A note on the critical length phenomenon for the Korteweg-de Vries equation

Roberto de A. Capistrano–Filho

Abstract Over the past 25 years, the critical length phenomenon for the Korteweg-de Vries (KdV) equation has been challenging. After Rosier's pioneering work in 1997, some authors tried to find boundary conditions that would guarantee the characterization of the critical set explicitly. In this chapter, we will give some results in this sense and present new results that complement previous results in the literature for the KdV equation considering Neuman boundary conditions.

1 Introduction

It is well known today that formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, there exist two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. See, for instance, [1, 3, 4, 17] and references therein for a rigorous justification. Moreover, another non-dimensional parameter μ appears, the Bond number, to measure the importance of gravitational forces compared to surface tension forces also appears.

Considering the physical condition $\delta \ll 1$ we can characterize the waves, called long waves or shallow water waves. In particular, considering the relations between ε and δ, we can have the KdV regime: $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$. Under this regime, Korteweg and de Vries [16][1] derived the following well-known equation as a central equation among other dispersive or shallow water wave models called the KdV equation from the equations for capillary-gravity waves:

Roberto de A. Capistrano–Filho
Universidade Federal de Pernambuco, Departamento de Matemática, Cidade Universitária, Recife (PE), Brazil e-mail: roberto.capistranofilho@ufpe.br

[1] This equation was first introduced by Boussinesq [5], and Korteweg and de Vries rediscovered it twenty years later.

$$\pm 2\eta_t + 3\eta\eta_x + \left(\frac{1}{3} - \mu\right)\eta_{xxx} = 0.$$

Today, it is well known that this equation has an important phenomenon that directly affects the control problem, the so-called *critical length phenomenon*. Let us briefly present the control problem, which makes the phenomenon of critical lengths emerge. The control problem was presented in a pioneering work of Rosier [19] that studied the following system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = 0,\ u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases} \tag{1}$$

where the boundary value function $g(t)$ is considered as a control input. Precisely, the author answered the following problem for the system (1), giving the origin of the critical length phenomenon for the KdV equation.

Exact controllability problem: *Given $T > 0$ and $u_0, u_T \in L^2(0, L)$, can one find an appropriate control input $g(t) \in L^2(0, T)$ such that the corresponding solution $u(x, t)$ of* (1) *satisfies*

$$u(x, 0) = u_0(x) \quad \textit{and} \quad u(x, T) = u_T(x)? \tag{2}$$

In this chapter, our motivation is, first, to present a review of the main results of the literature concerning the critical length phenomenon for the KdV equation. Moreover, we will announce new results in the study of this phenomenon for this kind of dispersive system under a suitable set of boundary conditions.

1.1 How does the critical length phenomenon appear?

To present the main results concerning the critical length phenomenon for the KdV equation we need to understand how this phenomenon appears. As well known in the literature [18] to prove exact controllability for the system (1) is equivalent to prove an *observability inequality* for the linearized system associated with (1). To prove it we use, in general, the multipliers method and compactness arguments which reduce the problem to show a unique continuation property for the state operator.

To show a unique continuation property, in our context, we reduce our problem to the study of the spectral problem associated with the linear operator under consideration. Specifically, after taking the Fourier transform, the issue is to establish when a certain quotient of entire functions still turns out to be an entire function. We then pick a polynomial function $q : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_\alpha : \mathbb{C} \times (0, \infty) \to \mathbb{C},$$

with $\alpha \in \mathbb{C} \setminus \{0\}$, whose restriction $N_\alpha(\cdot, L)$ is entire for each $L > 0$. Next, we consider a family of functions $f_\alpha(\cdot, L)$, defined by

$$f_\alpha(\mu, L) = \frac{N_\alpha(\mu, L)}{q(\mu)},$$

in its maximal domain. The problem is then reduced to determine $L > 0$ for which there exists $\alpha \in \mathbb{C}\setminus\{0\}$ such that $f_\alpha(\cdot, L)$ is entire. In some works, see for example [8, 9, 19], and the reference therein, this approach provides an explicit characterization of the set of critical lengths if it exists, however, in several cases, the set can not be obtained explicitly [2, 7, 13]. With this in mind, let us present a summary of the main results that have used this approach.

This chapter contains three sections including the introduction. Section 2 is devoted to giving a state-of-the-art critical set phenomenon for the KdV equation considering several sets of boundary conditions. Finally, in Section 3, we present two new results that improve previous results in the literature.

2 Overview of critical set phenomenon: KdV equation

The study of control (and stabilization) for the KdV equation began with the Russel and Zhang work's [21, 22, 23, 25] in which they studied internal control of the KdV equation posed on a finite domain $(0, L)$ with periodic boundary conditions. Since then, control and stabilization of the KdV equation have been intensively studied (see [11, 14, 15, 19, 20, 24] and references therein). So, in the next subsection, let us present the main results related to the control problem, and consequently, with the critical set phenomenon for the KdV equation.

2.1 Dirichelet-Neumann boundary conditions

Concerning the control problem introduced at the beginning of this work, Rosier [19] studied boundary control of the KdV equation posed on the finite domain $(0, L)$ with the Dirichlet boundary conditions (1). He considered first the associated linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0, \ u(L, t) = 0, \ u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases} \tag{3}$$

and discovered the so-called *critical length phenomenon*; whether the system (3) is exactly controllable depends on the length L of the spatial domain $(0, L)$.

Theorem 1 (Rosier [19]) *The linear system* (3) *is exactly controllable in the space* $L^2(0,L)$ *if and only if the length* L *of the spatial domain* $(0,L)$ *does not belong to the set*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2} \,:\, k,\, l \,\in \mathbb{N}^* \right\}. \tag{4}$$

The controllability result of the linear system was then extended to the nonlinear system when $L \notin \mathcal{N}$.

Theorem 2 (Rosier [19]) *Let* $T > 0$ *be given and assume* $L \notin \mathcal{N}$. *There exists* $\delta > 0$ *such for any* $u_0, u_T \in L^2(0,L)$ *with* $||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta$, *one can find a control input* $g \in L^2(0,T)$ *such that the nonlinear system* (1) *admits a unique solution* $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *satisfying* (2).

In the case of $L \in \mathcal{N}$, Rosier proved in [19] that the associated linear system (3) is not controllable; there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M} = \mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi \in \mathcal{M}$, $g \in L^2(0,T)$ and $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ satisfying (3) and $u(\cdot,0) = 0$, one has $u(\cdot,T) \neq \psi$. A spatial domain $(0,L)$ is called *critical* for the system (3) if its domain length $L \in \mathcal{N}$.

When the spatial domain $(0,L)$ is critical, one usually would not expect the corresponding nonlinear system (1) to be exactly controllable as the linear system (3) is not. It thus came as a surprise when Coron and Crépeau showed in [14] that the nonlinear system (1) is still locally exactly controllable even though its spatial domain is critical with its length $L = 2k\pi$ and $k \in \mathbb{N}^*$ satisfying

$$\nexists (m,n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n.$$

For those values of L, the unreachable space $\mathcal{M}$ of the associated linear system is a one-dimensional linear space generated by the function $1 - cos(x)$. As for the other types of critical domains, the nonlinear system (1) was shown later by Cerpa [10], and Cerpa and Crépeau in [12] to be local, large time exactly controllable.

Theorem 3 (Crépeau and Cerpa [10, 12]) *Let* $L \in \mathcal{N}$ *be given. There exists a* $T_L > 0$ *such that for any* $T > T_L$ *there exists* $\delta > 0$ *such for any* $u_0,\ u_T \in L^2(0,L)$ *with* $||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta$, *there exists* $g \in L^2(0,T)$ *such that the system* (1) *admits a unique solution* $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *satisfying* (2).

It is important to point out, that if we change the control of position in the boundary condition of (3), for example

$$u(0,t) = h(t),\ u(L,t) = 0,\ u_x(L,t) = 0 \quad \text{in } (0,T) \tag{5}$$

or

$$u(0,t) = 0,\ u(L,t) = f(t),\ u_x(L,t) = 0 \quad \text{in } (0,T), \tag{6}$$

we can not characterize explicitly the critical sets for the KdV equation with the boundary conditions (5) and (6). For details, we infer [11, 15].

2.2 Neumann boundary conditions

After 97′, some authors tried to prove the critical set phenomenon for the KdV equation with some boundary condition, we can cite, for example, [15, 13], and the references therein. However, for this set considered in these works, the authors were not allowed to characterize explicitly the set.

Twenty years later, in [8], another boundary condition was considered. The authors introduced the KdV equation with Neumann conditions. Capistran-Filho *et al.* investigated the following boundary control system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = h(t),\ u_{xx}(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L). \end{cases} \tag{7}$$

First, the authors studied the following linearized system associated with (7),

$$\begin{cases} u_t + (1+\beta)u_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = h(t),\ u_{xx}(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{8}$$

where β is a given real constant. For any $\beta \neq -1$, considering the following set

$$\mathcal{R}_\beta := \left\{ \frac{2\pi}{\sqrt{3(1+\beta)}}\sqrt{k^2+kl+l^2} \,:\, k,\ l\ \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{\beta+1}} \,:\, k \in \mathbb{N}^* \right\}. \tag{9}$$

The authors showed the following results:

Theorem 4 (Capistrano-Filho et al. [8])

(i) If $\beta \neq -1$, the linear system (8) is exactly controllable in the space $L^2(0,L)$ if and only if the length L of the spatial domain $(0,L)$ does not belong to the set $\mathcal{R}_\beta$.
(ii)If $\beta = -1$, then the system (8) is not exact controllable in the space $L^2(0,L)$ for any $L > 0$.

The next theorem addressing the controllability of the nonlinear system (7) is another result of the paper:

Theorem 5 (Capistrano-Filho et al. [8]) *Let $T > 0$, $\beta \neq -1$ and $L \notin \mathcal{R}_\beta$ be given. There exists a $\delta > 0$ such that for any $u_0, u_T \in L^2(0,L)$ with $||u_0 - \beta||_{L^2(0,L)} + ||u_T - \beta||_{L^2(0,L)} \leq \delta$, one can find a control input $h \in L^2(0,T)$ such that the system (7) admits unique solution $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ satisfying (2).*

Note that, as in [19], the set $\mathcal{R}_\beta$ is completely characterized. Moreover, when $\beta = 0$, $\mathcal{N}$ (see (4)) is a proper subset of $\mathcal{R}_0$. The linear system (8) has more critical length domains than that of the linear system (3). In the case of $\beta = -1$, every $L > 0$ is critical for the system (8). By contrast, removing the term u_x from the equation in (3), every $L > 0$ is not critical for the system (3).

As usual, Theorems 4 and 5 were shown using the same approach that Rosier [19] used to establish Theorems 1 and 2. However, in the case proved in [8], difficulties appear that demand special attention. The adjoint system of the linear system (8) is given by

$$\begin{cases} \psi_t + (1+\beta)\psi_x + \psi_{xxx} = 0 & \text{in } (0,L)\times(0,T), \\ (1+\beta)\psi(0,t) + \psi_{xx}(0,t) = 0 & \text{in } (0,T), \\ (1+\beta)\psi(L,t) + \psi_{xx}(L,t) = 0 & \text{in } (0,T), \\ \psi_x(0,t) = 0 & \text{in } (0,T), \\ \psi(x,T) = \psi_T(x) & \text{in } (0,L). \end{cases} \tag{10}$$

It is well known that the exact controllability of system (8) is equivalent to the following observability inequality for the adjoint system (10):

$$||\psi_T||_{L^2(0,L)} \leq C||\psi_x(L,\cdot)||_{L^2(0,T)}.$$

However, the usual multiplier method and compactness arguments as those used in dealing with the system (10) only lead to

$$||\psi_T||^2_{L^2(0,L)} \leq C_1||\psi_x(L,\cdot)||^2_{L^2(0,T)} + C_2||\psi(L,\cdot)||^2_{L^2(0,T)}. \tag{11}$$

One has to find a way to remove the extra term present in (11). For this, a technical lemma is necessary which gives the hidden regularities (or sharp trace regularities) for solutions of the adjoint system (10), for details see [8].

3 Neumann boundary conditions: New results

To end this chapter, we want to announce two results. First, observe that considering the results proposed in [8], that is, Theorems 4 and 5, for the linearized system (8), the natural question appears:

Question $\mathcal{A}$: *Given $T > 0$, $L = 2k\pi$, and $u_0, u_T \in L^2(0,L)$, can one find an appropriate control input $h(t) \in L^2(0,T)$ such that the corresponding solution $u(x,t)$ of the system (7) satisfies (2)?*

In this spirit, the following two results, presented in [6] give answers for Question $\mathcal{A}$. The first result ensures that for β near enough to 0 (small perturbations of 0), the system (7) is exactly controllable in a neighborhood of β in $L^2(0,2k\pi)$, the result is presented as follows:

Theorem 6 *Let $L = 2k\pi$. Then, there exists $\epsilon > 0$ such that for every $\beta \in (0,\epsilon]$ the system (8) is exactly controllable in $L^2(0,L)$ and, consequently, system (7) is exactly controllable in a neighborhood of β in $L^2(0,L)$.*

The second result is a generalization of the previous one. Precisely, given $L \in \mathcal{R}_\beta$ (see (9)), for d close enough to β, but not equal, L does not belong to the set $\mathcal{R}_d$. This generalization can be read below.

Theorem 7 *Let $T > 0$, $\beta \neq -1$ and $L \in \mathcal{R}_\beta$. There exists $\epsilon_\beta > 0$ such that, for every $d \in (\beta - \epsilon_\beta, \beta) \cup (\beta, \beta + \epsilon_\beta)$, $d \neq -1$, we have $L \notin \mathcal{R}_d$. Consequently, system* (8) *(with $\beta = d$) is exactly controllable in $L^2(0, L)$ and the system* (1) *is exactly controllable around of the steady state $u = d$ in $L^2(0, L)$.*

The proof of these theorems is given in a very simple way and is based on the topological properties of real numbers together with the Theorems 4 and 5, completing in some sense the previous results given in [8].

Acknowledgements Capistrano-Filho was supported by CAPES grants 88881.311964/2018-01 and 88881.520205/2020-01, CNPq grants 307808/2021-1 and 401003/2022-1, MATHAMSUD grant 21-MATH-03 and Propesqi (UFPE). This work was finished during the stay of the author at Centro de Modelamiento Matemático (Chile). The author thanks the host institution for their warm hospitality

References

1. B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math. 171:3 (2008), 485–541.
2. F. D. Araruna, R. A. Capistrano-Filho, and G. Doronin, *Energy decay for the modified Kawahara equation posed in a bounded domain*, Journal of Mathematical Analysis and Applications 385 (2012), 743–756.
3. J. L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear. Sci. 12 (2002), 283–318.
4. J. L. Bona, T. Colin, and D. Lannes, *Long wave approximations for water waves*, Arch. Ration. Mech. Anal. 178:3 (2005), 373–410.
5. J. M. Boussinesq, *Thórie de l'intumescence liquide, applelée onde solitaire ou de, translation, se propageant dans un canal rectangulaire*, C. R. Acad. Sci. Paris. 72 (1871), 755–759.
6. R. A. Capistrano-Filho and J. S. da Silva, *Boundary controllability of the Korteweg-de Vries equation: The Neumann case*, arXiv:2310.04977 [math.AP].
7. R. A. Capistrano-Filho, B. Chentouf, L. de Sousa, and V. H. Gonzalez Martinez, *Two stability results for the Kawahara equation with a time-delayed boundary control,* Zeitschrift für Angewandte Mathematik und Physik, 74:16 (2023), 1–26.
8. M. A. Caicedo, R. A. Capistrano-Filho, and B.-Y. Zhang, *Neumann boundary controllability of the Korteweg-de Vries equation on a bounded domain*, SIAM J. Control Optim., 55:6 (2017), 3503–3532.
9. R. A. Capistrano-Filho, A. F. Pazoto, and L. Rosier, *Control of Boussinesq system of KdV-KdV type on a bounded interval*, ESAIM-COCV 25: 58 (2019), 1-55.
10. E. Cerpa, *Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain*, SIAM J. Control Optim., 46 (2007), 877–899.
11. E. Cerpa, *Control of a Korteweg-de Vries equation: a tutorial*, Math. Control Relat. Fields, 4 (2014), 45–99.
12. E. Cerpa and E. Crépeau, *Boundary controllability for the nonlinear Korteweg-de Vries equation on any critical domain*, Ann. I.H. Poincaré, 26 (2009), 457–475.
13. E. Cerpa, I. Rivas, and B.-Y. Zhang, *Boundary controllability of the Korteweg-de Vries equation on a bounded domain*, SIAM J. Control Optim., 51:4 (2013), 2976–3010.
14. J.-M. Coron and E. Crépeau, *Exact boundary controllability of a nonlinear KdV equation with a critical length*, J. Eur. Math. Soc., 6 (2004), 367–398.

15. O. Glass and S. Guerrero, *Controllability of the Korteweg-de Vries equation from the right Dirichlet boundary condition*, Systems Control Lett., 59 (2010),390–395.
16. D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. 39(1895) 422–443.
17. D. Lannes, The water waves problem. Mathematical analysis and asymptotics. *Mathematical Surveys and Monographs*, 188. American Mathematical Society, Providence, RI, 2013. xx+321 pp. ISBN: 978–0–8218–9470–5
18. J.-L. Lions, *Exact controllability, stabilizability, and perturbations for distributed systems*, Siam Rev. 30 (1988), 1–68.
19. L. Rosier, *Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var., 2 (1997), 33–55.
20. L. Rosier and B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: Recent progress*, J. Syst Sci & Complexity, 22 (2009), 647–682.
21. D. L. Russell, *Computational study of the Korteweg-de Vries equation with localized control action,* Distributed Parameter Control Systems: New Trends and Applications, G. Chen, E. B. Lee, W. Littman, and L. Markus, eds., Marcel Dekker, New York, 1991.
22. D. L. Russell and B.-Y. Zhang, *Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain*, SIAM J. Control and Optimization, 31 (1993), 659–676.
23. D. L. Russell and B.-Y. Zhang, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc., 348 (1996), 3643–3672.
24. B.-Y. Zhang, *Exact boundary controllability of the Korteweg-de Vries equation*, SIAM J. Cont. Optim., 37 (1999), 543–565.
25. B.-Y. Zhang, *S*ome results for nonlinear dispersive wave equations with applications to control, Ph. D thesis, University of Wisconsin-Madison, 1990.