y = 0$, the solution corresponding to $u = u^*$ is an equilibrium point of (A.2.18). Therefore, instead of studying the behavior equation of (A.2.16) in the neighborhood of u^*, one can equivalently study the behavior of the equations (A.2.18) in the neighborhood of the origin. In the remainder of this subsection, we will assume that u is a solution to the system (A.2.16).

Definition A.2.26. The equilibrium point 0 is stable at t_0 if for any $R > 0$, there exists a positive scalar $r := r(R, t_0)$ such that

$$\|u(t_0)\| < r \Rightarrow \|u(t)\| < R, \quad \forall t \geq t_0.$$

Otherwise, the equilibrium point 0 is unstable.

Definition A.2.27. The equilibrium point 0 is asymptotically stable at a time t_0 if

- it is stable;
- $\exists r(t_0) > 0$ such that $\|u(t_0)\| < r(t_0) \rightarrow \|u(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

The concepts of stability and asymptotic stability presented here are for non-autonomous systems where the radius r of the initial ball, which is included in the definitions of this subsection, may depend on the initial time t_0. Here, asymptotic stability requires that there is an attractive region for each initial time t_0. The size of the attractive region and the trajectory convergence speed may depend on the initial time t_0. In the case of autonomous systems, the definitions differ in that r does not depend on the considered initial time $t_0 = 0$.

Definition A.2.28. The equilibrium point 0 is exponentially stable if there exist two positive numbers, α and λ, such that for sufficiently small $u(t_0)$,

$$\|u(t)\| \leq \alpha \|u(t_0)\| e^{-\lambda(t-t_0)}, \ \forall t \geq t_0.$$

The positive number λ is often called the *rate* of exponential convergence.

Definition A.2.29. The equilibrium point 0 is globally asymptotically stable if

$$u(t) \to 0 \ \text{ as } t \to \infty, \ \forall u(t_0).$$

Remark A.2.30. Throughout the next chapters, we will say that a system of differential equations is stable, asymptotically stable, exponentially stable, or globally asymptotically stable when the origin is an equilibrium point, respectively, stable, asymptotically stable, exponentially stable or globally asymptotically stable.

Given a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate an "energy-like" scalar function for the dynamical system and examine the time variation of this scalar function. In this way, conclusions can be drawn about the stability of the set of differential equations without using difficult stability definitions or requiring explicit knowledge of the solutions.

A.2.7.1 Lyapunov's direct method for autonomous systems

To introduce Lyapunov's direct method for autonomous systems, the first property to be formalized is the notion of positive definite functions, and the second is the concept of so-called Lyapunov functions.

Definition A.2.31. A continuous scalar function $V(x)$ is said to be locally positive definite if $V(0) = 0$ and, on a ball $B_R = \{x \in X; \|x\| \leq R\}$,

$$x \in B_R, \ x \neq 0 \ \to \ V(x) > 0.$$

If, on the other hand, the above properties hold for the entire state space, then $V(x)$ is said to be globally positive definite. A function $V(x)$ is said to be negative definite if $-V(x)$ is positive definite and a function $V(x)$ is globally negative definite if $-V(x)$ is globally positive definite.

Similarly to the previous definition, we have the following definition.

Definition A.2.32. A continuous scalar function $V(x)$ is said to be locally positive semi-definite if $V(0) = 0$ and, on a ball B_R,

$$x \neq 0 \;\rightarrow\; V(x) \geq 0.$$

If, on the other hand, the above properties hold for the entire state space, then $V(x)$ is said to be globally positive semi-definite. A function $V(x)$ is said to be negative semi-definite if $-V(x)$ is positive semi-definite and a function $V(x)$ is globally negative semi-definite if $-V(x)$ is globally positive semi-definite.

Definition A.2.33. If, on a ball B_R, the function $V(x)$ is positive definite and continuously differentiable (i.e. has continuous partial derivatives), and if it is time derivative along any path of state of the system (A.2.17) is negative semi-definite, that is, for a trajectory $u(t)$ of the system taking $V(t) := V(u(t))$, we have

$$V'(t) = \nabla V(u(t)) \cdot u'(t) < 0 \text{ (or } V'(x) = \nabla V(x) \cdot g(x) < 0, \forall x \in B_R)$$

then $V(x)$ is said to be a Lyapunov function for the system (A.2.17) .

The next two results ensure when the local and global stability holds.

Theorem A.2.34 (Lyapunov theorem for local stability)**.** *If, on a ball B_R, the function $V(x)$ is a Lyapunov function for the system (A.2.17), then the equilibrium point 0 is stable. If, in addition, $V'(x) < 0, \forall x \in B_R$, then the equilibrium point 0 is asymptotically stable.*

Theorem A.2.35 (Lyapunov theorem for global stability)**.** *Assume that there exists a scalar function V of the state x, with continuous first-order derivatives such that*

- *$V(x)$ is positive definite;*
- *$V'(x)$ is negative definite;*
- *$V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$,*

then the equilibrium at the origin is globally asymptotically stable.