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JACKELLYNY DASSY DO NASCIMENTO CARVALHO

CONTROL AND STABILIZATION FOR THE NONLINEAR SCHRÖDINGER EQUATION

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JACKELLYNY DASSY DO NASCIMENTO CARVALHO

CONTROL AND STABILIZATION FOR THE NONLINEAR SCHRÖDINGER EQUATION

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JACKELLYNY DASSY DO NASCIMENTO CARVALHO

"Control and Stabilization for the Nonlinear Schrödinger equation"

Tese apresentada ao Programa de Pós-graduação do Departamento de Matemática da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutorado em Matemática.

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RESUMO

Essa tese reúne alguns resultados relacionados à equação de Schrödinger não linear H^1-crítica em $\mathbb{R}^3$, em especial, um resultado de controlabilidade nula onde, usando estimativas de Strichartz, a controlabilidade do sistema linear (método HUM) e um argumento de perturbação, obtemos a controlabilidade do sistema não linear. Além disso, para a equação supracitada com um termo de perturbação, provamos decaimento exponencial para algumas soluções limitadas no espaço de energia, mas pequenas em uma norma específica. Esse resultado é consequência de uma decomposição de perfis obtida para soluções lineares e não lineares combinada com um resultado de propagação que envolve argumentos de análise microlocal, a saber, a teoria de medida de defeito. Após mostrar que uma sequência de soluções não lineares pode ser linearizada sob algumas condições, provamos uma estimativa de observabilidade que implica o resultado de estabilização.

Palavras-chaves: estabilização; expoente crítico; decomposição em perfis; observação; medida microlocal; controle; estimativas de Strichartz; caso de desfocagem.

ABSTRACT

This thesis brings together some results related to the nonlinear H^1-critical Schrödinger equation in $\mathbb{R}^3$, in particular, a null controllability result where, using Strichartz estimates, the controllability of the linear system (HUM method) and a perturbation argument, the controllability for the nonlinear system is achieved. Furthermore, for the aforementioned equation with a perturbation term, we prove exponential decay for some solutions that are bounded in energy space but small at a lower norm. This result is a consequence of a profile decomposition obtained for linear and nonlinear solutions combined with a propagation result that involves arguments from microlocal analysis, namely the defect measure theory. After showing that a sequence of nonlinear solutions can be linearized under some conditions, we prove an observability estimate that implies the stabilization result.

Keywords: stabilization; critical exponent; profile decomposition; observability; microlocal measures; control; Strichartz estimates; defocusing case.

SUMÁRIO

1 GENERAL INTRODUCTION

This thesis comprises the study of the nonlinear Schrödinger equation with a critical exponent in $\mathbb{R}^3$ defocusing case. Some mathematical aspects of the solutions to this equation are systematically addressed, such as well-posedness, decomposition into profiles, stability and control. Concerning the stabilization problem, we study the asymptotic behavior of solutions, i.e., through an analysis of the energy associated with the system, the question is: Is it possible to ensure that the solutions are asymptotically stable for arbitrarily large time t? To obtain an exponential decay rate, we use a profile decomposition to describe how linear and nonlinear solutions approach each other in some sense. To deal with the problem of exact controllability, we verify under what circumstances it is possible to appropriately choose control functions in order to direct the system to a desired state in a finite time.

To provide a minimum of the theory used in the course of the following chapters, we present a small sample of the history of the nonlinear Schrödinger equation, as well as some concepts which will be necessary for the development of this thesis.

1.1 ABOUT THE SCHRÖDINGER EQUATION

The Schrödinger equation was introduced by Erwin Schrödinger in 1925, an Austrian physicist, as part of the fundamental developments in quantum theory that emerged in the first half of the 20th century, for which he received the Nobel Prize in Physics in 1933. The history behind Schrödinger's equation is intrinsically linked to the period when physicists were trying to understand the behavior of electrons in atoms. Schrödinger's approach was based on an effort to find a suitable mathematical description for the energy states of electrons in atoms. At the time, Bohr's model for the atom, which described electrons in discrete orbits around the nucleus, was already known, but it had some limitations, especially when it came to more complex atoms.

After studying De Broglie's thesis in 1924, Schrödinger, inspired by De Broglie's ideas, began working on a new quantum theory that would combine wave and corpuscular characteristics. Schrödinger's central idea was to treat electrons as waves of matter. He proposed that, instead of tracking the precise trajectories of electrons as particles, we should describe their probabilistic distribution in terms of wave functions. The wave function, represented by

Ψ, contains information about the probability of finding an electron in a given position.

From a variational problem, Schrödinger deduced the wave equation for the hydrogen atom. In this deduction, presented in Schrödinger's first paper of 1926, his only "justification"is that the wave equation leads to the correct energy levels for the hydrogen atom. In 1926, Schrödinger published his fundamental equation, which describes the temporal evolution of the wave function of a quantum particle. The form of the Schrödinger equation depends on the system in question, but the general equation for a non-relativistic particle (i.e., particles that do not move at speeds close to the speed of light) is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V\Psi.$$

In this equation, i represents the imaginary unit $(i^2 = -1)$, $\hbar$ is the reduced Planck constant $(\hbar = \frac{h}{2\pi}$ where h is the Planck constant), Ψ is the wave function, t is the time, m is the mass of the particle, Δ is the Laplacian operator, which describes the divergence of the gradient, V is the potential. The solution of the Schrödinger equation is the wave function, which in turn can be used to calculate various observable properties of the particle, such as position and momentum. The Schrödinger equation is one of the cornerstones of quantum mechanics and has profound implications for understanding the behavior of subatomic particles. It is fundamental to describing the dual nature of particles, which can exhibit both particle and wave properties, depending on the experimental context.

Simultaneously, other physicists, such as Werner Heisenberg with matrix mechanics, were developing alternative mathematical formalisms to describe quantum systems. Schrödinger and Heisenberg's formalisms were shown to be equivalent, consolidating the quantum theory.

1.2 THE NONLINEAR SCHRÖDINGER EQUATION

The nonlinear Schrödinger equation is a generalization of the standard Schrödinger equation of quantum mechanics. Nonlinear versions arise in contexts where significant interactions between quantum particles are taken into account. In standard quantum mechanics, the interaction between particles is often described by a linear equation. However, in some cases, such as in extreme conditions of particle density or energy, interactions between particles can become more intense and nonlinear. Introducing nonlinear terms into the Schrödinger equation can lead to a variety of interesting and complex phenomena, often outside the scope of standard linear quantum mechanics. Some significant results associated with the nonlinear Schrödinger equation include, for instance, solitons and pulsons. Solitons are localized waves that maintain their shape and amplitude during propagation, even in the presence of nonlinearities. Pulsons are versions of solitons that are localized pulses of light.

Furthermore, nonlinearity in the Schrödinger equation can lead to self-focusing phenomena, where light pulses contract spatially due to nonlinear interaction. On the other hand, dispersion, which tends to spread the pulses, can counterbalance this effect in certain conditions. These are only two examples among others.

1.2.1 Critical exponent

The nonlinear Schrödinger equation with critical exponent is usually a specific form of nonlinear equation that appears in contexts such as nonlinear optics, soliton theory, and other physical phenomena. This equation can be written in the form

$$i\frac{\partial u}{\partial t} = -\Delta u \pm |u|^{p-1}u, \tag{1.1}$$

here $|u|^{p-1}u$ is the nonlinear term, where the exponent p is a real number. The term "critical exponent"in a nonlinear equation refers to the crucial role that the value of the exponent plays in the nature of the solutions and in the behavior of the associated system. The term "critical"suggests that there is a specific value of this exponent that marks a transition or critical point in the system's behavior.

For example, variations in the value of the exponent p can lead to different types of nonlinear behavior, from integrable behavior to chaotic or turbulent behavior. Moreover, the value of such exponent can also be related to the stability of the system's solutions. For certain critical values, solutions can become more or less stable, influencing the way the system evolves over time.

The nonlinear Schrödinger equations with defocusing and focusing terms refer to different types of nonlinearities present in the equation, which affect the behavior of the wave function. When the sign at the nonlinear term of (1.1) is positive, the nonlinear term is "defocusing", which means that the nonlinearity acts to disperse the wave function over time. This is often associated with solitons and stable spatial patterns. On the other hand, when the sign at the nonlinear term of (1.1) is negative, this indicates a "focusing"term, meaning that the nonlinearity acts to focus the wave function over time. This can lead to the formation of bright solitons.

Throughout this thesis, we will study the behavior of the nonlinear Schrödinger equation with critical exponent $p = 5$ in the defocusing case.

1.3 BASIC THEORY

In this section, we will address some definitions, concepts, and methods used in this thesis. For more details, check (ADAMS; FOURNIER, 2003), (BREZIS, 2011), (MEDEIROS; MIRANDA, 1989) and (SCHWARTZ, 1966).

1.3.1 Theory of distributions and Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ be a continuous function. The support of f is denoted by $supp(f) = \overline{\{x \in \Omega; f(x) \neq 0\}}$. Thus, $supp(f)$ is a closed subset of Ω. A n-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called a multi-index and its order is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We denote by D^α the derivation operator of order α defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}.$$

For $\alpha = (0, 0, \cdots, 0)$, we define

$$D^0 u := u.$$

Let $C_0^\infty(\Omega)$ be the vector space of all the functions defined in Ω which are infinitely differentiable and have compact support contained in Ω. A classic example of a function in $C_0^\infty(\Omega)$ is given below.

Example 1.3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set such that $B_1(0) = \{x \in \mathbb{R}^n; \|x\| < 1\}$ is compactly contained in Ω. Let $f : \Omega \to \mathbb{R}$ be a function such that*

$$f(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \text{if } \|x\| < 1, \\ 0, & \text{if } \|x\| \geq 1, \end{cases}$$

where $x = (x_1, \ldots, x_n)$ and $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ is the Euclidean norm of x. We have $f \in C^\infty(\Omega)$ and $supp(f) = \overline{B_1(0)}$ is compact, so $f \in C_0^\infty(\Omega)$.

Definition 1.3.1. *A sequence $(\varphi_n)_{n\in\mathbb{N}} \in C_0^\infty(\Omega)$ is said to be convergent to $\varphi \in C_0^\infty(\Omega)$ if the following conditions are satisfied:*

(i) There exists a compact $K \subset \Omega$ such that $supp(\varphi) \subset K$ and $supp(\varphi_n) \subset K, \ \forall n \in \mathbb{N}$;

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in $K,$ for all multi-indexes α.

*The space $C_0^\infty(\Omega)$ with this notion of convergence will be denoted by $\mathcal{D}(\Omega)$. It is called **the space of test functions on** Ω.*

A **distribution** over Ω is a continuous linear functional over $\mathcal{D}(\Omega)$. More precisely, a distribution over Ω is a functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ satisfying the following conditions:

(i) $T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi), \ \forall \alpha, \beta \in \mathbb{R}$ and $\forall \varphi, \psi \in \mathcal{D}(\Omega)$;

(ii) T is continuous in the sense of the convergence defined on $\mathcal{D}(\Omega)$, that is, if $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $\mathcal{D}(\Omega)$, then $(T(\varphi_n))_{n\in\mathbb{N}}$ converges to $T(\varphi)$ in $\mathbb{R}$.

It is common to denote the value of the distribution T in φ by $\langle T, \varphi \rangle$. Moreover, the set of all distributions over Ω with the usual operations is a vector space denoted by $D'(\Omega)$. The following examples of scalar distributions play a key role in the theory.

Example 1.3.2. *Let $u \in L^1_{loc}(\Omega)$. The functional $T_u : D(\Omega) \to \mathbb{R}$, defined by*

$$\langle T_u, \varphi \rangle = \int_\Omega u(x)\varphi(x)dx,$$

is a distribution over Ω uniquely determined by u. For this reason, u is identified as the distribution T_u defined by it and $L^1_{loc}(\Omega)$ is identified as a (proper) part of $\mathcal{D}'(\Omega)$.

Definition 1.3.2. *A sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{D}'(\Omega)$ is said to be convergent to T in $\mathcal{D}'(\Omega)$ when the numeric sequence $(\langle T_n, \varphi \rangle)_{n\in\mathbb{N}}$ converges to $\langle T, \varphi \rangle$ in $\mathbb{R}$, for all $\varphi \in \mathcal{D}(\Omega)$.*

Lemma 1.3.1 (Du Bois Raymond)**.** *Let $u \in L^1_{loc}(\Omega)$. Then,*

$$\int_\Omega u(x)\varphi(x)dx = 0, \ \forall \varphi \in \mathcal{D}(\Omega),$$

if and only if, $u = 0$ almost everywhere in Ω.

Example 1.3.3. *Consider $0 \in \Omega$ and the functional $\delta_0 : \mathcal{D}(\Omega) \to \mathbb{R}$ defined by*

$$\langle \delta_0, \varphi \rangle = \varphi(0).$$

*It can be shown that δ_0 is a distribution over Ω called **the Dirac distribution**. Furthermore, the distribution δ_0 is not defined by a function in $L^1_{loc}(\Omega)$.*

Definition 1.3.3. *Let T be a distribution over Ω and α be a multi-index. The derivative $D^\alpha T$ (in the sense of distributions) of T is the functional defined in $\mathcal{D}(\Omega)$ by*

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle,\ \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 1.3.1. *It follows from Definition 1.3.3 that each distribution T over Ω has derivatives of all orders.*

Remark 1.3.2. *Let $T \in \mathcal{D}'(\Omega)$. It is possible to show that $D^\alpha T$ is a distribution over Ω. In fact, it is easy to check that $D^\alpha T$ is linear. For the continuity, consider $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ in $\mathcal{D}(\Omega)$. One has*

$$|\langle D^\alpha T, \varphi_n\rangle - \langle D^\alpha T, \varphi\rangle| \leq |\langle T, D^\alpha \varphi_n - D^\alpha \varphi\rangle| \to 0$$

as $n \to \infty$.

Remark 1.3.3. *The map $D^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ such that $T \mapsto D^\alpha T$ is linear and continuous in the sense of convergence in $\mathcal{D}'(\Omega)$.*

Let $m > 0$ be an integer. The Sobolev space of order m over Ω is the set denoted by $W^{m,p}(\Omega),\ 1 \leq p \leq \infty$, of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$, for every multi-indexes α, with $|\alpha| \leq m$. The space $W^{m,p}(\Omega)$ is a vector space for all $1 \leq p < \infty$. For each $u \in W^{m,p}(\Omega)$, the norm of u is defined by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p dx\right)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} supess_{x\in\Omega}|D^\alpha u(x)|$$

if $p = \infty$. The Sobolev space $W^{m,p}(\Omega)$ endowed with the norm above is a Banach space.

Remark 1.3.4. *When $p = 2$, the space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which endowed with the inner product*

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

is a Hilbert space.

Denote by $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$ with respect to the norm of the space $H^m(\Omega)$. The set $H_0^m(\Omega)$ endowed with the induced inner product of $H^m(\Omega)$ is a vector subspace. Furthermore, it is possible to prove that the induced inner product of $H^m(\Omega)$ and the induced norm of $H^m(\Omega)$ are equivalent, respectively, to

$$((u,v))_{H^m(\Omega)} = \sum_{|\alpha|=m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

and

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^2 dx.$$

We have the following results:

Lemma 1.3.2 (Poincaré-Friedrichs inequality). *Let Ω be a bounded open subset of $\mathbb{R}^n$. If $u \in H_0^1(\Omega)$, there exists a constant $C > 0$, depending only on Ω, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}.$$

Lemma 1.3.3 (Gagliardo–Nirenberg inequality). *Let $I = (0,1)$, $1 \leq q < \infty$ and $1 < r \leq \infty$. Then,*

$$\|u\|_{L^\infty(I)} \leq C\|u\|^a_{W^{1,r}(I)}\|u\|^{1-a}_{L^q(I)}, \ \forall u \in W^{1,r}(I)$$

for some constant $C = C(q,r)$, where $0 < a < 1$ is defined by $a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}$.

Lemma 1.3.4 (Sobolev embedding). *Suppose $1 \leq p < n$ and consider*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n}.$$

For each $\varphi \in \mathcal{D}(\mathbb{R}^n)$, there exists $C = C(p,n) > 0$ such that

$$\|\varphi\|_{L^q(\mathbb{R}^n)} \leq C\sum_{i=1}^n \|D_i\varphi\|_{L^p(\mathbb{R}^n)}.$$

Lemma 1.3.5 (Sobolev embedding in a bounded open set). *Let Ω be a bounded open set of $\mathbb{R}^n$, Ω of class C^m and $1 \leq p < \infty$.*

(i) If $n > 2m$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, where $p \in \left[1, \frac{2n}{n-2m}\right]$;

(ii) If $n = 2m$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, where $p \in [1, +\infty[$;

(iii) If $n = 1$ and $m \geq 1$, then $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$.

Here the symbol $\hookrightarrow$ denotes a continuous embedding.

Lemma 1.3.6 (Rellich-Kondrachov). *Let Ω be a bounded open set of $\mathbb{R}^n$, Ω of class C^m.*

(i) If $n > 2m$, then $H^m(\Omega)$ is compactly embedding in $L^p(\Omega)$, where $p \in \left[1, \frac{2n}{n-2m}\right[$.

(ii) If $n = 2m$, then $H^m(\Omega)$ is compactly embedding in $L^p(\Omega)$, where $p \in [1, +\infty[$.

(iii) If $2m > n$ and $m \geq 1$, then $H^m(\Omega)$ is compactly embedding in $C^k(\overline{\Omega})$, where k is a non-negative integer such that $k < m - \frac{n}{2} \leq k+1$.

We denote by $L^p(0,T;X)$, with $1 \leq p < \infty$, the Banach space of (classes of) functions u defined in $]0,T[$ with values in X that are strongly measurable and $\|u(t)\|_X^p$ is Lebesgue integrable on $]0,T[$. The norm of $L^p(0,T;X)$ is defined by

$$\|u(t)\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}}.$$

Additionally, $L^\infty(0,T;X)$ represents the Banach space of (classes of) functions u defined in $]0,T[$ with values in X that are strongly measurable and $\|u(t)\|_X$ has essential supremum finite on $]0,T[$. The norm of $L^\infty(0,T;X)$ is defined by

$$\|u(t)\|_{L^\infty(0,T;X)} = supess_{t\in]0,T[}\|u(t)\|_X.$$

Remark 1.3.5. *If $p = 2$ and X is a Hilbert space, the space $L^2(0,T;X)$ is a Hilbert space with respect to the ineer product*

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X dt.$$

Consider the space $L^p(0,T;X),\ 1 < p < \infty$, where X is a separable Hilbert space. We identify

$$[L^p(0,T;X)]' \approx L^q(0,T;X'),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. When $p = 1$, we identify

$$[L^1(0,T;X)]' \approx L^\infty(0,T;X').$$

Let X be a Banach space. The vector space of linear and continuous maps of $\mathcal{D}(0,T)$ on X is called the space of vector distributions on $]0,T[$ with values in X. It is denoted by $\mathcal{D}'(0,T;X)$.

Example 1.3.4. *Given $u \in L^p(0,T;X), 1 \leq p < \infty$, and $\varphi \in \mathcal{D}(0,T)$, the application $Tu : \mathcal{D}(0,T) \to X$, defined by*

$$Tu(\varphi) = \int_0^T u(t)\varphi(t)dt$$

and usually called Bochner's integral on X, is linear and continuous in the sense of the convergence in $\mathcal{D}(0,T)$. Thus, T is a distribution. The map $u \mapsto Tu$ is injective, so we identify u with Tu and, in this sense,

$$L^p(0,T;X) \subset \mathcal{D}'(0,T;X).$$

Definition 1.3.4. *Given $S \in \mathcal{D}'(0,T;X)$. The derivative of order n of S is the vector distribution over $]0,T[$ with values in X given by*

$$\left\langle \frac{d^n S}{dt^n}, \varphi \right\rangle = (-1)^n \left\langle S, \frac{d^n \varphi}{dt^n} \right\rangle, \ \forall \varphi \in \mathcal{D}(0,T).$$

Consider the space

$$W^{m,p}(0,T;X) = \{u \in L^p(0,T;X); u^{(j)} \in L^p(0,T,X), j = 1, \ldots, m\},$$

where $u^{(j)}$ represents the $j-$th derivative of u in the sense of vector distributions endowed with the norm

$$\|u\|_{W^{m,p}(0,T;X)} = \left(\sum_{j=0}^{m} \|u^{(j)}\|^p_{L^p(0,T;X)} \right)^{\frac{1}{p}}.$$

The space $\left(W^{m,p}(0,T;X), \|\cdot\|_{W^{m,p}(0,T;X)}\right)$ is a Banach space.

Remark 1.3.6. *When $p = 2$ and X is a Hilbert space, the space $W^{m,p}(0,T;X)$ will be denoted by $H^m(0,T;X)$. Endowed with the inner product*

$$(u,v)_{H^m(0,T;X)} = \sum_{j=0}^{m} (u^{(j)}, v^{(j)})_{L^2(0,T;X)},$$

it is a Hilbert space. We denote by $H_0^m(0,T;X)$ the closure of $\mathcal{D}(0,T;X)$ in $H^m(0,T;X)$ and denote by $H^{-m}(0,T;X)$ the topological dual of $H_0^m(0,T;X)$.

The following lemma can be found in (AUBIN, 1963).

Lemma 1.3.7 (Aubin-Lions lemma)**.** *Let $X_0,\ X$ and X_1 be Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p, q \leq \infty$, let*

$$W = \{u \in L^p([0,T], X_0); u' \in L^q([0,T]; X_1)\}.$$

(i) If $p < \infty$, then the embedding of W into $L^p([0,T], X)$ is compact.

(ii) If $p = \infty$ and $q > 1$, then the embedding of W into $C([0,T]; X)$ is compact.

1.3.2 Interpolation of Sobolev spaces

The results that we will state from now on, as well as their demonstrations, can be found in (LIONS; MAGENES, 1968) and (TAYLOR, 2011).

Let X and Y be two separable Hilbert spaces with a continuous and dense embedding $X \hookrightarrow Y$. Let $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ be the inner products of X and Y, respectively. We denote by $D(S)$ the set of all functions u defined in X such that the application $v \longmapsto (u,v)_X, v \in X$, is continuous in the topology induced by Y. Moreover, the identification $(u,v)_X = (Su,v)_Y$ defines S as an unbounded operator on Y with domain $D(S)$ dense in Y. Since S is a self-adjoint and strictly positive operator, using the spectral decomposition of self-adjoint operators, we define $S^\theta,\ \theta \in \mathbb{R}$. In particular, we use $A = S^{\frac{1}{2}}$. The operator A is self-adjoint, positive and defined on Y with domain X and

$$(u,v)_X = (Au, Av)_Y, \forall u, v \in X.$$

Definition 1.3.5. *With the previous assumptions, we define the intermediate space*

$$[X,Y]_\theta = D \ (\textit{domain of } A^{1-\theta}),\ 0 \leq \theta \leq 1,$$

with norm

$$\|u\|_{[X,Y]_\theta} = (\|u\|_Y^2 + \|A^{1-\theta}u\|_Y^2)^{\frac{1}{2}}.$$

Note that

1. $X \hookrightarrow [X,Y]_\theta \hookrightarrow Y$;

2. $\|u\|_{[X,Y]_\theta} \leq \|u\|_X^{1-\theta}\|u\|_Y^\theta$;

3. If $0 < \theta_0 < \theta_1 < 1$, then $[X,Y]_{\theta_0} \hookrightarrow [X,Y]_{\theta_1}$;

4. $[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\theta = [X,Y]_{(1-\theta)\theta_0+\theta\theta_1}$.

Theorem 1.3.1. *Let* $s_1, s_2 \in \mathbb{R},\ s_1 \geq s_2$. *If* $s = (1-\theta)s_1 + \theta s_2$, *then*

$$[H^{s_1}(\mathbb{R}^3), H^{s_2}(\mathbb{R}^3)]_\theta = H^s(\mathbb{R}^3)$$

and

$$\|u\|_{[H^{s_1}(\mathbb{R}^3),H^{s_2}(\mathbb{R}^3)]_\theta} \leq \|u\|_{H^{s_1}(\mathbb{R}^3)}^{1-\theta}\|u\|_{H^{s_2}(\mathbb{R}^3)}^\theta.$$

1.3.3 Some Important Inequalities

Let us present a series of inequalities that will be used throughout this thesis. The results are classical and the proofs will be omitted. For more details see, e.g., (ADAMS; FOURNIER, 2003) and (BREZIS, 2011).

Lemma 1.3.8 (Young's Inequality)**.** *Let a and b be positive constants. If $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 1.3.9 (Generalized Young's Inequality)**.** *Let a and b be positive constants, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that*

$$ab \leq \epsilon a^p + C(\epsilon) b^q.$$

Lemma 1.3.10 (Cauchy-Schwarz's Inequality)**.** *Let $(E, \langle \cdot, \cdot \rangle)$ be a vector space with an inner product and $\| \cdot \|$ be its induced norm. One has*

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \ \forall x, y \in E.$$

Furthermore, the equality holds if, and only if, x and y are linearly dependent.

Lemma 1.3.11 (Hölder's Inequality)**.** *Let $f \in L^p(\Omega)$ e $g \in L^q(\Omega)$ with $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} = \int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

1.3.4 Semigroup theory

We state some results on semigroup theory. The results can be found in (PAZY, 2012). In what follows, we denote by $(X, \| \cdot \|_X)$ a Banach space.

Definition 1.3.6. *Let $\mathcal{L}(X)$ be the algebra of bounded linear operators over X. The application $S : \mathbb{R}^+ \to \mathcal{L}(X)$ is a C_0-semigroup of bounded operators on X if*

(i) $S(0) = I$, *where I is the identity operator on X;*

(ii) $S(t+s) = S(t)S(s)$, *for all $t, s \in \mathbb{R}^+$;*

(iii) $\lim_{t\to 0^+} \|(S(t)-I)x\|_X = 0$, *for all* $x \in X$.

Proposition 1.3.1. *If* $S : \mathbb{R}^+ \to \mathcal{L}(X)$ *is a* C_0*-semigroup, then*

$$\lim_{t\to\infty} \frac{\mathit{ln}\|S(t)\|_{\mathcal{L}(X)}}{t} = \inf_{t>0} \frac{\mathit{ln}\|S(t)\|_{\mathcal{L}(X)}}{t} = \omega_0.$$

Furthermore, for every $\omega > \omega_0$, *there exists a constant* $M \geq 1$ *such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \text{ for all } t \geq 0. \tag{1.2}$$

Remark 1.3.7. *If* $\omega_0 < 0$, *it follows by* (1.2) *that there exists* $M \geq 1$ *such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq M, \text{ for all } t \geq 0.$$

Moreover, when $M \leq 1$, *we call* $S : \mathbb{R}^+ \to \mathcal{L}(X)$ *a* C_0*-semigroup of contractions.*

Definition 1.3.7. *Let* $S : \mathbb{R}^+ \to \mathcal{L}(X)$ *be a* C_0*-semigroup. The operator*

$$A : D(A) \subset X \to X$$

with domain $D(A)$ *and value in* x *defined, respectively, by*

$$D(A) := \left\{ x \in X; \exists \lim_{h\to 0^+} \left(\frac{S(h)-I}{h} \right) x \right\}$$

and

$$Ax := \lim_{h\to 0} \left(\frac{S(h)-I}{h} \right) x,$$

is called the infinitesimal generator of the C_0*-semigroup* $S(t)$.

Remark 1.3.8. *It is easy to see that if* $D(A) \subset X$ *is a nonempty subset, then* $D(A)$ *is a subspace of* X *and* A *is a linear operator.*

Proposition 1.3.2. *Let* $S : \mathbb{R}^+ \to \mathcal{L}(X)$ *be a* C_0*-semigroup and* $A : D(A) \subset X \to X$ *its infinitesimal generator.*

(i) If $x \in D(A)$, *then* $S(t)x \in D(A)$, *for all* $t \geq 0$, *and*

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \ \forall t \geq 0.$$

(ii) If $x \in D(A)$, *then*

$$S(t)x - S(s)x = \int_s^t AS(\xi)x d\xi = \int_s^t S(\xi)Axd\xi, \ 0 \leq s \leq t.$$

(iii) If $x \in X$, then $\int_0^t S(\xi)xd\xi \in D(A)$ and

$$A\int_0^t S(\xi)xd\xi = S(t)x - x.$$

Definition 1.3.8. *Let $A : D(A) \subset X \to X$ be a linear operator. The resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, that is, $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1},\ \lambda \in \rho(A)$, of bounded linear operators is called the resolvent of A.*

1.3.4.1 The Hille-Yosida and Lumer-Phillips theorems

This subsection presents two theorems that establish necessary and sufficient conditions for a linear operator $A : D(A) \subset X \to X$ to generate a C_0-semigroup.

Theorem 1.3.2. *(Hille-Yosida) A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \geq 0$ if, and only if,*

(i) A is closed and $\overline{D(A)} = X$;

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}_+$ and, for all $\lambda > 0$, one has

$$\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Before presenting the next result, we need another concept. Let X be a Banach space and let X^* be its dual space. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$. For every $x \in X$, define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{x^*;\ x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2\right\}.$$

Remark 1.3.9. *From the Hahn-Banach theorem, it follows that $F(x) \neq \emptyset$, for all $x \in X$.*

Definition 1.3.9. *A linear operator $A : D(A) \subset X \to X$ is dissipative if, for all $x \in D(A)$, there exists $x^* \in F(X)$ such that $\langle Ax, x^*\rangle \leq 0$.*

Theorem 1.3.3. *(Lumer-Phillips) Let $A : D(A) \subset X \to X$ be a linear operator with $\overline{D(A)} = X$.*

(a) *If A is dissipative and there is $\lambda_0 > 0$ such that the range $Ran(\lambda_0 I - A)$ of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.*

(b) *If A is the infinitesimal generator of a C_0-semigroup of contractions on X, then $Ran(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover,*

$$\langle Ax, x^* \rangle \leq 0, \text{ for all } x \in D(A) \text{ and } x^* \in F(x).$$

Corolary 1.3.1. *Let $A : D(A) \subset X \to X$ be a linear closed operator with $\overline{D(A)} = X$. If both A and its adjoint A^* are dissipative, A is a generator of a C_0-semigroup of contractions on X.*

Definition 1.3.10. *A semigroup S of linear and bounded operators on a Hilbert space H is said to be a unitary semigroup if, for each $t \geq 0$, $S(t)$ is a unitary operator, that is, $S(t)^* = S(t)^{-1}$ for all $t \geq 0$.*

1.3.4.2 The abstract Cauchy problem: The linear case

Let $A : D(A) \subset X \to X$ be a linear operator. Given $u_0 \in X$, the abstract Cauchy problem for A with initial data u_0 consists of finding a solution $u(t)$ to the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Let us introduce a notion of solution to the problem (1.3).

Definition 1.3.11. *(Classical solution) A function $u : \mathbb{R}^+ \to X$ is a classical solution of problem (1.3) for all $t \geq 0$ if u is continuous for all $t \geq 0$, continuously differentiable on $\mathbb{R}^+$, $u(t) \in D(A)$ for all $t \in \mathbb{R}^+$, $u(0) = u_0$ and the equation in (1.3) is satisfied for all $t > 0$.*

Remark 1.3.10. *Let $S : \mathbb{R}^+ \to X$ be a C_0-semigroup. Due to Proposition 1.3.2, if $u_0 \in D(A)$ and A is its infinitesimal generator, $u(\cdot) = S(\cdot)u_0 : \mathbb{R}^+ \to D(A)$ is a classical solution of problem (1.3). Moreover, $S(\cdot)u_0$ is the unique solution of problem (1.3).*

Consider the inhomogeneous Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t), \ t > 0, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $f :^{+} \to X$ is a continuous function. Let A be the infinitesimal generator of a semigroup S of class C_0. Similarly to Definition 1.3.11, we have the following definition for the classical solution of the problem (1.4) with $T > 0$ a fixed constant.

Definition 1.3.12. *(Classical solution) A function $u : [0, T[\to X$ is a classical solution of problem (1.4) for all $t \in [0, T[$ if u is continuous on $[0, T[$, continuously differentiable on $]0, T[$, $u(t) \in D(A)$ for all $t \in]0, T[$, $u(0) = u_0$ and the equation in (1.4) is satisfied for all $t \in (0, T)$.*

Suppose that A is an infinitesimal generator of a C_0- semigroup S and $u(t)$ is a classical solution of problem (1.4). Then, $v(s) = S(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\frac{dv}{ds} = S(t-s)f(s). \tag{1.5}$$

Hence, if $f \in L^1(0, T; X)$, $S(t-s)f(s)$ is integrable on $[0, t]$, integrating equation (1.5) from 0 to t yields

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds. \tag{1.6}$$

As a consequence, the equation (1.6) has at most one solution $u \in C([0, T]; X)$. Moreover, it is natural to define a generalized solution of problem (1.4).

Definition 1.3.13. *Let $u_0 \in X$ and $f \in L^1([0, T]; X)$. The function $u \in C([0, T]; X)$ given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \ 0 \le t \le T,$$

is called the mild (generalized) solution of the inhomogeneous Cauchy problem (1.4) on $[0, T]$.

Remark 1.3.11. *In general, the homogeneous Cauchy problem (1.3) does not have a classical solution, since, in general, $u_0 \notin D(A)$. Taking $f \equiv 0$ in Definition 1.3.13, $u(\cdot) = S(\cdot)u_0$ is the mild solution of problem (1.3) since $u_0 \in X$. It is therefore clear that not every mild solution of problem (1.4) is a classical solution even in the case $f \equiv 0$.*

Let us present another notion of solution to the Cauchy problem (1.4):

Definition 1.3.14. *(Strong solution): Let u be an almost everywhere differentiable function on $[0, T]$ such that $\frac{du}{dt} \in L^1([0, T]; X)$. We say that u is a strong solution of the Cauchy problem (1.4) if $u(0) = u_0$ and*

$$\frac{du}{dt} = Au + f,$$

almost everywhere on $[0, T]$.

Remark 1.3.12. *Observe that if $A = 0$ and $f \in L^1([0,T];X)$, then the Cauchy problem (1.4) has usually no solution unless $f \in C([0,T];X)$. However, problem (1.4) has always a strong solution given by*

$$u(t) = u_0 + \int_0^t f(s)ds.$$

Moreover, it is easy to show that if u is a strong solution of problem (1.4) and $f \in L^1([0,T];X)$, then u is a mild solution as well.

1.3.4.3 The abstract Cauchy problem: The nonlinear case

Let $(X, \|.\|_X)$ be a reflexive Banach space. Consider the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(u(t)), \ t > 0, \\ u(0) = u_0, \end{cases} \tag{1.7}$$

where $F : X \to X$ is a continuous function and $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $S : \mathbb{R}^+ \to \mathcal{L}(X)$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq M, \forall t \geq 0$. If u is either a classical solution or a strong solution of problem (1.7), it is not difficult to see that u satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds,$$

and u is a mild solution.

Theorem 1.3.4. *Let $F : X \to X$ be a Lipschitz function, i.e., there exists $L > 0$ such that*

$$\|F(u) - F(v)\|_X \leq L\|u - v\|_X, \ \forall u, v \in X.$$

For all $u_0 \in X$, there exists an unique mild solution of problem (1.7) $u \in C(\mathbb{R}^+; X)$. Moreover,

(i) If $u_0, v_0 \in X$ are initial data and u, v are their respective mild solutions of problem (1.7), then

$$\|u(t) - v(t)\|_X \leq Me^{LMt}\|u_0 - v_0\|_X.$$

(ii) If $u_0 \in D(A)$, then u is a strong solution of problem (1.7) on $[0,T]$.

1.4 PSEUDODIFFERENTIAL OPERATORS AND MICROLOCAL ANALYSIS

1.4.1 Tempered distributions and the Schwartz space

Definition 1.4.1. *A function $\varphi \in C^\infty(\mathbb{R}^n)$ is said to be rapidly decreasing at infinity if for each $k \in \mathbb{N}$ we have*

$$p_k(\varphi) = \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} (1 + \|x\|^2)^k |D^\alpha \varphi(x)| < \infty, \text{ for all } \alpha \in \mathbb{N}.$$

This is equivalent to

$$\lim_{\|x\|\to\infty} p(x) D^\alpha \varphi(x) = 0$$

for any polynomial p of n real variables and for all $\alpha \in \mathbb{N}^n$. Let $\mathcal{S}(\mathbb{R}^n)$ be the vector space of rapidly decreasing functions at infinity. Define the following notion of convergence in $\mathcal{S}(\mathbb{R}^n)$: a sequence (φ_ν) of functions of $\mathcal{S}(\mathbb{R}^n)$ converges to zero if, for all $k \in \mathbb{N}$, the sequence $(p_k(\varphi_\nu))$ converges to zero in $\mathbb{R}$ (or $\mathbb{C}$). The sequence (φ_ν) converges to φ in $\mathcal{S}(\mathbb{R}^n)$ if $(p_k(\varphi_\nu - \varphi))$ converges to zero in $\mathbb{R}$ (or $\mathbb{C}$), for all $k \in \mathbb{N}$.

The linear forms defined in $\mathcal{S}(\mathbb{R}^n)$ which are continuous in the sense of the convergence defined in $\mathcal{S}(\mathbb{R}^n)$ are called tempered distributions. The vector space of all tempered distributions will be represented by $\mathcal{S}'(\mathbb{R}^n)$.

Remark 1.4.1.

- *If $1 \leq p \leq \infty$, then $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$;*
- *If $1 \leq p < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$;*
- *$C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.*

1.4.2 Fourier transform

Definition 1.4.2. *Given a function $u \in L^1(\mathbb{R}^n)$, its Fourier transform is function $\mathcal{F}u$ defined in $\mathbb{R}^n$ by*

$$(\mathcal{F}u)(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\ dx,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n$. The application $(\mathcal{F}^{-1}u)(\xi) = (2\pi)^{-\frac{n}{2}}(\mathcal{F}u)(-\xi)$, for all $\xi \in \mathbb{R}^n$, is called the inverse Fourier transform of u. One has $\overline{\mathcal{F}u} = \mathcal{F}^{-1}\overline{u}$, where $\overline{u}$ the complex conjugate of u. The Fourier's inversion formula is

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi)\ d\xi.$$

Since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}\varphi$ and $\mathcal{F}^{-1}\varphi$ are well defined and it is possible to show that they are rapidly decreasing at infinity. In addition, $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\overline{\mathcal{F}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ are continuous isomorphisms. For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\mathcal{F}(D^\alpha \varphi) = i^{|\alpha|} x^\alpha \mathcal{F}\varphi, \quad D^\alpha(\mathcal{F}\varphi) = \mathcal{F}((-i)^{|\alpha|} x^\alpha \varphi)$$

$$(\mathcal{F}\varphi, \mathcal{F}\psi)_{L^2(\mathbb{R}^n)} = (\varphi, \psi)_{L^2(\mathbb{R}^n)} = (\mathcal{F}^{-1}\varphi, \mathcal{F}^{-1}\psi)_{L^2(\mathbb{R}^n)}.$$

Theorem 1.4.1. *(Plancherel's Theorem) The applications $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\mathcal{F}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isomorphisms of Hilbert spaces such that*

$$(\mathcal{F}u, \mathcal{F}v)_{L^2(\mathbb{R}^n)} = (u, v)_{L^2(\mathbb{R}^n)} = (\mathcal{F}^{-1}u, \mathcal{F}^{-1}v)_{L^2(\mathbb{R}^n)},$$

for every pair $u, v \in L^2(\mathbb{R}^n)$.

The notations $\widehat{u}$ and $\check{u}$ will also be used to denote $\mathcal{F}u$ and $\mathcal{F}^{-1}u$, respectively.

1.4.3 Differential operators

From here we will follow the content explored in (BURQ; GÉRARD, 2002) and (CAVALCANTI; CAVALCANTI, 2014). In what follows, Ω is an open and nonempty subset of $\mathbb{R}^n$.

Definition 1.4.3. *A differential operator on Ω is a linear map $P : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ of the form*

$$Pu(x) := \sum_{|\alpha| \le m} a_\alpha(x) \partial_x^\alpha u(x), \tag{1.8}$$

where $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$ and the complex valued functions a_α are C^∞ in Ω. The greatest integer m for such that the functions a_α, $|\alpha| = m$, are not all zero is called the order of P.

As mentioned before, considering the Fourier inversion formula, we obtain

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{u}(\xi)\, d\xi,$$

since $\widehat{(\partial_x^\alpha) u}(\xi) = (i\xi)^\alpha \widehat{u}(\xi)$. Observing that $\partial_x^\alpha u \in \mathcal{S}(\mathbb{R}^n)$, we get

$$\begin{aligned} \partial^\alpha u(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \widehat{\partial_x^\alpha u}(\xi)\, d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} i^{|\alpha|} e^{ix\cdot\xi} \xi^\alpha \widehat{u}(\xi)\, d\xi \end{aligned}$$

which implies that expression in (1.8) can be rewritten as

$$\begin{aligned} Pu(x) &= \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha u(x) \\ &= \sum_{|\alpha|\leq m} a_\alpha(x) i^{|\alpha|}(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\xi^\alpha \widehat{u}(\xi)\ d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\sum_{|\alpha|\leq m} a_\alpha(x) i^{|\alpha|}\xi^\alpha \right) \widehat{u}(\xi)\ d\xi. \end{aligned}$$

The map $p : \Omega \times \mathbb{R}^n \to \mathbb{C}$ defined by

$$p(x,\xi) := \sum_{|\alpha|\leq m} a_\alpha(x)(i\xi)^\alpha$$

is called *symbol* of P. In other words, differential operators with C^∞ coefficients on Ω are the operators of the form

$$Pu(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{u}(\xi)\ d\xi, \tag{1.9}$$

where $p(x,\xi)$ is a polynomial in ξ with coefficients that are C^∞ functions of $x \in \Omega$ so that the above integral makes sense.

Remark 1.4.2. *Adopting the notation*

$$D = \frac{1}{i}\partial, D_j = \frac{1}{i}\partial_j \text{ and } D^\alpha = \frac{1}{i^{|\alpha|}}\partial^\alpha,$$

and introducing the symbolic multi-index $D = (D_1, ..., D_n)$*, where* $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$*, the operator* P *can be written as*

$$P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha = \sum_{|\alpha|\leq m} a_\alpha(x) i^{|\alpha|} D^\alpha = p(x, D).$$

Now, we list some results that help to characterize the differential operators.

Proposition 1.4.1. *If* $P = p(x, D)$ *and* $Q = q(x, D)$ *are differential operators on* Ω *of order* m *and* n*, respectively, then the composition* PQ *is a differential operator of order at most* $m + n$ *and its symbol is given by*

$$p\#q(x,\xi) = \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi) D^\alpha q(x,\xi),$$

where the sum is finite.

Proposition 1.4.2. *If $P = p(x, D)$ is a differential operator of order m on Ω, then there exists a differential operator P^* of order m on Ω such that, for all $u, v \in \mathcal{D}(\Omega)$,*

$$(Pu, v)_{L^2(\Omega)} = (u, P^* v)_{L^2(\Omega)}.$$

The symbol of P^* is given by the finite sum

$$p^*(x,\xi) = \sum_{|\alpha| \le m} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha \partial_\xi^\alpha \overline{p}(x,\xi).$$

Definition 1.4.4. *If P is a differential operator of order at most m and symbol p, we call the principal symbol of P, which we will denote by $\sigma_m(P)$, the homogeneous part of degree m in ξ of the polynomial function $p(x,\xi)$, namely*

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha, \quad \text{if } P = \sum_{|\alpha| \le m} a_\alpha(x)\partial_x^\alpha.$$

Remark 1.4.3. *Note that $\sigma_m(P)$ is a homogeneous polynomial of degree m in ξ, i.e., a polynomial such that $\sigma_m(P)(x, \lambda\xi) = \lambda^m \sigma_m(P)(x,\xi)$, $\forall \lambda \in \mathbb{R}$. As a result, we can reconstruct the principal symbol from its value at $|\xi| = 1$. Indeed, note that*

$$\sigma_m(P)(x,\xi) = |\xi|^m \sigma_m(P)\left(x, \frac{\xi}{|\xi|}\right), \quad \text{for all } (x,\xi),\ \xi \neq 0.$$

Therefore, it is enough to consider $\xi \in S^{n-1}$, where n is the space dimension.

Definition 1.4.5. *If f, g are C^∞ functions defined in an open set of $\mathbb{R}_x \times \mathbb{R}^n_\xi$, the Poisson bracket of the functions f and g is defined by*

$$\{f,g\}(x,\xi) = \sum_{j=1}^n \left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j} \right).$$

Propositions 1.4.1 and 1.4.2 lead us to the following corollary.

Corolary 1.4.1. *If P is of order m and Q of order n, then*

$$(i)\ \sigma_{m+n}(PQ) = \sigma_m(P)\sigma_n(Q);$$

$$(ii)\ \sigma_{m+n-1}([P,Q]) = \frac{1}{i}\{\sigma_m(P), \sigma_n(Q)\};$$

$$(iii)\ \sigma_m(P^*) = \overline{\sigma_m(P)},$$

where $[P,Q]$ means the commutator of operators, i.e, $[P,Q] = PQ - QP$ and $\{f,g\}$ is the Poisson bracket.

1.4.4 Pseudodifferential calculus

Definition 1.4.6. *Let $m \in \mathbb{R}$. A symbol of order at most m in Ω is a function $a : \Omega \times \mathbb{R} \to \mathbb{C}$ of class C^∞ verifying the following estimates: for all $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n$, there exists a constant $C_{\alpha\beta}$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1 + |\beta|)^{m-|\beta|}.$$

We denote by $S^m(\Omega \times \mathbb{R}^n)$ the vector space of symbols of order at most m in Ω.

Definition 1.4.7. *If $a \in S^m(\Omega \times \mathbb{R}^n)$, the formula*

$$Au(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\ d\xi$$

defines, for all $u \in \mathcal{D}(\Omega)$, an element Au of $\mathcal{D}(\Omega)$. The linear application A is called a pseudodifferential operator on Ω of symbol a. The set of all pseudodifferential operators of order m on Ω will be denoted by $\Psi^m(\Omega)$.

Definition 1.4.8. *An operator $A \in \Psi^m(\Omega)$ is essentially homogeneous if there exists a function $a_m = a_m(x,\xi)$, homogeneous of order m in ξ, smooth except at $\xi = 0$, and a function $\chi \in C^\infty(\mathbb{R}^n)$ being zero near 0 and 1 in the infinity such that*

$$a(x,\xi) = a_m(x,\xi)\chi(\xi) + r(x,\xi),$$

for some $r \in S^{m-1}(\Omega)$.

Proposition 1.4.3. *Let $A \in \Psi^m(\Omega)$ essentially homogeneous. Then, for all $u \in \mathcal{D}(\Omega)$, for all $\xi \in \mathbb{R}^n \backslash \{0\}$, and for all $x \in \Omega$,*

$$t^{-m} e^{-i(tx)\cdot\xi} A(ue_{t\xi})(x) \to a_m(x,\xi)u(x), \quad \text{as } t \to +\infty, \tag{1.10}$$

where $e_\xi = e^{ix\cdot\xi}$.

Definition 1.4.9. *Under the conditions of Proposition 1.4.3, we say that A admits a principal symbol of order m; the function a_m characterized by (1.10) is called the principal symbol of order m of A, and it is denoted by $\sigma_m(A)$.*

We state some theorems of symbolic calculus.

Theorem 1.4.2. *Let A be a pseudodifferential operator of symbol $a \in S^m(\Omega)$ and let $\chi \in C_0^\infty(\Omega)$ such that $\chi(x) = 1$ for the values close to the projection of x of the support of a. There exists a pseudodifferential operator A_χ^* on Ω such that*

$$(A(\chi u), v)_{L^2(\Omega)} = (u, A_\chi^* v)_{L^2(\Omega)},$$

for all $u, v \in \mathcal{D}(\Omega)$. In addition, A_χ^* admits a symbol $a_\chi^* \in S^m(\Omega)$ verifying

$$a_\chi^* - \sum_{|\alpha| \le N} \frac{1}{\alpha!} D_x^\alpha \partial_\xi^\alpha \overline{a} \in S^{m-N-1}(\Omega),$$

for all $N \in \mathbb{N}$. In particular, if A admits a principal symbol of order m, then it is the same of A^* and

$$\sigma_m(A_\chi^*) = \overline{\sigma_m(A)}.$$

Theorem 1.4.3. *Let A and B be pseudodifferential operators with symbols $a \in S^m(\Omega)$, $b \in S^n(\Omega)$, respectively. The composition AB is a pseudodifferential operator which admits a symbol $S^{m+n}(\Omega)$ verifying*

$$a\#b - \sum_{|\alpha| \le N} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \in S^{m+n-N-1}(\Omega),$$

for all $N \in \mathbb{N}$. In particular, if A admits a principal symbol of order m and B admits a principal symbol of order n, then AB admits a principal symbol of order $m + n$ and $[A, B]$ admits a principal symbol of order $m + n - 1$ given by

$$\sigma_{m+n}(AB) = \sigma_m(A)\sigma_m(B),$$

$$\sigma_{m+n-1}([A, B]) = \frac{1}{i}\{\sigma_m(A), \sigma_n(B)\}.$$

1.4.5 Microlocal defect measures

Let Ω be an open set of $\mathbb{R}^d$. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2_{loc}(\Omega)$, i.e.,

$$\sup_{n\in\mathbb{N}} \int_K |u_n(x)|^2 \, dx < +\infty,$$

for all compact set $K \subset \Omega$. We say that u_n converges weakly to $u \in L^2_{loc}(\Omega)$ if

$$\int_\Omega u_n(x) f(x) \, dx \longrightarrow \int_\Omega u(x) f(x) \, dx$$

as $n \to +\infty$, for all $f \in L^2_{comp}(\Omega) = \bigcup_K L^2_K(\Omega)$ (when K ranges over all compact subsets of Ω). Once $(u_n)_{n\in\mathbb{N}}$ converges weakly to u in $L^2_{loc}(\Omega)$, then $(u_n)_{n\in\mathbb{N}}$ converges in the distributional sense to u, namely, $u_n \to u$ in $\mathcal{D}'(\Omega)$. We are interested in a description of the loss of strong compactness in $L^2_{loc}(\Omega)$ for the set $\{u\} \cup \{u_n; n \in \mathbb{N}\}$. To address this subject we need the notion of *defect measure*.

Proposition 1.4.4. *The sequence*

$$\nu_n = |u_n - u|^2,$$

converges weakly to a positive Radon measure ν called the defect measure of $(u_n)_{n\in\mathbb{N}}$.

Remark 1.4.4. *The support of ν is the set of points in Ω near which u_n does not converge to u in the strong topology of L^2. This notion provides the first tool for the classification of defects of compactness. Thus ν is defined by*

$$(\varphi(u_n - u), u_n - u)_{L^2(\Omega)} \to \int \varphi \, d\nu, \quad \forall \varphi \in C_0^\infty(\Omega) \tag{1.11}$$

as $n \to +\infty$, and, consequently, $u_n \to u$ as $n \to +\infty$, strongly in L^2_{loc} if, and only if, $\nu \equiv 0$.

It is natural to look for a generalization of the formula (1.11) in which the multiplication by test functions φ is replaced by the testing operators, bounded on L^2, which are able to select the possible frequencies of the sequence (u_n). This can be achieved by using the class of pseudodifferential operators of order zero and the corresponding object is then a positive Radon measure μ on $T^1\Omega := \Omega \times S^{d-1}$, whose concept is introduced in Gérard (GÉRARD, 1991) and Tartar (TARTAR, 1990). This type of measure is called **microlocal defect measure**, since it provides microlocal quantitative information on the sequence (u_n).

1.5 OBSERVABILITY INEQUALITIES

The results of this subsection can be seen in (LIONS, 1988). To obtain the stabilization in which we are interested in this thesis, it will be necessary to use an observability inequality. We introduce this concept below.

Let $(X, \|\ \|)$ be a Banach space, X' the dual space of X, where $\langle\langle\ ,\ \rangle\rangle$ indicates the duality between X' and X, and let $A : D(A) \subset X \to X$ be a linear operator. Define

$$D(A^*) = \{u^* \in X'; \exists v^* \in X' \text{ such that } \langle\langle u^*, Au\rangle\rangle = \langle\langle v^*, u\rangle\rangle, \forall u \in D(A)\}.$$

When $D(A)$ is dense in X, the vector v^* corresponding to u^* is unique. This allows us to define the adjoint operator A^* as

$$\begin{array}{rcl} A^* : D(A^*) \subset X' & \longrightarrow & X' \\ u^* & \longmapsto & A^*u^* = v^*. \end{array}$$

If $(X, \langle\ ,\ \rangle)$ is a Hilbert space, its dual can be identified with the space X itself. In this case, the inner product on X represents this identification, i.e.,

$$\langle\langle u^*, v\rangle\rangle = \langle u^*, v\rangle.$$

So, the adjoint of the operator A is the operator A^* with domain

$$\mathcal{D}(A^*) = \{z \in X : \exists C \in \mathbb{R}^+; |\langle Ay, z\rangle_X| \leq C\|y\|_X, \forall y \in \mathcal{D}(A)\}$$

which is defined by

$$\langle Ay, z\rangle_X = \langle y, A^*z\rangle_X,\ \forall y \in \mathcal{D}(A),\ \forall z \in \mathcal{D}(A^*).$$

Furthermore, if A generates a continuous semigroup $(e^{tA})_{t\geq 0}$, then A^* also generates a continuous semigroup $(e^{tA^*})_{t\geq 0}$ satisfying

$$e^{tA^*} = (S(t))^*, \forall t \geq 0.$$

If $A^* = A$ (respectively $A^* = -A$), then the operator A is said to be self-adjoint (respectively skew-adjoint)[1].

Consider the abstract system

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \quad 0 < t < T, \\ y(0) = y_0, \end{cases}$$

where A generates a strongly continuous group on a Hilbert space X(state space) and $B \in L(X, X)$. Consider the adjoint system

$$\begin{cases} \varphi'(t) = -A^*\varphi(t), \quad 0 < t < T, \\ \varphi(T) = \varphi_T. \end{cases} \tag{1.12}$$

[1] A skew-adjoint operator generates a continuous group of isometries (e.g. (PAZY, 2012)).

Definition 1.5.1. *The system* (1.12) *is said to be* observable *in time* $T > 0$ *if there exists* $C > 0$ *such that*

$$\int_0^T \|B^*\varphi\| dt \geq C\|\varphi_T\|, \ \forall \varphi_T \in X, \tag{1.13}$$

where φ *is the solution of problem* (1.12)*. The inequality* (1.13) *is called the observability inequality of system* (1.12)*.*

Remark 1.5.1. *Inequality* (1.13) *is equivalent to the following unique continuation principle:*

$$B^*\varphi(t) = 0, \forall t \in [0, T] \Rightarrow \varphi_T = 0.$$

We finish this chapter with a subsection about Homogeneous Sobolev spaces, which will be much used in this work.

1.5.1 Homogeneous Sobolev spaces

Definition 1.5.2. *Let* $s \in \mathbb{R}$*. The homogeneous Sobolev space* $\dot{H}^s(\mathbb{R}^d)$ *is the space of tempered distributions* u *defined over* $\mathbb{R}^d$ *whose Fourier transform belongs to* $L^1_{loc}(\mathbb{R}^d)$ *and satisfies*

$$\|u\|^2_{\dot{H}^s} := \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi < \infty.$$

Remark 1.5.2. *The spaces* $\dot{H}^s$ *and* $\dot{H}^{s'}$*, where* $\dot{H}^{s'}$ *denotes the dual space of* $\dot{H}^s$*, cannot be compared for the inclusion. Moreover, by the Fourier-Plancherel formula, one has* $L^2 = \dot{H}^0$*.*

Proposition 1.5.1. *If* $s_0 \leq s \leq s_1$*, then,* $(\dot{H}^{s_0} \cap \dot{H}^{s_1}) \subset \dot{H}^s$ *and*

$$\|u\|_{\dot{H}^s} \leq \|u\|^{1-\theta}_{\dot{H}^{s_0}} \|u\|^{\theta}_{\dot{H}^{s_1}}, \qquad \text{with } s = (1-\theta)s_0 + \theta s_1.$$

Proposition 1.5.2. *The homogeneous Sobolev space* $\dot{H}^s(\mathbb{R}^d)$ *is a Hilbert space if, and only if,* $s < \frac{d}{2}$ $\cdot$

Proposition 1.5.3. *If* $s < \frac{d}{2}$*, then the space* $\mathcal{S}_0(\mathbb{R}^d)$ *of functions of* $\mathcal{S}(\mathbb{R}^d)$ *whose Fourier transform vanishes near the origin is dense in* $\dot{H}^s$*.*

The next proposition characterizes the dual space of $\dot{H}^s$.

Proposition 1.5.4. *If* $|s| < \frac{d}{2}$*, then the bilinear functional*

$$\begin{aligned} \mathcal{B} : \quad & \mathcal{S}_0 \times \mathcal{S}_0 \to \mathbb{C} \\ & (\phi, \varphi) \mapsto \int_{\mathbb{R}^d} \phi(x)\varphi(x) \, dx \end{aligned}$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s} \times \dot{H}^{s}$. *Moreover, if* L *is a continuous linear functional on* $\dot{H}^{s}$, *then there exists a unique tempered distribution* $u \in \dot{H}^{-s}$ *such that*

$$\forall \phi \in \dot{H}^{s}, \langle L, \phi \rangle = \mathcal{B}(u, \phi) \quad \text{and} \quad \|L\|_{(\dot{H}^{s})'} = \|u\|_{\dot{H}^{-s}}.$$

Denote the dual space of $\dot{H}^{s}(\mathbb{R}^d)$ by $\dot{H}^{-s}(\mathbb{R}^d)$. Now we state the embedding of $\dot{H}^{s}(\mathbb{R}^d)$ spaces in $L^p(\mathbb{R}^d)$ spaces.

Theorem 1.5.1. *If* $s \in [0, d/2[$, *then the space* $\dot{H}^{s}(\mathbb{R}^d)$ *is continuously embedded in* $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$.

Corolary 1.5.1. *If* $p \in]1, 2]$, *then* $L^p(\mathbb{R}^d)$ *is continuously embedded in* $\dot{H}^{s}(\mathbb{R}^d)$ *with* $s = \frac{d}{2} - \frac{d}{p}$.

2 PROBLEMS AND MAIN RESULTS OBTAINED

In this chapter, we present the well-posedness, stabilization and control problems for the quintic defocusing Schrödinger equation we are interested in. We summarize the main results obtained in this work and we clarify in which order these results appear in the text.

2.1 WELL-POSEDNESS AND STABILIZATION FOR THE NONLINEAR SCHRÖDINGER EQUATION

The first part of this thesis presents results of well-posedness and stability for the quintic defocusing Schrödinger equation in $\mathbb{R}^{3+1}$

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u(t,x)$ is a complex-valued field in spacetime $[0,+\infty) \times \mathbb{R}^3$. We also consider the following system

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u = 0, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3) \end{cases} \tag{2.2}$$

in $H^1(\mathbb{R}^3)$ which presents an energy identity that involves the full norm in $H^1(\mathbb{R}^3)$. We are mainly concerned with the following stabilization problem for system (2.2).

Stabilization problem: Can one find a feedback control law $f(x,t) = \mathcal{K}u$ so that the resulting closedloop system

$$i\partial_t u + \Delta u - u - |u|^4 u = \mathcal{K}u, (t,x) \in [0,+\infty) \times \mathbb{R}^3$$

is asymptotically stable as $t \to +\infty$?

Consider $a \in C^\infty(\mathbb{R}^3; [0,1])$ an almost everywhere non-negative satisfying

$$a(x) = \begin{cases} 0 & \text{if } |x| \leq R, \\ 1 & \text{if } |x| \geq R+1, \end{cases} \tag{2.3}$$

for some $R > 0$ and $\eta > 0$ such that

$$a(x) \geq \eta > 0, \quad \text{for} \quad |x| \geq R.$$

The stabilization system we consider is

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u - a(x)(1-\Delta)^{-1}a(x)\partial_t u = 0, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3). \end{cases} \tag{2.4}$$

First, we prove the well-posedness of system (2.4), using some Strichartz estimates.

Theorem 2.1. *Let $u_0 \in H^1(\mathbb{R}^3)$ and $a(x) \in C^\infty(\mathbb{R}^3)$ be a non-negative real valued function satisfying (2.3). There exists an unique $u \in C(\mathbb{R}_+, H^1(\mathbb{R}^3))$, solution of the system (2.4) satisfying*

$$\|u\|_{L^{10}([0,T]);L^{10}(\mathbb{R}^3)} < \infty, \quad \|\nabla u\|_{L^{\frac{10}{3}}([0,T]);L^{\frac{10}{3}}(\mathbb{R}^3)} < \infty$$

for all $T < \infty$.

Our main theorem states that it is possible to obtain exponential decay for some solutions of the perturbed system (2.4) which are bounded in the energy space but small in a lower norm.

Theorem 2.2. *Let $\lambda_0 > 0$. There exist $C, \gamma > 0$ and $\delta > 0$ such that for all u_0 in $H^1(\mathbb{R}^3)$ satisfying*

$$\|u_0\|_{H^1(\mathbb{R}^3)} \leq \lambda_0 \text{ and } \|u_0\|_{H^{-1}(\mathbb{R}^3)} \leq \delta,$$

the unique strong solution of problem (2.4) satisfies

$$E(u)(t) \leq Ce^{-\gamma t}E(u)(0), \quad \forall t \geq 0.$$

To prove the exponential decay for the energy of system (2.4), it is necessary to show an observability estimate obtained through propagation results for a microlocal defect measure, using the same strategy used by Dehman in (DEHMAN; LEBEAU; ZUAZUA, 2003). Before that, we need to prove that the solutions for the nonlinear system behave similarly to the solutions for the associated linear system. In this part of the work, we introduce a decomposition into profiles for both linear and nonlinear solutions, as in Keraani (KERAANI, 2001). Furthermore, we also use a scattering property of the system (2.1).

Even with a perturbation term, our approach will not undergo any significant modification since, with the change of variables $w = e^{it}u$, w is a solution of

$$\begin{cases} i\partial_t w = -\Delta w + |w|^4 w, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ w(0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{cases}$$

and we get the original system back. Therefore, it is possible to use the entire profile decomposition theory developed by Keraani in (KERAANI, 2001) for our new system and the scattering property through this change of variables.

2.2 CONTROL OF SCHRÖDINGER EQUATION IN $\mathbb{R}^3$: THE CRITICAL CASE

The second part of this thesis deals with the $\dot{H}^1$-level null controllability for the defocusing critical nonlinear Schrödinger equation on $\mathbb{R}^3$. Consider the system

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^3, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3), \end{cases} \tag{2.5}$$

where $u = u(t,x)$ is a complex-valued function of two real variables $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, where the function $f(t,x)$ is a control input. We are interested in answering the following question: **Control problem:** Let $T > 0$ be given. For any given $u_0 \in \dot{H}^1(\mathbb{R}^3)$, can one find a control $f(t,x)$ such that the system (2.5) admits a solution u in $C\big([0,T]; \dot{H}^1(\mathbb{R}^3)\big)$ satisfying $u(T,x) = 0$ in $\mathbb{R}^3$?

Firstly, we show the problem under consideration to be well-posed using Strichartz estimates and considering $f \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$ resulting in the following theorem.

Theorem 2.3. *Let $u_0 \in H^1(\mathbb{R}^3)$, with $\|u_0\|_{H^1}$ small enough. There exist $T > 0$ and an unique $u \in C(\mathbb{R}_+, H^1(\mathbb{R}^3))$ solution of the system (2.5) satisfying*

$$\|u\|_{L^{10}([0,T]);L^{10}(\mathbb{R}^3)} < \infty, \quad \|\nabla u\|_{L^{\frac{10}{3}}([0,T]);L^{\frac{10}{3}}(\mathbb{R}^3)} < \infty \quad \textit{and} \quad \|\nabla u\|_{L^{10}([0,T]);L^{\frac{30}{13}}(\mathbb{R}^3)} < \infty.$$

Through the Hilbert uniqueness method, we show the linear Schrödinger equation to be controllable. Finally, we use a perturbation argument and show local null controllability for the critical nonlinear Schrödinger equation obtaining a first answer to the control problem above. More specifically, consider the control system

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = \varphi(x) h(t,x), & (t,x) \in [0,T] \times \mathbb{R}^3, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3), \end{cases} \tag{2.6}$$

where the function φ satisfies the condition (2.3). Our result is as follows.

Theorem 2.4. *Let $T > 0$ be given. There exists $\delta > 0$ such that for any u_0 in $H^1(\mathbb{R}^3)$ satisfying $\|u_0\|_{H^1} \leq \delta$, one can find $h(t,x) \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ such that problem (2.6) admits a solution $u \in C([0,T]; H^1(\mathbb{R}^3))$ satisfying $u(T) = 0$.*

The reader may have noticed that the controllability result is obtained for the original system, without adding a perturbation. Since these are two different systems, we prove the well-posedness of each of them. In Chapter 3, we prove the well-posedness for the perturbed system (2.2) and, since these demonstrations will be similar, the proof of the Theorem 2.3 is given in the Appendix.

Remark 2.2.1. *The following observations are worth mentioning:*

i. Theorem 2.2 completes the analysis begun in (SILVA et al.,), where local controllability was shown.

ii. Note that $a \in C^\infty(\mathbb{R}^3)$ satisfying (2.3) act in $\omega := \left(\mathbb{R}^3 \backslash B_R(0)\right)$. Thus, as opposed to (LAURENT, 2010a), the function ω satisfies a unique geometrical assumption: There exists $T_0 > 0$ such that every geodesic travelling at speed 1 meets ω in a time $t < T_0$.

iii. As mentioned in (LAURENT, 2010b), the most physically relevant damping term for system (2.4) would be $ia(x)u$ instead of $a(x)(1-\Delta)^{-1}a(x)\partial_t u$, as used in the one-dimensional case (LAURENT, 2011). For this damping term, the analysis remains open.

3 WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION

3.1 INTRODUCTION

The theory of the Cauchy problem for the equation (1.1) has been extensively investigated, see, for instance, (CAZENAVE; WEISSLER, 1990; GRILLAKIS, 2000; BOURGAIN, 1999a; BOURGAIN, 1999b; GINIBRE; VELO, 1985; IBRAHIM, 1987). In (CAZENAVE; WEISSLER, 1990), the authors showed that when the initial data $u_0(x)$ possesses finite energy, the Cauchy problem is locally well-posed. This implies the existence of a local-in-time solution to (2.1) belonging to the space $C_t^0\dot{H}_x^1 \cap L_{t,x}^{10}$. Moreover, such a solution is unique within this class and the mapping taking initial data to its corresponding solution exhibits local Lipschitz continuity in these norms. In cases where the energy is small, the solution exists globally in time and scatters to a solution $u_{\pm}(t)$ of the free Schrödinger equation $(i\partial_t + \Delta)\, u_{\pm} = 0$. This scattering behavior is characterized by $\|u(t) - u_{\pm}(t)\|_{\dot{H}^1(\mathbb{R}^3)} \to 0$ as $t \to \pm\infty$.

For large finite energy data, particularly for those assumed to be radially symmetric, Bourgain (BOURGAIN, 1999a) proved global existence and scattering for (2.1) in $\dot{H}^1(\mathbb{R}^3)$. Subsequently, Grillakis (GRILLAKIS, 2000) presented an alternative argument that partially recovered the results of (BOURGAIN, 1999a), focusing on global existence from smooth, radial, finite energy data. Recently, Colliander *et al.* (COLLIANDER et al., 2008) obtained global well-posedness, scattering, and global L^{10} space-time bounds for energy class solutions to the quintic defocusing Schrödinger equation in $\mathbb{R}^{1+3}$, which is energy critical. In our case, we study the Schrödinger equation (1.1) with $p = 5$

$$\begin{cases} i\partial_t u = -\Delta u + |u|^4 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{cases} \tag{3.1}$$

in $\dot{H}^1(\mathbb{R}^3)$. The solution of problem (3.1) satisfies some integrability properties and Strichartz estimates (more details will be given later). Furthermore, equation (3.1) has a hamiltonian structure, namely

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t)|^2\, dx + \frac{1}{6}\int_{\mathbb{R}^3} |u(t)|^6\, dx \tag{3.2}$$

which is preserved by the flow (3.1). We shall often refer to it as the energy and write $E(u)$ for $E(u(t))$. Our interest here in the defocusing quintic equation (3.1) is motivated mainly by the fact that the problem concerning the energy norm is critical.

The first term of the right-hand side of energy (3.2) of the originally proposed equation (3.1) presents the norm of the homogeneous Sobolev space $\dot{H}^1$, a space in which there are not many known immersions and inclusions. For this reason, we replaced the equation by a perturbed formulation (2.2) presenting the complete energy

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |u(t)|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t)|^2 \, dx + \frac{1}{6}\int_{\mathbb{R}^3} |u(t)|^6 \, dx, \tag{3.3}$$

which we call H^1-energy, involving now the L^2-mass defined as $\|u(t)\|_{L^2}^2$, which is also preserved by the flow. In this case, we can use, for instance, the immersion of $H^1(\mathbb{R}^3)$ in $H^{-1}(\mathbb{R}^3)$, which is not available for the space $\dot{H}^1(\mathbb{R}^3)$, to complete the proof of the observability estimate giving the exponential decay of energy.

From here onwards, the stabilization system we consider is

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u - a(x)(1-\Delta)^{-1}a(x)\partial_t u = 0, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3), \end{cases} \tag{3.4}$$

where $a(x)$ satisfies (2.3). A solution $u = u(t,x)$ to problem (3.4) satisfies the energy identity

$$E(u)(t_2) - E(u)(t_1) = -2\int_{t_1}^{t_2} \left\| (1-\Delta)^{-\frac{1}{2}} a(x)\partial_t u \right\|_{L^2}^2 dt, \tag{3.5}$$

where $E(u)(t)$ is decreasing and, therefore, system (3.4) is dissipative. The well-posedness of systems (2.2) and (3.4) are the content of this chapter. We follow the approach from (KENIG; MERLE, 2006) for the case $N = 3$.

3.2 NOTATION AND AUXILIARY RESULTS

Before presenting the main results of the chapter, we give some definitions, notations, and auxiliary results, which can be seen in more details in (CAZENAVE, 2003). We begin by introducing the notion of an admissible pair.

Definition 3.2.1.

i) *A pair (q,r) is called L^2-admissible if $r \in [2,6]$ and q satisfies*

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}. \tag{3.6}$$

ii) *A pair (q,r) is called H^1-admissible if $r \in [6,+\infty)$ and q satisfies*

$$\frac{2}{q} + \frac{3}{r} = \frac{1}{2}. \tag{3.7}$$

Remark 3.2.1. *If (q, r) is a L^2-admissible pair, then $2 \leq q \leq \infty$. The pair $(\infty, 2)$ is always L^2-admissible. The pair $(2, \frac{2N}{N-2})$ is L^2- admissible if $N > 3$.*

The following estimates are essential for solving nonlinear Schrödinger equations and they are derived thoroughly in (CAZENAVE, 2003). The first estimates of this type were obtained by Strichartz (STRICHARTZ, 1977) as a Fourier restriction theorem. Strichartz's estimates were generalized by Ginibre and Velo (GINIBRE; VELO, 1985), who gave a remarkable elementary proof. Strichartz's estimates for the nonhomogeneous problem were obtained by Yajima (YAJIMA, 1987) and by Cazenave and Weissler (CAZENAVE; WEISSLER, 1990).

Lemma 3.2.1. *(Strichartz estimates) Let (q, r) be a L^2-admissible pair. We have*

$$\|e^{it\Delta}h\|_{L_t^q L_x^r} \leq c\|h\|_{L^2}, \tag{3.8}$$

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g \; d\tau \right\|_{L_t^q L_x^r} + \left\| \int_0^t e^{i(t-\tau)\Delta} g \; d\tau \right\|_{L_t^q L_x^r} \leq c\|g\|_{L_t^{q'} L_x^{r'}}, \tag{3.9}$$

and

$$\left\| \int_{-\infty}^{+\infty} e^{it\Delta} g(\tau) \; d\tau \right\|_{L_x^2} \leq C\|g\|_{L_t^{q'} L_x^{r'}}. \tag{3.10}$$

Additionally, we have

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g(\tau) \; d\tau \right\|_{L_t^q L_x^r} \leq C\|g\|_{L_t^{m'} L_x^{n'}} \tag{3.11}$$

where (q, r), (m, n) are any pair of L^2-admissible indices and q', r', m', n' are the conjugate exponents of q, r, m, n, respectively.

Lemma 3.2.2. *(Sobolev embedding) For $v \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$, we have*

$$\|v\|_{L_t^{10} L_x^{10}} \leq C\|\nabla v\|_{L_t^{10} L_x^{\frac{30}{13}}}.$$

Define the $S(I)$, $W(I)$, $Z(I)$ norms for an interval I by

$$\|u\|_{S(I)} = \|u\|_{L^{10}(I; L^{10}(\mathbb{R}^3))}, \quad \|u\|_{Z(I)} = \|u\|_{L^{10}(I; L^{\frac{30}{13}}(\mathbb{R}^3))} \text{ and } \quad \|u\|_{W(I)} = \|u\|_{L^{\frac{10}{3}}(I; L^{\frac{10}{3}}(\mathbb{R}^3))}.$$

Remark 3.2.2. *Note that $\left(10, \frac{30}{13}\right)$ and $\left(\frac{10}{3}, \frac{10}{3}\right)$ are L^2-admissible pairs.*

3.3 CAUCHY PROBLEM

In this section, we will study the well-posedness of the system

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u = g, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3), \end{cases} \tag{3.12}$$

where $g \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$, i.e., the H^1 critical defocusing Cauchy problem for the nonlinear Schrödinger equation with a perturbation term. Then, we replace the function g by the damping term $a(1-\Delta)^{-1} a\partial_t u$, where a satisfies (2.3), resulting in the system (3.4). Finally, we investigate the existence of solutions for this case as well.

Theorem 3.3.1. *Let $u_0 \in H^1(\mathbb{R}^3)$, with $\|u_0\|_{H^1}$ small enough. There exist an interval I and an unique $u \in C(\mathbb{R}_+, H^1(\mathbb{R}^3))$ solution of problem (3.12) with*

$$\|u\|_{S(I)} < \infty, \quad \|\nabla u\|_{W(I)} < \infty \ \textit{ and } \ \|\nabla u\|_{Z(I)} < \infty.$$

Demonstração. Assume, without loss of generality, that $I = [0,T]$, $T > 0$. The Cauchy problem is equivalent to the integral equation

$$u(t) = e^{it\Delta} u_0 - \int_0^t e^{i(t-\tau)\Delta}[u + |u|^4 u + g] \, d\tau$$

by Duhamel's formula. Consider the set X_I of functions with norm

$$\|u\|_{X_I} = \sup_{t\in I} \|\nabla u(t)\|_{L^2} + \sup_{t \in I} \|u(t)\|_{L^2} + \|u\|_{S(I)} + \|\nabla u\|_{W(I)} + \|\nabla u\|_{Z(I)}$$

finite. Let $R > 0$, which will be chosen later and denote $B_R = \left\{u \in X_I; \quad \|u\|_{X_I} \leq R\right\}$. Let $A > 0$ fixed, to be chosen later and assume $\|u_0\|_{H^1} < A$. Consider the operator

$$\Phi_{u_0}(u) = e^{it\Delta} u_0 - \int_0^t e^{i(t-\tau)\Delta}[u + |u|^4 u + g] \, d\tau. \tag{3.13}$$

We will show that it is possible to choose R, A so that Φ_{u_0} satisfies $\Phi_{u_0} : B_R \longrightarrow B_R$ and it is a contraction there. First, note that, by (3.13),

$$\begin{aligned} \|\Phi_{u_0}(u)\|_{L^2_x} &\leq \|e^{it\Delta} u_0\|_{L^2_x} + \left\| \int_0^t e^{i(t-\tau)\Delta}[u + |u|^4 u + g] \, d\tau \right\|_{L^2_x} \\ &\leq \|u_0\|_{L^2} + C\||u|^4 u\|_{L^1_t L^2_x} + C\|g\|_{L^1_t L^2_x} + C\|u\|_{L^1_t L^2_x} \\ &\leq CA + CT^{\frac{1}{2}} \|u\|^5_{S(I)} + CT\|g\|_{L^\infty_t H^1_x} + CT\|u\|_{L^\infty_t L^2_x} \\ &\leq 2CA + CR^5 + CTR, \end{aligned}$$

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{L^2_x} &\le \|\nabla e^{it\Delta}u_0\|_{L^2_x} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u+u^5+g]\ d\tau\right\|_{L^2_x}\\ &\le \|\nabla u_0\|_{L^2} + C\|\nabla |u|^4 u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|\nabla g\|_{L^1_tL^2_x} + C\|\nabla u\|_{L^1_tL^2_x}\\ &\le CA + C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + CT\|g\|_{L^\infty_t H^1_x} + CT\|\nabla u\|_{L^\infty_t L^2_x}\\ &\le 2CA + CR^5 + CTR.\end{aligned}$$

Choosing T such that $T < \min\left\{1, \frac{1}{4C}, (2^{\frac{13}{4}}C^{\frac{5}{4}}\|g\|_{L^\infty_t H^1_x})^{-1}\right\}$ and $\|u_0\|_{H^1} < A$ with $A \le \frac{R}{8C}$, we have

$$\|\Phi_{u_0}(u)\|_{L^2_x} + \|\nabla\Phi_{u_0}(u)\|_{L^2_x} \le \frac{R}{2} + CR^5.$$

Secondly, by identity (3.13),

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\le \|\nabla e^{it\Delta}u_0\|_{W(I)} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u+u^5+g]\ d\tau\right\|_{W(I)}\\ &\le \|\nabla u_0\|_{L^2} + C\|\nabla |u|^4 u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|\nabla g\|_{L^1_tL^2_x} + C\|\nabla u\|_{L^1_tL^2_x}.\end{aligned}$$

Using Hölder's inequality with $p=\frac{7}{4}, q=\frac{7}{3}$, $p=\frac{25}{12}$, $q=\frac{25}{13}$ and $p=\frac{5}{4}$, $q=5$, we have

$$\|\nabla |u|^4 u\|_{L^2_tL_x^{\frac{6}{5}}} \le C\|u\|^4_{S(I)}\|\nabla u\|_{Z(I)} \text{ and } \|\nabla |u|^4 u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} \le C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)}.$$

Indeed,

$$\begin{aligned}\|\nabla |u|^4 u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} &= \left(\int_0^T\int_{\mathbb{R}^3} |\nabla |u|^4 u|^{\frac{10}{7}}\ dxdt\right)^{\frac{7}{10}}\\ &= \left(\int_0^T\int_{\mathbb{R}^3} |u|^{\frac{40}{7}}|\nabla u|^{\frac{10}{7}}\ dxdt\right)^{\frac{7}{10}}\\ &\le \left(\int_0^T\left(\int_{\mathbb{R}^3}(|u|^{\frac{40}{7}})^{\frac{7}{4}}\ dx\right)^{\frac{4}{7}}\cdot\left(\int_{\mathbb{R}^3}(|\nabla u|^{\frac{10}{7}})^{\frac{7}{3}}\ dx\right)^{\frac{3}{7}}\ dt\right)^{\frac{7}{10}}\\ &\le \left(\int_0^T\left(\int_{\mathbb{R}^3}|u|^{10}\ dx\right)^{\frac{4}{7}}\cdot\left(\int_{\mathbb{R}^3}|\nabla u|^{\frac{10}{3}}\ dx\right)^{\frac{3}{7}}\ dt\right)^{\frac{7}{10}}\\ &\le \left(\int_0^T \|u\|^{\frac{40}{7}}_{L^{10}_x}\cdot\|\nabla u\|^{\frac{10}{7}}_{L_x^{\frac{10}{3}}}\ dt\right)^{\frac{7}{10}}\\ &\le \left(\left(\int_0^T(\|u\|^{\frac{40}{7}}_{L^{10}_x})^{\frac{7}{4}}\ dt\right)^{\frac{4}{7}}\cdot\left(\int_0^T(\|\nabla u\|^{\frac{10}{7}}_{L_x^{\frac{10}{3}}})^{\frac{7}{3}}\ dt\right)^{\frac{3}{7}}\right)^{\frac{7}{10}}\\ &\le \left(\|u\|^{\frac{40}{7}}_{L^{10}_tL^{10}_x}\cdot\|\nabla u\|^{\frac{10}{7}}_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}}}\right)^{\frac{7}{10}}\\ &\le \|u\|^4_{L^{10}_tL^{10}_x}\|\nabla u\|_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}}}.\end{aligned}$$

Additionally,

$$\begin{aligned}
\left\|\nabla|u|^4u\right\|_{L^2_tL^{\frac{6}{5}}_x} &= \left(\int_0^T \|u^4\cdot\nabla u\|^2_{L^{\frac{6}{5}}_x}\right)^{\frac{1}{2}}\\
&= \left(\int_0^T \left(\int_{\mathbb{R}^3}(|u|^4)^{\frac{6}{5}}\cdot|\nabla u|^{\frac{6}{5}}\,dx\right)^{\frac{5}{6}\cdot 2}\right)^{\frac{1}{2}}\\
&= \left(\int_0^T \left(\int_{\mathbb{R}^3}|u|^{\frac{24}{5}}\cdot|\nabla u|^{\frac{6}{5}}\,dx\right)^{\frac{5}{3}}\,dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^T \left(\int_{\mathbb{R}^3}|u|^{\frac{24}{5}\cdot\frac{25}{12}}\,dx\right)^{\frac{12}{25}\cdot\frac{5}{3}}\left(\int_{\mathbb{R}^3}|\nabla u|^{\frac{6}{5}\cdot\frac{25}{13}}\,dx\right)^{\frac{13}{25}\cdot\frac{5}{3}}\,dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^T \left(\int_{\mathbb{R}^3}|u|^{10}\,dx\right)^{\frac{4}{5}}\left(\int_{\mathbb{R}^3}|\nabla u|^{\frac{30}{13}}\,dx\right)^{\frac{13}{15}}\,dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^T \|u\|^8_{L^{10}_x}\,dx\cdot\|\nabla u\|^2_{L^{\frac{30}{13}}_x}\,dt\right)^{\frac{1}{2}}\\
&\leq \left(\left(\int_0^T \|u\|^{8\cdot\frac{10}{8}}_{L^{10}_x}\,dt\right)^{\frac{4}{5}}\left(\int_0^T \|\nabla u\|^{10}_{L^{\frac{30}{13}}_x}\,dx\right)^{\frac{1}{5}}\right)^{\frac{1}{2}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\left\|\nabla|u|^4u\right\|_{L^2_tL^{\frac{6}{5}}_x} &\leq \left(\|u\|^{4\cdot 2}_{L^{10}_tL^{10}_x}\cdot\|\nabla u\|^2_{L^{10}_tL^{\frac{30}{13}}_x}\right)^{\frac{1}{2}}\\
&\leq \|u\|^4_{L^{10}_tL^{10}_x}\|\nabla u\|_{L^{10}_tL^{\frac{30}{13}}_x}.
\end{aligned}$$

So

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\leq C\Big(\|\nabla u_0\|_{L^2}+\|u\|^4_{S(I)}\|\nabla u\|_{W(I)}+\|\nabla g\|_{L^1_tL^2_x}+\|\nabla u\|_{L^1_tL^2_x}\Big)\\
&\leq CA+CR^5+CT\|g\|_{L^\infty_tH^1_x}+CTR.
\end{aligned}$$

With the same choice of T and A as made previously, we have

$$\|\nabla\Phi_{u_0}(u)\|_{W(I)} \leq \frac{R}{2}+CR^5.$$

On the other hand, by (3.8), (3.9), and by (3.11), with q,r satisfying $q=10, r=\frac{30}{13}$ and

$m' = 2, n' = \frac{6}{5}$ and Hölder's inequality,

$$\begin{aligned}
\|\nabla \Phi_{u_0}(u)\|_{Z(I)} &\leq \|\nabla e^{it\Delta} u_0\|_{Z(I)} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u + u^5 + g]\, d\tau \right\|_{Z(I)} \\
&\leq C\|\nabla u_0\|_{L^2} + C\|\nabla |u|^4 u\|_{L_t^2 L_x^{\frac{6}{5}}} + C\|\nabla g\|_{L_t^1 L_x^2} + C\|\nabla u\|_{L_t^1 L_x^2} \\
&\leq C\|\nabla u_0\|_{L^2} + C\|\nabla u\|_{Z(I)} \|u\|_{S(I)}^4 + C\|\nabla g\|_{L_t^1 L_x^2} + C\|\nabla u\|_{L_t^1 L_x^2} \\
&\leq CA + CR^5 + CT\|g\|_{L_t^\infty H_x^1} + CT\|\nabla u\|_{L_t^\infty L_x^2} \\
&\leq 2CA + CR^5 + CT\|g\|_{L_t^\infty H_x^1} + CTR.
\end{aligned}$$

Choosing T such that $T < \min\left\{1, \frac{1}{4C}, (2^{\frac{13}{4}} C^{\frac{5}{4}} \|g\|_{L_t^\infty H_x^1})^{-1}\right\}$ and $\|u_0\|_{H^1} < A$ with $A \leq \frac{R}{8C}$, we have

$$\|\nabla \Phi_{u_0}(u)\|_{Z(I)} \leq \frac{R}{2} + CR^5.$$

Finally, by Sobolev's embedding,

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{S(I)} &\leq \|\nabla \Phi_{u_0}(u)\|_{Z(I)} \\
&\leq 2CA + CR^5 + CT\|g\|_{L_t^\infty H_x^1} + CTR,
\end{aligned}$$

and, with the same choice of T and A as before, we have

$$\|\Phi_{u_0}(u)\|_{S(I)} \leq \frac{R}{2} + CR^5.$$

Adding up all the estimates above,

$$\|\Phi_{u_0}(u)\|_{X_I} \leq \frac{R}{2} + CR^5 \leq R,$$

as long as $R < \frac{1}{(2C)^{\frac{1}{4}}}$.

Next, to show that Φ_{u_0} is a contraction, denote $f(u) = |u|^4 u$. By the definition of Φ_{u_0} (3.13),

$$\begin{aligned}
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L_x^2} &\leq C\|f(u) - f(v)\|_{L_t^1 L_x^2} + C\|u - v\|_{L_t^1 L_x^2} \\
&\leq CT^{\frac{1}{2}} \|u - v\|_{S(I)} \Big(\|u\|_{S(I)}^4 + \|v\|_{S(I)}^4\Big) + CT\|u - v\|_{L_t^\infty L_x^2} \\
&\leq CT^{\frac{1}{2}} R^4 \|u - v\|_{S(I)} + CT\|u - v\|_{L_t^\infty L_x^2} \\
&\leq (2CT^{\frac{1}{2}} R^4 + CT)\|u - v\|_{X_I},
\end{aligned}$$

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{L^2_x} &\leq C\|\nabla f(u)-\nabla f(v)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}+C\|\nabla u-\nabla v\|_{L^1_tL^2_x}\\
&\leq C\Bigg(\Big\||u|^4|\nabla u-\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}+\Big\||u-v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\quad+\Big\||u-v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\Bigg)+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u-\nabla v\|_{W(I)}+\|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|^3_{S(I)}+\\
&\quad+\|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|^3_{S(I)}\Bigg)+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq CR^4\|u-v\|_{S(I)}+CR^4\|\nabla u-\nabla v\|_{W(I)}+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq (2CR^4+CT)\|u-v\|_{X_I}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{W(I)} &\leq C\|\nabla f(u)-\nabla f(v)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}+C\|\nabla u-\nabla v\|_{L^1_tL^2_x}\\
&\leq C\Bigg(\Big\||u|^4|\nabla u-\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}+\Big\||u-v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\quad+\Big\||u-v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\Bigg)+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u-\nabla v\|_{W(I)}+\|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|^3_{S(I)}+\\
&\quad+\|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|^3_{S(I)}\Bigg)+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq CR^4\|u-v\|_{S(I)}+CR^4\|\nabla u-\nabla v\|_{W(I)}+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq (2CR^4+CT)\|u-v\|_{X_I}.
\end{aligned}$$

Following the same reasoning,

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{Z(I)} &\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u-\nabla v\|_{Z(I)}+\|u-v\|_{S(I)}\|\nabla v\|_{Z(I)}\|u\|^3_{S(I)}+\\
&\quad+\|u-v\|_{S(I)}\|\nabla v\|_{Z(I)}\|v\|^3_{S(I)}\Bigg)+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}+CT\|\nabla u-\nabla v\|_{L^\infty_tL^2_x}\\
&\leq (2CR^4+CT)\|u-v\|_{X_I}.
\end{aligned}$$

Moreover, by Sobolev's embedding,

$$\|\Phi_{u_0}(u)-\Phi_{u_0}(v)\|_{S(I)}\leq\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{Z(I)}\leq(2CR^4+CT)\|u-v\|_{X_I}.$$

Adding up,

$$\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_I} \leq C(R^4 + T^{\frac{1}{2}}R^4 + T)\|u - v\|_{X_I}.$$

Thus, choosing R and T such that $C(R^4 + T^{\frac{1}{2}}R^4 + T) < 1$, we conclude that Φ_{u_0} is a contraction. Therefore, there exists $u \in B_R$ satisfying $\Phi_{u_0}(u) = u$. □

Remark 3.3.1. *Observe that the solution $u = u(t,x)$ of problem (3.12) is globally well-defined in time. To verify this, first consider the energy defined by (3.3) which is conserved if $g = 0$. Multiplying equation (3.12) by $\partial_t \overline{u}$, integrating and taking the real part, we have*

$$\begin{aligned}
E(t) &\leq E(0) - Re\int_0^t \int_{\mathbb{R}^3} g\partial_t \overline{u}\, dxdt \\
&\leq E(0) - Re\int_0^t \int_{\mathbb{R}^3} g(i\Delta\overline{u} - i\overline{u} - i|u|^4\overline{u} - i\overline{g})\, dxdt \\
&\leq E(0) + C\int_0^t \|\nabla g(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\, d\tau + C\int_0^t \|g(\tau)\|_{L^2}\|u(\tau)\|_{L^2}\, d\tau \\
&\quad + C\int_0^t \|g(\tau)\|_{L^6}\|u(\tau)^5\|_{L^{\frac{6}{5}}}\, d\tau + \int_0^t \|g(\tau)\|_{L^2}^2\, d\tau \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}\sqrt{E(\tau)}\, d\tau + C\int_0^t \|g(\tau)\|_{L^6}(E(\tau))^{\frac{5}{6}}\, d\tau + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}\sqrt{E(\tau)}\, d\tau + C\int_0^t \|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\, d\tau + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

One has

$$\begin{aligned}
E(t) &\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}(E(\tau))^{-\frac{1}{3}}(E(\tau))^{\frac{5}{6}}\, d\tau + C\int_0^t \|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\, d\tau + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\, d\tau + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
&\leq E(0) + C\int_0^t \|g(\tau)\|_{H^1}\left(1 + (E(\tau))^{\frac{5}{6}}\right)\, d\tau + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

Therefore,

$$\max_{0\leq\tau\leq t} E(\tau) \leq E(0) + C\left(1 + \max_{0\leq\tau\leq t}(E(\tau))^{\frac{5}{6}}\right)\|g\|_{L^1([0,T];H^1(\mathbb{R}^3))} + \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.$$

Denoting $F(t) := \max_{0\leq\tau\leq t} E(\tau)$, $\alpha := \|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}$ and $\beta := C\|g\|_{L^1([0,T];H^1(\mathbb{R}^3))}$, we have

$$F(t)^6 \leq F(0)^6 + \alpha^6 + \beta^6 F(t)^5 + I + II,$$

where I and II are two extremely long terms that belong to the calculation of the sixth power. We omit their explicit expression for simplicity. Then,

$$F(t) \leq \frac{1}{F(t)^5}F(0)^6 + \frac{1}{F(t)^5}\alpha^6 + \beta^6 + \frac{1}{F(t)^5} \cdot I + \frac{1}{F(t)^5} \cdot II. \tag{3.14}$$

Assuming that $g \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$ and suposing that there exists $M > 0$ such that $F(t) \geq M$, this implies $\frac{1}{F(t)} \leq \frac{1}{M}$, $0 \leq t \leq T$. Hence, the last two terms of (3.14) are bounded. So,

$$\max_{0 \leq \tau \leq t} E(\tau) \leq C(1 + E(0)^6 + \|g\|^{12}_{L^2([0,T]\times\mathbb{R}^3)} + \|g\|^6_{L^1([0,T];H^1(\mathbb{R}^3))})$$

and finally

$$E(t) \leq C\Big(1 + E(0)^6 + \|g\|^{12}_{L^2([0,T]\times\mathbb{R}^3)} + \|g\|^6_{L^1([0,T];H^1(\mathbb{R}^3))}\Big).$$

This implies that the energy is bounded if $g \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$.

To finish this section, we prove the existence of solutions for the H^1 critical nonlinear Schrödinger equation with a modified damping term, that is, changing g by $a(x)(1-\Delta)^{-1}a(x)\partial_t u$ in the system (3.12). The local result is the following.

Theorem 3.3.2. *Let $T > 0$, $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\|_{H^1}$ small enough and $a(x) \in C^\infty(\mathbb{R}^3)$ a non-negative real valued function. There exists an unique $u \in C(\mathbb{R}_+, H^1(\mathbb{R}^3))$ solution of the system*

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u - a(x)(1-\Delta)^{-1}a(x)\partial_t u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ u(0) = u_0, \quad x \in \mathbb{R}^3, \end{cases} \tag{3.15}$$

with

$$\|u\|_{S([0,T])} < \infty, \quad \|\nabla u\|_{W([0,T])} < \infty \ \text{ and } \ \|\nabla u\|_{Z([0,T])} < \infty$$

for all $T < \infty$.

Demonstração. We claim that the operator $Jv = (1 - ia(x)(1-\Delta)^{-1}a(x))v$ is a pseudo-differential operator of order 0 which defines an isomorphism in $H^s(\mathbb{R}^3)$, for $s \in \mathbb{R}$, and also in $L^p(\mathbb{R}^3)$. Indeed, note that we can write J as $J = I + J_1$, where J_1 is an anti-self-adjoint operator in $L^2(\mathbb{R}^3)$. Thus, J is an isomorphism in $L^2(\mathbb{R}^3)$ and, due to the ellipticity, in $H^s(\mathbb{R}^3)$, for $s > 0$. Moreover, J^{-1} (considered, for example, acting in $L^2([0,T] \times \mathbb{R}^3)$) is a pseudodifferential operator of order 0 and satisfies $J^{-1} = 1 - J_1 J^{-1}$.

Denote $v = Ju$ and write system (3.15) as

$$\begin{cases} \partial_t v - i\Delta v - R_0 v + i|u|^4 u = 0, & (t,x) \in [0,T] \times \mathbb{R}^3, \\ v = Ju, \\ v(0) = v_0 = Ju_0, & x \in \mathbb{R}^3, \end{cases} \tag{3.16}$$

where $R_0 = -i\Delta J_1 J^{-1} + iJ^{-1}$ is a pseudodifferential operator of order 0. The Cauchy problem (3.16) is equivalent to the integral equation

$$v(t) = e^{it\Delta} v_0 + \int_0^t e^{i(t-\tau)\Delta}[R_0 v - i|u|^4 u]\ d\tau. \tag{3.17}$$

Let $I = [0,T]$ and consider the set X_I of functions having the norm

$$\|v\|_{X_I} = \sup_{t\in I} \|\nabla v(t)\|_{L^2} + \sup_{t \in I} \|v(t)\|_{L^2} + \|v\|_{S(I)} + \|\nabla v\|_{W(I)}$$

finite. Let $R > 0$, which will be chosen later and denote $B_R = \Big\{ v \in X_I; \quad \|v\|_{X_I} \leq R \Big\}$. Let $\|v_0\|_{H^1} < A$, with $A > 0$ small enough fixed (which will be chosen later as well). Define the functional

$$\Phi_{u_0}(v)(t) = e^{it\Delta} v_0 + \int_0^t e^{i(t-\tau)\Delta} R_0 v\ d\tau - \int_0^t e^{i(t-\tau)\Delta} i|u|^4 u\ d\tau. \tag{3.18}$$

Our goal is to show that this functional, defined in a suitable ball B_R, has a fixed point. We show that R may be chosen in such a way that $\Phi(v) : B_R \longrightarrow B_R$ is a contraction. First,

$$\begin{aligned} \|\nabla \Phi_{u_0}(v)\|_{L^2_x} &\leq \|\nabla e^{it\Delta} v_0\|_{L^2} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta} |u|^4 u\ d\tau \right\|_{L^2_x} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta} R_0 v\ d\tau \right\|_{L^2_x} \\ &\leq \|\nabla v_0\|_{L^2} + C\|\nabla |u|^4 u\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|\nabla R_0 v\|_{L^1_t L^2_x} \\ &\leq \|\nabla v_0\|_{L^2} + C\|u\|^4_{S(I)} \|\nabla u\|_{W(I)} + C\|[\nabla, R_0] v\|_{L^1_t L^2_x} + C\|R_0 \nabla v\|_{L^1_t L^2_x}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla u\|_{W(I)} &= \|\nabla J^{-1} v\|_{W(I)} \\ &= \|[\nabla, J^{-1}] v + J^{-1} \nabla v\|_{W(I)} \\ &\leq C\|v\|_{W(I)} + C\|\nabla v\|_{W(I)}. \end{aligned}$$

Then,

$$\begin{aligned}
\|\nabla\Phi_{u_0}(v)\|_{L^2_x} &\leq \|\nabla v_0\|_{L^2} + C\|v\|^4_{S(I)}\Big(\|v\|_{W(I)} + \|\nabla v\|_{W(I)}\Big) + C\|[\nabla, R_0]v\|_{L^1_tL^2_x} + \|R_0\nabla v\|_{L^1_tL^2_x} \\
&\leq \|\nabla v_0\|_{L^2} + C\|v\|^4_{S(I)}\Big(\|v\|_{W(I)} + \|\nabla v\|_{W(I)}\Big) + C\|v\|_{L^1_tL^2_x} + C\|\nabla v\|_{L^1_tL^2_x} \\
&\leq \|v_0\|_{H^1} + C\|v\|^4_{S(I)}\|v\|_{W(I)} + C\|v\|^4_{S(I)}\|\nabla v\|_{W(I)} \\
&\quad + CT\sup_{t\in I}\|v(t)\|_{L^2} + CT\sup_{t\in I}\|\nabla v(t)\|_{L^2}.
\end{aligned}$$

By interpolation,

$$\|v(t)\|_{L^{\frac{10}{3}}} \leq \|v(t)\|^{\frac{2}{5}}_{L^2}\|v(t)\|^{\frac{3}{5}}_{L^6}.$$

Then,

$$\begin{aligned}
\int_0^T \|v(t)\|^{\frac{10}{3}}_{L^{\frac{10}{3}}}\,dt &\leq \int_0^T \|v(t)\|^{\frac{4}{3}}_{L^2}\|v(t)\|^2_{L^6}\,dt \\
&\leq \sup_{t\in I}\|v(t)\|^{\frac{4}{3}}_{L^2}\int_0^T \|v(t)\|^2_{L^6}\,dt \\
&\leq T\sup_{t\in I}\|v(t)\|^{\frac{4}{3}}_{L^2}\sup_{t\in I}\|v(t)\|^2_{L^6} \\
&\leq T\|v\|^{\frac{4}{3}}_{X_I}\|v\|^2_{X_I} \\
&\leq T\|v\|^{\frac{10}{3}}_{X_I}
\end{aligned}$$

$$\Rightarrow \|v\|_{W(I)} \leq T^{\frac{3}{10}}\|v\|_{X_I}.$$

Hence,

$$\begin{aligned}
\|\nabla\Phi_{u_0}(v)\|_{L^2_x} &\leq \|\nabla v_0\|_{L^2} + CT^{\frac{3}{10}}\|v\|^4_{S(I)}\|v\|_{X_I} + C\|v\|^4_{S(I)}\|\nabla v\|_{W(I)} \\
&\quad + CT\sup_{t\in I}\|v(t)\|_{L^2} + CT\sup_{t\in I}\|\nabla v(t)\|_{L^2} \\
&\leq C\|v_0\|_{H^1} + CT^{\frac{3}{10}}\|v\|^5_{X_I} + C\|v\|^5_{X_I} + CT\|v\|_{X_I},
\end{aligned}$$

where, for these inequalities, we are using estimate (3.8) with $(q,r) = \left(\frac{10}{3}, \frac{10}{3}\right)$ and estimate (3.11) with $(q,r) = \left(\frac{10}{3}, \frac{10}{3}\right)$ and $(m,n) = (\infty, 2)$. Again,

$$\begin{aligned}
\|\Phi_{u_0}(v)\|_{L^2_x} &\leq \|e^{it\Delta}v_0\|_{L^2} + \left\|\int_0^t e^{i(t-\tau)\Delta}|u|^4u\,d\tau\right\|_{L^2_x} + \left\|\int_0^t e^{i(t-\tau)\Delta}R_0v\,d\tau\right\|_{L^2_x} \\
&\leq \|v_0\|_{L^2} + C\||u|^4u\|_{L^1_tL^2_x} + C\|R_0v\|_{L^1_tL^2_x} \\
&\leq \|v_0\|_{L^2} + CT^{\frac{1}{2}}\|u\|^5_{S(I)} + CT\sup_{t\in I}\|v(t)\|_{L^2} \\
&\leq C\|v_0\|_{H^1} + CT\|v\|^5_{X_I} + CT\|v\|_{X_I}.
\end{aligned}$$

Moreover,

$$\begin{aligned}\|\nabla\Phi_{u_0}(v)\|_{W(I)} &\leq \|\nabla e^{it\Delta}v_0\|_{W(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}|u|^4u\ d\tau\right\|_{W(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}R_0 v\ d\tau\right\|_{W(I)}\\ &\leq \|\nabla v_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|\nabla R_0 v\|_{L_t^1L_x^2}\\ &\leq \|\nabla v_0\|_{L^2} + C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + C\|[\nabla, R_0]v\|_{L_t^1L_x^2} + C\|R_0\nabla v\|_{L_t^1L_x^2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\nabla\Phi_{u_0}(v)\|_{W(I)} &\leq \|\nabla v_0\|_{L^2} + CT^{\frac{3}{10}}\|v\|^4_{S(I)}\|v\|_{X_I} + C\|v\|^4_{S(I)}\|\nabla v\|_{W(I)}\\ &\quad + CT\sup_{t\in I}\|v(t)\|_{L^2} + CT\sup_{t\in I}\|\nabla v(t)\|_{L^2}\\ &\leq C\|v_0\|_{H^1} + CT^{\frac{3}{10}}\|v\|^5_{X_I} + C\|v\|^5_{X_I} + CT\|v\|_{X_I}.\end{aligned}$$

Finally,

$$\begin{aligned}\|\Phi_{u_0}(v)\|_{S(I)} &\leq \|\nabla\Phi_{u_0}(v)\|_{Z(I)}\\ &\leq \|\nabla e^{it\Delta}v_0\|_{Z(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}|u|^4u\ d\tau\right\|_{Z(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}R_0 v\ d\tau\right\|_{Z(I)}\\ &\leq \|\nabla v_0\|_{L^2} + C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + C\|[\nabla, R_0]v\|_{L_t^1L_x^2} + C\|R_0\nabla v\|_{L_t^1L_x^2}\\ &\leq \|\nabla v_0\|_{L^2} + CT^{\frac{3}{10}}\|v\|^4_{S(I)}\|v\|_{X_I} + C\|v\|^4_{S(I)}\|\nabla v\|_{W(I)}\\ &\quad + CT\sup_{t\in I}\|v(t)\|_{L^2} + CT\sup_{t\in I}\|\nabla v(t)\|_{L^2}\\ &\leq C\|v_0\|_{H^1} + CT^{\frac{3}{10}}\|v\|^5_{X_I} + C\|v\|^5_{X_I} + CT\|v\|_{X_I},\end{aligned}$$

where, in the third inequality, we used estimate (3.8) with $(q,r) = \left(10, \dfrac{30}{13}\right)$ and estimate (3.11) with $(q,r) = \left(10, \frac{30}{13}\right)$, $(m,n) = \left(\frac{10}{3}, \frac{10}{3}\right)$ and again $(m,n) = (\infty, 2)$. Adding up, we have

$$\|\Phi_{u_0}(v)\|_{X_I} \leq C\|v_0\|_{H^1} + CT^{\frac{3}{10}}\|v\|^5_{X_I} + C\|v\|^5_{X_I} + CT\|v\|_{X_I}.$$

Choosing $T < \min\left\{1, \frac{1}{4C}\right\}$, $A < \frac{R}{8C}$ and $R < \frac{1}{(4C)^{\frac{1}{4}}}$, we conclude that Φ_{u_0} takes elements of B_R to elements of B_R. To prove that Φ_0 is a contraction, consider the two systems

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4u - a(1-\Delta)^{-1}a\partial_t u = 0, & (t,x) \in [0,T]\times\mathbb{R}^3,\\ u(0) = u_0, & x \in \mathbb{R}^3,\end{cases}$$

and

$$\begin{cases} i\partial_t z + \Delta z - z - |z|^4z - a(1-\Delta)^{-1}a\partial_t z = 0, & (t,x) \in [0,T]\times\mathbb{R}^3,\\ z(0) = u_0, & x \in \mathbb{R}^3,\end{cases}$$

performing the transformation $v = Ju$ and $w = Jz$, we have

$$\begin{cases} \partial_t v - i\Delta v - R_0 v + i|u|^4 u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ v = Ju, \\ v(0) = v_0 = Ju_0, \quad x \in \mathbb{R}^3, \end{cases} \tag{3.19}$$

and

$$\begin{cases} \partial_t w - i\Delta w - R_0 w + i|z|^4 z = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ w = Jz, \\ w(0) = w_0 = v_0 = Ju_0, \quad x \in \mathbb{R}^3. \end{cases} \tag{3.20}$$

Using Duhamel's formula, the difference between the systems (3.19) and (3.20) is

$$\Phi_{u_0}(v) - \Phi_{u_0}(w) = \int_0^t e^{i(t-\tau)\Delta} R_0(v-w)\, d\tau - \int_0^t e^{i(t-\tau)\Delta} i\Big(|u|^4 u - |z|^4 z\Big)\, d\tau.$$

Bound,

$$\begin{aligned}
\|\nabla\Phi_{u_0}(v) - \nabla\Phi_{u_0}(w)\|_{L^2_x} &\leq \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}(|u|^4 u - |z|^4 z)\, d\tau \right\|_{L^2_x} \\
&\quad + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta} R_0(v-w)\, d\tau \right\|_{L^2_x} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|\nabla R_0(v-w)\|_{L^1_t L^2_x} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|[\nabla, R_0](v-w)\|_{L^1_t L^2_x} \\
&\quad + C\|R_0 \nabla(v-w)\|_{L^1_t L^2_x} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|v-w\|_{L^1_t L^2_x} + C\|\nabla(v-w)\|_{L^1_t L^2_x} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + CT \sup_{t\in I} \|(v-w)(t)\|_{L^2} \\
&\quad + CT \sup_{t\in I} \|\nabla(v-w)(t)\|_{L^2}.
\end{aligned}$$

We have

$$\begin{aligned}
\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} &\leq C\Bigg(\|u\|^4_{S(I)} \|\nabla u - \nabla z\|_{W(I)} + \|u-z\|_{S(I)} \|\nabla z\|_{W(I)} \|u\|^3_{S(I)} + \\
&\quad + \|u-z\|_{S(I)} \|\nabla z\|_{W(I)} \|z\|^3_{S(I)} \Bigg) \\
&\leq C\Bigg(\|v\|^4_{S(I)} \|\nabla u - \nabla z\|_{W(I)} + \|v-w\|_{S(I)} \|\nabla z\|_{W(I)} \|v\|^3_{S(I)} + \\
&\quad + \|v-w\|_{S(I)} \|\nabla z\|_{W(I)} \|w\|^3_{S(I)} \Bigg).
\end{aligned}$$

But

$$\begin{aligned} \|\nabla(u-z)\|_{W(I)} &= \|\nabla J^{-1}(v-w)\|_{W(I)} \\ &\leq \|[\nabla, J^{-1}](v-w)\|_{W(I)} + \|J^{-1}\nabla(v-w)\|_{W(I)} \\ &\leq C\|v-w\|_{W(I)} + C\|\nabla(v-w)\|_{W(I)}. \end{aligned}$$

So,

$$\begin{aligned} \|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} &\leq C\Bigg(\|v\|_{S(I)}^4\|\nabla u-\nabla z\|_{W(I)} + \|v-w\|_{S(I)}\|\nabla z\|_{W(I)}\|v\|_{S(I)}^3 + \\ &\quad + \|v-w\|_{S(I)}\|\nabla z\|_{W(I)}\|w\|_{S(I)}^3\Bigg), \end{aligned}$$

i.e.,

$$\begin{aligned} \|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} &\leq C\Bigg(\|v\|_{S(I)}^4\|v-w\|_{W(I)} + \|v\|_{S(I)}^4\|\nabla(v-w)\|_{W(I)} \\ &\quad + \|v-w\|_{S(I)}\|w\|_{W(I)}\|v\|_{S(I)}^3 + \|v-w\|_{S(I)}\|\nabla w\|_{W(I)}\|v\|_{S(I)}^3 \\ &\quad + \|v-w\|_{S(I)}\|w\|_{W(I)}\|w\|_{S(I)}^3 + \|v-w\|_{S(I)}\|\nabla w\|_{W(I)}\|w\|_{S(I)}^3\Bigg) \\ &\leq C\Bigg(T^{\frac{3}{10}}R^4\|v-w\|_{X_I} + R^4\|v-w\|_{X_I} \\ &\quad + T^{\frac{3}{10}}R^4\|v-w\|_{X_I} + R^4\|v-w\|_{X_I} \\ &\quad + T^{\frac{3}{10}}R^4\|v-w\|_{X_I} + R^4\|v-w\|_{X_I}\Bigg) \\ &\leq CT^{\frac{3}{10}}R^4\|v-w\|_{X_I} + CR^4\|v-w\|_{X_I}. \end{aligned}$$

Hence,

$$\|\nabla\Phi_{u_0}(v)-\nabla\Phi_{u_0}(w)\|_{L_x^2} \leq CT^{\frac{3}{10}}R^4\|v-w\|_{X_I} + CR^4\|v-w\|_{X_I} + CT\|v-w\|_{X_I}.$$

Secondly,

$$
\begin{aligned}
\|\nabla\Phi_{u_0}(v)-\nabla\Phi_{u_0}(w)\|_{W(I)} &\leq \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}(|u|^4u-|z|^4z)\ d\tau\right\|_{W(I)} \\
&\quad + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}R_0(v-w)\ d\tau\right\|_{W(I)} \\
&\leq C\|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|\nabla R_0(v-w)\|_{L_t^1L_x^2} \\
&\leq C\|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|[\nabla,R_0](v-w)\|_{L_t^1L_x^2} \\
&\quad + C\|R_0\nabla(v-w)\|_{L_t^1L_x^2} \\
&\leq C\|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + C\|v-w\|_{L_t^1L_x^2} + C\|\nabla(v-w)\|_{L_t^1L_x^2} \\
&\leq C\|\nabla(|u|^4u-|z|^4z)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + CT\sup_{t\in I}\|(v-w)(t)\|_{L^2} \\
&\quad + CT\sup_{t\in I}\|\nabla(v-w)(t)\|_{L^2} \\
&\leq CT^{\frac{3}{10}}R^4\|v-w\|_{X_I} + CR^4\|v-w\|_{X_I} + CT\|v-w\|_{X_I}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|\Phi_{u_0}(v)-\Phi_{u_0}(w)\|_{L^2} &\leq \left\|\int_0^t e^{i(t-\tau)\Delta}(|u|^4u-|z|^4z)\ d\tau\right\|_{L^2} \\
&\quad + \left\|\int_0^t e^{i(t-\tau)\Delta}R_0(v-w)\ d\tau\right\|_{L^2} \\
&\leq C\||u|^4u-|z|^4z\|_{L_t^1L_x^2} + C\|R_0(v-w)\|_{L_t^1L_x^2} \\
&\leq C\||u|^4u-|z|^4z\|_{L_t^1L_x^2} + C\|v-w\|_{L_t^1L_x^2} \\
&\leq C\||u|^4u-|z|^4z\|_{L_t^1L_x^2} + CT\sup_{t\in I}\|(v-w)(t)\|_{L^2} \\
&\leq CT^{\frac{1}{2}}\|u-z\|_{S(I)}\Big(\|u\|_{S(I)}^4+\|z\|_{S(I)}^4\Big) + CT\sup_{t\in I}\|(v-w)(t)\|_{L^2} \\
&\leq CT^{\frac{1}{2}}\|v-w\|_{S(I)}\Big(\|v\|_{S(I)}^4+\|w\|_{S(I)}^4\Big) + CT\sup_{t\in I}\|(v-w)(t)\|_{L^2} \\
&\leq CT^{\frac{1}{2}}R^4\|v-w\|_{X_I} + CT\|v-w\|_{X_I}.
\end{aligned}
$$

Finally,

$$\begin{aligned}
\|\Phi_{u_0}(v) - \Phi_{u_0}(w)\|_{S(I)} &\leq \left\| \int_0^t e^{i(t-\tau)\Delta}(|u|^4 u - |z|^4 z)\, d\tau \right\|_{Z(I)} \\
&\quad + \left\| \int_0^t e^{i(t-\tau)\Delta} R_0(v-w)\, d\tau \right\|_{Z(I)} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|\nabla R_0(v-w)\|_{L_t^1 L_x^2} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|[\nabla, R_0](v-w)\|_{L_t^1 L_x^2} \\
&\quad + C\|R_0 \nabla(v-w)\|_{L_t^1 L_x^2} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + C\|v-w\|_{L_t^1 L_x^2} + C\|\nabla(v-w)\|_{L_t^1 L_x^2} \\
&\leq C\|\nabla(|u|^4 u - |z|^4 z)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + CT \sup_{t\in I} \|(v-w)(t)\|_{L^2} \\
&\quad + CT \sup_{t\in I} \|\nabla(v-w)(t)\|_{L^2} \\
&\leq CT^{\frac{3}{10}} R^4 \|v-w\|_{X_I} + CR^4 \|v-w\|_{X_I} + CT\|v-w\|_{X_I}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Phi_{u_0}(v) - \Phi_{u_0}(w)\|_{X_I} &\leq CT^{\frac{1}{2}} R^4 \|v-w\|_{X_I} + CT^{\frac{3}{10}} R^4 \|v-w\|_{X_I} \\
&\quad + CR^4 \|v-w\|_{X_I} + CT\|v-w\|_{X_I},
\end{aligned}$$

which provides the local existence if we take constants T, R satisfying $C(T^{\frac{1}{2}} R^4 + T^{\frac{3}{10}} R^4 + R^4 + T) < 1$. □

To prove global existence, notice that, since

$$E(t) \leq E(0), \quad \forall t \in I.$$

Thus, the energy is bounded for every $t \geq 0$. We use this property and the finite blow-up criterion below to prove that the maximal interval where the solution of system (3.15) is defined can not be finite.

Lemma 3.3.1 (Finite blow-up criterion)**.** *Let $T(u_0) > 0$ and $I_0 = [0, T(u_0)]$ be the maximal interval for which the solution u for system (3.15) is defined on I_0. If $T(u_0) < +\infty$, then*

$$\|u\|_{S([0,T(u_0)])} = +\infty.$$

Demonstração. We argue by contradiction. Assume that $T(u_0) < +\infty$ and $\|u\|_{S([0,T(u_0)])} < +\infty$. Let $\|u\|_{S([0,T(u_0)])} = M$ and, for $\varepsilon > 0$ which will be chosen below, we choose $N = N(\varepsilon)$ intervals I_j such that

$$\bigcup_{j=1}^{N} I_j = [0, T(u_0)]$$

with $\|u\|_{S(I_j)} \le \varepsilon$. The first step is to show that

$$\|u\|_{L^\infty[0,T(u_0)];\dot H^1(\mathbb{R}^3)}+\|u\|_{L^\infty[0,T(u_0)];L^2(\mathbb{R}^3)}+\|\nabla u\|_{W([0,T(u_0)])}+\|\nabla u\|_{Z([0,T(u_0)])} < +\infty. \quad (3.21)$$

We write the integral equation (3.17) on each interval I_j (or apply Proposition 4.4.2 to system (3.15) on each interval I_j) to obtain

$$\begin{aligned} & \sup_{t\in I_j}\|u(t)\|_{\dot H^1(\mathbb{R}^3)} + \sup_{t\in I_j}\|u(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla u\|_{W(I_j)} + \|\nabla u\|_{Z(I_j)} \\ \le\ & C\|u(t_j)\|_{\dot H^1(\mathbb{R}^3)} + C\|u\|^4_{S(I_j)}\|\nabla u\|_{Z(I_j)} + C\|u\|^4_{S(I_j)}\|u\|_{S(I_j)} \\ \le\ & C\|u(t_j)\|_{\dot H^1(\mathbb{R}^3)} + C\|u\|^4_{S(I_j)}\|\nabla u\|_{Z(I_j)} \\ \le\ & C\|u(t_j)\|_{\dot H^1(\mathbb{R}^3)} + C\varepsilon^4\|\nabla u\|_{Z(I_j)}, \end{aligned}$$

where t_j is a fixed point in I_j. The desired estimate (3.21) follows if we choose $\varepsilon > 0$ such that $C\varepsilon^4 < \frac{1}{2}$. For the second step, we choose a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to T(u_0)$ as $n \to \infty$. Let T_* be the length of the existence interval given by Theorem 3.3.2. Let n be large enough but fixed such that

$$T(u_0) - t_n = \varepsilon_0$$

with $\varepsilon_0 > 0$ satisfying $\varepsilon_0 = \frac{T_*}{2}$. Since $E(t_n) \le E(0)$ for all $t_n \ge 0$, Theorem 3.3.2 may be applied for the interval $[0, T(u_0)+\varepsilon]$ whose length is T_*. However, this contradicts the maximality of $T(u_0)$ and concludes the proof. □

Remark 3.3.2. *We have proved that for all u_0, g with $\|u_0\|_{H^1}+\|g\|_{L^\infty([0,T];H^1)}$ small enough, the solution u of system (3.12) satisfies*

$$\|u\|_{X_I} \le C\Big(\|u_0\|_{H^1} + \|g\|_{L^\infty([0,T];H^1)} + \|u\|^5_{X_I}\Big).$$

Hence,

$$\begin{aligned} \|u\|_{X_I} &\le C\Big(\|u_0\|_{H^1} + \|g\|_{L^\infty([0,T];H^1)}\Big) \\ &\le C(T,A), \end{aligned}$$

applying a classical bootstrap argument (Lemma 4.3.2).

Remark 3.3.3. *If u is a solution of system (3.15), then, using Proposition 4.4.2, one has*

$$\|u\|_{X_I} \le C\Big(\|u(a)\|_{H^1} + \|u\|^5_{X_I}\Big).$$

So, for $\|u_0\|_{H^1}$ small enough, one has

$$\|u\|_{X_I} \le C(T, \|u_0\|_{H^1}) \quad (3.22)$$

by a classical bootstrap argument (Lemma 4.3.2).

3.4 SCATTERING RESULT

In this section, we bring the scattering result obtained as a consequence of the existence of solutions in the Strichartz space, proven by Cazenave and Weissler in (CAZENAVE; WEISSLER, 1990).

Proposition 3.4.1. *Let u be a solution of*

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = 0 \ (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(t_0) = u_0, \end{cases}$$

where $u_0 \in H^1(\mathbb{R}^3)$ and supposed to be small enough, $u \in L^{10}(\mathbb{R}^4)$ and $u \in L^{10}(\mathbb{R}; L^{\frac{30}{13}}(\mathbb{R}^3))$. There exist $u_+, u_- \in \dot{H}^1(\mathbb{R}^3)$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{\dot{H}^1} = 0 \text{ and } \lim_{t\to-\infty} \|u(t) - e^{it\Delta}u_-\|_{\dot{H}^1}.$$

Demonstração. Note that

$$\begin{aligned} \left\| \int_t^{+\infty} \nabla e^{i(t-\tau)\Delta}|u|^4 u \ d\tau \right\|_{L^2} &\leq C\|\nabla|u|^4 u\|_{L^2(t,+\infty)L^{\frac{6}{5}}(\mathbb{R}^3)} \\ &\leq C\|u\|^4_{L^{10}(t,+\infty)L^{10}(\mathbb{R}^3)}\|\nabla u\|_{L^{10}(t,+\infty)L^{\frac{30}{13}}(\mathbb{R}^3)} \to 0 \end{aligned} \quad (3.23)$$

as $t \to +\infty$. Then, since

$$u(t) = e^{i(t-t_0)\Delta}u_0 + \int_{t_0}^t e^{i(t-\tau)\Delta}|u|^4 u \ d\tau,$$

taking

$$u_+ = e^{-it_0\Delta}u_0 + \int_{t_0}^{+\infty} e^{-i\tau\Delta}|u|^4 u \ d\tau$$

and

$$u_- = e^{-it_0\Delta}u_0 - \int_{-\infty}^{t_0} e^{-i\tau\Delta}|u|^4 u \ d\tau,$$

u_+ and u_- have the desired property. Indeed,

$$\begin{aligned} \|u(t) - e^{it\Delta}u_+\|_{\dot{H}^1} &= \Bigg\| e^{i(t-t_0)\Delta}u_0 + \int_{t_0}^t e^{i(t-\tau)\Delta}|u|^4 u \ d\tau \\ &\quad - e^{it\Delta}e^{-it_0\Delta}u_0 - \int_{t_0}^{+\infty} e^{it\Delta}e^{-i\tau\Delta}|u|^4 u \ d\tau \Bigg\|_{\dot{H}^1} \\ &= \left\| \int_{t_0}^t e^{i(t-\tau)\Delta}|u|^4 u \ d\tau - \int_{t_0}^{+\infty} e^{i(t-\tau)\Delta}|u|^4 u \ d\tau \right\|_{\dot{H}^1} \\ &= \left\| \int_t^{+\infty} e^{i(t-\tau)\Delta}|u|^4 u \ d\tau \right\|_{\dot{H}^1} \to 0 \end{aligned}$$

as $t \to +\infty$, by (3.23). Moreover,

$$
\begin{aligned}
\|u(t) - e^{it\Delta}u_-\|_{\dot{H}^1} &= \Bigg\| e^{i(t-t_0)\Delta}u_0 + \int_{t_0}^{t} e^{i(t-\tau)\Delta}|u|^4 u \, d\tau \\
&\quad - e^{it\Delta}e^{-it_0\Delta}u_0 + \int_{-\infty}^{t_0} e^{it\Delta}e^{-i\tau\Delta}|u|^4 u \, d\tau \Bigg\|_{\dot{H}^1} \\
&= \left\| \int_{t_0}^{t} e^{i(t-\tau)\Delta}|u|^4 u \, d\tau + \int_{-\infty}^{t_0} e^{i(t-\tau)\Delta}|u|^4 u \, d\tau \right\|_{\dot{H}^1} \\
&= \left\| \int_{-\infty}^{t} e^{i(t-\tau)\Delta}|u|^4 u \, d\tau \right\|_{\dot{H}^1} \to 0
\end{aligned}
$$

as $n \to -\infty$, by (3.23), concluding the proof. □

4 PROFILE DECOMPOSITION

4.1 INTRODUCTION

In this chapter, we consider the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ \qquad\qquad u(0,x) = \varphi(x), \quad x \in \mathbb{R}^3. \end{cases} \tag{4.1}$$

For $\varphi \in \dot{H}^1(\mathbb{R}^3)$, the solution of problem (4.1) is given explicitly by $v = e^{it\Delta}\varphi \in C(\mathbb{R}_t, \dot{H}^1(\mathbb{R}^3_x))$ and we have the conservation law

$$E_0(v)(t) := \int_{\mathbb{R}^3} |\nabla v(t)|^2 \, dx = E_0(\varphi). \tag{4.2}$$

The nonlinear $\dot{H}^1$-critical Schrödinger equation in three space dimensions associated to (4.1) is

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ \qquad\qquad u(0,x) = \varphi(x) \quad x \in \mathbb{R}^3. \end{cases} \tag{4.3}$$

The Cauchy problem (4.3) has the following properties: (see, e.g., (CAZENAVE; WEISSLER, 1990)).

i) For all $\varphi \in \dot{H}^1(\mathbb{R}^3)$, there exists a unique maximal solution $u(t,x)$ of problem (4.3) satisfying

$$u \in C((T_*, T^*); \dot{H}^1(\mathbb{R}^3)), \text{ and } \nabla u \in L^q_{loc}((T_*, T^*); L^r(\mathbb{R}^3)),$$

for every L^2-admissible pair(q,r).

ii) The solution u satisfies the conservation law

$$E_1(u)(t) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t)|^2 \, dx + \frac{1}{6}\int_{\mathbb{R}^3} |u(t)|^6 \, dx = E_1(\varphi).$$

iii) If either T_* or T^* is finite, then $\|\nabla u\|_{L^q((T_*,T^*);L^r(\mathbb{R}^3))} = \infty$ for all L^2-admissible pair (q,r) with $r > 2$.

Furthermore, the theory of small data explored in (CAZENAVE; WEISSLER, 1990) ensures that for $\|\varphi\|_{\dot{H}^1(\mathbb{R}^3)}$ small enough there exists a unique maximal solution $u(t,x)$ of the initial value problem (IVP) (4.3) satisfying

$$u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)), \quad u \in L^{10}(\mathbb{R}^4), \text{ and } \nabla u \in L^{\frac{10}{3}}(\mathbb{R}^4). \tag{4.4}$$

We prove that every sequence of solutions to the linear Schrödinger equation with bounded data in $\dot{H}^1(\mathbb{R}^3)$ can be written, up to a subsequence, as an almost orthogonal sum of sequences of the type $h_n^{-\frac{1}{2}}\varphi\left(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\right)$, where φ is a solution of the linear Schrödinger equation with a small remainder term in Strichartz norms. Using this decomposition, we prove a similar one for the defocusing $\dot{H}^1$-critical nonlinear Schrödinger equation (4.3), assuming that the initial data belong to a ball in the energy space where the equation is solvable. This implies, in particular, the existence of an estimate for the Strichartz norms in terms of the energy.

4.1.1 Notations

Throughout this chapter, C denotes a numerical constant that can be different from one step to another in the demonstrations. u_n (or v_n) denotes a sequence $(u_n)_{n\in\mathbb{N}}$ (or $(v_n)_{n\in\mathbb{N}}$).

Definition 4.1. Let $\varphi \in \dot{H}^1(\mathbb{R}^3)$ with $\|\varphi\|_{\dot{H}^1} < \lambda$ for $\lambda > 0$ small enough such that the global existence for the problem (4.3) holds with $u \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^3) \cap L^{10}(\mathbb{R}^4)$, $\nabla u \in L^{\frac{10}{3}}(\mathbb{R}^4)$. We define λ_0 as the supremum of these λ.

Remark 4.2. If $\|\varphi\|_{\dot{H}^1(\mathbb{R}^3)} < \lambda_0$, then system (4.3) admits a complete scattering theory relative to its associated linear problem. However, it is an open problem to prove that $\lambda_0 = \infty$, i.e., to prove global well-posedness for the IVP (4.3) for any initial data in $\dot{H}^1(\mathbb{R}^3)$[1].

The following definition will be useful in the first part of the proof of the linear profile decomposition, which consists of the extraction of the scales of oscillation h_n.

Definition 4.3.

i) We call **scale** every sequence $\underline{h} = (h_n)_{n\geq 0}$ of positive numbers converging to 0 and **core** every sequence $[\underline{x}, \underline{t}] = (x_n, t_n)_{n\geq 0} \subset \mathbb{R}^3 \times \mathbb{R}$. We denote a **scale-core** by $[\underline{h}, \underline{x}, \underline{t}]$.

ii) We say that two sequences of scale-core $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ and $[\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ are orthogonal if either

$$\frac{h_n^{(1)}}{h_n^{(2)}} + \frac{h_n^{(2)}}{h_n^{(1)}} \longrightarrow +\infty \text{ as } n \to \infty, \tag{4.5}$$

or $h_n^{(1)} = h_n^{(2)} = h_n$ and

$$\left|\frac{t_n^{(1)} - t_n^{(2)}}{h_n^2}\right| + \left|\frac{x_n^{(1)} - x_n^{(2)}}{h_n}\right| \longrightarrow +\infty \text{ as } n \to \infty. \tag{4.6}$$

[1] Bourgain solved this problem in the particular case of radially symmetric data (BOURGAIN, 1999a).

We denote $[\underline{h}^{(1)}, \underline{x}^{(1)}, \underline{t}^{(1)}] \perp [\underline{h}^{(2)}, \underline{x}^{(2)}, \underline{t}^{(2)}]$ and $(\underline{x}^{(1)}, \underline{t}^{(1)}) \perp_{h_n} (\underline{x}^{(2)}, \underline{t}^{(2)})$, if $\underline{h}^{(1)} = \underline{h}^{(2)} = h_n$.

4.1.2 Concentrating solutions

Now, we introduce the concept of concentration solution, which will be extremely important for the study of the asymptotic behavior of systems (4.1) and (4.3).

Definition 4.4.

i) Let $f \in L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$, $\underline{h} = h_n \in \mathbb{R}_+^*$, $\underline{x} = x_n \in \mathbb{R}^3$ and $\underline{t} = t_n \in \mathbb{R}$ such that $\lim_n(h_n, x_n, t_n) = (0, x_\infty, t_\infty)$. A **linear concentrating solution** associated to $[f, \underline{h}, \underline{x}, \underline{t}]$ is a sequence $(v_n)_{n\in\mathbb{N}}$ of solutions to

$$i\partial_t v_n + \Delta v_n = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \tag{4.7}$$

of the form

$$v_n(t,x) = \frac{1}{\sqrt{h_n}} f\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big); \tag{4.8}$$

ii) The associated **nonlinear concentrating solution** is a sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to

$$\begin{cases} i\partial_t u_n + \Delta u_n - |u_n|^4 u_n = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u_n(0) = v_n(0), & x \in \mathbb{R}^3, \end{cases} \tag{4.9}$$

of the form

$$u_n(t,x) = \frac{1}{\sqrt{h_n}} \overline{f}\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big),$$

where $\overline{f}(-t_n/h_n^2) = f(-t_n/h_n^2)$.

The next definition is the tool that will be used to "track back" the concentrations.

Definition 4.5. Let $x_\infty \in \mathbb{R}^3$, $t_\infty \in \mathbb{R}$, $\underline{h} = h_n \in \mathbb{R}_+^*$, $\underline{x} = x_n \in \mathbb{R}^3$, $\underline{t} = t_n \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$ such that $\lim_n(h_n, x_n, t_n) = (0, x_\infty, t_\infty)$. Given a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$, we write

$$D_{h_n} f_n \rightharpoonup f$$

if

$$h_n^{\frac{1}{2}} f_n(t_n + h_n^2 t, x_n + h_n x) \rightharpoonup f(t,x) \ \text{ weakly in } \dot{H}^1(\mathbb{R}^3)$$

as $n \to \infty$, for all $t \in \mathbb{R}$.

Of course, this definition depends on the core of concentration h_n, x_n and t_n. When several rates of concentration $[\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$, $j \in \mathbb{N}$, are used in a proof, we use the notation $D_h^{(j)}$ to distinguish them.

Now, we state a series of results related to the concept of concentration solutions.

Lemma 4.1.1. *$D_{h_n} f_n \rightharpoonup f$ is equivalent to*

$$\int_{\mathbb{R}^3} \nabla_x f_n(t_n + h_n^2 s) \cdot \nabla u_n(t_n + h_n^2 s)\ dx \longrightarrow \int_{\mathbb{R}^3} \nabla_y f(s) \cdot \nabla_y \varphi(s)\ dy \text{ as } n \to \infty,\ \forall s \in \mathbb{R},$$

where u_n is of the form $u_n(t,x) = \frac{1}{\sqrt{h_n}} \varphi\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big)$ with $\varphi \in L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$.

Demonstração. Assume $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$. We denote

$$L_n = \sqrt{h_n} \int_{\mathbb{R}^3} \nabla_y f_n(t_n + h_n^2 s, x_n + h_n y) \cdot \nabla_y \varphi(s,y)\ dy.$$

So, with the change of variables $x_n + h_n y = x$,

$$\begin{aligned} L_n &= \sqrt{h_n} \int_{\mathbb{R}^3} \nabla_y f_n(t_n + h_n^2 s, x) \cdot \nabla_y \varphi\Big(s, \frac{x-x_n}{h_n}\Big)\ \frac{dx}{h_n^3} \\ &= \int_{\mathbb{R}^3} \nabla_x f_n(t_n + h_n^2 s, x) \cdot \nabla_x \frac{1}{\sqrt{h_n}} \varphi\Big(s, \frac{x-x_n}{h_n}\Big)\ dx \\ &= \int_{\mathbb{R}^3} \nabla_x f_n(t_n + h_n^2 s) \cdot \nabla_x u_n(t_n + h_n^2 s)\ dx \end{aligned}$$

Therefore, L_n tends to $\displaystyle\int_{\mathbb{R}^3} \nabla_y f(s) \cdot \nabla_s \varphi(s)\ dy$ for all $s \in \mathbb{R}$ if, and only if, $\int_{\mathbb{R}^3} \nabla_x f_n(t_n + h_n^2 s) \cdot \nabla_x u_n(t_n + h_n^2 s)\ dx$ has the same limit. □

The previous lemma is directly linked to the concept of concentrating solutions.

Lemma 4.1.2. *If f_n is a linear concentrating solution associated to $[f, \underline{h}, \underline{x}, \underline{t}]$, then*

$$D_{h_n} f_n \rightharpoonup f.$$

Demonstração. Since f_n has the form

$$f_n(t,x) = \frac{1}{\sqrt{h_n}} f\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big),$$

the change of variables

$$\sqrt{h_n} f_n(t_n + h_n^2 s, x_n + h_n y) = f(s,y)$$

yields that

$$\begin{aligned} L_n &= \sqrt{h_n} \int_{\mathbb{R}^3} \nabla_y f_n(t_n + h_n^2 s, x_n + h_n y) \cdot \nabla_y \varphi(s,y)\ dx \\ &= \int_{\mathbb{R}^3} \nabla_y f(s) \cdot \nabla_y \varphi(s)\ dy. \end{aligned}$$

Thus, the same computation of Lemma 4.1.1 yields

$$\int_{\mathbb{R}^3} \nabla_x f_n(t_n + h_n^2 s) \cdot \nabla_x u_n(t_n + h_n^2 s)\ dx = \int_{\mathbb{R}^3} \nabla_y f(s) \cdot \nabla_y \varphi(s)\ dy,$$

which gives $D_{h_n} f_n \rightharpoonup f$. □

Lemma 4.1.3. *If u_n is a concentrating solution associated to $[\varphi, \underline{h}, \underline{x}, \underline{t}]$, then*

$$\|u_n\|_{L^\infty \dot{H}^1} = \|\varphi\|_{L^\infty \dot{H}^1}, \quad \|u_n\|_{L_t^{10} L_x^{10}} = \|\varphi\|_{L_t^{10} L_x^{10}} \ \text{ and } \ \|\nabla u_n\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} = \|\nabla \varphi\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}. \tag{4.10}$$

Demonstração. Using Definition 4.4 and the change of variables $\frac{t-t_n}{h_n^2} = s$ and $\frac{x-x_n}{h_n} = y$, we get

$$\begin{aligned}
\|\nabla u_n(t)\|_{L^2} &= \left(\int_{\mathbb{R}^3} |\nabla_x u_n(t,x)|^2\ dx \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{h_n}} \left(\int_{\mathbb{R}^3} \left| \nabla_x \varphi\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n} \Big) \right|^2\ dx \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{h_n}} \left(\int_{\mathbb{R}^3} |\nabla_x \varphi(s,y)|^2\ h_n^3 dy \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{R}^3} |\nabla_y \varphi(s,y)|^2\ dy \right)^{\frac{1}{2}} \\
&= \|\nabla \varphi(s)\|_{L^2}.
\end{aligned}$$

Second, through the same change of variables $\frac{t-t_n}{h_n^2} = s$ and $\frac{x-x_n}{h_n} = y$, one has

$$\begin{aligned}
\|u_n\|_{L_t^{10} L_x^{10}}^{10} &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} |u_n(t,x)|^{10}\ dxdt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left| \frac{1}{\sqrt{h_n}} \varphi\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n} \Big) \right|^{10}\ dxdt \\
&= \frac{1}{h_n^5} \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\varphi(s,y)|^{10}\ h_n^3 dy h_n^2 ds \\
&= \|\varphi\|_{L_t^{10} L_x^{10}}^{10}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\nabla u_n\|^{\frac{10}{3}}_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\nabla_x u_n(t,x)|^{\frac{10}{3}} \, dxdt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \left| \nabla_x \frac{1}{\sqrt{h_n}} \varphi\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n} \Big) \right|^{\frac{10}{3}} \, dxdt \\
&= \frac{1}{h_n^{\frac{5}{3}}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\nabla_x \varphi(s,y)|^{\frac{10}{3}} \, h_n^3 dy h_n^2 ds \\
&= \frac{1}{h_n^{\frac{5}{3}}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\nabla_x \varphi(s,y)|^{\frac{10}{3}} \, h_n^3 dy h_n^{-\frac{4}{3}} h_n^{\frac{10}{3}} ds \\
&= \frac{1}{h_n^{\frac{5}{3}}} \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\nabla_y \varphi(s,y)|^{\frac{10}{3}} \, h_n^3 dy h_n^{-\frac{4}{3}} ds \\
&= \|\nabla \varphi\|^{\frac{10}{3}}_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}.
\end{aligned}$$

□

4.1.3 Scales

On the Hilbert space $H^1(\mathbb{R}^3)$,we define the self-adjoint operator A by

$$D(A) = H^2(\mathbb{R}^3)$$

$$Au = (-\Delta)^{\frac{1}{2}} u.$$

The next definition and remarks can be found in (LAURENT, 2011).

Definition 4.6. Let A be a self-adjoint (unbounded) operator on a Hilbert space H. Let h_n be a sequence of positive numbers converging to 0. A bounded sequence (u_n) in H is said to be h_n-oscillatory with respect to A if

$$\limsup_{n\to\infty} \left\| 1_{|A| \geq \frac{R}{h_n}} u_n \right\|_H \longrightarrow 0 \text{ as } R \to \infty, \tag{4.11}$$

and (u_n) is said to be strictly h_n-oscillatory with respect to A if it satisfies (4.11) and

$$\limsup_{n\to\infty} \left\| 1_{|A| \leq \frac{\epsilon}{h_n}} u_n \right\|_H \longrightarrow 0 \text{ as } \epsilon \to 0. \tag{4.12}$$

Moreover, (u_n) is said to be h_n-singular with respect to A if

$$\left\| 1_{\frac{a}{h_n} \leq |A| \leq \frac{b}{h_n}} u_n \right\|_H \longrightarrow 0 \text{ as } n \to \infty, \text{ for all } a, b > 0. \tag{4.13}$$

Remark 4.1.1.

i) Let (h_n) be a scale. Let (f_n) and (g_n) be two bounded sequences in $L^2(\mathbb{R}^3)$ such that (f_n) is h_n-oscillatory and (g_n) is h_n-singular. Then, via Plancherel's inversion formula and Cauchy-Schwartz inequality,

$$\int_{\mathbb{R}^3} f_n(x)\bar{g}_n(x)\,dx \longrightarrow 0 \text{ as } n \to \infty.$$

Hence, it follows that

$$\|f_n + g_n\|^2_{L^2(\mathbb{R}^3)} = \|g_n\|^2_{L^2(\mathbb{R}^3)} + \|f_n\|^2_{L^2(\mathbb{R}^3)} + o(1), \quad \text{as } n \to \infty.$$

ii) Let (h_n) be a scale and (f_n) a bounded sequence in $L^2(\mathbb{R}^3)$, such that (f_n) is h_n-oscillatory. Then (f_n) is h'_n-singular for every scale h'_n orthogonal to h_n.

iii) We remark that a sequence is (strictly) h_n-oscillatory with respect to A if and only if it is (strictly) h_n^2-oscillatory with respect to A^2. So we can replace A by $-\Delta$.

The next result ensures that the Schrödinger equation conserves h_n-oscillation.

Proposition 4.1.1. *Let $T > 0$. Let φ_n be a bounded sequence of $H^1(\mathbb{R}^3)$ that is (strictly) h_n- oscillatory with respect to A. If u_n is the solution of*

$$\begin{cases} i\partial_t u_n + \Delta u_n = 0, \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ u_n(0) = \varphi_n, \quad x \in \mathbb{R}^3, \end{cases} \tag{4.14}$$

then, $(u_n(t))$ is (strictly) h_n-oscillatory with respect to A, uniformly on $[0,T]$. If (φ_n) is h_n-singular with respect to A, then $(u_n(t))$ is h_n-singular with respect to A, uniformly on $[0,T]$.

Demonstração. Consider the cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that $0 \le \chi(s) \le 1$ and $\chi(s) = 1$ for $|s| \le 1$. The h_n-oscillation (respectively strict oscillation) is equivalent to

$$\limsup_{n\to\infty} \left\| \nabla(1-\chi)\left(\frac{h_n^2\Delta}{R^2}\right)u_n \right\|_{L^2} \longrightarrow 0 \text{ as } R \to \infty$$

(resp. $\limsup_{n\to\infty} \|\nabla\chi(R^2h_n^2\Delta)u_n\|_{L^2} \longrightarrow 0$ as $R \to \infty$). Note that $v_n = (1-\chi)(\frac{h_n^2\Delta}{R^2})u_n$ is a solution of

$$\begin{cases} i\partial_t v_n + \Delta v_n = 0, \\ v_n(0) = (1-\chi)\left(\frac{h_n^2\Delta}{R^2}\right)\varphi_n, \end{cases}$$

and the conservation of energy gives

$$\begin{aligned} \|\nabla v_n(t)\|_{L^2} &= \|\nabla v_n(0)\|_{L^2} \\ &= \left\| \nabla(1-\chi)\left(\frac{h_n^2\Delta}{R^2}\right)\varphi_n \right\|_{L^2}. \end{aligned}$$

Therefore, taking the limsup in n we get the expected result uniformly in t for $0 \le t \le T$. The results for strict oscillation are proved similarly. Regarding the case of singularity, note that if u_n is a solution of (4.14) then $v_n(t) = \sigma_n(D)u_n(t)$ is also solution to the same equation, where $\sigma_n(\xi) = \mathbf{1}_{\frac{a}{h_n} \le |\xi| \le \frac{b}{h_n}}(\xi)$. Then, since

$$\|\nabla v_n\|_{L^2} = \|\nabla v_n(0)\|_{L^2},$$

we get the result. □

Definition 4.7. The Besov space $\dot{B}^0_{2,\infty}(\mathbb{R}^3)$ is defined by

$$\dot{B}^0_{2,\infty}(\mathbb{R}^3) := \Big\{ u = u(x) : \|u\|^2_{\dot{B}^0_{2,\infty}(\mathbb{R}^3)} = \sup_{k\in\mathbb{Z}} \int_{2^k \le |\xi| \le 2^{k+1}} |\widehat{u}(\xi)|^2 \, d\xi < +\infty \Big\}.$$

The following result gives us an estimation of Besov spaces.

Proposition 4.1.2. *For every (φ_n) bounded sequence of $H^1(\mathbb{R}^3)$, there exists $C_T > 0$ such that*

$$\limsup_{n\to\infty} \|\nabla u_n\|_{L^\infty([0,T];\dot{B}^0_{2,\infty}(\mathbb{R}^3))} \le C_T \limsup_{n\to\infty} \|\nabla \varphi_n\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^3)}$$

where u_n is the solution of system (4.14).

Demonstração. Since u_n is the solution of system (4.14), the function $\sigma_k(D)u_n$ is also a solution to the same system, where $\sigma_k(\xi) = \mathbf{1}_{2^k \le |\xi| \le 2^{k+1}}$. The conservation law for all $\sigma_k(D)u_n(t)$, $k \in \mathbb{Z}$, gives

$$\|\nabla u_n(t)\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^3)} = \|\nabla u_n(0)\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^3)} = \|\nabla \varphi_n\|_{\dot{B}^0_{2,\infty}(\mathbb{R}^3)},$$

showing the result. □

4.2 LINEAR PROFILE DECOMPOSITION

The main result of this section is a combination of theories developed by Bahouri and Gerard (BAHOURI, 2011), Keraani (KERAANI, 2001) and Laurent (LAURENT, 2011) and is given by the following theorem.

Theorem 4.8. *Let (v_n) be a sequence of solutions to the Schrödinger equation (4.14) on $[0,T]$ with initial data φ_n, at time $t = 0$, bounded in $H^1(\mathbb{R}^3)$ and such that $\limsup_{n\to\infty} \|\varphi_n\|_{H^1} < \lambda_0$, where λ_0 was given in Definition 4.1. Then, up to extraction, there exists a sequence* *of linear concentrating solutions* $(\underline{p}^{(j)})$ *associated to* $[\varphi^{(j)}, \underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$ *such that, for any* $l \in \mathbb{N}^*$,

$$v_n(t,x) = \sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x) \tag{4.15}$$

satisfies

$$\limsup_{n\to\infty} \|w_n^{(l)}\|_{L_t^\infty L_x^6 \cap L_t^{10} L_x^{10}} \longrightarrow 0 \text{ as } l \to \infty, \tag{4.16}$$

for all $T > 0$ *and*

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla p_n^{(j)}\|_{L^2}^2 + \|\nabla w_n^{(l)}\|_{L^2}^2 + o(1) \text{ as } n \to \infty. \tag{4.17}$$

Moreover, we have $(\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}) \perp (\underline{h}^{(k)}, \underline{x}^{(k)}, \underline{t}^{(k)})$ *for any* $j \neq k$*, according to Definition 4.3.*

Our goal in this section is to prove Theorem 4.8. We split its proof into four steps: the first one is the extraction of the scales $h_n^{(j)}$, where we decompose v_n in an infinite sum of sequences $v_n^{(j)}$ which are respectively $h_n^{(j)}$-oscillatory.

Demonstração. ***Step 1****. (Extraction of scales):* In this first step, we present the determination of the family of scales, where we perform the first decomposition we need. Before that, we establish the next result which will be necessary to obtain this decomposition.

Proposition 4.2.1. *Let* (f_n) *be a bounded sequence in* $L^2(\mathbb{R}^3)$*. Then, up to an extraction, there exists a family* (h_n^j) *of pairwise orthogonal scales and a family* (g_n^j) *of bounded sequences in* $L^2(\mathbb{R}^3)$ *such that*

i) for every j, g_n^j *is* h_n^j*-oscillatory;*

ii) for every $l \geq 1$ *and* $x \in \mathbb{R}^3$,

$$f_n(x) = \sum_{j=1}^{l} g_n^j(x) + R_n^l,$$

where (R_n^j) *is* h_n^j*-singular for every* $j \in 1, ..., l$*, and*

$$\limsup_{n\to\infty} \|R_n^l\|_{\dot{B}^0_{2,\infty}} \longrightarrow 0 \text{ as } l \to \infty;$$

iii) for every $l \geq 1$,

$$\|f_n\|_{L^2} = \sum_{j=1}^{l} \|g_n^j\|_{L^2}^2 + \|R_n^l\|_{L^2}^2 + o(1) \text{ as } n \to \infty.$$

The proof of Proposition 4.2.1 is found in (BAHOURI, 2011). With this in mind, let us present the following proposition.

Proposition 4.2.2. *Let $T > 0$. Let (φ_n) be a bounded sequence of $H^1(\mathbb{R}^3)$ and v_n the solution of*

$$\begin{cases} i\partial_t v_n + \Delta v_n = 0 \ (t,x) \in [0,T] \times \mathbb{R}^3, \\ v_n(0) = \varphi_n. \end{cases} \tag{4.18}$$

Then, up to an extraction, v_n can be decomposed in the following way: for any $l \in \mathbb{N}^*$,

$$v_n(t,x) = \sum_{j=1}^{l} v_n^{(j)}(t,x) + \rho_n^{(l)}(t,x), \tag{4.19}$$

where $v_n^{(j)}$ is a strictly $(h_n^{(j)})$-oscillatory solution of the linear Schrödinger equation (4.18) on $\mathbb{R}^3$. The scales $h_n^{(j)}$ satisfy $h_n^{(j)} \to 0$ as $n \to \infty$ and are pairwise orthogonal. Additionally, we have

$$\limsup_{n\to\infty} \|\rho_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))\cap L^{10}([0,T];L^{10}(\mathbb{R}^3))} \longrightarrow 0 \tag{4.20}$$

as $l \to \infty$ and

$$\|\nabla v_n(t)\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla v_n^{(j)}(t)\|_{L^2}^2 + \|\nabla \rho_n^{(l)}(t)\|_{L^2}^2 + o(1) \text{ as } n \to \infty. \tag{4.21}$$

Proof of Proposition 4.2.2. First, we prove this decomposition for the initial data through the Proposition 4.2.1. Then, using the propagation of h_n-oscillation, proved in Proposition 4.1.1, we extend it for all time.

Applying Proposition 4.2.1 to the sequence $(\nabla\varphi_n)_n$, we obtain a family of scales $h_n^{(j)}$ and a family $(\varphi_n^{(j)})$ of bounded sequences in $\dot{H}^1(\mathbb{R}^3)$ such that

$$\varphi_n(x) = \sum_{j=1}^{l} \varphi_n^{(j)}(x) + \Phi_n^{(l)}(x),$$

where $\varphi_n^{(j)}$ is h_n^j-oscillatory with respect to A for every $j \geq 1$. Moreover, $\Phi_n^{(l)}$ is $h_n^{(j)}$-singular with respect to A for every $j \in 1, 2, ..., l$, and

$$\limsup_{n\to\infty} \|\nabla\Phi_n^l\|_{\dot{B}^0_{2,\infty}} \longrightarrow 0 \text{ as } l \to \infty. \tag{4.22}$$

Furthermore, the following almost orthogonality identity

$$\|\nabla\varphi_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla\varphi_n^{(j)}\|_{L^2}^2 + \|\nabla\Phi_n^{(l)}\|_{L^2}^2 + o(1)$$

holds for all $l \geq 1$, and the $h_n^{(j)}$ are pairwise orthogonal. This decomposition for the initial data can be extended to the solution by

$$v_n(t,x) = \sum_{j=1}^{l} v_n^{(j)}(t,x) + \rho_n^{(l)}(t,x),$$

where each $v_n^{(j)}$ is a solution of

$$\begin{cases} i\partial_t v_n^{(j)} + \Delta v_n^{(j)} = 0 \text{ on } [0,T] \times \mathbb{R}^3, \\ v_n^{(j)}(0) = \varphi_n^{(j)}, \end{cases}$$

and $\rho_n^{(l)}$ is a solution to the same system with initial data equal to $\Phi_n^{(l)}$.

Due to Proposition 4.1.1, each $v_n^{(j)}(t)$ is strictly $h_n^{(j)}$-oscillatory and $\rho_n^{(l)}(t)$ is $h_n^{(j)}$-singular for $1 \leq j \leq l$. So,

$$\langle \nabla \rho_n^{(l)}(t), \nabla v_n^{(j)}(t) \rangle_{L^2} \longrightarrow 0$$

as $n \to \infty$, uniformly in $[0,T]$. This also holds for the product between $v_n^{(j)}$ and v_n^k, $j \neq k$ by the orthogonality of the scales, i.e.,

$$\langle \nabla v_n^{(j)}(t), \nabla v_n^{(k)}(t) \rangle_{L^2} \longrightarrow 0,$$

as $n \to \infty$. Then, we get

$$\|\nabla v_n(t)\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla v_n^{(j)}(t)\|_{L^2}^2 + \|\nabla \rho_n^{(l)}(t)\|_{L^2}^2 + o(1)$$

which is the desired equation (4.21).

Let us now prove convergence (4.20) for the remaining term in $L^\infty L^6$. The convergence (4.22) gives the convergence to zero of $\nabla \rho_n^{(l)}(0) = \nabla \Phi_n^{(l)}$ in $\dot{B}^0_{2,\infty}$. We extend this convergence for all time with Proposition 4.1.2 to get

$$\sup_{t \in [0,T]} \limsup_{n \to \infty} \|\nabla \rho_n^{(l)}(t)\|_{\dot{B}^0_{2,\infty}} \longrightarrow 0 \text{ as } l \to \infty.$$

The proof of the smallness of the remainder term is based on the following refined Sobolev inequality which can be found in (BAHOURI, 2011, Lemma 3.5).

Proposition 4.2.3. *For all $f \in \dot{H}^1(\mathbb{R}^3)$, there exists $C > 0$ such that*

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}} \|\nabla f\|_{\dot{B}^0_{2,\infty}}^{\frac{2}{3}}. \tag{4.23}$$

Using inequality (4.23), one has

$$\limsup_{n\to\infty}\|\rho_n^{(l)}(t)\|_{L^6}\le C\limsup_{n\to\infty}\|\nabla\rho_n^{(l)}(t)\|_{L^2}^{\frac{1}{3}}\limsup_{n\to\infty}\|\nabla\rho_n^{(l)}(t)\|_{\dot{B}^0_{2,\infty}}^{\frac{2}{3}}.$$

Observe that

$$\|\nabla\rho_n^{(l)}(t)\|_{L^2}^2\le\|\nabla v_n(t)\|_{L^2}^2\le\|\nabla\varphi_n\|_{L^2}^2\le C.$$

Therefore,

$$\limsup_{n\to\infty}\|\rho_n^{(l)}\|_{L_t^\infty L_x^6}\longrightarrow 0 \text{ as } l\to\infty.$$

By an interpolation inequality, we obtain

$$\|\rho_n^{(l)}\|_{L_t^{10}L_x^{10}}\le\|\rho_n^{(l)}\|_{L_t^\infty L_x^6}^{\alpha}\|\rho_n^{(l)}\|_{L_t^7L_x^{14}}^{\beta}. \tag{4.24}$$

But, since $\left(7,\frac{42}{17}\right)$ is a L^2-admissible pair and by Sobolev's embedding,

$$\begin{aligned}\|\rho_n^{(l)}\|_{L_t^7L_x^{14}} &\le \|\nabla e^{it\Delta}\Phi_n^{(l)}\|_{L_t^7L_x^{\frac{42}{17}}}\\ &\le \|\nabla\Phi_n^{(l)}\|_{L_t^7L_x^{\frac{42}{17}}}\\ &\le C\|\nabla\Phi_n^{(l)}\|_{L^2}\end{aligned}$$

which means

$$\limsup_{n\to\infty}\|\rho_n^{(l)}\|_{L_t^{10}L_x^{10}}\longrightarrow 0 \text{ as } l\to\infty.$$

This shows (4.20) and completes the proof of Proposition 4.2.2. □

Step 2. *(Description of linear concentrating solutions):* Now we describe the "non-reconcentration" property for linear concentrating solutions.

Lemma 4.9. *Let $\underline{v}=[\varphi,\underline{h},\underline{x},\underline{t}]$ be a linear concentrating solution. Consider the interval $I=[-T,T]$ of $\mathbb{R}$ containing t_∞. Then, if we set $I_n^{1,\Lambda}=[-T,t_n-\Lambda h_n]$ and $I_n^{3,\Lambda}=(t_n+\Lambda h_n,T]$, we have*

$$\limsup_{n\to\infty}\|v_n\|_{L^\infty(I_n^{1,\Lambda}\cup I_n^{3,\Lambda},L^6(\mathbb{R}^3))}\longrightarrow 0 \text{ as } \Lambda\to\infty \tag{4.25}$$

and

$$\limsup_{n\to\infty}\|v_n\|_{L^{10}(I_n^{1,\Lambda}\cup I_n^{3,\Lambda},L^{10}(\mathbb{R}^3))}\longrightarrow 0 \text{ as } \Lambda\to\infty. \tag{4.26}$$

Proof of Lemma 4.9. We show the convergence (4.25) and get (4.26) through an interpolation argument, similarly to what was done in (4.24). In order to prove (4.25), we argue by contradiction: Suppose that (4.25) is not valid. In this case, there exist a constant $C>0$, a real subsequence $(\Lambda_j)_j$ tending to $+\infty$ and a subsequence $(t_{n_j})_j$ of $(t_n)_n$ convergent to τ such that

$$|t_{n_j} - t_\infty| > \Lambda_j h_{n_j} \text{ and } \lim_j \|v_{n_j}(t_{n_j}, .)\|_{L^6(\mathbb{R}^3)} \longrightarrow C. \tag{4.27}$$

Let us consider separately the cases $\tau \neq t_\infty$ and $\tau = t_\infty$. If $\tau \neq t_\infty$, we have

$$\begin{cases} i\partial_t v_{n_j} + \Delta v_{n_j} = 0, \\ v_{n_j}(t_\infty) = \frac{1}{\sqrt{h_{n_j}}}\varphi\Big(\frac{x}{h_{n_j}}\Big). \end{cases}$$

Then,

$$v_{n_j}(t,x) = e^{i(t-t_\infty)\Delta}\frac{1}{\sqrt{h_{n_j}}}\varphi\Big(\frac{x}{h_{n_j}}\Big),$$

and so

$$v_{n_j}(t_{n_j},x) = e^{i(t_{n_j}-t_\infty)\Delta}\frac{1}{\sqrt{h_{n_j}}}\varphi\Big(\frac{x}{h_{n_j}}\Big),$$

with $t_{n_j} - t_\infty \to \tau - t_\infty = k$.

$$\begin{aligned} v_{n_j}(t_{n_j},x) &= e^{i(t_{n_j}-t_\infty)\Delta}\frac{1}{\sqrt{h_{n_j}}}\varphi\Big(\frac{x}{h_{n_j}}\Big) \\ &= (t_{n_j}-t_\infty)^{-\frac{3}{2}}\frac{1}{\sqrt{h_{n_j}}}\int_{\mathbb{R}^3} e^{i\frac{|y-x|^2}{2(t_{n_j}-t_\infty)}}\varphi\Big(\frac{y}{h_{n_j}}\Big)\,dy \\ &= (t_{n_j}-t_\infty)^{-\frac{3}{2}} h_{n_j}^{\frac{5}{2}}\int_{\mathbb{R}^3} e^{i\frac{|h_{n_j}z-x|^2}{2(t_{n_j}-t_\infty)}}\varphi(z)\,dz \\ &= (t_{n_j}-t_\infty)^{-\frac{3}{2}} h_{n_j}^{\frac{5}{2}}\int_{\mathbb{R}^3} e^{i\frac{h_{n_j}^2|z|^2}{2(t_{n_j}-t_\infty)}}\cdot e^{\frac{-ih_{n_j}\langle z,x\rangle}{(t_{n_j}-t_\infty)}}\cdot e^{\frac{i|x|^2}{2(t_{n_j}-t_\infty)}}\varphi(z)\,dz \\ &\leq C\Big|(t_{n_j}-t_\infty)^{-\frac{3}{2}} h_{n_j}^{\frac{5}{2}}\int_{\mathbb{R}^3} e^{i\frac{h_{n_j}^2|z|^2}{2(t_{n_j}-t_\infty)}}\cdot e^{\frac{-ih_{n_j}\langle z,x\rangle}{(t_{n_j}-t_\infty)}}\varphi(z)\,dz\Big|. \end{aligned}$$

Therefore

$$\begin{aligned} \|v_{n_j}(t_{n_j},x)\|_{L^6} &\leq C\Bigg(\int_{\mathbb{R}^3}\Big|(t_{n_j}-t_\infty)^{-\frac{3}{2}} h_{n_j}^{\frac{5}{2}}\int_{\mathbb{R}^3} e^{i\frac{h_{n_j}^2|z|^2}{2(t_{n_j}-t_\infty)}}\cdot e^{\frac{-ih_{n_j}\langle z,x\rangle}{(t_{n_j}-t_\infty)}}\varphi(z)\,dz\Big|^6\,dx\Bigg)^{\frac{1}{6}} \\ &\leq (t_{n_j}-t_\infty)^{-1}h_{n_j}^2\Bigg(\int_{\mathbb{R}^3}\Big|\int_{\mathbb{R}^3} e^{i\frac{\tilde{h}_j^2|z|^2}{2(t_{n_j}-t_\infty)}}\cdot e^{-i\langle z,w\rangle}\varphi(z)\,dz\Big|^6\,dw\Bigg)^{\frac{1}{6}} \\ &\simeq (t_{n_j}-t_\infty)^{-1}h_{n_j}^2\Bigg(\int_{\mathbb{R}^3}|\hat{\varphi}(w)|^6\,dw\Bigg)^{\frac{1}{6}} \to 0 \text{ as } j\to\infty, \end{aligned}$$

i.e., the right side of this inequality converges to 0 as j goes to ∞, which contradicts (4.27).

Now, if $\tau = t_\infty$, we set $\varepsilon_j^2 = |t_\infty - t_{n_j}|$, $\tilde{h}_j = \frac{h_{n_j}}{\varepsilon_j}$ and define the sequence

$$\tilde{f}_j(s,y) = \varepsilon_j^{\frac{1}{2}} v_{n_j}(t_\infty + \varepsilon_j^2 s, \varepsilon_j y).$$

Note that, since $|t_\infty - t_{n_j}| \geq \Lambda_j h_{n_j}$ and $\lim_j \Lambda_j = +\infty$, then $\lim_j \tilde{h}_j = 0$. The sequence $(\tilde{f}_j)$ is the solution of the system

$$\begin{cases} i\partial_s \tilde{f}_j + \Delta_y \tilde{f}_j = 0, \\ \tilde{f}_j(0) = \frac{1}{\sqrt{\tilde{h}_j}} \varphi\Big(\frac{y}{\tilde{h}_j}\Big). \end{cases}$$

To conclude the proof, it remains to show that $\lim_j \|\tilde{f}_j(1,.)\|_{L^6(\mathbb{R}^3)} = 0$. Note that

$$\begin{aligned} \tilde{f}_j(1,y) = e^{i\Delta} \frac{1}{\sqrt{\tilde{h}_j}} \varphi\Big(\frac{y}{\tilde{h}_j}\Big) &= \frac{1}{\sqrt{\tilde{h}_j}} \int_{\mathbb{R}^3} e^{i\frac{|y-x|^2}{2}} \varphi\Big(\frac{x}{\tilde{h}_j}\Big)\, dx \\ &= \tilde{h}_j^{\frac{5}{2}} \int_{\mathbb{R}^3} e^{i\frac{|\tilde{h}_j z - y|^2}{2}} \varphi(z)\, dz \\ &= \tilde{h}_j^{\frac{5}{2}} \int_{\mathbb{R}^3} e^{i\frac{\tilde{h}_j^2 |z|^2}{2}} \cdot e^{-i\tilde{h}_j \langle z,y\rangle} \cdot e^{\frac{i|y|^2}{2}} \varphi(z)\, dz \\ &\leq C\Big|\tilde{h}_j^{\frac{5}{2}} \int_{\mathbb{R}^3} e^{i\frac{\tilde{h}_j^2|z|^2}{2}} \cdot e^{-i\tilde{h}_j\langle z,y\rangle} \varphi(z)\, dz\Big|. \end{aligned}$$

Then

$$\begin{aligned} \|\tilde{f}_j(1,y)\|_{L^6} &\leq \bigg(\int_{\mathbb{R}^3} \Big|\tilde{h}_j^{\frac{5}{2}} \int_{\mathbb{R}^3} e^{i\frac{\tilde{h}_j^2|z|^2}{2}} \cdot e^{-i\tilde{h}_j\langle z,y\rangle} \varphi(z)\, dz\Big|^6 dy\bigg)^{\frac{1}{6}} \\ &\leq \tilde{h}_j^2 \bigg(\int_{\mathbb{R}^3} \Big|\int_{\mathbb{R}^3} e^{i\frac{\tilde{h}_j^2|z|^2}{2}} \cdot e^{-i\langle z,x\rangle} \varphi(z)\, dz\Big|^6 dx\bigg)^{\frac{1}{6}} \\ &\simeq \tilde{h}_j^2 \bigg(\int_{\mathbb{R}^3} |\hat{\varphi}(x)|^6\, dx\bigg)^{\frac{1}{6}} \to 0 \text{ as } j \to \infty. \end{aligned}$$

Hence, $\|\tilde{f}_j(1,y)\|_{L^6} \to 0$ as $j \to \infty$. Therefore, since

$$\|\tilde{f}_j(1,y)\|_{L^6} = \|v_{nj}(t_{nj},.)\|_{L^6},$$

this contradicts what was stated in (4.27) again, which finishes the proof of this lemma. □

Step 3. *(Extraction of times and cores of concentration):*

Let h_n be a fixed sequence in $\mathbb{R}_+^*$ converging to 0. This step focuses on demonstrating the next proposition, which gives us the profile decomposition for h_n-oscillatory sequences. By merging this decomposition with the one presented in Proposition 4.2.2, we obtain Theorem 4.8

Before presenting the main result of this step, we state and prove two auxiliary lemmas that will be important for the orthogonality of the cores.

Lemma 4.2.1. *Let* $(\underline{x}^{(1)}, \underline{t}^{(1)}) \not\perp_{h_n} (\underline{x}^{(2)}, \underline{t}^{(2)})$. *Let* v_n *be an (strictly)* h_n*-oscillatory sequence of solutions to the linear Schrödinger equation such that*

$$D_{h_n}^{(1)} v_n \rightharpoonup \varphi^{(1)} \tag{4.28}$$

as $n \to \infty$. *There exists* $\varphi^{(2)}$ *such that,*

$$D_{h_n}^{(2)} v_n \rightharpoonup \varphi^{(2)} \tag{4.29}$$

as $n \to \infty$. *Moreover,*

$$\|\varphi^{(1)}\|_{L^\infty \dot{H}^1} = \|\varphi^{(2)}\|_{L^\infty \dot{H}^1}. \tag{4.30}$$

Demonstração. Let $x_n^{(2)} = x_n^{(1)} + (\overrightarrow{D} + o(1))h_n$, $\overrightarrow{D} \in \mathbb{R}^3$ constant, and $t_n^{(2)} = t_n^{(1)} + (\overrightarrow{C} + o(1))h_n^2$, $\overrightarrow{C}$ constant. We have

$$\sqrt{h} v_n(t_n^{(1)} + h_n^2 s, x_n^{(1)} + h_n y) \rightharpoonup \varphi^{(1)}(s, y) \text{ as } n \to \infty, \ \forall s \in \mathbb{R}.$$

Then,

$$\begin{aligned} \sqrt{h_n} v_n(t_n^{(2)} + h_n^2 s, x_n^{(2)} + h_n y) &= \sqrt{h_n} v_n(t_n^{(1)} + (\overrightarrow{C} + o(1))h_n^2 + h_n^2 s, x_n^{(1)} + (\overrightarrow{D} + o(1))h_n + h_n y) \\ &= \sqrt{h_n} v_n(t_n^{(1)} + (\overrightarrow{C} + s)h_n^2, x_n^{(1)} + (\overrightarrow{D} + y)h_n) \\ &\rightharpoonup \varphi^{(1)}(\overrightarrow{C} + s, \overrightarrow{D} + y), \ (s + \overrightarrow{C}) \in \mathbb{R}. \end{aligned}$$

Taking $\varphi^{(1)}(\overrightarrow{C} + s, \overrightarrow{D} + y) = \varphi^{(2)}(s, y)$, one has

$$D_{h_n}^{(2)} v_n \rightharpoonup \varphi^{(2)}, \quad s \in \mathbb{R}.$$

Moreover,

$$\|\nabla \varphi^{(2)}(s)\|_{L^2} = \|\nabla \varphi^{(1)}(s + \overrightarrow{C})\|_{L^2} \leq \sup_{s' \in \mathbb{R}} \|\nabla \varphi^{(1)}(s')\|_{L^2} = \|\nabla \varphi^{(1)}(s)\|_{L^\infty L^2}$$

and

$$\begin{aligned} \|\nabla \varphi^{(1)}(s + \overrightarrow{C})\|_{L^2} &= \|\nabla \varphi^{(2)}(s)\|_{L^2} \\ &\leq \sup_{s \in \mathbb{R}} \|\nabla \varphi^{(2)}(s)\|_{L^2} \\ &= \|\nabla \varphi^{(2)}(s)\|_{L^\infty L^2}. \end{aligned}$$

This finishes the proof of Lemma 4.2.1. □

The second lemma is the following, where we keep the notation of the construction that allowed us to extract the scales and cores.

Lemma 4.2.2. *Let $\{j, j'\} \in \{1, ..., K\}^2$ be such that*

$$(\underline{x}^{(j)}, \underline{t}^{(j)}) \not\perp_{h_n} (\underline{x}^{(K+1)}, \underline{t}^{(K+1)}) \text{ and } (\underline{x}^{(j)}, \underline{t}^{(j)}) \perp_{h_n} (\underline{x}^{(j')}, \underline{t}^{(j')}).$$

If $D_{h_n}^{(K+1)} w_n^{(K+1)} \rightharpoonup 0$, then $D_{h_n}^{(j)} w_n^{(K+1)} \rightharpoonup 0$. Moreover, $D_{h_n}^{(j)} p_n^{(j')} \rightharpoonup 0$ for any concentrating solution $p_n^{(j')}$ associated with $[\varphi^{(j')}, \underline{h}, \underline{x}^{(j')}, \underline{t}^{(j')}]$.

Demonstração. The first result is a particular case of Lemma 4.2.1. So, it only remains to show that

$$D_{h_n}^{(j)} p_n^{(j')} \rightharpoonup 0 \text{ as } n \to \infty$$

or, equivalently,

$$\sqrt{h_n} p_n^{(j')}(t_n^{(j)} + h_n^2 s, x_n^{(j)} + h_n y) \rightharpoonup 0 \text{ in } \dot{H}^1(\mathbb{R}^3)$$

as $n \to \infty$. Since $p_n^{(j')}$ is a concentrating solution associated with $[\varphi^{(j')}, \underline{h}, \underline{x}^{(j')}, \underline{t}^{(j')}]$, we have

$$p_n^{(j')}(t, x) = \frac{1}{\sqrt{h_n}} \varphi^{(j')} \Big(\frac{t - t_n^{(j')}}{h_n^2}, \frac{x - x_n^{(j')}}{h_n} \Big)$$

and

$$\sqrt{h_n} p_n^{(j')}(h_n^2 s, x_n^{(j)} + h_n y) = \varphi^{(j')} \Big(\frac{t_n^{(j)} - t_n^{(j')}}{h_n^2} + s, \frac{x_n^{(j)} - x_n^{(j')}}{h_n} + y \Big).$$

Suposing that $(\underline{x}^{(j)}, \underline{t}^{(j)}) \perp_{h_n} (\underline{x}^{(j')}, \underline{t}^{(j')})$, we assume, without loss of generality, that $\varphi^{(j')}$ is continuous and compactly supported. Thus,

$$\int_{\mathbb{R}^3} \nabla \sqrt{h_n} p_n^{(j')}(t_n^{(j)} + h_n^2 s, x_n^{(j)} + h_n y) \cdot \nabla \psi(y)\, dy = \int_{\mathbb{R}^3} \nabla \varphi^{(j')} \Big(\frac{t_n^{(j)} - t_n^{(j')}}{h_n^2} + s, \frac{x_n^{(j)} - x_n^{(j')}}{h_n} + y \Big) \cdot \nabla \psi(y)\, dy$$

which tends to 0 as n tends to ∞ if $\left| \frac{t_n^{(j)} - t_n^{(j')}}{h_n^2} \right| \to \infty$ or $\left| \frac{x_n^{(j)} - x_n^{(j')}}{h_n} \right| \to \infty$ as $n \to \infty$, since $\varphi^{(j')}$ is compactly supported. This proves the lemma. □

Now we prove the main result of this step. Precisely, the following proposition will ensure the profile decomposition for h_n-oscillatory sequences.

Proposition 4.2.4. *Let $(v_n)_{n \in \mathbb{N}}$ be an (strictly) h_n-oscillatory sequence of solutions to the linear Schrödinger equation (4.18). Then, up to extraction, there exist linear concentrating solutions p_n^k, as defined in Definition 4.4, associated to $[\varphi^{(k)}, \underline{h}, \underline{x}^{(k)}, \underline{t}^{(k)}]$ such that, for any $l \in \mathbb{N}^*$ and up to a subsequence,*

$$v_n(t, x) = \sum_{j=1}^{l} p_n^{(j)}(t, x) + w_n^{(l)}(t, x), \tag{4.31}$$

$$\limsup_{n\to\infty} \|w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))} \longrightarrow 0 \text{ as } l \to \infty, \tag{4.32}$$

for all $T > 0$ and

$$\|\nabla v_n(t)\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla p_n(t)^{(j)}\|_{L^2}^2 + \|\nabla w_n(t)^{(l)}\|_{L^2}^2 + o(1) \text{ as } n \to \infty, \tag{4.33}$$

for $t \in [0,T]$. Moreover, for any $j \neq k$, we have $(\underline{x}^{(k)}, \underline{t}^{(k)}) \perp (\underline{x}^{(j)}, \underline{t}^{(j)})$.

Proof of Proposition 4.2.4. Using the notation of Definition 4.5, if $v_n \in L^\infty([0,T], \dot{H}^1(\mathbb{R}^3))$, consider $\widetilde{v}_n$ its extension in $\mathbb{R}$ by zero outside $[0,T]$ and denote

$$\delta(\underline{v}) = \sup_{(t_n,x_n)} \left\{ \|\nabla\varphi(0)\|_{L^2}^2; D_{h_n}\widetilde{v}_n \rightharpoonup \varphi, \text{up to a subsequence, } \varphi \in L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)) \right\},$$

where (t_n, x_n) are sequences in $[0,T] \times \mathbb{R}^3$. This means that $h_n.^{\frac{1}{2}}\widetilde{v}_n(t_n + h_n^2 t, x_n + h_n x) \rightharpoonup \varphi(t,x)$ in $\dot{H}^1(\mathbb{R}^3)$ as $n \to \infty$. In this scenario, we consider φ the weak limit of the translated sequence $\widetilde{v}_n$. Taking a linear concentrating solution associated to φ such that $p_n(t,x) = \frac{1}{\sqrt{h_n}}\varphi\left(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\right)$ and let $\widetilde{p}_n$ be its extension on $\mathbb{R}$ by zero outside $[0,T]$, we have $\widetilde{p}_n(t,x) = \frac{1}{\sqrt{h_n}}\varphi\left(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\right)$. Let $\mathcal{V}(v_n)$ be the set of such functions φ.

If $\delta(\underline{v}) = 0$, we take $p_n^{(j)} = \widetilde{p}_n^{(j)} = 0$, for all $j = 1, ..., l$.

If $\delta(\underline{v}) > 0$, we choose $\varphi^{(1)} \in \mathcal{V}(v_n)$ such that

$$\|\nabla\varphi^{(1)}(0)\|_{L^2} \geq \frac{1}{2}\delta(\underline{v}) > 0.$$

This means that there exists $(\underline{x}^{(1)}, \underline{t}^{(1)}) \in [0,T] \times \mathbb{R}^3 \to (x_\infty^{(1)}, t_\infty^{(1)})$ such that

$$D_{h_n}\widetilde{v}_n \rightharpoonup \varphi^{(1)} \text{ as } n \to \infty,$$

equivalently,

$$\sqrt{h_n}\widetilde{v}_n(t_n^{(1)} + h_n^2 s, x_n^{(1)} + h_n y) \rightharpoonup \varphi^{(1)}(s,y) \text{ as } n \to \infty,\ s \in \mathbb{R}.$$

Now, choose $p_n^{(1)}$ as the linear concentrating solution associated with $[\varphi^{(1)}, \underline{h}, \underline{x}^{(1)}, \underline{t}^{(1)}]$ and let $\widetilde{p}_n^{(1)}$ be its extension on $\mathbb{R}$ by zero outside $[0,T]$. Note that the assumption $t_n^{(1)} \in [0,T]$ ensures $t_\infty^{(1)} \in [0,T]$, which will always be the case for all the concentrating solutions we consider. To proceed, we first state a lemma that will be used for the orthogonality of energies.

Lemma 4.2.3. *Let $w_n^{(1)} = \widetilde{v}_n - \widetilde{p}_n^{(1)}$. One has*

$$\|\nabla\widetilde{v}_n(t)\|_{L^2}^2 = \|\nabla\widetilde{p}_n^{(1)}(t)\|_{L^2}^2 + \|\nabla w_n^{(1)}(t)\|_{L^2}^2 + o(1) \text{ as } n \to \infty. \tag{4.34}$$

Proof of Lemma 4.2.3. Observe that

$$\begin{aligned}\sqrt{h_n}w_n^{(1)}(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny) &= \sqrt{h_n}\widetilde{v}_n(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny)-\sqrt{h_n}\widetilde{p}_n^{(1)}(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny)\\ &= \sqrt{h_n}\widetilde{v}_n(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny)-\varphi^{(1)}(s,y)\rightharpoonup 0\end{aligned}$$

as $n\to\infty$, which means that $D_{h_n}w_n^{(1)}\rightharpoonup 0$. Then,

$$\|\nabla\widetilde{v}_n(t)\|_{L^2}^2=\|\nabla w_n^{(1)}(t)\|_{L^2}^2+2\langle\nabla w_n^{(1)}(t),\nabla\widetilde{p}_n^{(1)}(t)\rangle+\|\nabla\widetilde{p}_n^{(1)}(t)\|_{L^2}^2.$$

The change of variables $\frac{t-t_n}{h_n^2}=s$ and $\frac{x-x_n}{h_n}=y$ yields

$$\begin{aligned}\langle\nabla w_n^{(1)}(t),\nabla\widetilde{p}_n^{(1)}(t)\rangle &= \int_{\mathbb{R}^3}\nabla_x w_n^{(1)}(t,x)\cdot\nabla_x\widetilde{p}_n^{(1)}(t,x)\;dx\\ &= \int_{\mathbb{R}^3}\nabla_x w_n^{(1)}(t,x)\cdot\nabla_x\frac{1}{\sqrt{h_n}}\varphi^{(1)}\Big(\frac{t-t_n^{(1)}}{h_n^2},\frac{x-x_n^{(1)}}{h_n}\Big)\;dx\\ &= \int_{\mathbb{R}^3}\nabla_x w_n^{(1)}(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny)\cdot\nabla_x\frac{1}{\sqrt{h_n}}\varphi^{(1)}(s,y)\;h_n^3dy\\ &= \int_{\mathbb{R}^3}\nabla_y\sqrt{h_n}w_n^{(1)}(t_n^{(1)}+h_n^2s,x_n^{(1)}+h_ny)\cdot\nabla_y\varphi^{(1)}(s,y)\;dy\end{aligned}$$

which tends to 0 as $n\to\infty$, proving Lemma 4.2.3. □

The previous lemma ensures that we get the expansion of v_n announced in Proposition 4.2.4 by induction iterating the same process. To this end, let us assume that

$$\widetilde{v}_n(t,x)=\sum_{j=1}^{l}\widetilde{p}_n^{(j)}(t,x)+w_n^{(l)}(t,x).$$

Hence,

$$v_n(t,x)=\sum_{j=1}^{l}p_n^{(j)}(t,x)+w_n^{(l)}(t,x)$$

and

$$\|\nabla v_n(t)\|_{L^2}^2=\sum_{j=1}^{l}\|\nabla p_n^{(j)}(t)\|_{L^2}^2+\|\nabla w_n^{(l)}(t)\|_{L^2}^2+o(1)\text{ as }n\to\infty,\tag{4.35}$$

where $p_n^{(j)}$ is a linear concentrating solution associated with $[\varphi^{(j)},\underline{h},\underline{x}^{(j)},\underline{t}^{(j)}]$, which are mutually orthogonal due to Lemma 4.2.3.

We now argue as before: if $\delta(w_n^{(l)})=0$, we just choose $p_n^{(l+1)}=0$. If $\delta(\underline{w}^{(l)})>0$, choose $[\varphi^{(l+1)},\underline{x}^{(l+1)},\underline{t}^{(l+1)}]$ such that

$$\|\nabla\varphi^{(l+1)}(0)\|_{L^2}^2\geq\frac{1}{2}\delta(\underline{w}^{(l)})\tag{4.36}$$

and

$$D_{h_n}w_n^{(l)}\rightharpoonup\varphi^{(l+1)}\text{ as }n\to\infty.$$

Define $p_n^{(l+1)}$ as a linear concentrating solution associated with $[\varphi^{(l+1)}, \underline{h}, \underline{x}^{(l+1)}, \underline{t}^{(l+1)}]$. Again, Lemma 4.2.3 applied to $w_n^{(l)}$ and $\widetilde{p}_n^{(l+1)}$ gives estimates (4.33) with $w_n^{(l+1)} = w_n^{(l)} - \widetilde{p}_n^{(l+1)}$.

Let us now show the convergence (4.32). Using Lemma 4.1.3 and energy estimates, we have

$$\|\nabla\varphi^{(j)}(0)\|_{L^2}^2 = \|\nabla p_n^{(j)}(t_n^{(j)})\|_{L^2}^2 = \|\nabla p_n^{(j)}(0)\|_{L^2}^2.$$

Using (4.33), we have, for some $C > 0$ depending only on T,

$$\begin{aligned}\sum_{j=1}^{l}\|\nabla\varphi^{(j)}(0)\|_{L^2}^2 &= \sum_{j=1}^{l}\|\nabla p_n^{(j)}(0)\|_{L^2}^2 \\ &\leq \limsup_{n\to\infty}\|\nabla v_n(0)\|_{L^2}^2 \\ &\leq C.\end{aligned}$$

So, the series with general term $\|\nabla\varphi^{(j)}(0)\|_{L^2}^2$ converges and

$$\|\nabla\varphi^{(j)}(0)\|_{L^2}^2 \to 0 \text{ as } l \to \infty.$$

Using estimate (4.36), one obtains

$$\delta(\underline{w}^{(l)}) \to 0 \text{ as } l \to \infty.$$

To get the first part of Proposition 4.2.4, it remains to show

$$\limsup_{n\to\infty}\|w_n^{(l)}\|_{L_t^\infty L_x^6} \longrightarrow 0 \text{ as } l \to \infty.$$

We begin by introducing a family of functions $\chi_R(t,x) = \chi_R^1(t)\cdot\chi_R^2(x) \in C_0^\infty(\mathbb{R}\times\mathbb{R}^3)$ satisfying the following properties:

$$\begin{cases} |\widetilde{\chi_R^1}| + |\widehat{\chi_R^2}| \leq 2; \\ supp(\widehat{\chi_R^2}) \subset \left\{\frac{1}{2Rh_n} \leq |\xi| \leq \frac{2R}{h_n}\right\}; \\ \widehat{\chi_R^2} \equiv 1 \quad \text{for } \left\{\frac{1}{Rh_n} \leq |\xi| \leq \frac{R}{h_n}\right\}; \\ \widetilde{\chi_R^1}\Big(|\xi|^2\Big) = 1 \quad \text{on } supp(\widehat{\chi_R^2}); \\ supp(\chi_R^1) \subset [-T, 0], \end{cases}$$

where $\widetilde{}$ and $\widehat{}$ denote de Fourier transform in time and space, respectively. One has

$$\|w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))} \leq \|\chi_R * w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))} + \|(\delta - \chi_R) * w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))}, \quad (4.37)$$

where $*$ denotes the convolution in (t,x) and δ denotes the Dirac distribution.

1. Bound for $\|\chi_R * w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))}$

Note that

$$\|\chi_R * w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))} \leq \|\chi_R * w_n^{(l)}\|^{\frac{1}{3}}_{L^\infty([0,T];L^2(\mathbb{R}^3))} \cdot \|\chi_R * w_n^{(l)}\|^{\frac{2}{3}}_{L^\infty([0,T]\times\mathbb{R}^3)}. \qquad (4.38)$$

The function $\chi_R * w_n^{(l)}$ is a solution to the first equation of (4.18) on $\mathbb{R}$ and, in particular, the L^2-conservation law gives

$$\begin{aligned} \|\chi_R * w_n^{(l)}\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3))} &= \|(\chi_R * w_n^{(l)})(0)\|^2_{L^2_x} \\ &= \frac{1}{(2\pi)^3}\|\mathfrak{F}_{x\to\xi}((\chi_R * w_n^{(l)})(0))(\xi)\|^2_{L^2_\xi}. \end{aligned} \qquad (4.39)$$

On the other hand, we write

$$(\chi_R * w_n^{(l)})(0,x) = \int_{\mathbb{R}} \chi_R^1(-s)\int_{\mathbb{R}^3}\chi_R^2(x-y)w_n^{(l)}(s,y)\ dyds.$$

By the Plancherel inversion formula, we get

$$\begin{aligned} (\chi_R * w_n^{(l)})(0,x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}}\chi_R^1(-s)\int_{\mathbb{R}^3}\chi_R^2(x-y)\int_{\mathbb{R}^3} e^{iy\xi}\widehat{w_n^{(l)}(s)}(\xi)e^{-ix\xi}e^{ix\xi}\ d\xi dyds \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}}\chi_R^1(-s)\int_{\mathbb{R}^3} e^{-i(x-y)\xi}\chi_R^2(x-y)\int_{\mathbb{R}^3}\widehat{w_n^{(l)}(s)}(\xi)e^{ix\xi}\ d\xi dyds \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}}\chi_R^1(-s)\int_{\mathbb{R}^3}\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(s)}(\xi)e^{ix\xi}\ d\xi ds. \end{aligned}$$

Since $\widehat{w_n^{(l)}(s)}(\xi) = e^{is|\xi|^2}\widehat{w_n^{(l)}(0)}(\xi)$, we obtain

$$\begin{aligned} (\chi_R * w_n^{(l)})(0,x) &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}}\chi_R^1(-s)\int_{\mathbb{R}^3}\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi)e^{is|\xi|^2}e^{ix\xi}\ d\xi ds \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\widetilde{\chi_R^1}\Big(|\xi|^2\Big)\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi)e^{ix\xi}\ d\xi \\ &= \mathfrak{F}^{-1}_{\xi\to x}\Big[\widetilde{\chi_R^1}\Big(|\xi|^2\Big)\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi)\Big](x). \end{aligned}$$

Consequently

$$\mathfrak{F}_{x\to\xi}((\chi_R * w_n^{(l)})(0))(\xi) = \widetilde{\chi_R^1}\Big(|\xi|^2\Big)\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi). \qquad (4.40)$$

Using the properties of χ_R, (4.39) and (4.40), we get

$$\begin{aligned} \|\chi_R * w_n^{(l)}\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3))} &= \frac{1}{(2\pi)^3}\Big\|\widetilde{\chi_R^1}\Big(|\xi|^2\Big)\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi)\Big\|^2_{L^2_\xi} \\ &\leq C\frac{1}{(2\pi)^3}\int_{\frac{1}{2Rh_n}\leq|\xi|\leq\frac{2R}{h_n}}|\widehat{\chi_R^2}(\xi)\widehat{w_n^{(l)}(0)}(\xi)|^2\ d\xi \\ &\leq C_1(R)h_n^2\|\xi\widehat{w_n^{(l)}(0)}\|^2_{L^2} \\ &\leq C_1(R)h_n^2\|\nabla w_n^{(l)}(0)\|^2_{L^2_x}, \end{aligned} \qquad (4.41)$$

where $C_1(R)$ is a R-dependent constant. Now, observe that

$$\limsup_{n\to\infty} \|\chi_R * w_n^{(l)}\|_{L^\infty([0,T]\times\mathbb{R}^3)} = \sup_{(t_n,x_n)} \limsup_{n\to\infty} \Big|(\chi_R * w_n^{(l)})(t_n,x_n)\Big|.$$

Let $\varphi \in \mathcal{V}(w_n^{(l)})$ be such that

$$\sqrt{h_n} w_n^{(l)}(t_n + h_n^2 s, x_n + h_n y) \rightharpoonup \varphi(s,y) \text{ as } n \to \infty$$

and let $\widetilde{p}_n$ be the rescaled function $\widetilde{p_n}(t,x) = \frac{1}{\sqrt{h_n}}\varphi\Big(\frac{t}{h_n^2}, \frac{x}{h_n}\Big)$. We have that $\widetilde{p}_n$ satisfies the linear Schrödinger equation and

$$w_n^{(l)}(t_n + t, x_n + x) \rightharpoonup \widetilde{p}_n(t,x) \text{ as } n \to \infty.$$

Hence,

$$(\chi_R * w_n^{(l)})(t_n + t, x_n + x) \rightharpoonup (\chi_R * \tilde{p}_n)(t,x) \text{ as } n \to \infty$$

and

$$(\chi_R * w_n^{(l)})(t_n, x_n) \rightharpoonup (\chi_R * \tilde{p}_n)(0,0) \text{ as } n \to \infty.$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty} \|\chi_R * w_n^{(l)}\|_{L^\infty([0,T]\times\mathbb{R}^3)} &\le \sup\left\{\Big|(\chi_R * \widetilde{p}_n)(0,0)\Big|\right\} \\ &\le \sup\left\{\left|\int_{\mathbb{R}}\int_{\mathbb{R}^3} \chi_R(-t,-x)\widetilde{p}_n(t,x)\ dxdt\right|\right\}.\end{aligned}$$

Therefore, by Hölder's inequality, it follows that

$$\limsup_{n\to\infty} \|\chi_R * w_n^{(l)}\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le C_2(R) \sup\left\{\|\widetilde{p}_n\|_{L_t^\infty L_x^6}\right\},$$

where $C_2(R) = \|\chi_R\|_{L^1([0,T];L^{\frac{6}{5}}(\mathbb{R}^3))}$. Since

$$\|\widetilde{p}_n(t)\|_{L_x^6} \le \|\widetilde{p}_n(t)\|_{\dot{H}_x^1} = \|\widetilde{p}_n(0)\|_{\dot{H}_x^1} = \|\varphi(0)\|_{\dot{H}_x^1} \le C\delta(w_n^{(l)}),$$

it follows that

$$\|\chi_R * w_n^{(l)}\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le C_2(R)\delta(w_n^{(l)}),$$

for every $l \ge 1$. Putting these estimates together, we conclude that

$$\begin{aligned}\|\chi_R * w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))} &\le C_1(R) h_n^{\frac{1}{3}} \|\nabla w_n^{(l)}\|_{L^2}^{\frac{1}{3}} \cdot C_2(R)\delta(w_n^{(l)})^{\frac{2}{3}} \\ &\le C(R) h_n^{\frac{1}{3}} \delta(w_n^{(l)})^{\frac{2}{3}}, \qquad (4.42)\end{aligned}$$

which is the desired bound.

2. Bound for $\|(\delta-\chi_R)*w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))}$

The function $(\delta-\chi_R)*w_n^{(l)}$ is a solution to linear Schrödinger equation in $\mathbb{R}$. Therefore,

$$\|(\delta-\chi_R)*w_n^{(l)}\|^2_{L^\infty([0,T];L^6(\mathbb{R}^3))}\le C\|\nabla(\delta-\chi_R)*w_n^{(l)}(t)\|^2_{L^2}\le C\|\nabla(\delta-\chi_R)*w_n^{(l)}(0)\|^2_{L^2}.$$

By Plancherel's theorem and identity (4.40), one has

$$\|\nabla(\delta-\chi_R)*w_n^{(l)}(0)\|_{L^2}=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}|\xi|^2\left|\widehat{w_n^{(l)}(0)}(\xi)\left[1-\widetilde{\chi_R^1}\left(|\xi|^2\right)\widehat{\chi_R^2}(\xi)\right]\right|^2\,d\xi.$$

Observe that,

$$\left[1-\widetilde{\chi_R^1}\left(|\xi|^2\right)\widehat{\chi_R^2}(\xi)\right]=0,\ \text{for }\frac{1}{h_nR}\le|\xi|\le\frac{R}{h_n}$$

and it is bounded. Consequently,

$$\limsup_{n\to\infty}\|(\delta-\chi_R)*w_n^{(l)}\|^2_{L^\infty([0,T];L^6(\mathbb{R}^3))}\le C\limsup_{n\to\infty}\int_{\{|\xi|\le\frac{1}{h_nR}\}\cup\{|\xi|\ge\frac{R}{h_n}\}}|\xi|^2|\widehat{w_n^{(l)}(0)}|^2\,d\xi, \tag{4.43}$$

which is the desired bound for the second term on the right-hand side of inequality (4.37).

With these bounds in hand, let us analyze inequality (4.37). From estimates (4.42) and (4.43), one has

$$\limsup_{n\to\infty}\|w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))}\le C(R)\limsup_{n\to\infty}\left[h_n^{\frac{1}{3}}\delta(w_n^{(l)})^{\frac{2}{3}}+\int_{\{|\xi|\le\frac{1}{h_nR}\}\cup\{|\xi|\ge\frac{R}{h_n}\}}|\xi|^2|\widehat{w_n^{(l)}(0)}|^2\,d\xi\right].$$

So, taking l tending to infinity, then R tending to infinity, using the fact that $\delta(w_h^{(l)})\longrightarrow 0$ as $l\to\infty$ and $w_n^{(l)}$ is (strictly) h_n-oscillatory (Remark 4.2.1), it follows that

$$\limsup_{n\to\infty}\|w_n^{(l)}\|_{L^\infty([0,T];L^6(\mathbb{R}^3))}\longrightarrow 0\ \text{ as } l\to\infty.$$

Therefore, by interpolation, one gets

$$\limsup_{n\to\infty}\|w_n^{(l)}\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}\longrightarrow 0\ \text{ as } l\to\infty,$$

since $\|w_n^{(l)}\|_{L_t^7L_x^{14}}\le C\|w_n(0)\|_{\dot{H}^1}$. This completes the proof of the first part of Proposition 4.2.4. It remains only to show the orthogonality of cores. We show it by contradiction. To this end, assume that the index

$$j_K=\max\left\{j\in\{1,...,K\};(t_n^{(j)},x_n^{(j)})\not\perp_{h_n}(t_n^{(K+1)},x_n^{(K+1)})\right\}$$

exists. The following are consequences of the construction at the beginning of the demonstration of Proposition 4.2.4:

$$D_{h_n}^{(l+1)}w_n^{(l)}\rightharpoonup\varphi^{(l+1)}\ \text{with }\varphi^{(l+1)}\neq 0\text{ if } l\le K, \tag{4.44}$$

$$w_n^{(l)} = p_n^{(l+1)} + w_n^{(l+1)}, \tag{4.45}$$

and

$$w_n^{(j_K)} = \sum_{j=j_K+1}^{K+1} p_n^{(j)} + w_n^{(K+1)}. \tag{4.46}$$

Moreover, the definition of $p_n^{(l)}$ and Lemma 4.1.2 implies

$$D_{h_n}^{(l)} p_n^{(l)} \rightharpoonup \varphi^{(l)}.$$

Then, we get from (4.44) and (4.45) that

$$D_{h_n}^{(l+1)} w_n^{(l+1)} \rightharpoonup 0.$$

Applying this to $l+1 = j_K$ give us

$$D_{h_n}^{(K+1)} w_n^{(j_K)} \rightharpoonup 0,$$

due to the first part of Lemma 4.2.2 and the definition of j_K, since $(t_n^{(j_K)}, x_n^{(j_K)}) \not\perp_{h_n} (t_n^{(K+1)}, x_n^{(K+1)})$. The definition of j_K and the second part of Lemma 4.2.2 give

$$D_{h_n}^{(K+1)} p_n^{(l)} \rightharpoonup 0 \text{ for } j_K + 1 \leq l \leq K.$$

"Applying"$D_{h_n}^{(K+1)}$ to equality (4.46) one gets

$$\begin{aligned} D_{h_n}^{(K+1)} w_n^{(j_K)} &= \sum_{j=j_K+1}^{K+1} D_{h_n}^{(K+1)} p_n^{(j)} + D_{h_n}^{(K+1)} w_n^{(K+1)} \\ &= \sum_{j=j_K+1}^{K} D_{h_n}^{(K+1)} p_n^{(j)} + D_{h_n}^{(K+1)} p_n^{(K+1)} + D_{h_n}^{(K+1)} w_n^{(K+1)}. \end{aligned}$$

Therefore,

$$D_{h_n}^{(K+1)} w_n^{(j_K)} \rightharpoonup \varphi^{K+1} \neq 0,$$

while we have just proved

$$D_{h_n}^{(K+1)} w_n^{(j_K)} \rightharpoonup 0,$$

which is a contradiction and completes the proof of the Proposition 4.2.4. □

Remark 4.2.1. *Observe that $w_n^{(l)}$ is (strictly) h_n-oscillatory.*

Indeed, being $w_n^{(1)} = \widetilde{v}_n - \widetilde{p}_n^{(1)}$ for $l = 1$, we apply the operator $\sigma_R(D)$ to equation (4.31), where $\sigma_R = \mathbf{1}_{\{h_n|\xi| \leq \frac{1}{R}\} \cup \{h_n|\xi| \geq R\}}$, for $R > 0$. We get

$$\|\nabla \sigma_R(D) \widetilde{v}_n\|_{L^2}^2 = \|\nabla \sigma_R(D) \widetilde{p}_n^{(1)}\|_{L^2}^2 + \|\nabla \sigma_R(D) w_n^{(1)}\|_{L^2}^2 + o(1).$$

Iterating, we obtain

$$\|\nabla\sigma_R(D)\widetilde{v}_n\|_{L^2}^2 = \sum_{j=1}^{l}\|\nabla\sigma_R(D)\widetilde{p}_n^{(j)}\|_{L^2}^2 + \|\nabla\sigma_R(D)w_n^{(l)}\|_{L^2}^2 + o(1),$$

which means

$$\limsup_{n\to\infty}\int_{\{h_n|\xi|\le\frac{1}{R}\}\cup\{h_n|\xi|\ge R\}}|\xi|^2|\hat{w}_n^{(l)}(.,\xi)|^2\,d\xi \le \limsup_{n\to\infty}\int_{\{h_n|\xi|\le\frac{1}{R}\}\cup\{h_n|\xi|\ge R\}}|\xi|^2|\hat{v}_n(.,\xi)|^2\,d\xi.$$

Since $\widetilde{v}_n$ is a (strictly) h_n-oscillatory sequence, so it is $w_n^{(l)}$.

For the sake of completeness, before presenting the proof of the Theorem 4.8, let us revisit the result showed in (KERAANI, 2001, Lemma 2.7), which will be important in the proof of the aforementioned theorem.

Lemma 4.2.4. *Let $(\underline{h}^{(j)},\underline{x}^{(j)},\underline{t}^{(j)})$ be a family of pairwise orthogonal scales-cores and $(V^{(j)})$ a family of functions in $L^{10}(\mathbb{R},L^{10}(\mathbb{R}^3))$. Then*

$$\left\|\sum_{j=1}^{l}\frac{1}{\sqrt{h_n^{(j)}}}V^{(j)}\Big(\frac{.-t_n^{(j)}}{{h_n^{(j)}}^2},\frac{.-x_n^{(j)}}{h_n^{(j)}}\Big)\right\|_{L_t^{10}L_x^{10}}^{10} \longrightarrow \sum_{j=1}^{l}\|V^{(j)}\|_{L_t^{10}L_x^{10}}^{10} \quad \text{as } n\to\infty,$$

for every $l\ge 1$.

Demonstração. Denote

$$V_n^{(j)}(t,x) = \frac{1}{\sqrt{h_n^{(j)}}}V^{(j)}\Big(\frac{t-t_n^{(j)}}{{h_n^{(j)}}^2},\frac{x-x_n^{(j)}}{h_n^{(j)}}\Big).$$

Note that it is enough to show that

$$I_n = \int_{\mathbb{R}^4}V_n^{(j_1)}V_n^{(j_2)}V_n^{(j_3)}V_n^{(j_4)}V_n^{(j_5)}V_n^{(j_6)}V_n^{(j_7)}V_n^{(j_8)}V_n^{(j_9)}V_n^{(j_{10})}\,dxdt \longrightarrow 0 \text{ as } n\to\infty,$$

where $1\le j_k\le l$ and at least two j_k's different. Assume, for example, that $j_1\ne j_2$. By Hölder's inequality, we estimate

$$|I_n|\le C\|V_n^{(j_1)}V_n^{(j_2)}\|_{L_t^5L_x^5},$$

where $C=\prod_{k=3}^{10}\|V_n^{(j_k)}\|_{L_t^{10}L_x^{10}}$. Now, let us compute $\|V_n^{(j_1)}V_n^{(j_2)}\|_{L_t^5L_x^5}$:

$$\begin{aligned}&\|V_n^{(j_1)}V_n^{(j_2)}\|_{L_t^5L_x^5}^5\\ =\;& \frac{1}{(h_n^{(j_1)}h_n^{(j_2)})^{\frac{5}{2}}}\int_{\mathbb{R}}\int_{\mathbb{R}^3}|V^{(j_1)}|^5\left(\frac{t-t_n^{(j_1)}}{{h_n^{(j_1)}}^2},\frac{x-x_n^{(j_1)}}{h_n^{(j_1)}}\right)|V^{(j_2)}|\left(\frac{t-t_n^{(j_2)}}{{h_n^{(j_2)}}^2},\frac{x-x_n^{(j_2)}}{h_n^{(j_2)}}\right)dxdt.\end{aligned}$$

The orthogonality of $[h_n^{(j_1)},\underline{x}^{(j_1)},\underline{t}^{(j_1)}],[h_n^{(j_2)},\underline{x}^{(j_2)},\underline{t}^{(j_2)}]$ means that

either $\frac{h_n^{(j_1)}}{h_n^{(j_2)}}+\frac{h_n^{(j_2)}}{h_n^{(j_1)}}\longrightarrow+\infty$ or $h_n^{(j_1)}=h_n^{(j_2)}$ and $\left|\frac{t_n^{(j_1)}-t_n^{(j_2)}}{{h_n^{(j_1)}}^2}\right|+\left|\frac{x_n^{(j_1)}-x_n^{(j_2)}}{h_n^{(j_1)}}\right|\longrightarrow+\infty$ as $n\to\infty$.

Without loss of generality, we assume that V^{j_1},V^{j_2} are continuous and compactly supported.

- If $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} + \frac{h_n^{(j_2)}}{h_n^{(j_1)}} \longrightarrow +\infty$, then either $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} \longrightarrow +\infty$ or $\frac{h_n^{(j_2)}}{h_n^{(j_1)}} \longrightarrow +\infty$. We assume $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} \longrightarrow +\infty$ (the other case is symmetrical).

The change of variables $x = h_n^{(j_2)}y + x_n^{(j_2)}, t = (h_n^{(j_2)})^2 s + t_n^{(j_2)}$ gives

$$\begin{aligned}
&\|V_n^{(j_1)}V_n^{(j_2)}\|^5_{L^5_t L^5_x} \\
= &\left(\frac{h_n^{(j_2)}}{h_n^{(j_1)}}\right)^{\frac{5}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}^3} |V^{(j_1)}|^5 \left(\frac{t_n^{(j_2)} - t_n^{(j_1)}}{{h_n^{(j_1)}}^2} + s\left(\frac{h_n^{(j_2)}}{h_n^{(j_1)}}\right)^2, \frac{h_n^{(j_2)}}{h_n^{(j_1)}}y + \frac{x_n^{(j_2)} - x_n^{(j_1)}}{h_n^{(j_2)}}\right) |V^{(j_2)}(s,y)|^5 \, dyds.
\end{aligned}$$

Since $V^{(j_1)}, V^{(j_2)}$ are assumed to be continuous and compactly supported, we obtain

$$\|V_n^{(j_1)}V_n^{(j_2)}\|^5_{L^5_t L^5_x} \longrightarrow 0 \text{ as } n \to \infty.$$

- If $h_n^{(j_1)} = h_n^{(j_2)}$, with the same change of variables as above, we get

$$\begin{aligned}
&\|V_n^{(j_1)}V_n^{(j_2)}\|^5_{L^5_t L^5_x} \\
= &\int_{\mathbb{R}}\int_{\mathbb{R}^3} |V^{(j_1)}|^5 \left(\frac{t_n^{(j_2)} - t_n^{(j_1)}}{{h_n^{(j_1)}}^2} + s, y + \frac{x_n^{(j_2)} - x_n^{(j_1)}}{h_n^{(j_1)}}\right) \cdot |V^{(j_2)}(s,y)|^5 \, dyds.
\end{aligned}$$

Thus, the previous integral tends to 0 as n tends to ∞. This finishes the proof of Lemma 4.2.4.

□

Remark 4.2.2. *(KERAANI, 2001) Using the inequality*

$$\left|\left|\sum_{j=1}^{l} a_j\right|^{\frac{10}{3}} - \sum_{j=1}^{l} |a_j|^{\frac{10}{3}}\right| \le C_l \sum_{j\neq k} |a_j||a_k|^{\frac{7}{3}}$$

and arguing in the same way as in the proof of Lemma 4.2.4, we prove that

$$\left\|\nabla\left(\sum_{j=1}^{l} \frac{1}{\sqrt{h_n^{(j)}}} V^{(j)}\left(\frac{\cdot - t_n^{(j)}}{{h_n^{(j)}}^2}, \frac{\cdot - x_n^{(j)}}{h_n^{(j)}}\right)\right)\right\|^{\frac{10}{3}}_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \longrightarrow \sum_{j=1}^{l} \|\nabla V^{(j)}\|^{\frac{10}{3}}_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} \quad \text{as } n \to \infty.$$

Now we have what we need to establish the proof of the Theorem 4.8.

***Step 4**. (Proof of Theorem 4.8):* The idea is to combine the two decompositions we made. Denote by $v_n^{(j)}$ (and the rest $(\rho_n^{(l)})$) the $h_n^{(j)}$-oscillatory component obtained by decomposition (4.19) and $p_n^{(j,\alpha)}$ the concentrating solutions obtained from decomposition (4.31) (and the rest $w_n^{(j,A_j)}$). Adding everything up, one has

$$v_n(t,x) = \sum_{j=1}^{l}\left(\sum_{\alpha=1}^{A_j} p_n^{(j,\alpha)}(t,x) + w_n^{(j,A_j)}(t,x)\right) + \rho_n^{(l)}(t,x).$$

Rewrite this equation as

$$v_n(t,x) = \sum_{j=1}^{l} \Big(\sum_{\alpha=1}^{A_j} p_n^{(j,\alpha)}(t,x) \Big) + w_n^{(l,A_1,\ldots,A_l)}(t,x),$$

where

$$w_n^{(l,A_1,\ldots,A_l)}(t,x) = \sum_{j=1}^{l} w_n^{(j,A_j)}(t,x) + \rho_n^{(l)}(t,x),$$

for l and A_j fixed, $1 \leq j \leq l$. We enumerate this pairs by the bijection $\sigma : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\sigma(j,\alpha) < \sigma(k,\beta) \ \text{ if } j+\alpha < k+\beta \text{ or } j+\alpha = k+\beta \text{ and } j<k.$$

The almost orthogonality identity (4.17) is satisfied. Indeed, combining (4.21) and (4.33), we obtain

$$\begin{aligned}
\|\nabla v_n\|_{L^2}^2 &= \sum_{j=1}^{l} \|\nabla v_n^{(j)}\|_{L^2}^2 + \|\nabla \rho_n^{(l)}\|_{L^2}^2 + o(1) \\
&= \sum_{j=1}^{l} \Big(\sum_{\alpha=1}^{A_j} \|\nabla p_n^{(j,\alpha)}\|_{L^2}^2 + \|\nabla w_n^{(j,A_j)}\|_{L^2}^2 \Big) + \|\nabla \rho_n^{(l)}\|_{L^2}^2 + o(1) \\
&= \sum_{j=1}^{l} \Big(\sum_{\alpha=1}^{A_j} \|\nabla p_n^{(j,\alpha)}\|_{L^2}^2 \Big) + \sum_{j=1}^{l} \|\nabla w_n^{(j,A_j)}\|_{L^2}^2 + \|\nabla \rho_n^{(l)}\|_{L^2}^2 + o(1),
\end{aligned}$$

but

$$\begin{aligned}
\|\nabla w_n^{(l,A_1,\ldots,A_l)}\|_{L^2}^2 &= \left\| \nabla \Big(\sum_{j=1}^{l} w_n^{(j,A_j)} + \rho_n^{(l)} \Big) \right\|_{L^2}^2 \\
&= \sum_{j=1}^{l} \|\nabla w_n^{(j,A_j)}\|_{L^2}^2 + \|\nabla \rho_n^{(l)}\|_{L^2}^2,
\end{aligned}$$

since $w_n^{(j,A_j)}$ is h_n^j-oscillatory and $\rho_n^{(l)}$ is h_n^j-singular for all $1 \leq j \leq l$. Therefore,

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{l} \sum_{\alpha=1}^{A_j} \|\nabla p_n^{(j,\alpha)}\|_{L^2}^2 + \|\nabla w_n^{(l,A_1,\ldots,A_l)}\|_{L^2}^2 + o(1) \text{ as } n \to \infty. \tag{4.47}$$

The last point that remains to be checked is the convergence of the remainder $w_n^{(l,A_1,\ldots,A_l)}$ to zero in the Strichartz norm. To this end, let $\varepsilon > 0$ be a small arbitrary number. To get the result, it suffices to prove that for l_0 large enough,

$$\|w_n^{(l,A_1,\ldots,A_l)}\|_{L_t^\infty L_x^6} \leq \varepsilon \tag{4.48}$$

for all $(l, A_1, ..., A_l)$ satisfying $l \geq l_0$ and $\sigma(j, A_j) \geq \sigma(l_0, 1)$. To prove this, first choose l_0 such that, for every $l \geq l_0$,

$$\limsup_{n\to\infty} \|\rho_n^{(l)}\|_{L_t^\infty L_x^6} \leq \varepsilon. \tag{4.49}$$

Note that the existence of such l_0 is ensured by (4.20). Moreover, by (4.32), for every $l \geq l_0$, there exists B_l such that $A_j \geq B_l$, for every $j \in \{1, ..., l\}$ and

$$\limsup_{n\to\infty} \|w_n^{(j,A_j)}\|_{L_t^\infty L_x^6} \leq \frac{\varepsilon}{l}. \tag{4.50}$$

Furthermore, the expression (4.47) implies that the series with general term

$$\sum_{(j,\alpha)} \limsup_{n\to\infty} \|\nabla p_n^{(j,\alpha)}(0)\|_{L^2}^2$$

is convergent. In particular, we may also assume, increasing l_0 if necessary, that l_0 is such that

$$\sum_{\sigma(j,\alpha)>\sigma(l_0,1)} \limsup_{n\to\infty} \|\nabla p_n^{(j,\alpha)}(0)\|_{L^2}^2 \leq \varepsilon. \tag{4.51}$$

Now, the remainder term can be rewritten in the form

$$w_n^{(l,A_1,\dots,A_l)} = \rho_n^{(l)} + \sum_{1\leq j\leq l} w_n^{(j,\max(A_j,B_l))} + S_n^{(l,A_1,\dots,A_l)},$$

where

$$S_n^{(l,A_1,\dots,A_l)} = \sum_{1\leq j\leq l, A_j<B_l} w_n^{(j,A_j)} - w_n^{(j,B_l)}.$$

However, one has

$$\begin{aligned} w_n^{(j,A_j)} - w_n^{(j,B_l)} &= \sum_{\alpha=1}^{B_l} p_n^{(j,\alpha)} - \sum_{\alpha=1}^{A_j} p_n^{(j,\alpha)} \\ &= \sum_{A_j<\alpha\leq B_l} p_n^{(j,\alpha)}. \end{aligned}$$

Hence,

$$S_n^{(l,A_1,\dots,A_l)} = \sum_{1\leq j\leq l, A_j<B_l} \sum_{A_j<\alpha\leq B_l} p_n^{(j,\alpha)}.$$

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L_t^\infty L_x^6} &\leq \limsup_{n\to\infty} \|\rho_n^{(l)}\|_{L_t^\infty L_x^6} + \limsup_{n\to\infty} \sum_{j=1}^{l} \|w_n^{(j,\max(A_j,B_l))}\|_{L_t^\infty L_x^6} \\ &\quad + \limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^\infty L_x^6} \\ &\leq 2\varepsilon + \limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^\infty L_x^6}. \end{aligned}$$

Since $S_n^{(l,A_1,\dots,A_l)}$ is a solution of the linear Schrödinger equation, we have

$$\begin{aligned} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^\infty L_x^6} &\leq C\|\nabla S_n^{(l,A_1,\dots,A_l)}\|_{L^2} \\ &\leq C\|\nabla S_n^{(l,A_1,\dots,A_l)}(0)\|_{L^2} \\ &\leq C \sum_{1\leq j\leq l, A_j<B_l} \sum_{A_j<\alpha\leq B_l} \|\nabla p_n^{(j,\alpha)}(0)\|_{L^2} \\ &\leq C\varepsilon, \end{aligned}$$

because the sum is restricted to some $\sigma(j,\alpha)$ satisfying $\sigma(j,\alpha) > \sigma(j,\alpha_j) > \sigma(l_0,1)$ and it is indeed smaller than $C\varepsilon$ due to inequality (4.51). Therefore, $\limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L_t^\infty L_x^6}$ is smaller than $(2+C)\varepsilon$ for all $(l, A_1, \dots, A_l)$ satisfying $l \geq l_0$ and $\sigma(j,A_j) \geq \sigma(l_0,1)$. Through the same procedure, we get the same estimates for the $L^{10}(L^{10})$ norm, that is,

$$\begin{aligned}\limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}} &\leq \limsup_{n\to\infty} \|\rho_n^{(l)}\|_{L_t^{10}L_x^{10}} + \limsup_{n\to\infty} \sum_{j=1}^{l} \|w_n^{(j,\max(A_j,B_l))}\|_{L_t^{10}L_x^{10}} \\ &\quad + \limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}} \\ &\leq 2\varepsilon + \limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}}.\end{aligned}$$

Moreover,

$$\limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}}^{10} = \limsup_{n\to\infty} \Big\| \sum_{(j,\alpha)} p_n^{(j,\alpha)} \Big\|_{L_t^{10}L_x^{10}}^{10}$$

and, rescaling,

$$p_n^{(j,\alpha)}(t,x) = \frac{1}{\sqrt{h_n}} \psi^{(j,\alpha)}\Big(\frac{t - t_n^{(j,\alpha)}}{h_n^2}, \frac{x - x_n^{(j,\alpha)}}{h_n}\Big),$$

where $\psi^{(j,\alpha)} \in L^\infty(\mathbb{R}; \dot{H}^1(\mathbb{R}^3))$. So, by Lemma 4.2.4,

$$\limsup_{n\to\infty} \Big\| \sum_{(j,\alpha)} p_h^{(j,\alpha)} \Big\|_{L_t^{10}L_x^{10}}^{10} = \sum_{(j,\alpha)} \|\psi^{(j,\alpha)}\|_{L_t^{10}L_x^{10}}^{10}.$$

Furthermore, through Strichartz estimates and Lemma 4.1.3, one gets

$$\begin{aligned}\sum_{(j,\alpha)} \|\psi^{(j,\alpha)}\|_{L_t^{10}L_x^{10}}^{10} &= \sum_{(j,\alpha)} \|p_n^{(j,\alpha)}\|_{L_t^{10}L_x^{10}}^{10} \\ &\leq C \sum_{(j,\alpha)} \Big(\|\nabla p_n^{(j,\alpha)}(0)\|_{L^2}^2 \Big)^5. \end{aligned} \tag{4.52}$$

On the other hand, by (4.47), $\sum_{(j,\alpha)} \|\nabla p_n^{(j,\alpha)}(0)\|_{L^2}^2$ is convergent, and so the right-hand side of (4.52) is finite. Thus

$$\Big(\sum_{\sigma(j,\alpha) > \sigma(l_0,1)} \|\psi^{(j,\alpha)}\|_{L_t^{10}L_x^{10}}^{10} \Big)^{\frac{1}{10}} \leq \varepsilon.$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}} &\leq 2\varepsilon + \limsup_{n\to\infty} \|S_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}} \\ &\leq 2\varepsilon + \Big(\sum_{(j,\alpha)} \|\psi^{(j,\alpha)}\|_{L_t^{10}L_x^{10}}^{10} \Big)^{\frac{1}{10}} \\ &= 3\varepsilon.\end{aligned}$$

Since ε is an arbitrary small number, we conclude that

$$\limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L_t^{10}L_x^{10}} \longrightarrow 0 \text{ as } n \to \infty, \tag{4.53}$$

which proves Theorem 4.8. □

To finish this section, we establish the following lemma, which is a consequence of the construction we made during the proof of Proposition 4.2.4.

Lemma 4.2.5. *Consider the notation and assumptions of Theorem 4.8. For any $l \in \mathbb{N}$ and $1 \leq j \leq l$, we have*

$$D^{(j)}_{h_n} w^{(l)}_n \rightharpoonup 0.$$

Demonstração. Assuming that $D^{(j)}_{h_n} w^{(l)}_n \rightharpoonup \varphi$, we can directly use the decomposition of Theorem 4.8 to write, for $L > l$,

$$w^{(l)}_n = \sum_{i=l+1}^{L} p^{(i)}_n + w^{(L)}_n.$$

In case of scale orthogonality of $h^{(j)}_n$ and $h^{(i)}_n$, for $l+1 \leq i \leq L$, we have $D^{(j)}_{h_n} p^{(i)}_n \rightharpoonup 0$. Indeed, by hypothesis, $p^{(i)}_n$ is a concentrating solution and so

$$p^{(i)}_n(t,x) = \frac{1}{\sqrt{h^{(i)}_n}} \varphi^{(i)}\Big(\frac{t - t^{(i)}_n}{{h^{(i)}_n}^2}, \frac{x - x^{(i)}_n}{h^{(i)}_n}\Big),$$

which means that

$$\sqrt{h^{(j)}_n} p^{(i)}_n(t^{(j)}_n + (h^{(j)}_n)^2 s, x^{(j)}_n + h^{(j)}_n y) = \frac{\sqrt{h^{(j)}_n}}{\sqrt{h^{(i)}_n}} \varphi^{(i)}\Big(\frac{t^{(j)}_n - t^{(i)}_n}{{h^{(i)}_n}^2} + s\Big(\frac{h^{(j)}_n}{h^{(i)}_n}\Big)^2, \frac{x^{(j)}_n - x^{(i)}_n}{h^{(i)}_n} + y\frac{h^{(j)}_n}{h^{(i)}_n}\Big).$$

Without loss of generality, we may assume that $\varphi^{(i)}$ is continuous and compactly supported. Thus, for a compactly supported function ψ, one has

$$\int_{\mathbb{R}^3} \nabla \sqrt{h^{(j)}_n} p^{(i)}_n(t^{(j)}_n + (h^{(j)}_n)^2 s, x^{(j)}_n + h^{(j)}_n y) \cdot \nabla\psi(y)\, dy$$
$$= \frac{\sqrt{h^{(j)}_n}}{\sqrt{h^{(i)}_n}} \int_{\mathbb{R}^3} \nabla \varphi^{(i)}\Big(\frac{t^{(j)}_n - t^{(i)}_n}{{h^{(i)}_n}^2} + s\Big(\frac{h^{(j)}_n}{h^{(i)}_n}\Big)^2, \frac{x^{(j)}_n - x^{(i)}_n}{h^{(i)}_n} + y\frac{h^{(j)}_n}{h^{(i)}_n}\Big) \cdot \nabla\psi(y)\, dy$$

and the orthogonality of $h^{(j)}_n$ and $h^{(i)}_n$ means that

$$\frac{h^{(j)}_n}{h^{(i)}_n} + \frac{h^{(i)}_n}{h^{(j)}_n} \longrightarrow +\infty.$$

If $\frac{h^{(i)}_n}{h^{(j)}_n} \longrightarrow +\infty$, we have

$$\frac{\sqrt{h^{(j)}_n}}{\sqrt{h^{(i)}_n}} \int_{\mathbb{R}^3} \nabla \varphi^{(i)}\Big(\frac{t^{(j)}_n - t^{(i)}_n}{{h^{(i)}_n}^2} + s\Big(\frac{h^{(j)}_n}{h^{(i)}_n}\Big)^2, \frac{x^{(j)}_n - x^{(i)}_n}{h^{(i)}_n} + y\frac{h^{(j)}_n}{h^{(i)}_n}\Big) \cdot \nabla\psi(y)\, dy \to 0$$

as $n \to \infty$, as done in Lemma 4.2.4. If $\frac{h^{(j)}_n}{h^{(i)}_n} \longrightarrow +\infty$, we make the change of variables

$$\frac{x^{(j)}_n - x^{(i)}_n}{h^{(i)}_n} + y\frac{h^{(j)}_n}{h^{(i)}_n} = x$$

to get

$$\frac{\sqrt{h_n^{(j)}}}{\sqrt{h_n^{(i)}}}\int_{\mathbb{R}^3}\nabla\varphi^{(i)}\Big(\frac{t_n^{(j)}-t_n^{(i)}}{h_n^{(i)2}}+s\Big(\frac{h_n^{(j)}}{h_n^{(i)}}\Big)^2,x\Big)\cdot\nabla\psi\Big(\frac{h_n^{(i)}}{h_n^{(j)}}x-\frac{x_n^{(j)}-x_n^{(i)}}{h_n^{(i)}}\Big)\frac{h_n^{(i)}}{h_n^{(j)}}\,dx.$$

Hence,

$$\frac{\sqrt{h_n^{(i)}}}{\sqrt{h_n^{(j)}}}\int_{\mathbb{R}^3}\nabla\varphi^{(i)}\Big(\frac{t_n^{(j)}-t_n^{(i)}}{h_n^{(i)2}}+s\Big(\frac{h_n^{(j)}}{h_n^{(i)}}\Big)^2,x\Big)\cdot\nabla\psi\Big(\frac{h_n^{(i)}}{h_n^{(j)}}x-\frac{x_n^{(j)}-x_n^{(i)}}{h_n^{(i)}}\Big)\,dx=O\Bigg(\Big(\frac{h_n^{(i)}}{h_n^{(j)}}\Big)^{\frac{1}{2}}\Bigg)\to 0$$

as $n\to\infty$, since ψ is assumed to be compactly supported, which gives the desired result $D_{h_n}^{(j)}p_n^{(i)}\rightharpoonup 0$.

Otherwise, in case $h_n^{(j)}=h_n^{(i)}$ and $(\underline{x}^{(j)},\underline{t}^{(j)})\perp_{h_n}(\underline{x}^{(i)},\underline{t}^{(i)})$, the second part of Lemma 4.2.2 gives the same result. Therefore, in both cases one has

$$D_{h_n}^{(j)}w_n^{(L)}\rightharpoonup\varphi.$$

Since, by Theorem 4.8, $\limsup_{n\to\infty}\|w_n^{(L)}\|_{L_t^\infty L_x^6}\to 0$, we have $\varphi=0$, proving the lemma. □

4.3 NONLINEAR PROFILE DECOMPOSITION

In this section, we establish a decomposition into profiles, similar to the one carried out in the previous section, but this time for the sequence of nonlinear solutions to system (4.3). The main objective of this subsection is to prove the following theorem. We follow in detail what was done by Keraani in (KERAANI, 2001).

Theorem 4.10. *Let u_n be the sequence of solutions to nonlinear Schrödinger equation (4.3) with initial data φ_n bounded in $\dot{H}^1(\mathbb{R}^3)$ and satisfying $\limsup_{n\to\infty}\|\varphi_n\|_{\dot{H}^1}<\lambda_0$. Let $p_n^{(j)}$ be the linear concentrating solution given by Theorem 4.8 and $q_n^{(j)}$ the associated nonlinear concentrating solution. Then, up to extraction, we have*

$$u_n(t,x)=\sum_{j=1}^{l}q_n^{(j)}(t,x)+w_n^{(l)}(t,x)+r_n^{(l)}(t,x)\tag{4.54}$$

and

$$\limsup_{n\to\infty}\big(\|\nabla r_n^{(l)}\|_{L^{\frac{10}{3}}([0,T];L^{\frac{10}{3}}(\mathbb{R}^3))}+\|r_n^{(l)}\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}+\|r_n^{(l)}\|_{L^\infty([0,T];\dot{H}^1(\mathbb{R}^3))}\big)\longrightarrow 0\ \text{ as } l\to\infty.\tag{4.55}$$

The following notations will be often used in this section

$$\beta(z)=|z|^4z,$$

$$W_n^{(l)} = \sum_{j=1}^{l} q_n^{(j)}$$

and

$$f_n^{(l)} = \sum_{j=1}^{l} \beta(q_n^{(j)}) - \beta\Big(\sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)}\Big).$$

Before getting such decomposition, we make sure that nonlinear concentration solutions behave similarly to linear concentration solutions, at least in a specific type of interval.

4.3.1 Behavior of nonlinear concentrating solutions

As we saw at the beginning of Section 4.1, in the ball $\|u_0\|_{\dot{H}^1(\mathbb{R}^3)} < \lambda_0$, the evolution problem

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = 0 \ \text{ on } \mathbb{R} \times \mathbb{R}^3, \\ u(0) = u_0 \ \in \dot{H}^1(\mathbb{R}^3) \end{cases}$$

admits a complete scattering theory with respect to the linear problem. The main theorem of this subsection is a consequence of this scattering property. In order to obtain it, we use the following two lemmas from Keraani (KERAANI, 2001).

Lemma 4.3.1. *Let $I = [a, b]$. The solution $v \in C([a, b]; \dot{H}^1(\mathbb{R}^3))$ of the equation*

$$i\partial_t v + \Delta v = f, \quad I \times \mathbb{R}^3,$$

with $\nabla f \in L^{\frac{10}{7}}(I \times \mathbb{R}^3)$, satisfies

$$|||v|||_I + \sup_{t \in I} \|\nabla v(t)\|_{L^2} \leq C\Big(\|\nabla v(a)\|_{L^2} + \|\nabla f\|_{L^{\frac{10}{7}}(I \times \mathbb{R}^3)}\Big).$$

Lemma 4.3.2. *Let $M = M(t)$ be a positive continuous function on $[0, T]$ such that $M(0) = 0$ and, for all $t \in [0, T]$, we have*

$$M(t) \leq c\Big(a + \sum_{\alpha=2}^{5} M^{\alpha}(t)\Big)$$

with $0 < a < a_0 = a_0(c)$. One has

$$M(t) \leq 2ca,$$

for all $t \in [0, T]$.

The next theorem is a consequence of the scattering property from Proposition 3.4.1.

Theorem 4.11. *Let u_n be a nonlinear concentrating solution. There exist two linear concentrating solutions denoted by $[\varphi_i, \underline{h}, \underline{x}, \underline{t}]$, $i = 1, 2$, such that for all interval $[-T, T]$ containing t_∞, one has*

$$\limsup_{n\to\infty} \Big(\|u_n - [\varphi_1, \underline{h}, \underline{x}, \underline{t}]\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)} + \|u_n - [\varphi_1, \underline{h}, \underline{x}, \underline{t}]\|_{L^\infty(I_n^{1,\Lambda};\dot{H}^1(\mathbb{R}^3))} \Big) \longrightarrow 0, \quad (4.56)$$

and

$$\limsup_{n\to\infty} \Big(\|u_n - [\varphi_2, \underline{h}, \underline{x}, \underline{t}]\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} + \|u_n - [\varphi_2, \underline{h}, \underline{x}, \underline{t}]\|_{L^\infty(I_n^{3,\Lambda};\dot{H}^1(\mathbb{R}^3))} \Big) \longrightarrow 0 \quad (4.57)$$

as $\Lambda \to \infty$. Here, $I_n^{1,\Lambda} = [-T, t_n - \Lambda h_n^2]$ and $I_n^{3,\Lambda} = (t_n + \Lambda h_n^2, T]$.

Demonstração. We consider the case $\frac{t_n}{h_n} \to \infty$. The other cases are analogous. First, let us show (4.56). The proof is based on Strichartz's inequalities and the absorption Lemma 4.3.2.

For the sake of simplicity, we take $I_n^{1,\Lambda} = [0, t_\infty - \Lambda h_n^2]$. We know that u_n is a solution to

$$\begin{cases} i\partial_t u_n + \Delta u_n - |u_n|^4 u_n = 0 \ \text{ on } [0,T]\times\mathbb{R}^3, \\ u_n(0) = \varphi \ \in \dot{H}^1(\mathbb{R}^3). \end{cases}$$

Since $u_n(t,x)$ is a nonlinear concentrating solution, one has

$$u_n(t,x) = \frac{1}{\sqrt{h_n}} u\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big),$$

where u satisfies

$$i\partial_s u + \Delta u - |u|^4 u = 0 \ \text{ on } \mathbb{R}\times\mathbb{R}^3.$$

Using the scattering theory of Proposition 3.4.1, there exists v, solution of the linear system

$$\begin{cases} i\partial_s v + \Delta v = 0 \ \text{ on } \mathbb{R}\times\mathbb{R}^3, \\ v(0) = \varphi^1, \end{cases}$$

such that

$$\|\nabla u(s) - \nabla v(s)\|_{L^2} \longrightarrow 0 \ \text{ as } s \to -\infty.$$

Let

$$v_n(t,x) = \frac{1}{\sqrt{h_n}} v\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big)$$

satisfying

$$\begin{cases} i\partial_t v_n + \Delta v_n = 0 \ \text{ on } [0,T]\times\mathbb{R}^3, \\ v_n(t_n) = \frac{1}{\sqrt{h_n}}\varphi^1. \end{cases}$$

We should prove that

$$\limsup_{n\to\infty}\Big(\|u_n-v_n\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}+\|u_n-v_n\|_{L^\infty(I_n^{1,\Lambda};\dot H^1(\mathbb{R}^3))}\Big)\longrightarrow 0$$

when $\Lambda\to\infty$. To this end, define $w_n:=u_n-v_n$. Thus, w_n satisfies the system

$$\begin{cases} i\partial_t w_n+\Delta w_n=|w_n+v_n|^4(w_n+v_n),\\ \qquad w_n(0)=u_n(0)-v_n(0).\end{cases}\tag{4.58}$$

Using Lemma 4.3.1 and denoting $|||.|||_I:=\|.\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}+\|\nabla.\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}$, we get

$$|||w_n|||_{I_n^{1,\Lambda}}+\|\nabla w_n\|_{L^\infty(I_n^{1,\Lambda};L^2(\mathbb{R}^3))}\;\le\; c\Big(\|\nabla w_n(0)\|_{L^2}+\|\nabla(w_n+v_n)^4(w_n+v_n)\|_{L^{\frac{10}{7}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big).$$

On the other hand, one has

$$\begin{aligned}\|\nabla w_n(0)\|_{L^2}&=\|\nabla(u_n(0)-v_n(0))\|_{L^2}\\ &=\Big\|\nabla u\Big(-\frac{t_n}{h_n^2}\Big)-\nabla v\Big(-\frac{t_n}{h_n^2}\Big)\Big\|_{L^2}\longrightarrow 0\end{aligned}$$

as $n\to\infty$. Therefore

$$\begin{aligned}|||w_n|||_{I_n^{1,\Lambda}}+\|\nabla w_n\|_{L^\infty(I_n^{1,\Lambda};L^2(\mathbb{R}^3))}\;&\le\; c\Big(\|\nabla w_n(0)\|_{L^2}\\ &\quad+\|w_n+v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\|\nabla(w_n+v_n)\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big)\\ &\le\; c\Big(\|\nabla w_n(0)\|_{L^2}+\|w_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\|\nabla w_n\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\\ &\quad+\|w_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\|\nabla v_n\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\\ &\quad+\|v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\|\nabla w_n\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\\ &\quad+\|v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\|\nabla v_n\|_{L^{\frac{10}{3}}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big).\end{aligned}$$

Using Lemma 4.9, one gets

$$\limsup_{n\to\infty}\|v_n\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\longrightarrow 0 \text{ as } \Lambda\to\infty.$$

Hence,

$$\limsup_{n\to\infty}\Big(\|\nabla w_n(0)\|_{L^2}+\|v_n\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big)\longrightarrow 0 \text{ as } \Lambda\to\infty.$$

Given $\delta>0$, there exists a real number Λ_0 such that for all $\Lambda\ge\Lambda_0$ and for any integer $n\ge n_0(\Lambda)$, one has $\|\nabla w_n(0)\|_{L^2}+\|v_n\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}<\delta$. Therefore, choosing δ such that $\delta^4<\frac{1}{2c}$, one has

$$\begin{aligned}|||w_n|||_{I_n^{1,\Lambda}}+\|\nabla w_n\|_{L^\infty(I_n^{1,\Lambda};L^2(\mathbb{R}^3))}\;&\le\; C\Big(\|\nabla w_n(0)\|_{L^2}+|||w_n|||^5_{I_n^{1,\Lambda}}+|||w_n|||^4_{I_n^{1,\Lambda}}\\ &\quad+\|v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big).\end{aligned}$$

In order to use Lemma 4.3.2, we denote

$$M_n(t) := \|w_n\|_{L^{10}([0,t]\times\mathbb{R}^3)} + \|\nabla w_n\|_{L^{\frac{10}{3}}([0,t]\times\mathbb{R}^3)} + \|\nabla w_n\|_{L^\infty([0,t];L^2(\mathbb{R}^3))}.$$

Then

$$M_n(t) \;\leq\; C\Big(\|\nabla w_n(0)\|_{L^2} + \|v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)} + \sum_{\alpha=2}^{5} M_n(t)^\alpha\Big),$$

for all $t \in I_n^{1,\Lambda}$. The bootstrap Lemma 4.3.2 enables us to deduce that, for any $\Lambda \geq \Lambda_0$ and $n \geq n_0(\Lambda)$, we have

$$M_n(t) \leq 2C\Big(\|\nabla w_n(0)\|_{L^2} + \|v_n\|^4_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\Big) \longrightarrow 0 \text{ as } n \to \infty, \;\; \forall t \in I_n^{1,\Lambda}.$$

Hence,

$$\limsup_{n\to\infty} \Big(\|u_n - v_n\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)} + \|\nabla(u_n - v_n)\|_{L^\infty(I_n^{1,\Lambda};L^2(\mathbb{R}^3))}\Big) \longrightarrow 0 \text{ as } \Lambda \to \infty,$$

showing the convergence (4.56). Now, we prove (4.57) following the same procedure. We know that u_n is a solution to

$$\begin{cases} i\partial_t u_n + \Delta u_n - |u_n|^4 u_n = 0 \;\text{ on } (t_n + \Lambda h_n^2, T] \times \mathbb{R}^3, \\ u_n(0) = \varphi \;\in \dot{H}^1(\mathbb{R}^3). \end{cases}$$

Since $u_n(t,x)$ is a nonlinear concentrating solution, one has $u_n(t,x) = \frac{1}{\sqrt{h_n}} u\Big(\dfrac{t-t_n}{h_n^2}, \dfrac{x-x_n}{h_n}\Big)$, where u satisfies

$$i\partial_s u + \Delta u - |u|^4 u = 0 \;\text{ on } \mathbb{R}\times\mathbb{R}^3.$$

However, using again the scattering theory we know that there exists a solution v of the linear problem

$$\begin{cases} i\partial_s v + \Delta v = 0 \;\text{ on } \mathbb{R}\times\mathbb{R}^3, \\ v(0) = \varphi^2 \end{cases}$$

such that

$$\|\nabla u(s) - \nabla v(s)\|_{L^2} \longrightarrow 0 \text{ as } s \to +\infty.$$

Let $v_n(t,x) = \frac{1}{\sqrt{h_n}} v\Big(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n}\Big)$. It satisfies

$$\begin{cases} i\partial_t v_n + \Delta v_n = 0 \;\text{ on } I_n^{3,\Lambda} \times \mathbb{R}^3, \\ v_n(t_n) = \frac{1}{\sqrt{h_n}}\varphi^2. \end{cases}$$

We must show now that

$$\limsup_{n\to\infty}\left(\|u_n - v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty(I_n^{3,\Lambda};\dot{H}^1(\mathbb{R}^3))}\right) \longrightarrow 0 \text{ as } \Lambda\to\infty.$$

We will do the same procedure using

$$\|\nabla(u_n - v_n)(t_n + h_n^2\Lambda)\|_{L^2} = \|\nabla u(\Lambda) - \nabla v(\Lambda)\|_{L^2} \longrightarrow 0 \text{ as } \Lambda\to\infty.$$

Define $w_n := u_n - v_n$. So, w_n is a solution of the system

$$\begin{cases} i\partial_t w_n + \Delta w_n = |w_n + v_n|^4(w_n + v_n), \\ w_n(t_n + h_n^2\Lambda) = u_n(t_n + h_n^2\Lambda) - v_n(t_n + h_n^2\Lambda). \end{cases} \tag{4.59}$$

Using Lemma 4.3.1, one has

$$\begin{aligned} |||w_n|||_{I_n^{3,\Lambda}} + \|\nabla w_n\|_{L^\infty(I_n^{3,\Lambda};L^2(\mathbb{R}^3))} \;\le\; & c\Big(\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} \\ & + \|\nabla(w_n + v_n)^4(w_n + v_n)\|_{L^{\frac{10}{7}}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} |||w_n|||_{I_n^{3,\Lambda}} + \|\nabla w_n\|_{L^\infty(I_n^{3,\Lambda};L^2(\mathbb{R}^3))} \;\le\; & c\Big(\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} + \|w_n\|^4_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\|\nabla w_n\|_{L^{\frac{10}{3}}(I_n^{3,\Lambda}\times\mathbb{R}^3)} \\ & + \|w_n\|^4_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\|\nabla v_n\|_{L^{\frac{10}{3}}(I_n^{3,\Lambda}\times\mathbb{R}^3)} \\ & + \|v_n\|^4_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\|\nabla w_n\|_{L^{\frac{10}{3}}(I_n^{3,\Lambda}\times\mathbb{R}^3)} \\ & + \|v_n\|^4_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\|\nabla v_n\|_{L^{\frac{10}{3}}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\Big). \end{aligned}$$

By Lemma 4.9, we have

$$\limsup_{n\to\infty}\|v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } \Lambda\to\infty.$$

Hence,

$$\limsup_{n\to\infty}\left(\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} + \|v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\right) \longrightarrow 0 \text{ as } \Lambda\to\infty.$$

Moreover, given $\delta > 0$, there exists a real number Λ_0 such that for all $\Lambda \ge \Lambda_0$ and for any integer $n \ge n_0(\Lambda)$ the quantity $\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} + \|v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} < \delta$. Therefore, choosing δ such that $\delta^4 < \frac{1}{2c}$, one has

$$\begin{aligned} |||w_n|||_{I_n^{3,\Lambda}} + \|\nabla w_n\|_{L^\infty(I_n^{3,\Lambda};L^2(\mathbb{R}^3))} \;\le\; & C\Big(\|\nabla w_n(0)\|_{L^2} + |||w_n|||^5_{I_n^{3,\Lambda}} + |||w_n|||^4_{I_n^{3,\Lambda}} \\ & + \|v_n\|^4_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\Big). \end{aligned}$$

Denote

$$M_n(t) := \|w_n\|_{L^{10}([t_n+h_n^2\Lambda,t]\times\mathbb{R}^3)} + \|\nabla w_n\|_{L^{\frac{10}{3}}([t_n+h_n^2\Lambda,t]\times\mathbb{R}^3)} + \|\nabla w_n\|_{L^\infty([t_n+h_n^2\Lambda,t];L^2(\mathbb{R}^3))}$$

with $t \in I_n^{3,\Lambda}$, we have

$$M_n(t) \le c\Big(\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} + \|v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} + \sum_{\alpha=2}^{5} M_n(t)^\alpha\Big),$$

$t \in I_n^{3,\Lambda}$. The bootstrap Lemma 4.3.2 enables us to deduce that, for any $\Lambda \ge \Lambda_0$ and $n \ge n_0(\Lambda)$

$$M_n(t) \le 2c\Big(\|\nabla w_n(t_n + h_n^2\Lambda)\|_{L^2} + \|v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\Big) \longrightarrow 0 \text{ as } n \to \infty, \ \forall t \in I_n^{3,\Lambda}.$$

Hence,

$$\limsup_{n\to\infty}\Big(\|u_n - v_n\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)} + \|\nabla(u_n - v_n)\|_{L^\infty(I_n^{3,\Lambda};L^2(\mathbb{R}^3))}\Big) \longrightarrow 0 \text{ as } \Lambda \to \infty.$$

showing (4.57). □

4.3.2 Auxiliary results

From now on, we state and prove several results that will be paramount for the proof of Theorem 4.10. Let us begin with the following lemma.

Lemma 4.3.3. *There exists $\delta_0 > 0$ such that, if v is a solution of linear Schrödinger equation satisfying*

$$|||v|||_{\mathbb{R}} \le \delta_0$$

and u is a solution of the nonlinear Schrödinger equation satisfying $v(T,x) = u(T,x)$, for some $T \in [-\infty,+\infty]$, then

$$|||u|||_{\mathbb{R}} \le 3|||v|||_{\mathbb{R}}.$$

Proof of Lemma 4.3.3. Suppose that $\|\nabla(u-v)(x,-\infty)\|_{L^2} = 0$ (the other cases can be handled similarly). Let $(T_n)_n$ be a sequence of numbers converging to $+\infty$ as $n \to +\infty$. Set

$$J_n = [-T_n, T_n].$$

The difference $w = u - v$ satisfies

$$\begin{cases} i\partial_t w + \Delta w = |w+v|^4(w+v) \text{ on } \mathbb{R}, \\ w(-T_n) = (u-v)(-T_n). \end{cases}$$

From Lemma 4.3.1, it follows that

$$\begin{aligned}
|||w|||_{J_n} &\leq C\Big(||\nabla(u-v)(-T_n)||_{L^2} + \|\nabla(w+v)^5\|_{L^{\frac{10}{7}}(J_n\times\mathbb{R}^3)}\Big) \\
&\leq C\Big(||\nabla(u-v)(-T_n)||_{L^2} + \|w+v\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla(w+v)\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)}\Big) \\
&\leq C\Big(||\nabla(u-v)(-T_n)||_{L^2} + (\|w\|^4_{L^{10}(J_n\times\mathbb{R}^3)} + \|v\|^4_{L^{10}(J_n\times\mathbb{R}^3)})\|\nabla(w+v)\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)}\Big) \\
&\leq C\Big(|\nabla(u-v)(-T_n)||_{L^2} + \|w\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla w\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)} \\
&\quad + \|w\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla v\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)} + \|v\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla w\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)} \\
&\quad + \|v\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla v\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)}\Big).
\end{aligned}$$

Now, let $\delta_0 > 0$ such that $\delta_0^4 < \frac{1}{2C}$, $\delta_0^5 < \frac{a_0}{2}$ and $\delta_0 < 1$ (a_0 is the small constant from Lemma 4.3.2). Therefore,

$$\begin{aligned}
\|w\|_{L^{10}(J_n\times\mathbb{R}^3)} + \|\nabla w\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)} &\leq 2C\Big(\|\nabla(u-v)(-T_n)||_{L^2} + |||w|||^5_{J_n} \\
&\quad + \|w\|^4_{L^{10}(J_n\times\mathbb{R}^3)}\|\nabla v\|_{L^{\frac{10}{3}}(J_n\times\mathbb{R}^3)} + |||v|||^5_{J_n}\Big) \\
&\leq 2C\Big(\|\nabla(u-v)(-T_n)||_{L^2} + |||w|||^5_{J_n} \\
&\quad + |||w|||^4_{L^{10}(J_n\times\mathbb{R}^3)} + |||v|||^5_{J_n}\Big).
\end{aligned}$$

Using the fact that $\|\nabla(u-v)(x,-T_n)\|_{L^2} \to 0$ as n tends to infinity, we get, for large n,

$$\|\nabla(u-v)(x,-T_n)\|_{L^2} + |||v|||^5_{J_n} \leq a_0.$$

Thus, for large n, the function $M : s \mapsto |||w|||_{[-T_n,s]}$ satisfies the conditions of Lemma 4.3.2 on $[-T_n, T_n]$, so that

$$M(T_n) = |||w|||_{J_n} \leq 4C(\|\nabla(u-v)(x,-T_n)\|_{L^2} + |||v|||^5_{J_n})$$

for n large. Taking $n \to +\infty$, we obtain

$$|||w|||_{\mathbb{R}} \leq 4C|||v|||^5_{\mathbb{R}}.$$

Hence

$$|||u|||_{\mathbb{R}} \leq |||w|||_{\mathbb{R}} + |||v|||_{\mathbb{R}} \leq (4C|||v|||^4_{\mathbb{R}} + 1)|||v|||_{\mathbb{R}}.$$

Since $2C\delta_0^4 < 1$, we conclude the proof of this lemma. □

With the previous lemma in hand, the following result holds.

Proposition 4.3.1. *There exists* $C>0$ *such that*

$$\limsup_{n\to\infty} |||W_n^{(l)}+w_n^{(l)}|||_I \leq C, \tag{4.60}$$

for all $l \geq 1$.

Proof of Proposition 4.3.1. First of all, observe that, using (4.17),

$$\limsup_{n\to\infty} |||w_n^{(l)}|||_I \leq C \limsup_{n\to\infty} \|\nabla w_n^{(l)}(0)\|_{L^2} \leq C \limsup_{n\to\infty} \|\nabla v_n(0)\|_{L^2} \leq C$$

for all $l \geq 1$. Thereby, to obtain (4.60), it suffices to prove that

$$\limsup_{n\to\infty} |||W_n^{(l)}|||_I \leq C,$$

for all $l \geq 1$. By definition, $p_n^{(j)}$ and $q_n^{(j)}$ satisfy

$$p_n^{(j)}(t,x) = \frac{1}{\sqrt{h_n^{(j)}}}\varphi^{(j)}\left(\frac{t-t_n^{(j)}}{(h_n^{(j)})^2}, \frac{x-x_n^{(j)}}{h_n^{(j)}}\right)$$

and

$$q_n^{(j)}(t,x) = \frac{1}{\sqrt{h_n^{(j)}}}\psi^{(j)}\left(\frac{t-t_n^{(j)}}{(h_n^{(j)})^2}, \frac{x-x_n^{(j)}}{h_n^{(j)}}\right),$$

respectively, with φ and ψ belonging to $L^\infty(\mathbb{R};\dot{H}^1(\mathbb{R}^3))$. Lemma 4.2.4 and Remark 4.2.2 ensure that, for all l,

$$\|W_n^{(l)}\|^{10}_{L^{10}(I\times\mathbb{R}^3)} \to \sum_{j=1}^{l} \|\psi^{(j)}\|^{10}_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} \text{ and } \|\nabla W_n^{(l)}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)} \to \sum_{j=1}^{l} \|\nabla\psi^{(j)}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)}$$

as $n\to\infty$. So, we have to prove that the series $\sum_{j\geq 1} |||\psi^{(j)}|||_{\mathbb{R}}^{\frac{10}{3}}$ is convergent. To this end, first note that (4.17) and Lemma 4.1.3 imply

$$\begin{aligned} \sum_{j\geq 1} |||\varphi^{(j)}|||_{\mathbb{R}}^{\frac{10}{3}} &= \sum_{j\geq 1} |||p_n^{(j)}|||_I^{\frac{10}{3}} \\ &\leq C\sum_{j\geq 1} \|\nabla p_n^{(j)}(0)\|_{L^2_x}^{\frac{10}{3}} \leq C \end{aligned} \tag{4.61}$$

where we have used the fact that the series $\sum_{j\geq 1} \|\nabla p_n^{(j)}(0)\|_{L^2_x}^{\frac{10}{3}}$ is convergent. Thus, if

$$|||\psi^{(j)}|||_{\mathbb{R}} \leq C|||\varphi^{(j)}|||_{\mathbb{R}}, \tag{4.62}$$

for large enough j, then the series $\sum_{j\geq 1} |||\psi^{(j)}|||_{\mathbb{R}}^{\frac{10}{3}}$ is convergent. But, from (4.61), one has that

$$|||\varphi^{(j)}|||_{\mathbb{R}} \leq \delta_0,$$

for large enough j, since $|||\varphi^{(j)}|||_{\mathbb{R}}$ is the general term of a convergent series, where δ_0 is as in Lemma 4.3.3. Moreover,

$$\|\nabla(\psi^{(j)} - \varphi^{(j)})(-t_n^{(j)}/(h_n^{(j)})^2)\|_{L_x^2} = \|\nabla(q_n^{(j)} - p_n^{(j)})(0)\|_{L_x^2} = 0.$$

Consequently, $\psi^{(j)}$ and $\varphi^{(j)}$ satisfy the conditions of Lemma 4.3.3 for large j, then we get (4.62) and, therefore, $\limsup_{n\to\infty} |||W_n^{(l)}|||_{\mathbb{R}} \leq C$, for all $l \geq 1$. This finishes the proof of Proposition 4.3.1. □

Let us present now a technical proposition.

Proposition 4.3.2. *For every $\varepsilon > 0$, there exists an n-dependent finite partition of $I = [0, T]$*

$$[0, T] = \bigcup_{i=1}^{p} I_n^i \tag{4.63}$$

such that

$$\limsup_{n\to\infty} \|W_n^{(l)} + w_n^{(l)}\|_{L^{10}(I_n^i\times\mathbb{R}^3)} \leq \varepsilon, \tag{4.64}$$

for all $1 \leq i \leq p$, $l \geq 1$.

Proof of Proposition 4.3.2. Since

$$\limsup_{n\to\infty} \|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty,$$

given $\varepsilon > 0$ be a small fixed number, there exists $l_1 \geq 1$ such that

$$\limsup_{n\to\infty} \|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} \leq \frac{\varepsilon}{2} \tag{4.65}$$

if $l \geq l_1$. Moreover, by Lemma 4.2.4, there exists $l_2 \geq 1$ such that

$$\begin{aligned}
\limsup_{n\to\infty} \|W_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} &= \limsup_{n\to\infty} \Big\| \sum_{j=1}^{l} q_n^{(j)} \Big\|_{L^{10}(I\times\mathbb{R}^3)} \\
&= \limsup_{n\to\infty} \Big\| \sum_{j=1}^{l_2} q_n^{(j)} + \sum_{j=l_2+1}^{l} q_n^{(j)} \Big\|_{L^{10}(I\times\mathbb{R}^3)} \\
&\leq \limsup_{n\to\infty} \|W_n^{(l_2)}\|_{L^{10}(I\times\mathbb{R}^3)} + \Big(\sum_{j=l_2+1}^{l} \|\psi^{(j)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}^{10} \Big)^{\frac{1}{10}},
\end{aligned} \tag{4.66}$$

for all $l \geq l_2 \geq 1$. Recall that the series $\sum_{j\geq 1} \|\psi^{(j)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}^{10}$ is convergent, so we may choose l_2 such that

$$\Big(\sum_{j\geq l_2} \|\psi^{(j)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}^{10} \Big)^{\frac{1}{10}} \leq \frac{\varepsilon}{4}. \tag{4.67}$$

Putting together estimates (4.65) and (4.67), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\|W_n^{(l)}+w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} &\leq \limsup_{n\to\infty}\|W_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)}+\limsup_{n\to\infty}\|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)}\\ &\leq \limsup_{n\to\infty}\|W_n^{(l_2)}\|_{L^{10}(I\times\mathbb{R}^3)}+\Big(\sum_{j=l_2+1}^{l}\|\psi^{(j)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}^{10}\Big)^{\frac{1}{10}}+\frac{\varepsilon}{2}\\ &\leq \limsup_{n\to\infty}\|W_n^{(l_3)}\|_{L^{10}(I\times\mathbb{R}^3)}+\frac{3\varepsilon}{4}, \end{aligned} \tag{4.68}$$

for every $l\geq l_3=\sup(l_1,l_2)$. Considering the natural number l_3, the idea is to construct l_3 partial finite partitions of I for every $1\leq j\leq l_3$, and the global decomposition is obtained by intersecting all the partial ones. Note that the partition (4.63) is needed for n large. Therefore, in the next construction, we take n large enough.

For $j=1$, we split the interval $[0,T]=I_n^{1,\Lambda}\cup I_n^{2,\Lambda}\cup I_n^{3,\Lambda}$ according to Theorem 4.11.

i. For $(I_n^{1,\Lambda})$: Using Theorem 4.11 and Lemma 4.9, there exists $p_n^{(1)}$ linear concentrating solution such that

$$\limsup_{n\to\infty}\|q_n^{(1)}\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\leq\|q_n^{(1)}-p_n^{(1)}\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}+\|p_n^{(1)}\|_{L^{10}(I_n^{1,\Lambda}\times\mathbb{R}^3)}\leq\frac{\varepsilon}{4l_3}.$$

ii. For $(I_n^{3,\Lambda})$: Analogously,

$$\limsup_{n\to\infty}\|q_n^{(1)}\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\leq\|q_n^{(1)}-p_n^{(1)}\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}+\|p_n^{(1)}\|_{L^{10}(I_n^{3,\Lambda}\times\mathbb{R}^3)}\leq\frac{\varepsilon}{4l_3}.$$

iii. For $(I_n^{2,\Lambda})$: We have $I_n^{2,\Lambda}=[t_n^{(1)}-(h_n^{(1)})^2\Lambda,t_n^{(1)}+(h_n^{(1)})^2\Lambda]$. Therefore,

$$\|q_n^{(1)}\|_{L^{10}(I_n^{2,\Lambda}\times\mathbb{R}^3)}=\|\psi^{(1)}\|_{L^{10}([-\Lambda,\Lambda]\times\mathbb{R}^3)}.$$

Once Λ is fixed, we may divide the interval $[-\Lambda,\Lambda]$ in a finite number of intervals $I^{(i),\Lambda}$ such that

$$\|\psi^{(1)}\|_{L^{10}(I^{(i),\Lambda}\times\mathbb{R}^3)}\leq\frac{\varepsilon}{4l_3}.$$

Therefore

$$\|q_n^{(1)}\|_{L^{10}(I_n^{(i),\Lambda}\times\mathbb{R}^3)}=\|\psi^{(1)}\|_{L^{10}(I^{(i),\Lambda}\times\mathbb{R}^3)}\leq\frac{\varepsilon}{4l_3}.$$

This gives the decomposition for $j=1$. Analogously, we construct a partial decomposition for every $j=2,...,l_3$. Finally, the global decomposition is obtained by intersecting all the partial ones. Hence,

$$\begin{aligned}\limsup_{n\to\infty}\|W_n^{(l)}+w_n^{(l)}\|_{L^{10}(I_n^i\times\mathbb{R}^3)} &\leq \limsup_{n\to\infty}\|W_n^{(l_3)}\|_{L^{10}(I_n^i\times\mathbb{R}^3)}+\frac{3\varepsilon}{4}\\ &\leq \limsup_{n\to\infty}\Big\|\sum_{j=1}^{l_3}q_n^{(j)}\Big\|_{L^{10}(I_n^i\times\mathbb{R}^3)}+\frac{3\varepsilon}{4}\\ &\leq \limsup_{n\to\infty}\sum_{j=1}^{l_3}\|q_n^{(j)}\|_{L^{10}(I_n^i\times\mathbb{R}^3)}+\frac{3\varepsilon}{4}\\ &\leq \sum_{j=1}^{l_3}\frac{\varepsilon}{4l_3}+\frac{3\varepsilon}{4}=\varepsilon.\end{aligned}$$

The Proposition 4.3.2 is proven. □

The next technical lemma will be important for the next proposition.

Lemma 4.3.4. *Let $\mathcal{B}$ be a compact set of $\mathbb{R}\times\mathbb{R}^3$. For every $\varepsilon>0$, there exists a constant $C(\varepsilon)$ such that*

$$\|\nabla v\|_{L^2(\mathcal{B})}\leq C(\varepsilon)\|v\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}+\varepsilon\|\nabla v(0)\|_{L^2(\mathbb{R}^3)}, \tag{4.69}$$

for all solutions v of the linear Schrödinger equation.

Proof of Lemma 4.3.4. We argue by contradiction. Suppose that (4.69) does not hold. Then, there exist $\varepsilon>0$ and a sequence (v_m) of solutions of the linear Schrödinger equation such that

$$\|\nabla v_m\|_{L^2(\mathcal{B})}>m\|v_m\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}+\varepsilon\|\nabla v_m(0)\|_{L^2(\mathbb{R}^3)}.$$

Define $\tilde{v}_m:=v_m/\|\nabla v_m\|_{L^2(\mathcal{B})}$. One has

$$1>m\|\tilde{v}_m\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}+\varepsilon\|\nabla\tilde{v}_m(0)\|_{L^2(\mathbb{R}^3)}.$$

Note that $\|\nabla\tilde{v}_m(0)\|_{L^2(\mathbb{R}^3)}$ is bounded and

$$m\|\tilde{v}_m\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} \quad < \quad -\varepsilon\|\nabla\tilde{v}_m(0)\|_{L^2(\mathbb{R}^3)}+1,$$

thus

$$\|\tilde{v}_m\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}\longrightarrow 0 \text{ as } m\to\infty. \tag{4.70}$$

By Strichartz estimates,

$$\|\nabla\tilde{v}_m\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)}\leq C\|\nabla\tilde{v}_m(0)\|_{L^2(\mathbb{R}^3)}.$$

So, we conclude that $\|\nabla\tilde{v}_m\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)}$ is also bounded. In view of (4.70), there exists a subsequence of $(\tilde{v}_m)$, also denoted by $(\tilde{v}_m)$, such that

$$\nabla\tilde{v}_m \rightharpoonup 0 \quad \text{weakly in} \quad L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3) \text{ as } m\to\infty. \tag{4.71}$$

We need the following lemma (see Lemma 3.23 in (MERLE; VEGA, 1998) for two spatial dimensions).

Lemma 4.3.5. *Let (φ_m) and φ be in $L^2(\mathbb{R}^3)$. The following statements are equivalent.*

i) $\varphi_m \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R}^3)$;

ii) $e^{it\Delta}\varphi_m \rightharpoonup e^{it\Delta}\varphi$ in $L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)$.

Continuing with the proof of Lemma 4.3.4, we set $\psi_m = \nabla\tilde{v}_m(0,.)$. One has

$$\begin{aligned}\|e^{it\Delta}\psi_m\|_{L^2(\mathcal{B})} &= \|e^{it\Delta}\nabla\tilde{v}_m(0)\|_{L^2(\mathcal{B})}\\ &= \frac{\|\nabla v_m(0)\|_{L^2(\mathcal{B})}}{\|\nabla v_m(t)\|_{L^2(\mathcal{B})}}\\ &= 1.\end{aligned}$$

But, up to a subsequence,

$$\psi_m \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^3) \text{ as } m\to\infty.$$

This fact contradicts the compactness of the operator $\psi \mapsto U(t)\psi$ from $L^2(\mathbb{R}^3)$ to $L^2_{loc}(\mathbb{R}^4)$. Therefore, (4.69) holds. □

The previous lemma ensures the following proposition, which guarantees the smallness of $\delta_n^{(l)}$, for large n and l, where

$$\delta_n^{(l)} = \left\|\nabla\left[\beta(W_n^{(l)}+w_n^{(l)})-\beta(W_n^{(l)})\right]\right\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} + \left\|\nabla\left(\sum_{j=1}^{l}\beta(q_n^{(j)})-\beta(W_n^{(l)})\right)\right\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)}.$$

Proposition 4.3.3. *We have that*

$$\limsup_{n\to\infty}\delta_n^{(l)} \longrightarrow 0 \text{ as } l\to\infty. \tag{4.72}$$

Proof of Proposition 4.3.3. We split the proof into two parts. The first one is devoted to proving that for every $l\geq 1$, one has

$$\left\|\nabla\left(\sum_{j=1}^{l}\beta(q_n^{(j)})-\beta(W_n^{(j)})\right)\right\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } n\to\infty. \tag{4.73}$$

In the second part, we shall prove that

$$\limsup_{n\to\infty} \|\nabla(\beta(W_n^{(l)} + w_n^{(l)}) - \beta(W_n^{(l)}))\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty. \tag{4.74}$$

Part 1. Note that

$$\left\|\nabla\Big(\sum_{j=1}^{l} \beta(q_n^{(j)}) - \beta(W_n^{(j)})\Big)\right\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \le CD_n,$$

i.e., the left-hand side of (4.73) is bounded by a sum of quantities

$$D_n = \|\nabla(q_n^{(j_1)}q_n^{(j_2)}q_n^{(j_3)}q_n^{(j_4)}q_n^{(j_5)})\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)},$$

with at least two differents j_k, for $k = 1,2,3,4,5$. Arguing in the same way as in the proof of Lemma 4.2.4, we want to prove that

$$D_n \longrightarrow 0 \text{ as } n \to \infty.$$

Assuming, for example, $j_1 \neq j_2$, we have

$$\begin{aligned}
D_n^{\frac{10}{7}} &= \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)} (q_n^{(j_k)})^3|^{\frac{10}{7}}\, dxdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla (q_n^{(j_1)} q_n^{(j_2)}) (q_n^{(j_k)})^3 + q_n^{(j_1)} q_n^{(j_2)} \nabla (q_n^{(j_k)})^3|^{\frac{10}{7}}\, dxdt \\
&\le C\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |q_n^{(j_k)}|^{\frac{30}{7}}\, dxdt \\
&\quad + C\int_{\mathbb{R}}\int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |\nabla (q_n^{(j_k)})^3|^{\frac{10}{7}} dxdt.
\end{aligned} \tag{4.75}$$

To bound the first integral on the right-hand side of the inequality above, use Hölder's inequality to get

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |q_n^{(j_k)}|^{\frac{30}{7}}\, dxdt &\le C\left(\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{4}}\, dxdt\right)^{\frac{4}{7}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}^3} |q_n^{(j_k)}|^{10}\, dxdt\right)^{\frac{3}{7}} \\
&\le C\|q_n^{(j_k)}\|^3_{L^{10}(\mathbb{R}^4)} \left(\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{4}}\, dxdt\right)^{\frac{4}{7}} \\
&\le C\left(\int_{\mathbb{R}}\int_{\mathbb{R}^3} |\nabla q_n^{(j_1)} q_n^{(j_2)}|^{\frac{5}{2}}\, dxdt\right)^{\frac{4}{7}}.
\end{aligned}$$

This last term can be written as

$$\frac{1}{(h_n^{(j_1)}h_n^{(j_2)})^{\frac{5}{7}}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}^3} \left|\nabla_x\psi^{(j_1)}\left(\frac{t-t_n^{(j_1)}}{(h_n^{(j_1)})^2}, \frac{x-x_n^{(j_1)}}{h_n^{(j_1)}}\right)\psi^{(j_2)}\left(\frac{t-t_n^{(j_2)}}{(h_n^{(j_2)})^2}, \frac{x-x_n^{(j_2)}}{h_n^{(j_2)}}\right)\right|^{\frac{5}{2}} dxdt\right)^{\frac{4}{7}}.$$

The orthogonality of $[h_n^{(j_1)}, \underline{x}^{(j_1)}, \underline{t}^{(j_1)}]$ and $[h_n^{(j_2)}, \underline{x}^{(j_2)}, \underline{t}^{(j_2)}]$ means that

$$\text{either } \frac{h_n^{(j_1)}}{h_n^{(j_2)}} + \frac{h_n^{(j_2)}}{h_n^{(j_1)}} \longrightarrow +\infty \text{ or } h_n^{(j_1)} = h_n^{(j_2)} \text{ and } \left| \frac{t_n^{(j_1)} - t_n^{(j_2)}}{{h_n^{(j_1)}}^2} \right| + \left| \frac{x_n^{(j_1)} - x_n^{(j_2)}}{h_n^{(j_1)}} \right| \longrightarrow +\infty,$$

as $n \to \infty$. Without loss of generality, we may assume ψ^{j_1}, ψ^{j_2} to be continuous and compactly supported and analyze the possible cases:

- If $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} + \frac{h_n^{(j_2)}}{h_n^{(j_1)}} \longrightarrow +\infty$, then either $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} \longrightarrow +\infty$ or $\frac{h_n^{(j_2)}}{h_n^{(j_1)}} \longrightarrow +\infty$. We assume $\frac{h_n^{(j_1)}}{h_n^{(j_2)}} \longrightarrow +\infty$ (the other case is analogous).

Using the change of variables $t = s(h_n^{(j_2)})^2 + t_n^{(j_2)}$, $x = yh_n^{(j_2)} + x_n^{(j_2)}$, we have

$$\begin{aligned}
&\frac{1}{(h_n^{(j_1)} h_n^{(j_2)})^{\frac{5}{7}}} \left(\int_{\mathbb{R}^4} \left| \nabla_x \psi^{(j_1)} \left(\frac{t_n^{(j_2)} - t_n^{(j_1)}}{(h_n^{(j_1)})^2} + s\frac{(h_n^{(j_2)})^2}{(h_n^{(j_1)})^2}, \frac{x_n^{(j_2)} - x_n^{(j_1)}}{h_n^{(j_1)}} + y\frac{h_n^{(j_2)}}{h_n^{(j_1)}} \right) \psi^{(j_2)}(s,y) \right|^{\frac{5}{2}} dyds(h_n^{(j_2} \right. \\
= \; & \frac{(h_n^{(j_2)})^{\frac{5}{7}}}{(h_n^{(j_1)})^{\frac{5}{7}}} \left(\int_{\mathbb{R}^4} \left| \nabla_y \psi^{(j_1)} \left(\frac{t_n^{(j_2)} - t_n^{(j_1)}}{(h_n^{(j_1)})^2} + s\frac{(h_n^{(j_2)})^2}{(h_n^{(j_1)})^2}, \frac{x_n^{(j_2)} - x_n^{(j_1)}}{h_n^{(j_1)}} + y\frac{h_n^{(j_2)}}{h_n^{(j_1)}} \right) \psi^{(j_2)}(s,y) \right|^{\frac{5}{2}} dyds \right)^{\frac{4}{7}} \\
\to \; & 0
\end{aligned}$$

as $n \to \infty$, since that ψ^{j_1}, ψ^{j_2} are continuous and compactly supported.

- If $h_n^{(j_1)} = h_n^{(j_2)}$, with the same change of variables as above, we get

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^3} \left| \nabla_y \psi^{(j_1)} \left(\frac{t_n^{(j_2)} - t_n^{(j_1)}}{(h_n^{(j_1)})^2} + s\frac{(h_n^{(j_2)})^2}{(h_n^{(j_1)})^2}, \frac{x_n^{(j_2)} - x_n^{(j_1)}}{h_n^{(j_1)}} + y\frac{h_n^{(j_2)}}{h_n^{(j_1)}} \right) \psi^{(j_2)}(s,y) \right|^{\frac{5}{2}} dyds \right)^{\frac{4}{7}}.$$

Since $\left| \frac{t_n^{(j_1)} - t_n^{(j_2)}}{{h_n^{(j_1)}}^2} \right| + \left| \frac{x_n^{(j_1)} - x_n^{(j_2)}}{h_n^{(j_1)}} \right| \longrightarrow +\infty$ as $n \to \infty$, the previous integral tends to 0, which ensures that the first integral on the right-hand side of (4.75) converges to 0.

Now, we examine the second integral on the right-hand side of (4.75). Again, Hölder's inequality ensures that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |\nabla (q_n^{(j_k)})^3|^{\frac{10}{7}} dxdt \;&\leq\; C \int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |(q_n^{(j_k)})^2 \nabla q_n^{(j_k)}|^{\frac{10}{7}} dxdt \\
&\leq\; C \int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^{\frac{10}{7}} |q_n^{(j_k)}|^{\frac{20}{7}} |\nabla q_n^{(j_k)}|^{\frac{10}{7}} dxdt \\
&\leq\; C \|q_n^{(j_k)}\|_{L^{10}(\mathbb{R}^4)}^{\frac{20}{7}} \|\nabla q_n^{(j_k)}\|_{L^{\frac{10}{3}}(\mathbb{R}^4)}^{\frac{10}{7}} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^5 \; dxdt \Big)^{\frac{2}{7}} \\
&\leq\; C \Big(\int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^5 \; dxdt \Big)^{\frac{2}{7}}
\end{aligned}$$

and

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}^3} |q_n^{(j_1)} q_n^{(j_2)}|^5 \, dxdt \right)^{\frac{2}{7}}$$
$$= \frac{1}{(h_n^{(j_1)} h_n^{(j_2)})^{\frac{5}{7}}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}^3} \left| \psi^{(j_1)} \left(\frac{t - t_n^{(j_1)}}{(h_n^{(j_1)})^2}, \frac{x - x_n^{(j_1)}}{h_n^{(j_1)}} \right) \psi^{(j_2)} \left(\frac{t - t_n^{(j_2)}}{(h_n^{(j_2)})^2}, \frac{x - x_n^{(j_2)}}{h_n^{(j_2)}} \right) \right|^5 dxdt \right)^{\frac{2}{7}}.$$

Analogously to the previous case, one concludes that the second integral on the right-hand side of (4.75) converges to 0 as well, which shows the convergence (4.73).

Part 2. By Leibnitz formula and Hölder's inequality, we get

$$\begin{aligned} \|\nabla(\beta(W_n^{(l)} + w_n^{(l)}) - \beta(W_n^{(l)}))\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} &\leq C\Big(\|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} |||W_n^{(l)} + w_n^{(l)}|||_I^4 \\ &\quad + |||W_n^{(l)}|||_I^3 \|W_n^{(l)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \Big). \end{aligned}$$

Since that (4.16) and (4.60) hold, if we prove that

$$\limsup_{n\to\infty} \|W_n^{(l)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty, \tag{4.76}$$

the proof of (4.74) is complete. Indeed, the convergence of the series $\sum_{j\geq 1} \|\psi^{(j)}\|^{10}_{L^{10}(\mathbb{R}\times\mathbb{R}^3)}$ implies that, for every $\varepsilon > 0$, there exists $l(\varepsilon)$ such that

$$\sum_{j\geq l(\varepsilon)} \|\psi^{(j)}\|^{10}_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} \leq \varepsilon^{10}. \tag{4.77}$$

In particular, using Hölder's inequality with $p = 4$ and $q = \frac{4}{3}$,

$$\begin{aligned} \limsup_{n\to\infty} \left\| \Big(\sum_{j=l(\varepsilon)}^{l} q_n^{(j)} \Big) \nabla w_n^{(l)} \right\|^{10}_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} &= \limsup_{n\to\infty} \left\| \sum_{j=l(\varepsilon)}^{l} q_n^{(j)} \right\|^{10}_{L^{10}(I\times\mathbb{R}^3)} \limsup_{n\to\infty} \|\nabla w_n^{(l)}\|^{10}_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)} \\ &\leq \sum_{j\geq l(\epsilon)} \|\psi^{(j)}\|^{10}_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} \limsup_{n\to\infty} \|\nabla w_n^{(l)}\|^{10}_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)} \\ &\leq C\varepsilon^{10}, \end{aligned}$$

where the last inequality follows from the fact that $\|\nabla w_n^{(l)}\|^{10}_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)}$ is uniformly bounded, by Strichartz estimates. Therefore,

$$\begin{aligned} \limsup_{n\to\infty} \|W_n^{(l)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} &= \limsup_{n\to\infty} \left\| \Big(\sum_{j=1}^{l} q_n^{(j)} \Big) \nabla w_n^{(l)} \right\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \\ &\leq \limsup_{n\to\infty} \left\| \Big(\sum_{j=1}^{l(\varepsilon)} q_n^{(j)} + \sum_{j=l(\varepsilon)}^{l} q_n^{(j)} \Big) \nabla w_n^{(l)} \right\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \\ &\leq \limsup_{n\to\infty} \left\| \Big(\sum_{j=1}^{l(\varepsilon)} q_n^{(j)} \Big) \nabla w_n^{(l)} \right\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \\ &\quad + \limsup_{n\to\infty} \left\| \Big(\sum_{j=l(\varepsilon)}^{l} q_n^{(j)} \Big) \nabla w_n^{(l)} \right\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \\ &\leq \limsup_{n\to\infty} \|W_n^{(l(\varepsilon))} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} + C\varepsilon, \end{aligned}$$

for $l \geq l(\varepsilon)$. Hence, our problem is reduced to prove that

$$\limsup_{n\to\infty} \|W_n^{(l_0)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty$$

for every fixed $l_0 \geq 1$. Since $W_n^{(l_0)} = \sum_{j=1}^{l_0} q_n^{(j)}$, we have to show that

$$\limsup_{n\to\infty} \|q_n^{(j)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty, \tag{4.78}$$

for every $l_0 \geq j \geq 1$, i.e.,

$$\limsup_{n\to\infty} \left\| \frac{1}{\sqrt{h_n^{(j)}}} \psi^{(j)} \Big(\frac{t - t_n^{(j)}}{(h_n^{(j)})^2}, \frac{x - x_n^{(j)}}{h_n^{(j)}} \Big) \nabla w_n^{(l)} \right\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty.$$

To this end, change variables to $y = \dfrac{x - x_n^{(j)}}{h_n^{(j)}}$, $s = \dfrac{t - t_n^{(j)}}{(h_n^{(j)})^2}$ to get

$$\|q_n^{(j)} \nabla w_n^{(l)}\|_{L^{\frac{5}{2}}(I\times\mathbb{R}^3)} = \|\psi^{(j)} \nabla \tilde{w}_n^{(l)}\|_{L^{\frac{5}{2}}(\mathbb{R}\times\mathbb{R}^3)},$$

where

$$\tilde{w}_n^{(l)}(s,y) = \sqrt{h_n^{(j)}}\, w_n^{(l)}(t_n^{(j)} + (h_n^{(j)})^2 s, x_n^{(j)} + h_n^{(j)} y).$$

Observe that, by Lemma 4.1.3,

$$\|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} = \|\tilde{w}_n^{(l)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} \quad \text{and} \quad \|\nabla w_n^{(l)}\|_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)} = \|\nabla \tilde{w}_n^{(l)}\|_{L^{\frac{10}{3}}(\mathbb{R}\times\mathbb{R}^3)}.$$

By density, we can take $\psi^{(j)} \in C_0^\infty(\mathbb{R}^4)$. Using Hölder's inequality, one sees that it is enough to prove that

$$\limsup_{n\to\infty} \|\nabla \tilde{w}_n^{(l)}\|_{L^2(\mathcal{B})} \longrightarrow 0 \text{ as } l \to \infty, \tag{4.79}$$

where $\mathcal{B}$ is a fixed compact of $\mathbb{R}\times\mathbb{R}^3$. Indeed, let ν_n^l be the function defined by

$$\tilde{\nu}_n^l(t,x) = \begin{cases} \tilde{w}_n^{(l)}(t,x), & \text{if } (t,x) \in \mathcal{B}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, ν_n^l is a solution for the linear Schrödinger equation and we get, by Strichartz estimates,

$$\begin{aligned}
\limsup_{n\to\infty} \|\psi^{(j)} \nabla \tilde{w}_n^{(l)}\|_{L^{\frac{5}{2}}(\mathbb{R}\times\mathbb{R}^3)} &\leq \limsup_{n\to\infty} \|\psi^{(j)} \nabla \tilde{w}_n^{(l)}\|_{L^{\frac{5}{2}}(\mathcal{B})} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{w}_n^{(l)}\|_{L^{\frac{10}{3}}(\mathcal{B})} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{\nu}_n^{(l)}\|_{L^{\frac{10}{3}}(\mathcal{B})} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{\nu}_n^{(l)}\|_{L^{\frac{10}{3}}(\mathbb{R}^4)} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{\nu}_n^{(l)}\|_{L^2(\mathbb{R}^3)} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{\nu}_n^{(l)}\|_{L^2(\mathcal{B})} \\
&\leq \limsup_{n\to\infty} \|\psi^{(j)}\|_{L^{10}(\mathcal{B})} \|\nabla \tilde{w}_n^{(l)}\|_{L^2(\mathcal{B})}.
\end{aligned}$$

Applying Lemma 4.3.4 to $\tilde{w}_n^{(l)}$ gives

$$\|\nabla \tilde{w}_n^{(l)}\|_{L^2(\mathcal{B})} \leq C(\varepsilon)\|\tilde{w}_n^{(l)}\|_{L^{10}(\mathbb{R}\times\mathbb{R}^3)} + \varepsilon\|\nabla \tilde{w}_n^{(l)}(0)\|_{L^2(\mathbb{R}^3)}.$$

The invariance of the L^{10} and $\dot{H}^1$ norms by the change of variables gives

$$\|\nabla \tilde{w}_n^{(l)}\|_{L^2(\mathcal{B})} \leq C(\varepsilon)\|w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)} + \varepsilon\|\nabla w_n^{(l)}(0)\|_{L^2(\mathbb{R}^3)}.$$

So, it follows that

$$\limsup_{l\to\infty} \|\nabla \tilde{w}_n^{(l)}\|_{L^2(\mathcal{B})} \leq C\varepsilon.$$

Since ε is arbitrary, (4.79) holds. This concludes the proof of Proposition 4.3.3. □

4.3.3 Proof of the nonlinear decomposition

We finally prove Theorem 4.10 following the ideas introduced in (KERAANI, 2001). First of all, note that the nonlinear profile $q_n^{(j)}$ is globally well-defined. Indeed, for a bounded sequence (φ_n) in $\dot{H}(\mathbb{R}^3)$ such that $\limsup_{n\to\infty}\|\varphi_n\|_{\dot{H}^1} < \lambda_0$ (where λ_0 is given by Definition 4.1), and (v_n) (respectively (u_n)) the sequence of solutions to the linear equation (respectively to the nonlinear equation) with initial data φ_n, Theorem 4.8 provides a decomposition of v_n for a subsequence (still denoted as v_n) in the form

$$v_n(t,x) = \sum_{j=1}^{l} p_n^{(j)}(t,x) + w_n^{(l)}(t,x)$$

where $p_n^{(j)}$ is a family of linear concentrating solutions associated with $[\varphi^{(j)}, h^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}]$and the remainder term $w_n^{(l)}$ satisfies

$$\limsup_{n\to\infty} \|w_n^{(l)}\|_{L_t^\infty L_x^6 \cap L_t^{10} L_x^{10}} \longrightarrow 0 \text{ as } l \to \infty,$$

for all $T > 0$ and $(\underline{h}^{(j)}, \underline{x}^{(j)}, \underline{t}^{(j)}) \perp (\underline{h}^{(k)}, \underline{x}^{(k)}, \underline{t}^{(k)})$, for any $j \neq k$. Also, the following almost orthogonality identity holds

$$\|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{l} \|\nabla p_n^{(j)}\|_{L^2}^2 + \|\nabla w_n^{(l)}\|_{L^2}^2 + o(1) \text{ as } n \to \infty.$$

Let $q_n^{(j)}$ be the nonlinear concentrating solution associated to $p_n^{(j)}$ for every $j \geq 1$. Observe that, given the almost orthogonality identity,

$$\|\nabla q_n^{(j)}(0)\|_{L^2}^2 = \|\nabla p_n^{(j)}(0)\|_{L^2}^2 \leq \limsup_{n\to\infty} \|\nabla v_n(0)\|_{L^2}^2 \leq \|\nabla \varphi_n\|_{L^2}^2 \leq \lambda_0^2,$$

and then the nonlinear profile $q_n^{(j)}$ is globally well defined.

Proof of Theorem 4.10. Consider

$$r_n^{(l)}(t,x) = u_n(t,x) - \sum_{j=1}^{l} q_n^{(j)}(t,x) - w_n^{(l)}(t,x).$$

We need to prove the convergence

$$\limsup_{n\to\infty}(\|\nabla r_n^{(l)}\|_{L^{\frac{10}{3}}([0,T];L^{\frac{10}{3}}(\mathbb{R}^3))} + \|r_n^{(l)}\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))} + \|r_n^{(l)}\|_{L^\infty([0,T];\dot{H}^1(\mathbb{R}^3))}) \longrightarrow 0 \text{ as } l\to\infty.$$

To this end, recall the notation used before,

$$\beta(z) = |z|^4 z,$$
$$W_n^{(l)} = \sum_{j=1}^{l} q_n^{(j)},$$

and

$$f_n^{(l)} = \sum_{j=1}^{l} \beta(q_n^{(j)}) - \beta\Big(\sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)}\Big).$$

The function $r_n^{(l)}$ satisfies the equation

$$\begin{cases} i\partial_t r_n^{(l)} + \Delta r_n^{(l)} = f_n^{(l)}, \\ r_n^{(l)}(0) = \sum_{j=1}^{l}(p_n^{(j)} - q_n^{(j)})(0) = 0. \end{cases}$$

Introduce the norm

$$|||g|||_I = \|g\|_{L^{10}(I\times\mathbb{R}^3)} + \|\nabla g\|_{L^{\frac{10}{3}}(I\times\mathbb{R}^3)}.$$

Note that, by Strichartz estimates, for any v solution of linear Schrödinger equation with initial data $\varphi \in \dot{H}^1$, one has

$$\begin{aligned} |||v|||_I = \|v\|_{L_t^{10}L_x^{10}} + \|\nabla v\|_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}}} &\leq C\|\nabla e^{it\Delta}\varphi\|_{L_x^2} \\ &\leq C\|\nabla\varphi\|_{L_x^2}. \end{aligned}$$

From now on, we use the notation

$$\gamma_n^{(l)}(a) = \|\nabla r_n^{(l)}(a)\|_{L_x^2},$$

for every $a \in [0,T]$. Applying Lemma 4.3.1 to $r_n^{(l)}$ on $I = [0,T]$, we obtain

$$|||r_n^{(l)}|||_I + \sup_{t\in I}\|\nabla r_n^{(l)}(t)\|_{L^2} \leq C\Big(\|\nabla f_n^{(l)}\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)}\Big). \tag{4.80}$$

We estimate the right-hand side of inequality (4.80) by

$$\begin{aligned} \|\nabla f_n^{(l)}\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \leq\ & \Big\|\nabla\Big(\sum_{j=1}^{l}\beta(q_n^{(j)}) - \beta(W_n^{(l)})\Big)\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \\ &+ \Big\|\nabla\Big[\beta(W_n^{(l)} + w_n^{(l)}) - \beta(W_n^{(l)})\Big]\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \\ &+ \Big\|\nabla\Big[\beta(W_n^{(l)} + w_n^{(l)} + r_n^{(l)}) - \beta(W_n^{(l)} + w_n^{(l)})\Big]\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)}. \end{aligned} \tag{4.81}$$

Furthermore, a combination of Leibnitz formula and Hölder's inequality gives that

$$\Big\|\nabla\Big[\beta(W_n^{(l)}+w_n^{(l)}+r_n^{(l)})-\beta(W_n^{(l)}+w_n^{(l)})\Big]\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)} \tag{4.82}$$

$$\leq C\Big(|||W_n^{(l)}+w_n^{(l)}|||_I^3\|W_n^{(l)}+w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)}|||r_n^{(l)}|||_I + \sum_{\alpha=2}^{5}|||W_n^{(l)}+w_n^{(l)}|||_I^{5-\alpha}|||r_n^{(l)}|||_I^{\alpha}\Big).$$

Denote

$$\delta_n^{(l)}=\Big\|\nabla\Big[\beta(W_n^{(l)}+w_n^{(l)})-\beta(W_n^{(l)})\Big]\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)}+\Big\|\nabla\Big(\sum_{j=1}^{l}\beta(q_n^{(j)})-\beta(W_n^{(l)})\Big)\Big\|_{L^{\frac{10}{7}}(I\times\mathbb{R}^3)}. \tag{4.83}$$

Using (4.81), (4.82) and (4.83) into (4.80), it follows that

$$|||r_n^{(l)}|||_I+\sup_{t\in I}\|\nabla r_n^{(l)}(t)\|_{L^2} \leq C\Big(\delta_n^{(l)}+\sum_{\alpha=2}^{5}|||W_n^{(l)}+w_n^{(l)}|||_I^{5-\alpha}|||r_n^{(l)}|||_I^{\alpha} + |||W_n^{(l)}+w_n^{(l)}|||_I^3\|W_n^{(l)}+w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)}|||r_n^{(l)}|||_I\Big) \tag{4.84}$$

In view of bound (4.84) and Proposition 4.3.1, we get

$$|||r_n^{(l)}|||_I+\sup_{t\in I}\|\nabla r_n^{(l)}(t)\|_{L^2}\leq C\Big(\gamma_n^{(l)}(a)+\delta_n^{(l)}+\sum_{\alpha=2}^{5}|||r_n^{(l)}|||_I^{\alpha}+\|W_n^{(l)}+w_n^{(l)}\|_{L^{10}(I\times\mathbb{R}^3)}|||r_n^{(l)}|||_I\Big), \tag{4.85}$$

for all $l\geq 1$ and $n\geq N(l)$. The Proposition 4.3.2 shows that under a suitable finite partition of $[0,T]$, one can absorb the linear term in $|||r_n^{(l)}|||_I$ in the right-hand side of (4.85). Applying (4.85) on an interval I_n^i , provided by Proposition 4.3.2, one gets

$$|||r_n^{(l)}|||_{I_n^i}+\sup_{t\in I_n^i}\|\nabla r_n^{(l)}(t)\|_{L^2}\leq C\Big(\gamma_n^{(l)}(a_n^i)+\delta_n^{(l)}+\sum_{\alpha=2}^{5}|||r_n^{(l)}|||_{I_n^i}^{\alpha}+2\varepsilon|||r_n^{(l)}|||_{I_n^i}\Big),$$

for all $l\geq 1$ and $n\geq N(l)$. So, choosing ε so that $C\epsilon<\frac{1}{4}$, we obtain

$$|||r_n^{(l)}|||_{I_n^i}+\sup_{t\in I_n^i}\|\nabla r_n^{(l)}(t)\|_{L^2}\leq C\Big(\gamma_n^{(l)}(a_n^i)+\delta_n^{(l)}+\sum_{\alpha=2}^{5}|||r_n^{(l)}|||_{I_n^i}^{\alpha}\Big). \tag{4.86}$$

Now, we use an iterative process to achieve the result. For $i=1$, (4.86) reads

$$|||r_n^{(l)}|||_{I_n^1}+\sup_{t\in I_n^1}\|\nabla r_n^{(l)}(t)\|_{L^2}\leq C\Big(\gamma_n^{(l)}(0)+\delta_n^{(l)}+\sum_{\alpha=2}^{5}|||r_n^{(l)}|||_{I_n^1}^{\alpha}\Big). \tag{4.87}$$

Recall that, in view of definition of $\gamma_n^{(l)}$, we have

$$\gamma_n^{(l)}(0)=\|\nabla r_n^{(l)}(0)\|_{L^2}=\Big\|\nabla\Big(\sum_{j=1}^{l}(p_n^{(j)}-q_n^{(j)})(0)\Big)\Big\|_{L^2}=0, \tag{4.88}$$

for all $l \geq 1$. Due to (4.72) and (4.88), it follows that, for all large enough l, there exists $N(l)$ such that, if $n \geq N(l)$, then

$$\gamma_n^{(l)}(0) + \delta_n^{(l)} \leq a_0(c). \tag{4.89}$$

Denote by M_n^l the function defined on $I_n^1 = [0, a_n^1]$ by

$$M_n^l(s) = |||r_n^{(l)}|||_{[0,s]} + \frac{s}{a_n^1} \sup_{t\in[0,s]} \|\nabla r_n^{(l)}(t)\|_{L^2}.$$

It is clear that (4.86) still holds if we replace $I_n^1 = [0, a_n^1]$ by $[0, s]$ for all $s \in I_n^1$ .Thus,

$$M_n^l(s) \leq C\Big(\gamma_n^{(l)}(0) + \delta_n^{(l)} + \sum_{\alpha=2}^{5} (M_n^l)^\alpha(s)\Big).$$

Hence, the function M_n^l satisfies the conditions of Lemma 4.3.2 for large l and $n \geq N(l)$. So

$$M_n^l(a_n^1) = |||r_n^{(l)}|||_{I_n^1} + \sup_{t\in I_n^1} \|\nabla r_n^{(l)}(t)\|_{L^2} \leq 2c(\gamma_n^{(l)}(0) + \delta_n^{(l)}), \tag{4.90}$$

for large l and $n \geq N(l)$. Using (4.72), (4.88) and (4.90), one obtains

$$\limsup_{n\to\infty} \Big(|||r_n^{(l)}|||_{I_n^1} + \sup_{t\in I_n^1} \|\nabla r_n^{(l)}(t)\|_{L^2}\Big) \longrightarrow 0 \text{ as } l \to \infty.$$

On the other hand, we have

$$\gamma_n^{(l)}(a_n^1) \leq \sup_{t\in I_n^1} \|\nabla r_n^{(l)}(t)\|_{L^2},$$

which gives

$$\limsup_{n\to\infty} \gamma_n^{(l)}(a_n^1) \longrightarrow 0 \text{ as } l \to \infty.$$

This allows us to repeat the same argument on the interval $I_n^2 = [a_n^1, a_n^2]$. We get, similarly,

$$|||r_n^{(l)}|||_{I_n^2} + \sup_{t\in I_n^2} \|\nabla r_n^{(l)}(t)\|_{L^2} \leq c(\gamma_n^{(l)}(a_n^1) + \delta_n^{(l)}).$$

Thus

$$\limsup_{n\to\infty} \Big(|||r_n^{(l)}|||_{I_n^2} + \sup_{t\in I_n^2} \|\nabla r_n^{(l)}(t)\|_{L^2}\Big) \longrightarrow 0 \text{ as } l \to \infty.$$

Iterating this process, we get

$$\limsup_{n\to\infty} \Big(|||r_n^{(l)}|||_{I_n^i} + \sup_{t\in I_n^i} \|\nabla r_n^{(l)}(t)\|_{L^2}\Big) \longrightarrow 0 \text{ as } l \to \infty,$$

for all $1 \leq i \leq p$. Since p does not depend on n and l, one has

$$\limsup_{n\to\infty} \Big(|||r_n^{(l)}|||_{[0,T]} + \sup_{t\in[0,T]} \|\nabla r_n^{(l)}(t)\|_{L^2}\Big) \longrightarrow 0 \text{ as } l \to \infty,$$

which concludes the proof. □

4.4 APPLICATIONS

In this section, we bring some properties satisfied by the solutions of the nonlinear equation. Some of these results will be useful in demonstrating the Theorem 2.2, one of the two main results obtained in this thesis.

4.4.1 Some estimates for the nonlinear evolution solution

Our first result is a consequence of Theorem 4.10.

Proposition 4.4.1. *[Corollary 1.14, (KERAANI, 2001)] There exists a nondecreasing function* $A : [0, \lambda_0 \to [0, +\infty[$ *such that, for every solution* u *to system (4.3) with* $\|\nabla u(0, .)\|_{L^2(\mathbb{R}^3)} <$ λ_0*, we have*

$$\|\nabla u\|_{L^{\frac{10}{3}}([0,T]\times\mathbb{R}^3)} + \|u\|_{L^{10}([0,T]\times\mathbb{R}^3)} \leq A(\|\nabla u(0, .)\|_{L^2(\mathbb{R}^3)}). \tag{4.91}$$

Demonstração. We argue by contradiction: Assume that the estimate (4.91) fails. Then, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to system (4.3) such that

$$\sup_{n\in\mathbb{N}} \|\nabla u_n(0, .)\|_{L^2(\mathbb{R}^3)} < \lambda_0 \tag{4.92}$$

and

$$\|\nabla u_n\|_{L^{\frac{10}{3}}([0,T]\times\mathbb{R}^3)} + \|u_n\|_{L^{10}([0,T]\times\mathbb{R}^3)} \longrightarrow \infty \tag{4.93}$$

as $n \to \infty$. Applying Theorem 4.10 to the sequence $(u_n(0, .))_{n\in\mathbb{N}}$,we get that there exists a subsequence (still denoted by $(u_n)_{n\in\mathbb{N}}$) such that

$$u_n(t, x) = \sum_{j=1}^{l} q_n^{(j)}(t, x) + w_n^{(l)}(t, x) + r_n^{(l)}(t, x),$$

with

$$\limsup_{n\to\infty} |||w_n^{(l)} + r_n^l|||_{[0,T]} \leq C,$$

for all $l \geq 1$. Hence,

$$\limsup_{n\to\infty} |||u|||_{[0,T]} \leq \limsup_{n\to\infty} |||w_n^{(l)} + r_n^{(l)}|||_{[0,T]} + \sum_{j=1}^{l} |||\Psi^{(j)}|||_{\mathbb{R}} < +\infty,$$

which contradicts (4.93) and proves the existence of a function A satisfying (4.91). □

The next proposition is a consequence of Strichartz estimates.

Proposition 4.4.2. *Let $u \in C([a,b]; H^1(\mathbb{R}^3))$ be a solution of the damped Schrödinger equation*

$$i\partial_t v + \Delta v - v - a(1-\Delta)^{-1} a\partial_t v = f$$

on $I=[a,b]$ with $\nabla f \in L^2(I; L^{\frac{6}{5}}(\mathbb{R}^3))$ and $f \in L^1(I; L^2(\mathbb{R}^3))$. Thus, the following inequality holds

$$\begin{aligned}\|\nabla v\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}}(\mathbb{R}^3))} + \|\nabla v\|_{L^{10}(I;L^{\frac{30}{13}}(\mathbb{R}^3))} + \sup_{t\in I}\|v(t)\|_{L^2} + \sup_{t\in I}\|\nabla v(t)\|_{L^2} \\ \le C\Big(\|v(a)\|_{H^1} + \|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + \|f\|_{L^1(I;L^2(\mathbb{R}^3))}\Big).\end{aligned}$$

Demonstração. The solution v satisfies

$$v(t) = e^{it\Delta}v(a) + \int_a^t e^{i(t-\tau)\Delta} f \; d\tau + \int_a^t e^{i(t-\tau)\Delta}[v + a(1-\Delta)^{-1}a\partial_t v] \; d\tau.$$

Applying Strichartz's estimates,

$$\begin{aligned}\|v(t)\|_{L^2} &\le C\|v(a)\|_{L^2} + \left\|\int_a^t e^{i(t-\tau)\Delta} f \; d\tau\right\|_{L^2} + \left\|\int_a^t e^{i(t-\tau)\Delta}[v + a(1-\Delta)^{-1}a\partial_t v] \; d\tau\right\|_{L^2} \\ &\le C\|v(a)\|_{H^1} + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|v\|_{L^1(I;L^2(\mathbb{R}^3))} + \|a(1-\Delta)^{-1}a\partial_t v\|_{L^1(I;L^2(\mathbb{R}^3))} \\ &\le C\|v(a)\|_{H^1} + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))} + C_I \sup_{t\in I}\|v(t)\|_{L^2(\mathbb{R}^3)} \\ &\quad + C\|a(1-\Delta)^{-1}aJ^{-1}\Delta v\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|a(1-\Delta)^{-1}aJ^{-1}f\|_{L^1(I;L^2(\mathbb{R}^3))} \\ &\le C\|v(a)\|_{H^1} + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))} + C_I \sup_{t\in I}\|v(t)\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla v(t)\|_{L^2} &\le C\|\nabla v(a)\|_{L^2} + \left\|\int_a^t \nabla e^{i(t-\tau)\Delta} f \; d\tau\right\|_{L^2} + \left\|\int_a^t \nabla e^{i(t-\tau)\Delta}[v + a(1-\Delta)^{-1}a\partial_t v] \; d\tau\right\|_{L^2} \\ &\le C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + \|\nabla v\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|\nabla a(1-\Delta)^{-1}a\partial_t v\|_{L^1(I;L^2(\mathbb{R}^3))} \\ &\le C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I}\|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I}\|v(t)\|_{L^2(\mathbb{R}^3)} \\ &\quad + C\|\nabla a(1-\Delta)^{-1}aJ^{-1}\Delta v\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|\nabla a(1-\Delta)^{-1}aJ^{-1}f\|_{L^1(I;L^2(\mathbb{R}^3))} \\ &\le C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I}\|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I}\|v(t)\|_{L^2(\mathbb{R}^3)} \\ &\quad + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))}.\end{aligned}$$

Additionally, we get

$$\begin{aligned}
\|v\|_{L^{10}(I\times\mathbb{R}^3)} &\leq \|\nabla v\|_{L^{10}(I;L^{\frac{30}{13}}(\mathbb{R}^3))}\\
&\leq C\|\nabla v(a)\|_{L^2} + \left\| \int_a^t \nabla e^{i(t-\tau)\Delta} f \, d\tau \right\|_{L_t^{10} L_x^{\frac{30}{13}}}\\
&\quad + \left\| \int_a^t \nabla e^{i(t-\tau)\Delta} [v + a(1-\Delta)^{-1} a\partial_t v] \, d\tau \right\|_{L_t^{10} L_x^{\frac{30}{13}}}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C\|\nabla v\|_{L^1(I;L^2(\mathbb{R}^3))}\\
&\quad + C\|\nabla a(1-\Delta)^{-1} a\partial_t v\|_{L^1(I;L^2(\mathbb{R}^3))}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I} \|v(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad + C\|\nabla a(1-\Delta)^{-1} a J^{-1} \Delta v\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|\nabla a(1-\Delta)^{-1} a J^{-1} f\|_{L^1(I;L^2(\mathbb{R}^3))}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I} \|v(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\nabla v\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} &\leq C\|\nabla v(a)\|_{L^2} + \left\| \int_a^t \nabla e^{i(t-\tau)\Delta} f \, d\tau \right\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}\\
&\quad + \left\| \int_a^t \nabla e^{i(t-\tau)\Delta} [v + a(1-\Delta)^{-1} a\partial_t v] \, d\tau \right\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad + C\|\nabla a(1-\Delta)^{-1} a\partial_t v\|_{L^1(I;L^2(\mathbb{R}^3))}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I} \|v(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad + C\|\nabla a(1-\Delta)^{-1} a J^{-1} \Delta v\|_{L^1(I;L^2(\mathbb{R}^3))} + C\|\nabla a(1-\Delta)^{-1} a J^{-1} f\|_{L^1(I;L^2(\mathbb{R}^3))}\\
&\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C_I \sup_{t\in I} \|\nabla v(t)\|_{L^2(\mathbb{R}^3)} + C_I \sup_{t\in I} \|v(t)\|_{L^2(\mathbb{R}^3)}\\
&\quad + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))}.
\end{aligned}$$

Putting together these inequalities, one obtains

$$\|\nabla v\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}}(\mathbb{R}^3))} + \|\nabla v\|_{L^{10}(I;L^{\frac{30}{13}}(\mathbb{R}^3))} + \sup_{t\in I} \|v(t)\|_{L^2} + \sup_{t\in I} \|\nabla v(t)\|_{L^2}$$

$$\leq C\|v(a)\|_{H^1} + C\|\nabla f\|_{L^2(I;L^{\frac{6}{5}}(\mathbb{R}^3))} + C\|f\|_{L^1(I;L^2(\mathbb{R}^3))} + C_I \sup_{t\in I} \|v(t)\|_{L^2} + \sup_{t\in I} C_I \|\nabla v(t)\|_{L^2}.$$

Now, we absorb the additional terms to obtain the desired estimate for I with small enough length. Reiterating the process, it is possible to get the result for large times. □

Remark 4.4.1. *Using the same reasoning it is possible to show a similar result.*
Let $u \in C([a,b]; H^1(\mathbb{R}^3))$ *be a solution of the nohomogeneous Schrödinger equation*

$$i\partial_t v + \Delta v - v = f$$

on $I = [a,b]$*, with* $\nabla f \in L^2(I; L^{\frac{6}{5}}(\mathbb{R}^3))$ *and* $f \in L^1(I; L^2(\mathbb{R}^3))$*. The following inequality holds*

$$\|\nabla v\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}}(\mathbb{R}^3))} + \|\nabla v\|_{L^{10}(I;L^{\frac{30}{13}}(\mathbb{R}^3))} + \sup_{t\in I}\|v(t)\|_{L^2} + \sup_{t\in I}\|\nabla v(t)\|_{L^2}$$
$$\leq C\Big(\|v(a)\|_{H^1} + \|\nabla f\|_{L^{\frac{10}{7}}(I;L^{\frac{10}{7}}(\mathbb{R}^3))} + \|f\|_{L^1(I;L^2(\mathbb{R}^3))}\Big).$$

The next lemma guarantees the L^2-norm dependence of the solution on the initial data.

Lemma 4.4.1. *Let* $T > 0$*. There exists* $C > 0$ *such that any solution* u *to*

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u = a(1-\Delta)^{-1} a\partial_t u \ \text{ on } [0,T]\times\mathbb{R}^3, \\ u(0) = u_0, \\ \|u_0\|_{H^1} \leq \lambda_0, \ \text{where } \lambda_0 \text{ is giving by Definition 4.1,} \end{cases} \tag{4.94}$$

satisfies

$$\|u\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq C\|u_0\|_{L^2(\mathbb{R}^3)}.$$

Demonstração. First, notice that $u \in L^7([0,T]; L^{14}(\mathbb{R}^3))$. Indeed, write

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}[u + |u|^4 u + a(1-\Delta)^{-1}a\partial_t u]\ d\tau.$$

Through Sobolev embedding and Strichartz estimates, one has

$$\begin{aligned} \|u\|_{L^7_t L^{14}_x} &\leq \|\nabla u\|_{L^7_t L^{\frac{42}{17}}_x} \\ &\leq C\|\nabla u(0)\|_{L^2} + C\|\nabla |u|^4 u\|_{L^{\frac{10}{7}}([0,T]\times\mathbb{R}^3)} + C\|\nabla u\|_{L^1([0,T];L^2(\mathbb{R}^3))} \\ &\quad + C\|\nabla a(1-\Delta)^{-1}a\partial_t u\|_{L^1([0,T];L^2(\mathbb{R}^3))} \\ &\leq C\|\nabla u_0\|_{L^2} + C\|u\|^4_{L^{10}_t L^{10}_x}\|\nabla u\|_{L^{\frac{10}{3}}_t L^{\frac{10}{3}}_x} + CT\sup_{t\in[0,T]}\|\nabla u(t)\|_{L^2} \\ &\quad + CT\sup_{t\in[0,T]}\|u(t)\|_{L^2} + \|u\|^5_{L^{10}([0,T];L^{10}(\mathbb{R}^3))} \\ &\leq C, \end{aligned}$$

since $\left(7, \frac{42}{17}\right)$ is a L^2- admissible pair. By interpolation, we get

$$\|u(t)\|_{L^{12}} \leq \|u(t)\|^{\frac{5}{12}}_{L^{10}}\|u(t)\|^{\frac{7}{12}}_{L^{14}}.$$

So,

$$\|u(t)\|^4_{L^{12}} \leq \|u(t)\|^{\frac{5}{3}}_{L^{10}}\|u(t)\|^{\frac{7}{3}}_{L^{14}}.$$

Thereafter,

$$\begin{aligned}
\int_0^T \|u(t)\|_{L^{12}}^4\ dt &\le \int_0^T \|u(t)\|_{L^{10}}^{\frac{5}{3}} \|u(t)\|_{L^{14}}^{\frac{7}{3}}\ dt \\
&\le \Big(\int_0^T \|u(t)\|_{L^{10}}^{\frac{10}{3}}\ dt\Big)^{\frac{1}{2}} \Big(\int_0^T \|u(t)\|_{L^{14}}^{\frac{14}{3}}\ dt\Big)^{\frac{1}{2}} \\
&\le \|u\|_{L^{\frac{10}{3}}([0,T];L^{10}(\mathbb{R}^3))}^{\frac{5}{3}} \|u\|_{L^{\frac{14}{3}}([0,T];L^{14}(\mathbb{R}^3))}^{\frac{7}{3}} \\
&\le C\|u\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}^{\frac{5}{3}} \|u\|_{L^{7}([0,T];L^{14}(\mathbb{R}^3))}^{\frac{7}{3}} \\
&\le C.
\end{aligned}$$

Thus, $u \in L^4([0,T]; L^{12}(\mathbb{R}^3))$. Multiplying the first equation of system (4.94) by $\overline{u}$, integrating in x and taking its imaginary part, one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u(t)|^2\ dx &= Im\Big(\int_{\mathbb{R}^3} a(1-\Delta)^{-1} a\partial_t u \cdot \overline{u}\ dx\Big) \\
&\le \int_{\mathbb{R}^3} |a(1-\Delta)^{-1} a\partial_t u \cdot \overline{u}|\ dx.
\end{aligned}$$

Integrating from 0 to t,

$$\begin{aligned}
\frac{1}{2}\|u(t)\|_{L^2}^2 - \frac{1}{2}\|u(0)\|_{L^2}^2 &\le \int_0^t \int_{\mathbb{R}^3} |a(1-\Delta)^{-1} a\partial_t u \cdot \overline{u}|\ dxdt \\
&\le \int_0^t \|a(1-\Delta)^{-1} a\partial_t u\|_{L^2} \|u(t)\|_{L^2}\ dt.
\end{aligned}$$

On one hand,

$$\begin{aligned}
\|a(1-\Delta)^{-1} a\partial_t u\|_{L^2} &= \|a(1-\Delta)^{-1} a(-iJ^{-1}(1-\Delta)u - iJ^{-1}|u|^4 u)\|_{L^2} \\
&\le \|a(1-\Delta)^{-1} aJ^{-1}(1-\Delta)u\|_{L^2} + \|a(1-\Delta)^{-1} aJ^{-1}|u|^4 u\|_{L^2} \\
&\le C\|u(t)\|_{L^2} + \||u|^4 u\|_{H^{-1}}.
\end{aligned}$$

Observe that taking $V = |u|^4$, since $u \in L^4([0,T]; L^{12}(\mathbb{R}^3))$, we have $V \in L^1([0,T]; L^3(\mathbb{R}^3))$. Also, using Sobolev's embedding and Hölder's inequality,

$$\begin{aligned}
\||u|^4 u\|_{H^{-1}} &\le C\|Vu\|_{L^{\frac{6}{5}}} \\
&\le C\|V\|_{L^3} \|u\|_{L^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\|u(t)\|_{L^2}^2 &\leq \int_0^t \|a(1-\Delta)^{-1}a\partial_t u\|_{L^2}\|u(t)\|_{L^2}\ dt + \frac{1}{2}\|u(0)\|_{L^2}^2 \\
&\leq C\int_0^t \Big(\|u(t)\|_{L^2} + \||u|^4 u\|_{H^{-1}}\Big)\|u(t)\|_{L^2}\ dt + \frac{1}{2}\|u(0)\|_{L^2}^2 \\
&\leq C\int_0^t \Big(\|u(t)\|_{L^2} + \|V\|_{L^3}\|u(t)\|_{L^2}\Big)\|u(t)\|_{L^2}\ dt + \frac{1}{2}\|u(0)\|_{L^2}^2 \\
&\leq C\int_0^t \Big(1 + \|V\|_{L^3}\Big)\|u(t)\|_{L^2}^2\ dt + \frac{1}{2}\|u(0)\|_{L^2}^2 \\
&\leq \|u\|_{L^\infty([0,t];L^2)}^2 C\int_0^t (1 + \|V\|_{L^3})\ dt + \frac{1}{2}\|u(0)\|_{L^2}^2.
\end{aligned}$$

Consequently,

$$\|u\|_{L^\infty([0,t];L^2)}^2 \leq 2C(t + \|V\|_{L^1([0,t];L^3)})\|u\|_{L^\infty([0,t];L^2)}^2 + \|u(0)\|_{L^2}^2.$$

We can divide the interval $[0,T]$ into a finite number of intervals $[a_i, a_{i+1}]$, $i = 1, ..., N$, such that $2C(t + \|V\|_{L^1([a_i,a_{i+1}];L^3)}) < 1/4$. In each of these intervals, we have

$$\|u\|_{L^\infty([a_i,a_{i+1}];L^2)}^2 \leq C\|u(a_i)\|_{L^2}^2.$$

The desired result is obtained by iteration. The final constant C only depends on λ_0 and T. □

As a consequence of the previous result, we have the following corollary.

Corolary 4.4.1. *Let $T > 0$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that any solution u satisfying system* (4.94) *and*

$$\|u_0\|_{H^{-1}} \leq \delta$$

satisfies

$$\|u(T)\|_{H^{-1}} \leq \varepsilon.$$

Demonstração. By Lemma 4.4.1, we have

$$\|u(T)\|_{H^{-1}} \leq C\|u(T)\|_{L^2} \leq C\|u(0)\|_{L^2}.$$

However, by an interpolation argument between $H^s(\mathbb{R}^3)$ spaces, $s \in \mathbb{R}$, one has

$$\begin{aligned}
\|u(0)\|_{L^2} &\leq \|u(0)\|_{H^{-1}}^{\frac{1}{2}}\|u(0)\|_{H^1}^{\frac{1}{2}} \\
&\leq \lambda_0^{\frac{1}{2}}\|u(0)\|_{H^{-1}}^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$\|u(T)\|_{H^{-1}} \leq C\|u(0)\|_{L^2} \leq C\lambda_0^{\frac{1}{2}}\|u(0)\|_{H^{-1}}^{\frac{1}{2}}.$$

Taking $\delta = \frac{\varepsilon^2}{C^2\lambda_0}$, we conclude that

$$\|u(T)\|_{H^{-1}} \leq \varepsilon.$$

□

The next lemma ensures an approximation between sequences of solutions under some conditions.

Lemma 4.4.2. *Let $u_n, \tilde{u}_n$ be two sequences of solutions for*

$$\begin{cases} i\partial_t u_n + \Delta u_n - u_n - |u_n|^4 u_n = a(1-\Delta)^{-1} a\partial_t u_n \quad \textit{on } [0,T]\times\mathbb{R}^3, \\ u_n(0) = u_{0,n} \quad \textit{bounded in } H^1(\mathbb{R}^3), \ \textit{with } \|u_{0,n}\|_{H^1} < \lambda_0 \end{cases}$$

and

$$\begin{cases} i\partial_t \tilde{u}_n + \Delta \tilde{u}_n - \tilde{u}_n - |\tilde{u}_n|^4 \tilde{u}_n = 0 \quad \textit{on } [0,T]\times\mathbb{R}^3, \\ \tilde{u}_n(0) = \tilde{u}_{0,n} \quad \textit{bounded in } H^1(\mathbb{R}^3), \ \textit{with } \|\tilde{u}_{0,n}\|_{H^1} < \lambda_0, \end{cases}$$

respectively, with $\|u_{n,0} - \tilde{u}_{n,0}\|_{H^1} \to 0$ and $\|(1-\Delta)^{-\frac{1}{2}} a\partial_t u_n\|_{L^2([0,T];L^2(\mathbb{R}^3))} \longrightarrow 0$ as $n\to\infty$. Then,

$$\|u_n - \tilde{u}_n\|_{L^{10}([0,T]\times\mathbb{R}^3)} + \|\nabla(u_n-\tilde{u}_n)\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} + \sup_{t\in[0,T]} \|\nabla(u_n-\tilde{u}_n)\|_{L^2} + \sup_{t\in[0,T]} \|u_n - \tilde{u}_n\|_{L^2} \longrightarrow 0$$

as $n\to\infty$.

Demonstração. Let $r_n = u_n - \tilde{u}_n$. It satisfies the system

$$\begin{cases} i\partial_t r_n + \Delta r_n - r_n - |u_n|^4 u_n + |\tilde{u}_n|^4 \tilde{u}_n = a(1-\Delta)^{-1} a\partial_t u_n \quad \text{on } [0,T]\times\mathbb{R}^3, \\ r_n(0) = u_{0,n} - \tilde{u}_{0,n}. \end{cases}$$

Denote

$$|||.|||_{[0,T]} = \|.\|_{L^{10}([0,T]\times\mathbb{R}^3)} + \|\nabla .\|_{L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}} + \|\nabla .\|_{L_t^{10} L_x^{\frac{30}{13}}}.$$

Strichartz's estimates give us

$$\begin{aligned}|||r_n|||_{[0,T]} + \sup_{t\in[0,T]}\|\nabla r_n(t)\|_{L^2} + \sup_{t\in[0,T]}\|r_n(t)\|_{L^2} &\leq \|r_n(0)\|_{H^1} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}(u_n^5 - \tilde{u}_n^5)\ d\tau\right\|_{L_t^{10}L_x^{\frac{30}{13}}} \\ &\quad + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta} a(1-\Delta)^{-1}a\partial_t u_n\ d\tau\right\|_{L_t^{10}L_x^{\frac{30}{13}}} \\ &\leq \|r_n(0)\|_{H^1} + \|\nabla(u_n^5 - \tilde{u}_n^5)\|_{L_t^2L_x^{\frac{6}{5}}} \\ &\quad + \|a(1-\Delta)^{-1}a\partial_t u_n\|_{L_t^1H_x^1} \\ &\quad + \|u_n^5 - \tilde{u}_n^5\|_{L_t^1L_x^2}. \end{aligned} \tag{4.95}$$

Observe that

$$\begin{aligned}\|a(1-\Delta)^{-1}a\partial_t u_n\|_{L_t^1H_x^1} &\leq C\|a(1-\Delta)^{-1}a\partial_t u_n\|_{L_t^2H_x^1} \\ &= \|a(1-\Delta)^{-\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}}a\partial_t u_n\|_{L_t^2H_x^1} \\ &\leq C\|(1-\Delta)^{-\frac{1}{2}}a\partial_t u_n\|_{L_t^2L_x^2} \longrightarrow 0 \end{aligned} \tag{4.96}$$

as $n\to\infty$. On the other hand,

$$\begin{aligned}\|\nabla(u_n^5 - \tilde{u}_n^5)\|_{L_t^2L_x^{\frac{6}{5}}} &\leq \|u_n\|^4_{L_t^{10}L_x^{10}}\|\nabla u_n - \nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}} + \|u_n - \tilde{u}_n\|_{L_t^{10}L_x^{10}}\|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|u_n\|^3_{L_t^{10}L_x^{10}} + \\ &\quad + \|u_n - \tilde{u}_n\|_{L_t^{10}L_x^{10}}\|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\tilde{u}_n\|^3_{L_t^{10}L_x^{10}} \\ &\leq C\|\nabla u_n - \nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\nabla u_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} \\ &\quad + C\|\nabla u_n - \nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\left(\|\nabla\tilde{u}_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\nabla u_n\|^3_{L_t^{10}L_x^{\frac{30}{13}}}\right) \\ &\leq C\|\nabla u_n - \nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\left(\|\nabla\tilde{u}_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla u_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}}\right. \\ &\quad \left. + \|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\nabla u_n\|^3_{L_t^{10}L_x^{\frac{30}{13}}}\right) \\ &\leq C\|\nabla r_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\left(\|\nabla\tilde{u}_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla u_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\nabla u_n\|^3_{L_t^{10}L_x^{\frac{30}{13}}}\right)\end{aligned}$$

and

$$\begin{aligned}\|u_n^5 - \tilde{u}_n^5\|_{L_t^1L_x^2} &\leq \|u_n - \tilde{u}_n\|_{L_t^5L_x^{10}}\left(\|u_n\|^4_{L_t^5L_x^{10}} + \|\tilde{u}_n\|^4_{L_t^5L_x^{10}}\right) \\ &\leq C\|u_n - \tilde{u}_n\|_{L_t^{10}L_x^{10}}\left(\|u_n\|^4_{L_t^{10}L_x^{10}} + \|\tilde{u}_n\|^4_{L_t^{10}L_x^{10}}\right) \\ &\leq C\|\nabla r_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\left(\|\nabla\tilde{u}_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla u_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}}\right).\end{aligned}$$

So, dividing the interval $[0,T]$ in a finite number of intervals $I_{i,n} = [a_{i,n}, a_{i+1,n}]$, $1\leq i\leq N$ such that $C\left(\|\nabla u_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla\tilde{u}_n\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla\tilde{u}_n\|_{L_t^{10}L_x^{\frac{30}{13}}}\|\nabla u_n\|^3_{L_t^{10}L_x^{\frac{30}{13}}}\right) \leq \frac{1}{2}$, the terms of inequality (4.95) can be absorbed. We iterate this estimate N times, which gives the result. □

4.4.2 Profile decomposition of the limit energy

Let u be a solution of the nonlinear Schrödinger equation (4.97). Denote its nonlinear energy density by

$$e(t)(t,x) = \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6.$$

For a sequence u_n of solutions with bounded initial data in $\dot{H}^1(\mathbb{R}^3)$, the corresponding nonlinear energy density is bounded in $L^\infty([0,T],L^1)$ and so, by Sobolev embedding, in the space of bounded measures on $[0,T]\times\mathbb{R}^3$. This allows one to consider, up to a subsequence, its weak* limit. The following theorem proves that the energy limit can be decomposed into profiles as u_n. It will be a crucial argument that will allow the use of a microlocal defect measure on each profile and then apply a linearization argument.

Theorem 4.12. *Let u_n be a sequence of solutions to*

$$i\partial_t u_n + \Delta u_n - |u_n|^4 u_n = 0, \tag{4.97}$$

with $u_n(0)$ convergent to 0 in $L^2(\mathbb{R}^3)$. The nonlinear energy density limit of u_n (up to a subsequence) is

$$e(t,x) = \sum_{j=1}^{\infty} e^{(j)}(t,x) + e_f(t,x),$$

where $e^{(j)}$ is the limit energy limit density of $q_n^{(j)}$ (following the notation of Theorem 4.10) and

$$e_f = \lim_{l\to\infty}\lim_{n\to\infty} e(w_n^{(l)}),$$

where the two limits are considered up to a subsequence and in the weak sense. In particular, e_f can be written as*

$$e_f(t,x) = \int_{\xi\in S^2} \mu(t,x,d\xi).$$

Moreover, e is also the limit of the linear energy density

$$e_{lim}(u_n)(t,x) = \frac{1}{2}|\nabla u_n(t,x)|^2.$$

Demonstração. The proof of this result is a direct consequence of Theorem 4.10. Indeed, since $\|u_n\|_{L^{10}([0,T]\times\mathbb{R}^3)} \le C$, it follows, by an interpolation argument, that

$$\|u_n\|_{L^2([0,T]\times\mathbb{R}^3)} \to 0 \Longrightarrow \|u_n\|_{L^6([0,T]\times\mathbb{R}^3)} \to 0 \text{ as } n\to\infty.$$

Therefore, e is the limit of $b(u_n,u_n)$, with

$$b(f,g) = \nabla f(t,x)\cdot\overline{\nabla g(t,x)}.$$

Now, we have to compute the limit of $b(u_n, u_n)$ using the decomposition of Theorem 4.10. We set, for each $l \in \mathbb{N}$,

$$s_n^{(l)} = \sum_{j=1}^{l} q_n^{(j)}$$

and so

$$b(u_n, u_n) = b(s_n^{(l)}, s_n^{(l)}) + b(w_n^{(l)}, w_n^{(l)}) + 2b(s_n^{(l)}, w_n^{(l)}) + 2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}).$$

The convergence (4.55) gives

$$\limsup_{n\to\infty} \|2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)})\|_{L^1([0,T]\times\mathbb{R}^3)} \to 0$$

as $l \to \infty$. So, if we define $e_r^{(l)} = w * \lim_{n\to\infty}(2b(u_n, r_n^{(l)}) - b(r_n^{(l)}, r_n^{(l)}))$, the weak* limit, we have

$$e_r^{(l)} \longrightarrow 0 \ \text{ as } l \to \infty.$$

Let $\varphi(t,x) = \varphi_1(t)\cdot\varphi_2(x) \in C_0^\infty((0,T)\times\mathbb{R}^3)$. It remains to estimate

$$\int_0^T \int_{\mathbb{R}^3} \varphi b(s_n^{(l)}, w_n^{(l)}) = \sum_{j=1}^{l} \int_0^T \varphi_1 \int_{\mathbb{R}^3} \varphi_2 b(q_n^{(j)}, w_n^{(l)}),$$

for each fixed l. To this end, first note that, since $b(q_n^{(j)}, w_n^{(l)})$ is bounded in $L^\infty((0,T), L^1(\mathbb{R}^3))$, we can assume, up to an arbitrary small error, that φ_1 is supported in $\{t < t_\infty^{(j)}\}$ or $\{t > t_\infty^{(j)}\}$ (replace φ_1 by $(1-\Psi)(t)\varphi_1$ with $\Psi(t_\infty^{(j)}) = 1$ and $\|\Psi\|_{L^1(0,T)}$ small). On each interval, Theorem 4.11 allows to replace $q_n^{(j)}$ by a linear concentrating solution. Then, by Lemma 4.2.5, we get the weak convergence to zero of $b(s_n^{(l)}, w_n^{(l)})$, for each fixed l. Indeed, by Lemma 4.2.5, $D_{h_n}^{(j)} w_n^{(l)} \rightharpoonup 0$, $1 \le j \le l$, which means,

$$\sqrt{h_n^{(j)}}\, w_n^{(l)}(t_n^{(j)} + (h_n^{(j)})^2 s, x_n^{(j)} + h_n^{(j)} y) \rightharpoonup 0 \text{ in } \dot{H}^1(\mathbb{R}^3) \text{ as } n \to \infty.$$

It is enough to compute $\int_{\mathbb{R}^3} \nabla_x w_n^{(l)}(t,x)\cdot\nabla_x p_n^{(j)}(t,x)\,dx$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla_x w_n^{(l)}(t,x)\cdot\nabla_x p_n^{(j)}(t,x)\,dx \\
=\ &\int_{\mathbb{R}^3} \nabla_x w_n^{(l)}(t,x)\cdot\nabla_x \frac{1}{\sqrt{h_n^{(j)}}}\varphi^{(j)}\left(\frac{t-t_n^{(j)}}{(h_n^{(j)})^2}, \frac{x-x_n^{(j)}}{h_n^{(j)}}\right) dx \\
=\ &\int_{\mathbb{R}^3} \nabla_x w_n^{(l)}(t_n^{(j)} + (h_n^{(j)})^2 s, x_n^{(j)} + h_n^{(j)} y)\cdot\nabla_x \frac{1}{\sqrt{h_n^{(j)}}}\varphi^{(j)}(s,y)\ (h_n^{(j)})^3 dy \\
=\ &\int_{\mathbb{R}^3} \nabla_y \sqrt{h_n^{(j)}}\, w_n^{(l)}(t_n^{(j)} + (h_n^{(j)})^2 s, x_n^{(j)} + h_n^{(j)} y)\cdot\nabla_y\varphi^{(j)}(s,y)\ dy \longrightarrow 0
\end{aligned}$$

as $n \to \infty$. Lemma 4.2.2 and the orthogonality of the cores of concentration yields

$$D_{h_n}^{(j)} p_n^{(j')} \rightharpoonup 0,$$

for $j \neq j'$ and $p_n^{(j')}$ a concentrating solution at rate $[\underline{h}^{(j')}, \underline{x}^{(j')}, \underline{t}^{(j')}]$. Then, the same argument as before gives

$$b(s_n^{(l)}, s_n^{(l)}) \rightharpoonup^* \sum_{j=1}^{l} e^{(j)}.$$

So we have proved that, for any $l \in \mathbb{N}$,

$$b(u_n, u_n) \rightharpoonup^* e = \sum_{j=1}^{l} e^{(j)} + e_w^{(l)} + e_r^{(l)} \text{ as } n \to \infty,$$

where $e_w^{(l)}$ is the weak* limit of $b(w_n^{(l)}, w_n^{(l)})$ and $e_r^{(l)}$ satisfies $e_r^{(l)} \to 0$ as $l \to \infty$. Since $e_w^{(l)}$ is the weak* limit of a sequence of solutions of the linear Schrödinger equation with initial data convergent to zero in L^2, we can use Proposition .2 (Appendix) to conclude that $b(w_n^{(l)}, w_n^{(l)})$ converges (locally) to a positive measure e_f. Hence,

$$e = \sum_{j=1}^{\infty} e^{(j)} + e_f$$

and the result is proven. □

Remark 4.4.2. *The previous result only holds locally, since the Proposition .2 is valid only in compact sets. The reader will note later that this will be enough to achieve our purpose.*

4.5 CONCLUSION

In this chapter, we closely followed the work from in (KERAANI, 2001) and (LAURENT, 2011), combining the methods used in both. We followed the decompositions carried out by Keraani in (KERAANI, 2001), but with the significant difference that in this thesis we consider h_n as a scale of positive numbers converging to zero, in the spirit of Laurent in (LAURENT, 2011), since Keraani considers h_n to be a constant sequence equals to 1. This combination of techniques produces some new results such as, for example, Lemma 4.9, Theorem 4.11, and the results from the previous section.

5 STABILIZABILITY OF NONLINEAR PERTURBED SCHRÖDINGER EQUATION

This chapter is devoted to the proof of Theorem 2.2. To show the desired stability result, we need to obtain the following observability estimate

$$E(u)(t) \leq C \int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a\partial_t u|^2 \, dxdt. \tag{5.1}$$

The proof of Stabilizability consists in the analysis of possible sequences contradicting this observability estimate or, more precisely, a weak version of this estimate. In our case, specifically, to obtain (5.1) we need a weaker observability involving the initial data in a lower norm. Then, considering this small enough data, we obtain the desired strong observability. The first step is to prove that such sequences are linearizable in the sense that their behavior is close to the behavior of solutions of the linear equation.

5.1 LINEARIZATION

From now on, we consider $a \in C^\infty(\mathbb{R}^3)$ satisfying (2.3). So, denoting $\omega := \left(\mathbb{R}^3 \backslash B_R(0)\right)$, ω satisfies the following geometric control condition. There exists $T_0 > 0$ such that every geodesic travelling at speed 1 meets ω in a time $t < T_0$; Let us present now the following linearization lemma.

Lemma 5.1.1. *Let $T > T_0$ and u_n be a sequence of solutions to*

$$\begin{cases} i\partial_t u_n + \Delta u_n - u_n - |u_n|^4 u_n - a(1-\Delta)^{-1} a\partial_t u_n = 0, & \text{on } [0,T]\times\mathbb{R}^3, \\ u_n(0) = u_{0,n} & \text{in } H^1(\mathbb{R}^3) \end{cases} \tag{5.2}$$

satisfying

$$u_{0,n} \to 0 \text{ in } L^2(\mathbb{R}^3) \text{ as } n \to \infty \tag{5.3}$$

and

$$\int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a\partial_t u_n|^2 \, dxdt \longrightarrow 0 \text{ as } n \to \infty. \tag{5.4}$$

Consider the profile decomposition according to Theorem 4.10 of u_n in a subinterval $[t_0, t_0 + L] \subset [0,T]$ with $T_0 < L$. Then, for any $0 < \varepsilon < L - T_0$, this decomposition does not contain any nonlinear concentrating solution with $t_\infty^{(j)} \in [t_0, t_0+\varepsilon]$ and u_n is linearizable in $[t_0, t_0+\varepsilon]$, i.e.,

$$\|u_n - v_n\|_{L^{10}([t_0,t_0+\varepsilon]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([t_0,t_0+\varepsilon];H^1(\mathbb{R}^3))} \longrightarrow 0 \text{ as } n \to \infty,$$

where v_n is the solution of

$$\begin{cases} i\partial_t v_n + \Delta v_n - v_n = 0 & \text{on } [0,T]\times\mathbb{R}^3, \\ v_n(0) = u_{0,n} & \text{in } H^1(\mathbb{R}^3). \end{cases} \tag{5.5}$$

Demonstração. With no loss of generality, we consider the interval $[0, L]$ instead of $[t_0, t_0 + L]$ to keep the notation simple.

Claim 1: The sequence u_n is convergent to 0 in $L^2([0,T]\times\mathbb{R}^3)$.

Indeed, multiplying the first equation of (5.2) by $\overline{u}_n$ and taking its imaginary part, we obtain the estimate

$$\frac{1}{2}\|u_n(t)\|_{L^2} \leq \frac{1}{2}\|u_n(0)\|_{L^2} + \int_0^t \|a(1-\Delta)^{-1}a\partial_t u_n\|_{L^2}\|u_n\|_{L^2}\, ds.$$

Since $(1-\Delta)^{-\frac{1}{2}}a(x)\partial_t u_n$ tends to 0 in $L^2(\mathbb{R}^3)$ and $\|u_n(t)\|_{L^2}$ is bounded, by convergence (5.3), we obtain Claim 1.

Claim 2: The sequence u_n is convergent to 0 in $L^2_{loc}((0,L);\dot{H}^1_{loc}(\omega))$.

Since, by hypothesis,

$$\|(1-\Delta)^{-\frac{1}{2}}a\partial_t u_n\|_{L^2(([0,L];\mathbb{R}^3)} \to 0 \text{ as } \; n\to\infty$$

one has

$$\|(1-\Delta)^{-\frac{1}{2}}a(-iJ^{-1}(I-\Delta)u_n - iJ^{-1}|u_n|^4u_n)\|_{L^2([0,L];\mathbb{R}^3)} \to 0 \text{ as } n\to\infty.$$

Observe that,

$$\begin{aligned} & \|(1-\Delta)^{-\frac{1}{2}}aiJ^{-1}(I-\Delta)u_n\|_{L^2((0,L);\mathbb{R}^3)} \\ = \; & \|(1-\Delta)^{-\frac{1}{2}}a(iJ^{-1}(I-\Delta)u_n - iJ^{-1}|u_n|^4u_n + iJ^{-1}|u_n|^4u_n)\|_{L^2} \\ \leq \; & \|(1-\Delta)^{-\frac{1}{2}}a(iJ^{-1}(I-\Delta)u_n + iJ^{-1}|u_n|^4u_n)\|_{L^2([0,L];\mathbb{R}^3)} \\ & \quad + \|(1-\Delta)^{-\frac{1}{2}}aiJ^{-1}|u_n|^4u_n)\|_{L^2([0,L];\mathbb{R}^3)} \\ \leq \; & \|(1-\Delta)^{-\frac{1}{2}}a(iJ^{-1}(I-\Delta)u_n + iJ^{-1}|u_n|^4u_n)\|_{L^2([0,L];\mathbb{R}^3)} \\ & \quad + \|u_n^5\|_{L^2([0,L];H^{-1}(\mathbb{R}^3))} \to 0 \text{ as } n\to\infty, \end{aligned}$$

due to the convergence

$$\begin{aligned}\|u_n^5\|^2_{L^2((0,L);H^{-1}(\mathbb{R}^3))} &\leq c\|u_n^5\|^2_{L^2((0,L);L^{\frac{6}{5}}(\mathbb{R}^3))}\\ &\leq c\int_0^L \|u_n(t)\|^{10}_{L^6(\mathbb{R}^3)}\,dt\\ &\leq \int_0^L \|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n(t)\|^{\frac{25}{3}}_{L^{10}}\,dt\\ &\leq \sup_{t\in[0,L]}\|u_n(t)\|^{\frac{5}{3}}_{L^2}\int_0^L \|u_n(t)\|^{\frac{25}{3}}_{L^{10}}\,dt\\ &\leq \sup_{t\in[0,L]}\|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n\|^{\frac{3}{25}}_{L_t^{\frac{25}{3}}L_x^{10}}\\ &\leq \sup_{t\in[0,L]}\|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n\|^{\frac{3}{25}}_{L_t^{10}L_x^{10}}\\ &\to 0\end{aligned}$$

as $n\to\infty$ and using the interpolation

$$\|u_n(t)\|_{L^6}\leq \|u_n(t)\|^{\frac{1}{6}}_{L^2}\|u_n(t)\|^{\frac{5}{6}}_{L^{10}}.$$

Hence, for every $\chi\in C_0^\infty((0,L)\times\mathbb{R}^3)$, we have

$$\|(1-\Delta)^{-\frac{1}{2}}aJ^{-1}(I-\Delta)\chi u_n\|_{L^2(0,L)\times\mathbb{R}^3)}\to 0$$

as $n\to\infty$, which is equivalent to

$$\begin{aligned}&\left\langle(1-\Delta)^{-\frac{1}{2}}aJ^{-1}(I-\Delta)\chi u_n,(1-\Delta)^{-\frac{1}{2}}aJ^{-1}(I-\Delta)\chi u_n\right\rangle_{L^2((0,L)\times\mathbb{R}^3)}\to 0\\ \Rightarrow\;&\left\langle(1-\Delta)^{-1}aJ^{-1}(I-\Delta)\chi u_n,aJ^{-1}(I-\Delta)\chi u_n\right\rangle_{L^2((0,L)\times\mathbb{R}^3)}\to 0\\ \Rightarrow\;&\left\langle a(1-\Delta)^{-1}aJ^{-1}(I-\Delta)\chi u_n,J^{-1}(I-\Delta)\chi u_n\right\rangle_{L^2((0,L)\times\mathbb{R}^3)}\to 0\\ \Rightarrow\;&\left\langle(J^{-1})^*a(1-\Delta)^{-1}aJ^{-1}(I-\Delta)\chi u_n,(I-\Delta)\chi u_n\right\rangle_{L^2((0,L)\times\mathbb{R}^3)}\to 0\\ \Rightarrow\;&\left\langle(I-\Delta)(J^{-1})^*a(1-\Delta)^{-1}aJ^{-1}(I-\Delta)\chi u_n,\chi u_n\right\rangle_{L^2((0,L)\times\mathbb{R}^3)}\to 0\end{aligned}$$

as $n\to\infty$. This means, using Proposition .2 (Appendix), that

$$\int_{(0,L)\times\mathbb{R}^3\times S^3}\frac{(1+|\xi|^2)a^2}{1+|\xi|^2}(1+|\xi|^2)\,d\mu(t,x,\xi)=0.$$

Thus

$$\int_{(0,L)\times\omega\times S^3}1+|\xi|^2\,d\mu(t,x,\xi)=0,$$

i.e.,

$$u_n\longrightarrow 0 \text{ in } L^2_{loc}((0,L);\dot{H}^1_{loc}(\omega)) \text{ as } n\to\infty,$$

showing Claim 2.

Now, let $\widetilde{u}_n$ be a solution to

$$\begin{cases} i\partial_t \widetilde{u}_n + \Delta \widetilde{u}_n - \widetilde{u}_n - |\widetilde{u}_n|^4 \widetilde{u}_n = 0 \text{ on } [0,T] \times \mathbb{R}^3, \\ \widetilde{u}_n(0) = u_{0,n} \in H^1(\mathbb{R}^3). \end{cases}$$

By the convergence (5.4) and Lemma 4.4.2 we get

$$\widetilde{u}_n \longrightarrow 0 \text{ in } L^2_{loc}((0,L); \dot{H}^1_{loc}(\omega)) \text{ as } n \to \infty.$$

Let $w_n = e^{it}\widetilde{u}_n$. It satisfies

$$\begin{cases} i\partial_t w_n + \Delta w_n - |w_n|^4 w_n = 0 \text{ on } [0,T] \times \mathbb{R}^3, \\ w_n(0) = u_{0,n} \end{cases}$$

and

$$w_n \longrightarrow 0 \text{ in } L^2_{loc}((0,L); \dot{H}^1_{loc}(\omega)) \text{ as } n \to \infty$$

and, consequently,

$$|\nabla w_n(t)|^2 \longrightarrow 0 \text{ in } L^1 L^1 \text{ as } n \to \infty.$$

Using the notation of Theorem 4.12, this gives $e = 0$ in $(0,L) \times \omega$ (locally). Since all the measures in the decomposition of e are positive, we get the same result for any nonlinear concentrating solution in the decomposition of w_n, that is, $e_j = 0$ in $(0,L) \times \omega$ (locally), and

$$|\nabla q_n^{(j)}|^2 \rightharpoonup 0 \text{ in } L^1_{loc}((0,L) \times \omega) \text{ as } n \to \infty$$

which give us

$$\int_0^L \int_\omega \varphi |\nabla q_n^{(j)}|^2 \longrightarrow 0 \text{ as } n \to \infty,$$

for all $\varphi \in C_0^\infty$. Therefore,

$$q_n^{(j)} \longrightarrow 0 \text{ in } L^2_{loc}((0,L); \dot{H}^1_{loc}(\omega)) \text{ as } n \to \infty$$

and if $\mu^{(j)}$ is the microlocal defect measure of $q_n^{(j)}$, we have

$$\mu^{(j)} \equiv 0 \text{ in } (0,L) \times \omega \times S^3. \tag{5.6}$$

Assume that $t_\infty^{(j)} \in [0,\varepsilon]$ for some $j \in \mathbb{N}$, so that the interval $(t_\infty^{(j)}, L]$ has lenght greater than T_0. Denote by $p_n^{(j)}$ the linear concentrating solution approaching $q_n^{(j)}$ in the interval

$I_n^{3,\Lambda} = (t_n^{(j)} + \Lambda(h_n^{(j)})^2, L]$, according to the notation of Theorem 4.11, so that, for any $t_\infty^{(j)} < t < L$, we have

$$\|q_n^{(j)} - p_n^{(j)}\|_{L^{10}([t,L]\times\mathbb{R}^3)} + \|q_n^{(j)} - p_n^{(j)}\|_{L^\infty([t,L];\dot{H}^1(\mathbb{R}^3))} \longrightarrow 0 \text{ as } n \to \infty.$$

In particular, $\mu^{(j)}$ is also attached to $p_n^{(j)}$ on the time interval $(t_\infty^{(j)}, L]$.

Claim 3: $p_n^{(j)}$ is bounded in $\dot{H}^1(\mathbb{R}^3)$ and $\|p_n^{(j)}(t)\|_{L^2} \to 0$ as $n \to \infty$.

In fact, remember that $p_n^{(j)}$ is a solution of the linear Schrödinger equation. If $p_n^{(j)}$ is a linear concentrating solution, we may consider

$$p_n^{(j)}(t,x) = \frac{1}{\sqrt{h_n^{(j)}}}\varphi^{(j)}\Big(\frac{t-t_n^{(j)}}{(h_n^{(j)})^2}, \frac{x-x_n^{(j)}}{h_n^{(j)}}\Big),$$

and so, with the change of variables $\frac{x-x_n}{h_n} = y$,

$$\begin{aligned}\|p_n^{(j)}(t)\|_{L^2} &= \frac{1}{\sqrt{h_n^{(j)}}}\Big(\int_{\mathbb{R}^3} |\varphi^{(j)}(s,y)|^2 (h_n^{(j)})^3\, dy\Big)^{\frac{1}{2}} \\ &= h_n^{(j)}\|\varphi^{(j)}(s)\|_{L^2} \\ &\leq Ch_n^{(j)}\|\varphi^{(j)}(s)\|_{L^6} \to 0\end{aligned}$$

as $n \to \infty$, since we can consider $\varphi^{(j)}(s) \in C_0^\infty(\mathbb{R}^3)$. Thus, $p_n^{(j)}$'s measure propagates along the geodesics of $\mathbb{R}^3$ and we have

$$\mu^{(j)} \equiv 0 \text{ in } (t_\infty^{(j)}, L) \times \mathbb{R}^3 \times S^3,$$

since $|L - t_\infty^{(j)}| > T_0$ ensure that the geometric control condition is still verified in the interval $[t_\infty^{(j)}, L]$ when combined with (5.6). This means that

$$p_n^{(j)} \to 0 \text{ in } L^2_{loc}((t_\infty^{(j)}, L); H^1_{loc}(\mathbb{R}^3)) \text{ as } n \to \infty,$$

showing Claim 3.

Finally, solving the equation satisfied by $p_n^{(j)}$ with initial data $p_n^{(j)}(t_0)$, where $t_0 \in (t_\infty^{(j)}, L)$ is such that $\|p_n^{(j)}(t_0)\|_{H^1} \to 0$ as $n \to \infty$, one has the strong convergence $p_n^{(j)} \to 0$ in the space $L^\infty([t_\infty^{(j)}, L], H^1_{loc}(\mathbb{R}^3))$ as $n \to \infty$.

In particular, $p_n^{(j)}(t_\infty^{(j)}) \to 0$ in $\dot{H}^1_{loc}(\mathbb{R}^3)$ as $n \to \infty$, so the measure $\mu^{(j,\infty)}$ associated to $p_n^{(j)}(t_\infty^{(j)})$ satisfies $\mu^{(j,\infty)} \equiv 0$ in $\mathbb{R}^3 \times S^2$. On the other hand, since $p_n^{(j)}(t_\infty^{(j)}) = \frac{1}{\sqrt{h_n}}\varphi^{(j)}\Big(\frac{x-x_\infty^{(j)}}{h_n}\Big)$, we can calculate $\mu^{(j,\infty)}$ directly. To this end, note that

$$\begin{aligned}
& \langle A(x,D_x)\nabla p_n^{(j)}(t_\infty^{(j)}), \nabla p_n^{(j)}(t_\infty^{(j)})\rangle_{L^2} \\
= & \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a(x,\xi)e^{i(x-y)\xi}|\xi|^2 p_n^{(j)}(t_\infty^{(j)})(y)\overline{p_n^{(j)}(t_\infty^{(j)})(x)}\ dydxd\xi \\
= & \frac{1}{(2\pi)^3}\frac{1}{h_n}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a(x,\xi)e^{i(x-y)\xi}|\xi|^2\varphi^{(j)}\Big(\frac{y-x_\infty^{(j)}}{h_n}\Big)\overline{\varphi^{(j)}\Big(\frac{y-x_\infty^{(j)}}{h_n}\Big)}\ dydxd\xi \\
= & \frac{h_n^5}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a(h_n\tilde{x}+x_\infty^{(j)},\xi)e^{ih_n(\tilde{x}-\tilde{y})\xi}|\xi|^2\varphi^{(j)}(\tilde{y})\overline{\varphi^{(j)}(\tilde{x})}\ d\tilde{y}d\tilde{x}d\xi \\
= & \frac{h_n^2}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a(h_n\tilde{x}+x_\infty^{(j)},\frac{\tilde{\xi}}{h_n})e^{i(\tilde{x}-\tilde{y})\tilde{\xi}}\Big|\frac{\tilde{\xi}}{h_n}\Big|^2\varphi^{(j)}(\tilde{y})\overline{\varphi^{(j)}(\tilde{x})}\ d\tilde{y}d\tilde{x}d\tilde{\xi} \\
= & \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} a(h_n\tilde{x}+x_\infty^{(j)},\tilde{\xi})e^{i(\tilde{x}-\tilde{y})\tilde{\xi}}|\tilde{\xi}|^2\varphi^{(j)}(\tilde{y})\overline{\varphi^{(j)}(\tilde{x})}\ d\tilde{y}d\tilde{x}d\tilde{\xi} \\
= & \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} a(h_n\tilde{x}+x_\infty^{(j)},\tilde{\xi})|\tilde{\xi}|^2|\widehat{\varphi^{(j)}}(\tilde{\xi})|^2\ d\tilde{\xi} \to \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} a(x_\infty^{(j)},\tilde{\xi})|\tilde{\xi}|^2|\widehat{\varphi^{(j)}}(\tilde{\xi})|^2\ d\tilde{\xi}
\end{aligned}$$

as $n\to\infty$. Using polar coordinates, we get

$$\mu^{(j,\infty)} = \delta_{x-x_\infty^{(j)}} \otimes \Phi(\theta)\ d\theta,$$

where $\Phi(\theta) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}|r\theta|^2|\widehat{\varphi^{(j)}}(r\theta)|^2 r^2\ dr$. Therefore, $p_n^{(j)}(t_\infty^{(j)}) \equiv 0$, and the conservation of the energy yields

$$\|p_n^{(j)}(t)\|_{\dot{H}^1(\mathbb{R}^3)} = \|p_n^{(j)}(t_\infty^{(j)})\|_{\dot{H}^1(\mathbb{R}^3)} = 0,$$

for all $t\in(t_\infty^{(j)},L]$. Moreover,

$$\|q_n^{(j)}(t)\|_{\dot{H}^1(\mathbb{R}^3)} \to 0 \text{ as } n\to\infty,$$

for all $t\in(t_\infty^{(j)},L]$. Arguing in the same way as before, one obtains $q_n^{(j)}\equiv 0$ in $(t_\infty^{(j)},L]$ as expected, since $q_n^{(j)}(t_\infty^{(j)}) = \frac{1}{\sqrt{h_n}}\psi^{(j)}\Big(\frac{x-x_\infty^{(j)}}{h_n}\Big)$. Then, for the profile decomposition of w_n in the interval $[0,L]$, namely,

$$w_n = \sum_{j=1}^{l} q_n^{(j)} + w_n^{(l)} + r_n^{(l)},$$

we have proved that $t_n^{(j)}\in(\varepsilon,L]$, since assuming $t_n^{(j)}\in[0,\varepsilon]$ implies $q_n^{(j)}\equiv 0$. Thus, Theorem 4.11 provides a linear concentrating solution $p_n^{(j)}$ such that

$$\limsup_{n\to\infty}\Big(\|q_n^{(j)}-p_n^{(j)}\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} + \|q_n^{(j)}-p_n^{(j)}\|_{L^\infty([0,\varepsilon];\dot{H}^1(\mathbb{R}^3))}\Big) = 0$$

while Lemma 4.9 gives

$$\limsup_{n\to\infty}\|p_n^{(j)}\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} = 0.$$

Moreover, Theorems 4.8 and 4.10 ensure

$$\limsup_{n\to\infty} \|w_n^{(l)} + r_n^{(l)}\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } l \to \infty.$$

Therefore,

$$\limsup_{n\to\infty} \|w_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} = 0$$

and, hence,

$$\limsup_{n\to\infty} \|\widetilde{u}_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} = 0.$$

Thus,

$$\|\nabla|\widetilde{u}_n|^4\widetilde{u}_n\|_{L^{\frac{10}{7}}([0,\varepsilon];L^{\frac{10}{7}}(\mathbb{R}^3))} \longrightarrow 0 \text{ as } n \to \infty.$$

Since

$$\|\nabla|\widetilde{u}_n|^4\widetilde{u}_n\|_{L^{\frac{10}{7}}([0,\varepsilon];L^{\frac{10}{7}}(\mathbb{R}^3))} \leq \|\widetilde{u}_n\|^4_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)}\|\nabla\widetilde{u}_n\|_{L^{\frac{10}{3}}([0,\varepsilon];L^{\frac{10}{3}}(\mathbb{R}^3))}, \tag{5.7}$$

we have that $\widetilde{u}_n$ is linearizable on $[0,\varepsilon]$. Indeed, using Remark 4.4.1 and Remark 3.3.2 (or Proposition 4.4.1), note that

$$\begin{aligned} \|\widetilde{u}_n - v_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} + \|\widetilde{u}_n - v_n\|_{L^\infty([0,\varepsilon];H^1(\mathbb{R}^3))} &\leq \|\nabla|\widetilde{u}_n|^4\widetilde{u}_n\|_{L^{\frac{10}{7}}([0,\varepsilon];L^{\frac{10}{7}}(\mathbb{R}^3))} + \|\widetilde{u}_n^5\|_{L^1([0,\varepsilon];L^2(\mathbb{R}^3))} \\ &\leq C\|\widetilde{u}_n\|^4_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)}\|\nabla\widetilde{u}_n\|_{L^{\frac{10}{3}}([0,\varepsilon];L^{\frac{10}{3}}(\mathbb{R}^3))} \\ &\quad + C\|\widetilde{u}_n\|^5_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} \\ &\to 0, \end{aligned} \tag{5.8}$$

as $n \to \infty$, where v_n is a sequence of solution to system (5.5). It follows that,

$$\begin{aligned} \|u_n - v_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,\varepsilon];H^1(\mathbb{R}^3))} &\leq \|u_n - \widetilde{u}_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} + \|u_n - \widetilde{u}_n\|_{L^\infty([0,\varepsilon];H^1(\mathbb{R}^3))} \\ &\quad + \|\widetilde{u}_n - v_n\|_{L^{10}([0,\varepsilon]\times\mathbb{R}^3)} \\ &\quad + \|\widetilde{u}_n - v_n\|_{L^\infty([0,\varepsilon];H^1(\mathbb{R}^3))} \\ &\to 0, \end{aligned}$$

as $n \to \infty$, due to (5.8), (5.4) and Lemma 4.4.2. □

The next proposition assures that a sequence of solutions of the nonlinear system is close to the solutions of the linear system.

Proposition 5.1.1. *Under the assumptions of Lemma 5.1.1, we have that u_n is linearizable on $[0,t]$, for any $t < T - T_0$, that is*

$$\|u_n - v_n\|_{L^{10}([0,t]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,t];H^1(\mathbb{R}^3))} \longrightarrow 0 \text{ as } n \to \infty,$$

where v_n *is the solution of*

$$\begin{cases} i\partial_t v_n + \Delta v_n - v_n = 0 & \textit{on } [0,T]\times\mathbb{R}^3, \\ v_n(0) = u_{0,n} & \textit{in } H^1(\mathbb{R}^3). \end{cases} \tag{5.9}$$

Demonstração. Let

$$t_* = \sup\{s\in[0,T]; \lim_n \|u_n - v_n\|_{L^{10}([0,s]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,s];H^1(\mathbb{R}^3))} = 0\}.$$

We claim that $t_* \geq T - T_0$. Indeed, suppose, by contradiction, that this does not hold, so we can find an interval $[t_*-\varepsilon, t_*-\varepsilon+L]\subset[0,T]$ with $T_0 < L$ and $0 < 2\varepsilon < L-T_0$ (if $t_* = 0$, take the interval $[0,L]\subset[0,T]$). It follows from Lemma 5.1.1 that u_n is linearizable in $[t_*-\varepsilon, t_*+\varepsilon]$. The definition of t_* gives $\lim_n \|u_n - v_n\|_{L^{10}([0,t_*-\varepsilon]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,t_*-\varepsilon];H^1(\mathbb{R}^3))} = 0$ and we have proved that $\lim_n \|u_n - \tilde{v}_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} + \|u_n - \tilde{v}_n\|_{L^\infty([t_*-\varepsilon,t_*+\varepsilon];H^1(\mathbb{R}^3))} = 0$ where $\tilde{v}_n$ is a solution of

$$i\partial_t\tilde{v}_n + \Delta\tilde{v}_n - \tilde{v}_n = 0, \quad \tilde{v}_n(t_*-\varepsilon) = u_n(t_*-\varepsilon).$$

This yields $\lim_n \|u_n - v_n\|_{L^{10}([0,t_*+\varepsilon]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,t_*+\varepsilon];H^1(\mathbb{R}^3))} = 0$. Indeed, we have

$$\sup_{t\in[0,t_*+\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} \leq \sup_{t\in[0,t_*-\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} + \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)},$$

where the first term on the right-hand side converges to 0 as n tends to ∞. For the second term, we have

$$\begin{aligned}
& \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} \\
\leq & \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - \tilde{v}_n(t)\|_{H^1(\mathbb{R}^3)} + \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|\tilde{v}_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} \\
\leq & \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - \tilde{v}_n(t)\|_{H^1(\mathbb{R}^3)} + \|\tilde{v}_n(t_*-\varepsilon) - v_n(t_*-\varepsilon)\|_{H^1(\mathbb{R}^3)} \\
\leq & \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - \tilde{v}_n(t)\|_{H^1(\mathbb{R}^3)} + \|u_n(t_*-\varepsilon) - v_n(t_*-\varepsilon)\|_{H^1(\mathbb{R}^3)} \\
\leq & \sup_{t\in[t_*-\varepsilon,t_*+\varepsilon]} \|u_n(t) - \tilde{v}_n(t)\|_{H^1(\mathbb{R}^3)} + \sup_{t\in[0,t_*-\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} \to 0
\end{aligned}$$

as $n\to\infty$. Now, we estimate the L^{10} norm as

$$\begin{aligned}
\|u_n - v_n\|^{10}_{L^{10}([0,t_*+\varepsilon]\times\mathbb{R}^3)} &= \int_0^{t_*+\varepsilon} \|u_n - v_n\|^{10}_{L^{10}(\mathbb{R}^3)}\,dt \\
&\leq \int_0^{t_*-\varepsilon} \|u_n - v_n\|^{10}_{L^{10}(\mathbb{R}^3)}\,dt + \int_{t_*-\varepsilon}^{t_*+\varepsilon} \|u_n - v_n\|^{10}_{L^{10}(\mathbb{R}^3)}\,dt,
\end{aligned}$$

where the first term on the right-hand side converges to 0 as n tends to ∞. For the second term, we have

$$\begin{aligned}
\|u_n - v_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} &\leq \|u_n - \widetilde{v}_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} + \|\widetilde{v}_n - v_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} \\
&\leq \|u_n - \widetilde{v}_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} + \|\widetilde{v}_n(t_*-\varepsilon) - v_n(t_*-\varepsilon)\|_{H^1(\mathbb{R}^3)} \\
&\leq \|u_n - \widetilde{v}_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} + \|u_n(t_*-\varepsilon) - v_n(t_*-\varepsilon)\|_{H^1(\mathbb{R}^3)} \\
&\leq \|u_n - \widetilde{v}_n\|_{L^{10}([t_*-\varepsilon,t_*+\varepsilon]\times\mathbb{R}^3)} + \sup_{t\in[0,t_*-\varepsilon]} \|u_n(t) - v_n(t)\|_{H^1(\mathbb{R}^3)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, using Strichartz estimates, which contradicts the definition of t_*. □

5.2 WEAK OBSERVABILITY ESTIMATE

In order to show that the observability estimate (5.1) holds in some sense, we need the following weak observability estimate.

Theorem 5.2.1. *Let $T > T_0$ and $\lambda_0 > 0$ from Definition 4.1. There exists $C > 0$ such that any solution u of the system*

$$\begin{cases} i\partial_t u + \Delta u - u - |u|^4 u - a(1-\Delta)^{-1} a\partial_t u = 0, \quad \textit{on } [0,T]\times\mathbb{R}^3, \\ u(0) = u_0 \in H^1(\mathbb{R}^3), \\ \|u_0\|_{H^1} \leq \lambda_0, \end{cases} \tag{5.10}$$

satisfies

$$E(u)(0) \leq C\Big(\int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a\partial_t u|^2 \, dxdt + \|u_0\|_{H^{-1}(\mathbb{R}^3)} E(u)(0)\Big). \tag{5.11}$$

Demonstração. Remember that

$$E(u)(t) = \frac{1}{2}\|u(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{6}\|u(t)\|_{L^6}^6.$$

We argue by contradiction. Suppose that (5.11) does not hold. So, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to system (5.10) does not satisfying inequality (5.11), that is,

$$\Big(\int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a\partial_t u_n|^2 \, dxdt + \|u_{0,n}\|_{H^{-1}(\mathbb{R}^3)} E(u_n)(0)\Big) \leq \frac{1}{n} E(u_n)(0). \tag{5.12}$$

Let $\alpha_n = \Big(E(u_n)(0)\Big)^{\frac{1}{2}}$. Sobolev's embedding for the L^6 norm ensures that $\alpha_n \leq C(\lambda_0)$. So, up to a subsequence, we may assume that $\alpha_n \to \alpha \geq 0$ as $n \to \infty$. We divide the analysis into two possible cases $\alpha > 0$ and $\alpha = 0$.

- *Case 1:* $\alpha_n \to \alpha > 0$

Note that $\|u_{0,n}\|_{H^{-1}(\mathbb{R}^3)} \to 0$ as $n \to \infty$, by the second part of estimate (5.12). Hence, using the inequality

$$\|u_{0,n}\|_{L^2(\mathbb{R}^3)} \leq \|u_{0,n}\|^{\frac{1}{2}}_{H^{-1}(\mathbb{R}^3)} \|u_{0,n}\|^{\frac{1}{2}}_{H^1(\mathbb{R}^3)},$$

one obtains $\|u_{0,n}\|_{L^2(\mathbb{R}^3)} \to 0$ as $n \to \infty$. Therefore, taking into account the first part of the estimate (5.12) as well, we are in a position to apply Proposition 5.1.1 and conclude that u_n is linearizable in an interval $[0, L]$ with $L > T_0$, i.e.,

$$\|u_n - v_n\|_{L^{10}([0,L]\times\mathbb{R}^3)} + \|u_n - v_n\|_{L^\infty([0,L];H^1(\mathbb{R}^3))} \to 0 \text{ as } n \to \infty,$$

where v_n is a solution to

$$\begin{cases} i\partial_t v_n + \Delta v_n - v_n = 0 \text{ on } [0,T] \times \mathbb{R}^3, \\ v_n(0) = u_{0,n}. \end{cases}$$

Since $u_{0,n} \to 0$ in $L^2(\mathbb{R}^3)$, we get $\|u_n(t)\|_{L^2} \to 0$, $\forall t \in [0,T]$. Hence, $\|v_n(t)\|_{L^2} \to 0$ as $n \to \infty$, $\forall t \in [0, L]$.

Claim 1: The sequence u_n converges to zero in $L^2_{loc}\Big((0,L); H^1_{loc}(\omega)\Big)$.

Indeed, note that

$$\partial_t u_n = -iJ^{-1}\Big((1-\Delta)u_n + |u_n|^4 u_n\Big) = -iJ^{-1}(I-\Delta)u_n - iJ^{-1}(|u_n|^4 u_n),$$

where J given as in the proof of Theorem 3.3.2. By hypothesis,

$$\|(1-\Delta)^{-\frac{1}{2}} a\partial_t u_n\|_{L^2((0,L);\mathbb{R}^3)} \to 0 \text{ as } n \to \infty,$$

which means that

$$\|(1-\Delta)^{-\frac{1}{2}} a(-iJ^{-1}(I-\Delta)u_n - iJ^{-1}(|u_n|^4 u_n))\|_{L^2((0,L);\mathbb{R}^3)} \to 0 \text{ as } n \to \infty.$$

So, similarly to Claim 2 in the proof of Lemma 5.1.1, we have

$$\begin{aligned} & \|(1-\Delta)^{-\frac{1}{2}} aJ^{-1}(I-\Delta)u_n\|_{L^2((0,L);\mathbb{R}^3)} \\ = \; & \|(1-\Delta)^{-\frac{1}{2}} a(J^{-1}(I-\Delta)u_n - J^{-1}|u_n|^4 u_n + J^{-1}|u_n|^4 u_n)\|_{L^2((0,L);\mathbb{R}^3)} \\ \leq \; & \|(1-\Delta)^{-\frac{1}{2}} a(J^{-1}(I-\Delta)u_n + J^{-1}|u_n|^4 u_n)\|_{L^2((0,L);\mathbb{R}^3)} \\ & + \|(1-\Delta)^{-\frac{1}{2}} aJ^{-1}|u_n|^4 u_n)\|_{L^2((0,L);\mathbb{R}^3)} \\ \leq \; & \|(1-\Delta)^{-\frac{1}{2}} a(J^{-1}(I-\Delta)u_n + J^{-1}|u_n|^4 u_n)\|_{L^2((0,L);\mathbb{R}^3)} \\ & + \|u_n^5\|_{L^2((0,L);H^{-1}(\mathbb{R}^3))} \to 0 \end{aligned}$$

as $n \to \infty$, due to the convergence

$$\begin{aligned}
\|u_n^5\|_{L^2((0,L);H^{-1}(\mathbb{R}^3))} &\leq c\|u_n^5\|_{L^2((0,L);L^{\frac{6}{5}}(\mathbb{R}^3))} \\
&\leq c\int_0^L \|u_n(t)\|_{L^6(\mathbb{R}^3)}^{10}\, dt \\
&\leq \int_0^L \|u_n(t)\|_{L^2}^{\frac{5}{3}}\|u_n(t)\|_{L^{10}}^{\frac{25}{3}}\, dt \\
&\leq \sup_{t\in[0,L]} \|u_n(t)\|_{L^2}^{\frac{5}{3}} \int_0^L \|u_n(t)\|_{L^{10}}^{\frac{25}{3}}\, dt \\
&\leq \sup_{t\in[0,L]} \|u_n(t)\|_{L^2}^{\frac{5}{3}} \|u_n\|_{L_t^{\frac{25}{3}} L_x^{10}}^{\frac{3}{25}} \\
&\leq \sup_{t\in[0,L]} \|u_n(t)\|_{L^2}^{\frac{5}{3}} \|u_n\|_{L_t^{10} L_x^{10}}^{\frac{3}{25}} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

where we used the interpolation

$$\|u_n(t)\|_{L^6} \leq \|u_n(t)\|_{L^2}^{\frac{1}{6}}\|u_n(t)\|_{L^{10}}^{\frac{5}{6}}.$$

Hence, for every $\chi \in C_0^\infty((0,L)\times\mathbb{R}^3)$,

$$\|(1-\Delta)^{-\frac{1}{2}} aJ^{-1}(I-\Delta)\chi u_n\|_{L^2((0,L)\times\mathbb{R}^3)} \to 0$$

as $n \to \infty$, which means that

$$\begin{aligned}
&\left\langle (1-\Delta)^{-\frac{1}{2}} aJ^{-1}(I-\Delta)\chi u_n, (1-\Delta)^{-\frac{1}{2}} aJ^{-1}(I-\Delta)\chi u_n \right\rangle_{L^2((0,L)\times\mathbb{R}^3)} \to 0 \\
\Rightarrow &\left\langle (1-\Delta)^{-1} aJ^{-1}(I-\Delta)\chi u_n, aJ^{-1}(I-\Delta)\chi u_n \right\rangle_{L^2((0,L)\times\mathbb{R}^3)} \to 0 \\
\Rightarrow &\left\langle a(1-\Delta)^{-1} aJ^{-1}(I-\Delta)\chi u_n, J^{-1}(I-\Delta)\chi u_n \right\rangle_{L^2((0,L)\times\mathbb{R}^3)} \to 0 \\
\Rightarrow &\left\langle (J^{-1})^* a(1-\Delta)^{-1} aJ^{-1}(I-\Delta)\chi u_n, (I-\Delta)\chi u_n \right\rangle_{L^2((0,L)\times\mathbb{R}^3)} \to 0 \\
\Rightarrow &\left\langle (I-\Delta)(J^{-1})^* a(1-\Delta)^{-1} aJ^{-1}(I-\Delta)\chi u_n, \chi u_n \right\rangle_{L^2((0,L)\times\mathbb{R}^3)} \to 0
\end{aligned}$$

as $n \to \infty$. By Proposition .2 (Appendix), we get

$$\int_{(0,L)\times\mathbb{R}^3\times S^3} \frac{(1+|\xi|^2)a^2}{1+|\xi|^2}(1+|\xi|^2)\, d\mu(t,x,\xi) = 0,$$

or equivalently,

$$\int_{(0,L)\times\omega\times S^3} 1+|\xi|^2\, d\mu(t,x,\xi) = 0,$$

i.e.,

$$u_n \longrightarrow 0 \text{ in } L^2_{loc}((0,L);H^1_{loc}(\omega)) \text{ as } n \to \infty,$$

concluding the proof of Claim 1.

As a consequence of the previous statement, one has

$$v_n \longrightarrow 0 \text{ in } L^2_{loc}((0,L);H^1_{loc}(\omega)) \text{ as } n\to\infty.$$

Summarizing everything we know about the sequence v_n, we have

1. v_n is bounded in $L^\infty([0,L];H^1(\mathbb{R}^3))$;
2. v_n satisfies $i\partial_t v_n + \Delta v_n - v_n = 0$;
3. $\sup_{t\in[0,L]} \|v_n(t)\|_{L^2} \to 0$ as $n\to\infty$;
4. $v_n \to 0$ in $L^2_{loc}\Big((0,L);H^1_{loc}(\mathbb{R}^3\backslash B_{R+1}(0))\Big)$ as $n\to\infty$.

Therefore, we are able to use Corollary .3 (Appendix), to get

$$v_n \to 0 \text{ in } L^2_{loc}((0,L);H^1_{loc}(\mathbb{R}^3)) \text{ as } n\to\infty. \tag{5.13}$$

For the second part, note that

$$\|v_n\|_{L^2([0,T];H^1(\mathbb{R}^3\backslash B_{R+1}(0)))} \leq \|v_n-u_n\|_{L^2([0,T];H^1(\mathbb{R}^3\backslash B_{R+1}(0)))} + \|u_n\|_{L^2([0,T];H^1(\mathbb{R}^3\backslash B_{R+1}(0)))}.$$

On the other hand,

$$\begin{aligned}
&\|u_n\|_{L^2([0,T];H^1(\mathbb{R}^3\backslash B_{R+1}(0)))}\\
=\ &\|au_n\|_{L^2([0,T];H^1(\mathbb{R}^3\backslash B_{R+1}(0)))}\\
\leq\ &\|au_n\|_{L^2([0,T];H^1(\mathbb{R}^3))}\\
\leq\ &\|a(1-\Delta)^{-1}JJ^{-1}(1-\Delta)u_n\|_{L^2([0,T];H^1(\mathbb{R}^3))}\\
\leq\ &\Big\|[a,(1-\Delta)^{-1}J]J^{-1}(1-\Delta)u_n + (1-\Delta)^{-1}JaJ^{-1}(1-\Delta)u_n\Big\|_{L^2([0,T];H^1(\mathbb{R}^3))}\\
\leq\ &\Big\|[a,(1-\Delta)^{-1}J]J^{-1}(1-\Delta)u_n\Big\|_{L^2([0,T];H^1(\mathbb{R}^3))}\\
&+\Big\|(1-\Delta)^{-1}JaJ^{-1}(1-\Delta)u_n\Big\|_{L^2([0,T];H^1(\mathbb{R}^3))}\\
\leq\ &C\|u_n\|_{L^2([0,T]\times\mathbb{R}^3)} + \Big\|(1-\Delta)^{-1}JaJ^{-1}(1-\Delta)u_n\Big\|_{L^2([0,T];H^1(\mathbb{R}^3))}.
\end{aligned}$$

Additionally,

$$
\begin{aligned}
& \left\|(1-\Delta)^{-1} J a J^{-1}(1-\Delta) u_n\right\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
\leq\ & \left\|(1-\Delta)^{-1} J a\Big(i\partial_t u_n - J^{-1}|u_n|^4 u_n\Big)\right\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
\leq\ & \|(1-\Delta)^{-1} J a \partial_t u_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
& + \|(1-\Delta)^{-1} J a J^{-1}|u_n|^4 u_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
\leq\ & \|(1-\Delta)^{-1} J (1-\Delta)^{\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}} a \partial_t u_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
& + C\|u_n^5\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}.
\end{aligned}
$$

So,

$$
\begin{aligned}
& \|u_n\|_{L^2([0,T];H^1(\mathbb{R}^3\setminus B_{R+1}(0)))} \\
\leq\ & \|(1-\Delta)^{-1} J (1-\Delta)^{\frac{1}{2}}(1-\Delta)^{-\frac{1}{2}} a \partial_t u_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} + C\|u_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
& + C\|u_n^5\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
\leq\ & C\|(1-\Delta)^{-\frac{1}{2}} a \partial_t u_n\|_{L^2([0,T]\times\mathbb{R}^3)} + C\|u_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
& + C\|u_n^5\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
\leq\ & C\|(1-\Delta)^{-\frac{1}{2}} a \partial_t u_n\|_{L^2([0,T]\times\mathbb{R}^3)} + C\|u_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
& + C\|u_n\|^5_{L^{10}([0,T];L^6(\mathbb{R}^3))}
\end{aligned}
$$

and we get

$$
\int_0^L \|u_n(t)\|^2_{H^1(\mathbb{R}^3\setminus B_{R+1}(0))}\ dt \longrightarrow 0
$$

as $n \to \infty$, using the interpolation

$$
\|u_n(t)\|_{L^6} \leq \|u_n(t)\|^{\frac{1}{6}}_{L^2}\|u_n(t)\|^{\frac{5}{6}}_{L^{10}}
$$

and the convergence

$$
\begin{aligned}
\int_0^L \|u_n(t)\|^{10}_{L^6}\ dt \quad & \leq \quad \int_0^L \|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n(t)\|^{\frac{25}{3}}_{L^{10}}\ dt \\
& \leq \quad \sup_{t\in[0,L]} \|u_n(t)\|^{\frac{5}{3}}_{L^2} \int_0^L \|u_n(t)\|^{\frac{25}{3}}_{L^{10}}\ dt \\
& \leq \quad \sup_{t\in[0,L]} \|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n\|^{\frac{3}{25}}_{L_t^{\frac{25}{3}} L_x^{10}} \\
& \leq \quad \sup_{t\in[0,L]} \|u_n(t)\|^{\frac{5}{3}}_{L^2}\|u_n\|^{\frac{3}{25}}_{L_t^{10} L_x^{10}} \\
& \to \quad 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore,

$$v_n \to 0 \text{ in } L^2((0,L);H^1(\mathbb{R}^3\backslash B_{R+1}(0))) \text{ as } n\to\infty. \tag{5.14}$$

Hence, by convergences (5.13) and (5.14), one has

$$v_n \to 0 \text{ in } L^2_{loc}((0,L);H^1(\mathbb{R}^3)) \text{ as } n\to\infty.$$

Finally, choosing $t_0\in(0,L)$ such that $\|v_n(t_0)\|_{H^1(\mathbb{R}^3)}\to 0$ as $n\to\infty$ and solving the equation satisfied by v_n, we obtain

$$\|v_n(t)\|_{H^1(\mathbb{R}^3)} = \|v_n(t_0)\|_{H^1(\mathbb{R}^3)} \to 0 \text{ as } n\to\infty,$$

for all $t\in[0,L]$. So

$$v_n\to 0 \text{ in } L^\infty([0,L];H^1(\mathbb{R}^3)) \text{ as } n\to\infty$$

which implies that

$$v_n(0)\to 0 \text{ in } H^1(\mathbb{R}^3) \text{ as } n\to\infty.$$

But this means that $u_{n,0}$ converges to zero in $H^1(\mathbb{R}^3)$, in other words, $\alpha_n=\Big(E(u_n)(0)\Big)^{\frac{1}{2}}\to 0$ as $n\to\infty$, a contradiction.

- *Case 2:* $\alpha_n\to 0$

The first part of estimate (5.12) ensures that

$$\int_0^T\int_{\mathbb{R}^3}|(1-\Delta)^{-\frac{1}{2}}a\partial_t u_n|^2\,dxdt \le \frac{1}{n}E(u_n)(0).$$

Define $w_n=\frac{u_n}{\alpha_n}$, where the sequence $(w_n)_{n\in\mathbb{N}}$ satisfies

$$i\partial_t w_n+\Delta w_n - w_n-\alpha_n^4|w_n|^4w_n - a(1-\Delta)^{-1}a\partial_t w_n=0 \tag{5.15}$$

and

$$\int_0^T\int_{\mathbb{R}^3}|(1-\Delta)^{-\frac{1}{2}}a\partial_t w_n|^2\,dxdt \le \frac{1}{n}. \tag{5.16}$$

For a large enough constant $C>0$ and for all $t\in[0,T]$, we have

$$\frac{1}{C}\|u_n(t)\|^2_{H^1}\le E(u_n)(t)\le C\|u_n(t)\|^2_{H^1}.$$

Consequently, we get

$$\|w_n(t)\|_{H^1}=\frac{\|u_n(t)\|_{H^1}}{\sqrt{E(u_n)(0)}}\le C\frac{\sqrt{E(u_n)(t)}}{\sqrt{E(u_n)(0)}}\le C$$

and

$$\|w_n(0)\|_{H^1} = \frac{\|u_n(0)\|_{H^1}}{\sqrt{E(u_n)(0)}} \geq \frac{1}{\sqrt{C}} \frac{\|u_n(0)\|_{H^1}}{\|u_n(0)\|_{H^1}} \geq \frac{1}{\sqrt{C}}. \tag{5.17}$$

So,

$$\|w_n(0)\|_{H^1} \approx 1 \tag{5.18}$$

and w_n is bounded in $L^\infty([0,T]; H^1(\mathbb{R}^3))$. Applying Strichartz estimates (Proposition 4.4.2) to equation (5.15), there exists $C > 0$ such that

$$\begin{aligned} \|\nabla w_n\|_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))} &\leq C\Big(\|w_n(0)\|_{H^1} + \alpha_n^4 \|\nabla w_n\|_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))} \|w_n\|^4_{L^{10}([0,T];L^{10}(\mathbb{R}^3))} \\ &\quad + \alpha_n^4 \|w_n\|^5_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}\Big) \\ &\leq C\Big(1 + \alpha_n^4 \|\nabla w_n\|^5_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))}\Big). \end{aligned}$$

Using a bootstrap argument, we deduce that $\|\nabla w_n\|_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))}$ is bounded and, thus, $\|w_n\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}$ is bounded, due to Sobolev's embedding. Additionally, if we consider the sequence $(w_n)_{n\in\mathbb{N}}$ satisfying the Cauchy problem

$$\begin{cases} i\partial_t \tilde{w}_n + \Delta \tilde{w}_n - \tilde{w}_n - a(1-\Delta)^{-1} a \partial_t \tilde{w}_n = 0 \ \text{ on } [0,T] \times \mathbb{R}^3, \\ \tilde{w}_n(0) = w_n(0), \end{cases} \tag{5.19}$$

an application of Proposition 4.4.2 gives

$$\begin{aligned} &\|w_n - \tilde{w}_n\|_{L^{10}([0,T];L^{10}(\mathbb{R}^3))} + \|w_n - \tilde{w}_n\|_{L^\infty([0,T];H^1(\mathbb{R}^3))} \\ \leq\ & C\Big(\alpha_n^4 \|\nabla |w_n|^4 w_n\|_{L^2([0,T];L^{\frac{6}{5}}(\mathbb{R}^3))} \\ &+ \alpha_n^4 \||w_n|^4 w_n\|_{L^1([0,T];L^2(\mathbb{R}^3))}\Big) \\ \leq\ & C\Big(\alpha_n^4 \|\nabla w_n\|_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))} \|w_n\|^4_{L^{10}([0,T];L^{10}(\mathbb{R}^3))} \\ &+ \alpha_n^4 \|w_n\|^5_{L^{10}([0,T];L^{10}(\mathbb{R}^3))}\Big) \\ \leq\ & C\Big(\alpha_n^4 \|\nabla w_n\|^5_{L^{10}([0,T];L^{\frac{30}{13}}(\mathbb{R}^3))}\Big) \to 0 \end{aligned}$$

as $n \to \infty$.

Now, we need to prove that

$$\|(1-\Delta)^{-\frac{1}{2}} a \partial_t \tilde{w}_n\|_{L^2([0,T];L^2(\mathbb{R}^3))} \to 0 \text{ as } n \to \infty. \tag{5.20}$$

In fact,

$$\begin{aligned}\|(1-\Delta)^{-\frac{1}{2}}a\partial_t\tilde{w}_n\|_{L^2([0,T];L^2(\mathbb{R}^3))} &\leq \|(1-\Delta)^{-\frac{1}{2}}a\partial_t\tilde{w}_n-(1-\Delta)^{-\frac{1}{2}}a\partial_t w_n\|_{L^2([0,T];L^2(\mathbb{R}^3))}\\ &\quad +\|(1-\Delta)^{-\frac{1}{2}}a\partial_t w_n\|_{L^2([0,T];L^2(\mathbb{R}^3))}\\ &\leq \|(1-\Delta)^{-\frac{1}{2}}a(\partial_t\tilde{w}_n-\partial_t w_n)\|_{L^2([0,T];L^2(\mathbb{R}^3))}\\ &\quad +\|(1-\Delta)^{-\frac{1}{2}}a\partial_t w_n\|_{L^2([0,T];L^2(\mathbb{R}^3))}\\ &\leq \|\partial_t\tilde{w}_n-\partial_t w_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}+\|(1-\Delta)^{-\frac{1}{2}}a\partial_t w_n\|_{L^2([0,T];L^2(\mathbb{R}^3))}\end{aligned}$$

and

$$\begin{aligned}&\|\partial_t\tilde{w}_n-\partial_t w_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}\\ \leq\ &\|-iJ^{-1}(I-\Delta)\tilde{w}_n+iJ^{-1}(I-\Delta)w_n+iJ^{-1}\alpha_n^4|w_n|^4w_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}\\ \leq\ &\|J^{-1}(I-\Delta)(\tilde{w}_n-w_n)\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}+\alpha_n^4\|J^{-1}w_n^5\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}\\ \leq\ &C\|\tilde{w}_n-w_n\|_{L^2([0,T];H^1(\mathbb{R}^3))}+C\alpha_n^4\|w_n\|^5_{L^{10}([0,T];L^6(\mathbb{R}^3))}\\ \to\ &0\end{aligned}$$

as $n\to\infty$, where J is the same as in the proof of Theorem 3.3.2.

Now, since $\tilde{w}_n$ is bounded in $L^\infty([0,T];H^1(\mathbb{R}^3))$, we can extract a subsequence (still denoted by $\tilde{w}_n$) such that $\tilde{w}_n(t)\rightharpoonup w(t)$ weakly. Passing to the limit in the system (5.19) and taking into account the convergence (5.20), the function w satisfies

$$\begin{cases} i\partial_t w+\Delta w-w=0 \ \text{ on } (0,T)\times\mathbb{R}^3,\\ \partial_t w=0 \ \text{ on } (0,T)\times\mathbb{R}^3\backslash B_{R+1}(0).\end{cases}$$

Let $v=\partial_t w$. Taking the derivative with respect to time in the first equation of the system above, we have that v satisfies

$$\begin{cases} i\partial_t v+\Delta v-v=0 \ \text{ on } (0,T)\times\mathbb{R}^3,\\ v=0 \ \text{ on } (0,T)\times\mathbb{R}^3\backslash B_{R+1}(0).\end{cases}$$

By Proposition .4, $v\in C^\infty\big((0,T)\times\mathbb{R}^4\big)$. By an unique continuation property (see (MERCADO; OSSES; ROSIER, 2008)), $v\equiv 0$ in $(0,T)\times\mathbb{R}^3$. Therefore, $\partial_t w\equiv 0$ in $(0,T)\times\mathbb{R}^3$ and

$$\Delta w-w=0.$$

Multiplying the equation above by $\overline{w}$ and integrating by parts, we get

$$\int_{\mathbb{R}^3}|\nabla w|^2\,dx+\int_{\mathbb{R}^3}|w|^2\,dx=0,$$

which implies $w \equiv 0$. Therefore, $\tilde{w}_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$.

Summarizing everything we know about the sequence $\tilde{w}_n$, we have

1. $\tilde{w}_n$ is bounded in $L^\infty([0,T]; H^1(\mathbb{R}^3))$;

2. $\tilde{w}_n$ satisfies $i\partial_t \tilde{w}_n + \Delta \tilde{w}_n - \tilde{w}_n \to 0$ in $L^2([0,T]; H^1(\mathbb{R}^3))$ as $n \to \infty$;

3. $\sup_{t\in[0,T]} \|\chi \tilde{w}_n(t)\|_{L^2} \to 0$ as $n \to \infty$, for every $\chi \in C_0^\infty(\mathbb{R}^3)$;

4. $\tilde{w}_n \to 0$ in $L^2_{loc}\Big((0,T); H^1_{loc}(\mathbb{R}^3 \backslash B_{R+1}(0))\Big)$ as $n \to \infty$.

Let us remember how item 4 above is proven. In fact, due to the convergence

$$\|(1-\Delta)^{-\frac{1}{2}} a \partial_t \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0$$

we get

$$\|(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0$$

or, equivalently,

$$\|\chi(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } n \to \infty,$$

for $\chi \in C_0^\infty([0,T]\times\mathbb{R}^3)$ and J given as in the proof of Theorem 3.3.2. Thus,

$$\|(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \chi \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } n \to \infty, \tag{5.21}$$

since

$$\begin{aligned}
& \|(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \chi \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
= \; & \|[(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta), \chi] \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
& \; + \|\chi(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq \; & \|\chi_B \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} + \|\chi(1-\Delta)^{-\frac{1}{2}} a J^{-1}(I-\Delta) \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

Therefore, by (5.21),

$$\Big\langle (1-\Delta)(J^{-1})^* a (1-\Delta)^{-1} a J^{-1} (1-\Delta) \chi \tilde{w}_n, \chi \tilde{w}_n \Big\rangle_{L^2((0,T)\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } \; n \to \infty.$$

Hence, Proposition .2 (Appendix) gives us

$$\int_{(0,T)\times\mathbb{R}^3\times S^2} \frac{(1+|\xi|^2)a^2}{1+|\xi|^2}(1+|\xi|^2)\, d\mu = 0,$$

which means that

$$\int_{(0,T)\times\omega\times S^2} 1+|\xi|^2 \, d\mu = 0.$$

Thus, property 4 is verified.

Since the sequence $\tilde{w}_n$ satisfies the four conditions mentioned above, Corollary .3 (Appendix) ensures that

$$\tilde{w}_n \to 0 \text{ in } L^2_{loc}\Big((0,T); H^1_{loc}(\mathbb{R}^3)\Big) \text{ as } n \to \infty. \tag{5.22}$$

On the other hand, since $\|(1-\Delta)^{-\frac{1}{2}} a\partial_t \tilde{w}_n\|_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0$ as $n \to \infty$, we get

$$\|a\partial_t \tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \longrightarrow 0 \text{ as } n \to \infty.$$

Let $\chi_\omega \in C^\infty(\mathbb{R}^3)$ such that $\chi_\omega = 1$ on supp(a). Then,

$$\begin{aligned}
\|ai\partial_t \tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} &= \|aJ^{-1}(1-\Delta)\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\geq \eta\|\chi_\omega J^{-1}(1-\Delta)\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\geq \eta\|J^{-1}\chi_\omega(1-\Delta)\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\geq C\|\chi_\omega(1-\Delta)\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\geq C\|(1-\Delta)\chi_\omega\tilde{w}_n - [(1-\Delta),\chi_\omega]\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\geq C\|(1-\Delta)\chi_\omega\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} - C\|[(1-\Delta),\chi_\omega]\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))}.
\end{aligned}$$

Note that

$$\begin{aligned}
\|[(1-\Delta),\chi_\omega]\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} &= \|[(1-\Delta),\chi_\omega]\chi_B\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\leq C\|\chi_B\tilde{w}_n\|_{L^2([0,T];L^2(\mathbb{R}^3))} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, for $\chi_B \in C_0^\infty(\mathbb{R}^3)$. Consequently,

$$\begin{aligned}
\|(1-\Delta)\chi_\omega\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} &\leq C\|[(1-\Delta),\chi_\omega]\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} + C\|ai\partial_t\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Then,

$$\begin{aligned}
\|\chi_\omega\tilde{w}_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} &= \|(1-\Delta)^{-1}(1-\Delta)\chi_\omega\tilde{w}_n\|_{L^2([0,T];H^1(\mathbb{R}^3))} \\
&\leq \|(1-\Delta)\chi_\omega\tilde{w}_n\|_{L^2([0,T];H^{-1}(\mathbb{R}^3))} \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This means that

$$\tilde{w}_n \longrightarrow 0 \text{ in } L^2\Big([0,T]; H^1(\mathbb{R}^3 \backslash B_{R+1}(0))\Big) \text{ as } n \to \infty. \tag{5.23}$$

By convergences (5.22) and (5.23), we conclude that

$$\tilde{w}_n \longrightarrow 0 \text{ in } L^2_{loc}\Big((0,T); H^1(\mathbb{R}^3)\Big) \text{ as } n \to \infty.$$

So, choosing $t_0 \in (0,T)$ such that $\|\tilde{w}_n(t_0)\|_{H^1} \to 0$ as $n \to \infty$, and solving the equation satisfied by the sequence $\tilde{w}_n$ with $\tilde{w}_n(t_0)$ as initial data, we have

$$\tilde{w}_n(t) = e^{i(t-t_0)(\Delta - I)}\tilde{w}_n(t_0) + \int_{t_0}^{t} e^{i(t-\tau)(\Delta-I)} a(1-\Delta)^{-1} a \partial_t \tilde{w}_n \, d\tau.$$

Hence,

$$\begin{aligned} \|\tilde{w}_n(t)\|_{H^1} &\leq c\|\tilde{w}_n(t_0)\|_{H^1} + c\|a(1-\Delta)^{-1}a\partial_t\tilde{w}_n\|_{L^1([0,T];H^1)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$\tilde{w}_n \longrightarrow 0 \text{ in } L^\infty\Big([0,T]; H^1(\mathbb{R}^3)\Big) \text{ as } n \to \infty$$

and

$$\|w_n(0)\|_{H^1} = \|\tilde{w}_n(0)\|_{H^1} \to 0 \text{ as } n \to \infty,$$

which is a contradiction with (5.18). □

5.3 PROOF OF THEOREM 2.2

This subsection is devoted to the proof of the stabilizability of system (2.4), where we get the observability estimate (5.1) for solutions with initial data satisfying a special condition which we will make clear below.

First, by the decreasing of the energy and Sobolev's embedding, there exists a constant $C(\lambda_0)$ such that the assumption $\|u_0\|_{H^1} \leq \lambda_0$ implies

$$E(u)(t) \leq C(\lambda_0) \text{ and } \|u(t)\|_{H^1} \leq C(\lambda_0), \tag{5.24}$$

for all $t \geq 0$. Fix $T > 0$ such that Theorem 5.2.1 applies. Then, there exists $\varepsilon > 0$ such that, for any u_0 satisfying

$$\|u_0\|_{H^1} \leq \lambda_0 \text{ and } \|u_0\|_{H^{-1}} \leq \varepsilon, \tag{5.25}$$

the observability estimate

$$E(u)(0) \leq C \int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a \partial_t u|^2 \, dxdt$$

holds for any solution of the damped equation (2.4). This means that there exists $B > 0$ such that any solution of the damped equation satisfying (5.25) fulfills

$$E(u)(T) \leq (1-B)E(u)(0), \tag{5.26}$$

since

$$\begin{aligned} E(u)(T) &= E(u)(0) - \int_0^T \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a \partial_t u|^2 \, dxdt \\ &\leq E(u)(0) - C^{-1} E(u)(0) \\ &= E(u)(0)(1 - C^{-1}), \end{aligned}$$

where $C^{-1} = B$, $0 \leq B \leq 1$.

Choose $N \in \mathbb{N}$ large enough such that $(1-B)^N C(\lambda_0) \leq \varepsilon^2$. Lemma 4.4.1, Corollary 4.4.1 and (5.24) allow us to choose $\delta > 0$ small enough such that the assumptions

$$\|u_0\|_{H^1} \leq R_0 \text{ and } \|u_0\|_{H^{-1}} \leq \delta$$

imply

$$\|u(nT)\|_{H^{-1}} \leq \varepsilon, \text{ for } 0 \leq n \leq N. \tag{5.27}$$

So, with that choice, we have $E(u)(NT) \leq (1-B)^N E(u)(0)$. In fact,

$$\begin{aligned} E(u)(NT) &= E(u)((N-1)T) - \int_{(N-1)T}^{NT} \int_{\mathbb{R}^3} |(1-\Delta)^{-\frac{1}{2}} a \partial_t u|^2 \, dxdt \\ &\leq E(u)((N-1)T) - BE(u)((N-1)T) \\ &\leq E(u)(0)(1-B)^N. \end{aligned}$$

Then, by the decreasing of energy, for all $t \geq NT$, we have

$$\begin{aligned} \|u(t)\|_{H^{-1}}^2 &\leq C\|u(t)\|_{H^1}^2 \\ &\leq CE(u)(t) \\ &\leq CE(u)(NT) \\ &\leq C(1-B)^N E(u)(0) \\ &\leq C(R_0)(1-B)^N \\ &\leq \varepsilon^2. \end{aligned}$$

Therefore, the decay estimate (5.26) holds in each interval $[nT,(n+1)T]$, $n \in \mathbb{N}$, and we have

$$E(u)(nT) \leq (1-B)^n E(u)(0).$$

If $t \in [nT,(n+1)T]$, taking $t = nT + r$ with $0 \leq r \leq T$, we have, since $E(u)(t)$ is decreasing,

$$\begin{aligned} E(u)(t) \leq E(u)(t-r) &= E(u)(nT) \\ &\leq (1-B)^n E(u)(0) \\ &\leq A^n E(u)(0) \\ &\leq A^{\frac{t-r}{T}} E(u)(0), \end{aligned}$$

where $0 < A < 1$. Observe that

$$\begin{aligned} A^{\frac{t-r}{T}} = A^{\frac{t}{T}} A^{\frac{-r}{T}} = e^{ln[A^{\frac{t}{T}}]} e^{ln[A^{\frac{-r}{T}}]} &= e^{\frac{t}{T} lnA} e^{-\frac{r}{T} lnA} \\ &= e^{\frac{lnA}{T} t} e^{-\frac{lnA}{T} r}. \end{aligned}$$

Taking $\gamma = -\frac{lnA}{T}$ and $C = e^{-\frac{lnA}{T} r}$, we obtain

$$E(u)(t) \leq C e^{-\gamma t} E(u)(0).$$

This completes the proof of the Theorem 2.2.

5.4 CONCLUSION

This chapter presented the stabilizability for the perturbed nonlinear quintic defocusing Schrödinger equation, for solutions that are bounded in the energy space but small in a lower norm. This perturbation term appears here to make it possible to work with an energy identity that presents a norm in H^1 and, thus, allows the use of Sobolev space embeddings.

Finally, we would like to point out that we do not know if this exponential decay is not valid for the original equation (2.1) since there is no counter-example so far. Therefore, this problem is still open and lies in our future research perspectives.

6 CONTROLLABILITY OF THE NONLINEAR SCHRÖDINGER EQUATION

In this chapter, we turn our efforts to the problem of null controllability for the nonlinear Schrödinger equation with critical exponent defocusing case, i.e., the original system (2.5). Our purpose is to prove the second main theorem of this thesis, Theorem 2.4. To get this result, we use a duality strategy, which reduces the controllability problem associated with system (2.5) to prove an observability inequality by using the Hilbert Uniqueness Method (LIONS, 1988) for solutions of the linear system

$$\begin{cases} i\partial_t u + \Delta u = \varphi(x)h(t,x), \quad x \in \mathbb{R}^3, \ t \in (0,T), \\ u(0) = u_0, \end{cases} \tag{6.1}$$

where φ satisfies

$$\varphi(x) = \begin{cases} 0, \ if \ |x| \le R, \\ 1, \ if \ |x| \ge R+1, \end{cases} \tag{6.2}$$

for some $R > 0$. Our first step is to prove exact controllability for the system (6.1), i.e., to solve the following linear control problem.

Theorem 6.1. *For every initial data $u_0 \in H^1(\mathbb{R}^3)$ and every $T > 0$, there exists $R > 0$ and a control $h(x,t) \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ with support in $\mathbb{R} \times (\mathbb{R}^3 \backslash B_R(0))$ such that the unique solution of the linear system associated to (2.6) satisfies $u(T,\cdot) = 0$.*

6.1 CONTROL OF THE LINEAR SCHRÖDINGER EQUATION

Our aim in this section is to prove Theorem 6.1. We proceed similarly to Rosier in (ROSIER; ZHANG, 2009). The exact controllability of system (6.1) follows from the observability inequality

$$\|v_0\|^2_{H^{-1}} \le c \int_0^T \|\varphi v(t)\|^2_{H^{-1}} \, dt, \tag{6.3}$$

where $v(t,x)$ is a solution to the adjoint system associated to (6.1), namely,

$$\begin{cases} i\partial_t v + \Delta v = 0 \ on \ \mathbb{R} \times \mathbb{R}^3, \\ v(0) = v_0 \in H^{-1}(\mathbb{R}^3). \end{cases} \tag{6.4}$$

The observability inequality (6.3) is given by the following result.

Proposition 6.1.1. *Let φ be a C^∞ real function on $\mathbb{R}^3$ as in (6.2). Then, for every $T > 0$, there exists a constant $C = C(T) > 0$ such that inequality (6.3) holds for every solution $v(t,x)$ of system (6.4).*

Demonstração. We split the proof into several steps.

First step: *H^1–observability.*

Lemma 6.1.1. *Consider the system*

$$\begin{cases} i\partial_t w + \Delta w = 0, \quad x \in \mathbb{R}^3, \ t \in (0,T), \\ w(0) = w_0 \in H^1(\mathbb{R}^3). \end{cases} \tag{6.5}$$

There exists a constant $C > 0$ such that for each $w_0 \in H^1(\mathbb{R}^3)$, the solution $w(t)$ to system (6.5) satisfies

$$\|w_0\|^2_{H^1(\mathbb{R}^3)} \leq C \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\, dt. \tag{6.6}$$

Proof of Lemma 6.1.1. Let $q \in C_0^\infty(\mathbb{R}^3)$ such that

$$q(x) = \begin{cases} x, \ if \ |x| \leq R+2, \\ 0, \ if \ |x| \geq R+3. \end{cases}$$

Multiplying the equation in (6.5) by $q \cdot \nabla \overline{w} + \frac{1}{2}\overline{w}(div_x q)$, taking the real part and integrating by parts, the same computations as in (MACHTYNGIER, 1994) (Lemma 2.2) yield

$$\begin{aligned} \frac{1}{2} Im \int_{\mathbb{R}^3} (wq \cdot \nabla \overline{w})\ dx \Big|_0^T + \frac{1}{2} Re \int_0^T \int_{\mathbb{R}^3} w \nabla(div_x q) \cdot \nabla \overline{w}\ dxdt \\ + Re \int_0^T \int_{\mathbb{R}^3} \sum_{j,k=1}^3 \left(\frac{\partial q_k}{\partial x_j} \frac{\partial \overline{w}}{\partial x_k} \frac{\partial w}{\partial x_j} \right) dxdt = 0, \end{aligned} \tag{6.7}$$

where we have used the fact that the function $q(x)$ has a compact support. Notice that system (6.5) is forward and backward well-posed in $H^1(\mathbb{R}^3)$, so, for any $t_0 \in [0,T]$, there exists a constant $c > 0$ such that

$$\|w(t_0)\|^2_{H^1(\mathbb{R}^3)} \leq c \int_0^T \|w(t_0)\|^2_{H^1(\mathbb{R}^3)}\ dt = c \int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\ dt. \tag{6.8}$$

Thus, it follows from (6.7) and (6.8) that

$$\begin{aligned} \int_0^T \int_{B_{R+2}(0)} |\nabla w|^2\ dxdt \ \leq \ C_\varepsilon \bigg(\int_0^T \int_{B_{R+3}(0) \setminus B_{R+2}(0)} |\nabla w|^2\ dxdt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\ dt \bigg) \\ + \varepsilon \int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\ dt, \end{aligned}$$

for any $\varepsilon > 0$ and some constant $C_\varepsilon > 0$. We also have

$$\|w(t)\|^2_{H^1(\mathbb{R}^3)} \leq C\bigg(\int_{B_{R+2}(0)} |\nabla w|^2 \, dx + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\bigg).$$

Indeed, observe that

$$\begin{aligned}
\|w(t)\|^2_{H^1(\mathbb{R}^3)} &= \|w(t)\|^2_{H^1(B_{R+1}(0))} + \|w(t)\|^2_{H^1(\mathbb{R}^3\setminus B_{R+1}(0))} \\
&= \|w(t)\|^2_{H^1(B_{R+1}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3\setminus B_{R+1}(0))} \\
&\leq \|w(t)\|^2_{H^1(B_{R+2}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \\
&\leq \|\nabla w(t)\|^2_{L^2(B_{R+2}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)},
\end{aligned}$$

showing the previous claim. Moreover, if ε is small enough, we obtain

$$\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt \leq C\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt + \int_0^T \|w(t)\|^2_{L^2(\mathbb{R}^3)}\bigg), \tag{6.9}$$

since

$$\begin{aligned}
\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt &\leq C\bigg(\int_0^T \|\nabla w(t)\|^2_{L^2(B_{R+2}(0))} \, dt + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt\bigg) \\
&\leq C_\varepsilon\bigg(\int_0^T \int_{B_{R+3}(0)\setminus B_{R+2}(0)} |\nabla w|^2 \, dxdt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)} \, dt \\
&\quad + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt\bigg) + \varepsilon \int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt \\
&\leq C_\varepsilon\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} dt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)} \, dt \\
&\quad + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt\bigg) + \varepsilon \int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt \\
&\leq C_\varepsilon\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} dt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)} \, dt\bigg) \\
&\quad + \varepsilon \int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)} \, dt.
\end{aligned}$$

So, it remains to show

$$\int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)} \, dt \leq c \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} dt \tag{6.10}$$

to achieve the proof of Lemma 6.1.1. To this end, let us argue by contradiction, that is, suppose that (6.10) does not hold. If this is the case, there exists a sequence $(w_{n,0})_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^3)$ such that the corresponding sequence of solutions $(w_n)_{n\in\mathbb{N}}$ to system (6.5) satisfies

$$1 = \int_0^T \|w_n(t)\|_{L^2(\mathbb{R}^3)} \, dt \geq n \int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)} dt, \quad n = 1, 2, ... \tag{6.11}$$

Due to inequalities (6.9) and (6.11), we get

$$\int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \le C\bigg(\int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w_n(t)\|^2_{L^2(\mathbb{R}^3)}\bigg) \le C,$$

and so the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$. Hence, the sequence $(w_n(0) = w_{n,0})_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$ by (6.8). Extracting a subsequence, still denoting it by $(w_{n,0})_{n\in\mathbb{N}}$, we may assume that

$$w_{n,0} \rightharpoonup w_0 \text{ weakly in } H^1(\mathbb{R}^3) \text{ as } n\to\infty$$

and

$$w_n \rightharpoonup w \text{ weakly in } L^2\big((0,T);H^1(\mathbb{R}^3)\big) \text{ as } n\to\infty,$$

where $w \in C\big([0,T];H^1(\mathbb{R}^3)\big)$ is a solution to system (6.5). By inequality (6.11), $\varphi w_n \to 0$ in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$ strongly as $n\to\infty$. Since $\varphi w_n \rightharpoonup 0$ in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$ weakly as $n\to\infty$, we conclude that $\varphi w \equiv 0$ on $(0,T)\times\mathbb{R}^3$. Therefore,

$$w\equiv 0, \quad |x| > R+1, \ \forall t \in (0,T).$$

According to Proposition .4 (Remark .5), one has $w \in C^\infty(\mathbb{R}^3\times(0,T))$. Now, we are in a position to use the unique continuation property for the Schrödinger equation showed in (MERCADO; OSSES; ROSIER, 2008) to conclude that

$$w\equiv 0 \text{ on } \mathbb{R}^3\times(0,T).$$

Since $\varphi w_n \to 0$ strongly in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$ as $n\to\infty$, we get

$$w_n \to 0 \text{ strongly in } L^2\big((0,T);H^1(\mathbb{R}^3\backslash B_{R+1}(0))\big) \text{ as } n\to\infty. \tag{6.12}$$

On the other hand, taking into account (6.5) and (6.9), we obtain

$$\begin{aligned}
\int_0^T \|w_n(t)\|^2_{H^1(B_{R+1}(0))}\,dt &\le \int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \\
&\le C\Big(\int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w_n(t)\|^2_{L^2(\mathbb{R}^3)}\,dt\Big),
\end{aligned}$$

$$\begin{aligned}
\int_0^T \|\partial_t w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt &= \int_0^T \|-\Delta w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&= \int_0^T \|(1-\Delta)w_n(t) - w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\le \int_0^T \|(1-\Delta)w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\quad + \int_0^T \|(1-\Delta)w_n(t)\|_{H^{-1}(B_{R+1}(0))}\|w_n(t)\|_{H^{-1}(B_{R+1}(0))}\,dt \\
&\quad + \int_0^T \|w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\le C\int_0^T \|w_n(t)\|^2_{H^1(B_{R+1}(0))}
\end{aligned}$$

and

$$\int_0^T \|\nabla w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \leq C\int_0^T \|\nabla w_n(t)\|^2_{L^2(B_{R+1}(0))}\,dt$$
$$\leq C\int_0^T \|\nabla w_n(t)\|^2_{L^2(\mathbb{R}^3))}\,dt$$
$$\leq C\int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3))}\,dt.$$

Therefore, from the previous inequalities,

$$w_n \text{ is bounded in } L^2\Big((0,T);H^1(B_{R+1}(0))\Big)\cap H^1\Big((0,T);H^{-1}(B_{R+1}(0))\Big).$$

Due to Aubin's lemma (see (SIMON, 1986)) and the convergence (6.12), we conclude that, for a subsequence still denoted by $(w_n)_{n\in\mathbb{N}}$,

$$w_n \to w = 0 \text{ strongly in } L^2\Big((0,T);L^2(\mathbb{R}^3)\Big) \text{ as } n\to\infty,$$

which contradicts (6.11). So, the estimate (6.6) follows from (6.8), (6.9) and (6.10) as

$$\|w(0)\|^2_{H^1(\mathbb{R}^3)} \leq C\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt$$
$$\leq C\Big(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w(t)\|^2_{L^2(\mathbb{R}^3)}\,dt\Big)$$
$$\leq C\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt,$$

showing the lemma. □

Second step: *Weak observability inequality.*

We prove now a bound which is weaker than the observability inequality (6.3).

Lemma 6.1.2. *Let v be the solution of system (6.4) with $v_0\in H^{-1}(\mathbb{R}^3)$. Then,*

$$\|v_0\|^2_{H^{-1}} \leq C\left(\int_0^T \|\varphi v(t)\|^2_{H^{-1}}\,dt + \|(1-\varphi(x/2))v_0\|^2_{H^{-2}}\right). \tag{6.13}$$

Proof of Lemma 6.1.2. Again, let us argue by contradiction. If inequality (6.13) is not verified, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of solutions to problem (6.4) in $C([0,T];H^{-1}(\mathbb{R}^3))$ such that

$$1 = \|v_n(0)\|^2_{H^{-1}} \geq n\left(\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\,dt + \|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}}\right). \tag{6.14}$$

Up to a subsequence, we may assume that

$$v_n \rightharpoonup v \;\text{ in }\; L^\infty((0,T);H^{-1}(\mathbb{R}^3)) \text{ weak* as } n\to\infty$$

and

$$v_n(0) \rightharpoonup v(0) \text{ in } H^{-1}(\mathbb{R}^3) \text{ weak as } n \to \infty, \tag{6.15}$$

where $v \in C([0,T]; H^{-1}(\mathbb{R}^3))$ is a solution of problem (6.4). By inequality (6.14), one has

$$\varphi v_n \to 0 \text{ (strongly) in } L^2((0,T); H^{-1}(\mathbb{R}^3)) \text{ as } n \to \infty.$$

On the other hand, since

$$\varphi v_n \rightharpoonup \varphi v \text{ in } L^\infty((0,T); H^{-1}(\mathbb{R}^3)) \text{ weak* as } n \to \infty,$$

we conclude that $\varphi v \equiv 0$. Therefore, $v(t,x) = 0$ for $|x| > R+1$ and $t \in (0,T)$. So, using the unique continuation property as in Step 1, we get that $v \equiv 0$. In particular, $v(0) = 0$.

Now, we claim that

$$\|\varphi(x/2)v_n(0)\|^2_{H^{-2}} \leq C \int_0^T \|\varphi v_n(t)\|^2_{H^{-1}} \, dt. \tag{6.16}$$

To prove (6.16), introduce the sequence of functions $\tilde{v}_n(x,t) = \varphi(x/2)v_n(x,t)$. It satisfies

$$i\partial_t \tilde{v}_n + \Delta \tilde{v}_n = f_n,$$

since

$$i\partial_t \tilde{v}_n = i\varphi(x/2)\partial_t v_n,$$

$$\Delta \tilde{v}_n = \Delta\Big(\varphi(x/2)v_n\Big) = [\Delta\varphi(x/2)]v_n + 2\nabla\varphi(x/2)\nabla v_n + \varphi(x/2)[\Delta v_n],$$

and

$$i\partial_t \tilde{v}_n + \Delta \tilde{v}_n = f_n,$$

where $f_n = [\Delta\varphi(x/2)]v_n + 2\nabla\varphi(x/2)\nabla v_n$. Thus, one has

$$\|\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} \leq c\left(\int_0^T \|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} \, dt + \int_0^T \|f_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} \, dt\right).$$

Indeed, write the sequence $\tilde{v}_n(t)$ as

$$e^{it\Delta}\tilde{v}_n(0) = \tilde{v}_n(t) - \int_0^t e^{i(t-\tau)\Delta} f_n \, d\tau.$$

By the parallelogram law, we have

$$\left\|\tilde{v}_n(t) + \int_0^t e^{i(t-\tau)\Delta} f_n \, d\tau\right\|^2_{H^{-2}(\mathbb{R}^3)} + \|e^{it\Delta}\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} = 2\|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} + 2\left\|\int_0^t e^{i(t-\tau)\Delta} f_n \, d\tau\right\|^2_{H^{-2}(\mathbb{R}^3)},$$

which implies that

$$\begin{aligned}\|e^{it\Delta}\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} &\leq 2\|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} + 2\left\|\int_0^t e^{i(t-\tau)\Delta} f_n\, d\tau\right\|^2_{H^{-2}(\mathbb{R}^3)} \\ &\leq 2\|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} + C\|f_n\|^2_{L^1([0,T];H^{-2}(\mathbb{R}^3))} \\ &\leq C\Big(\|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)} + \|f_n\|^2_{L^2([0,T];H^{-2}(\mathbb{R}^3))}\Big).\end{aligned}$$

Since the semigroup is unitary, we have $\left\|e^{it\Delta}\tilde{v}_n(0)\right\|^2_{H^{-2}(\mathbb{R}^3)} = \|\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)}$. The fact that $supp[\varphi(x/2)] \subset \{\varphi = 1\}$ yields

$$\begin{aligned}\|\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} \leq& c\left(\int_0^T \|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)}\, dt + \int_0^T \|f_n(t)\|^2_{H^{-2}(\mathbb{R}^3)}\, dt\right) \\ \leq& c\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}(\mathbb{R}^3)},\end{aligned}$$

giving (6.16). Now, note that,

$$\|v_n(0)\|^2_{H^{-2}}+\|2\varphi(x/2)v_n(0)-v_n(0)\|^2_{H^{-2}} = 2\Big(\|\varphi(x/2)v_n(0)\|^2_{H^{-2}}+\|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}}\Big), \tag{6.17}$$

by parallelogram law again. So, using (6.18), (6.14) and (6.16), one has

$$\begin{aligned}\|v_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} &\leq 2\Big(\|\varphi(x/2)v_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} + \|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}(\mathbb{R}^3)}\Big) \\ &\leq c\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}(\mathbb{R}^3)}\, dt + 2\|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} \to 0\end{aligned}$$

as $n \to \infty$, that is,

$$v_n(0) \to 0 \text{ strongly in } H^{-2}(\mathbb{R}^3) \text{ as } n \to \infty. \tag{6.18}$$

Let $w_n = (1-\Delta)^{-1}v_n$. Then, $w_n \in C([0,T]; H^1(\mathbb{R}^3))$ is a solution of the equation (6.5). By the convergences (6.15) and (6.18), we can ensure that

$$w_n(0) \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^3) \text{ weakly } \text{ as } n \to \infty$$

and

$$w_n \to 0 \text{ in } C([0,T]; L^2(\mathbb{R}^3)) \text{ strongly as } n \to \infty. \tag{6.19}$$

Now, split φw_n as

$$\varphi w_n = (1-\Delta)^{-1}(\varphi v_n) - (1-\Delta)^{-1}[\varphi, (1-\Delta)]w_n.$$

Observe that the operator $[\varphi, (1-\Delta)]$ maps $L^2(\mathbb{R}^3)$ continuously into $H^{-1}(\mathbb{R}^3)$. So, due to the convergence (6.19), we get that

$$(1-\Delta)^{-1}[\varphi, (1-\Delta)]w_n \to 0 \text{ in } C([0,T]; H^1(\mathbb{R}^3)) \text{ as } n \to \infty. \tag{6.20}$$

On the other hand, by (6.14),

$$(1-\Delta)^{-1}(\varphi v_n) \to 0 \text{ in } L^2((0,T);H^1(\mathbb{R}^3)) \text{ as } n\to\infty. \tag{6.21}$$

Therefore, by the convergences (6.20) and (6.21), it follows that

$$\varphi w_n \to 0 \text{ in } L^2((0,T);H^1(\mathbb{R}^3)) \text{ as } n\to\infty.$$

Since w_n satisfies (6.5), using Lemma 6.1.1, more precisely, the observability inequality (6.6), we conclude that

$$w_n(0) \to 0 \text{ in } H^1(\mathbb{R}^3) \text{ strongly } \text{ as } n\to\infty$$

and so

$$v_n(0) \to 0 \text{ in } H^{-1}(\mathbb{R}^3) \text{ strongly as } n\to\infty,$$

which is a contradiction with the fact that $\|v_n(0)\|^2_{H^{-1}} = 1$, for all n. This finishes the proof. □

Third step: *Proof of the observability inequality* (6.3).

Now, to conclude the proof of Proposition 6.1.1, we argue by contradiction once more. If (6.3) is false, then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of solutions to (6.4) in $C([0,T];H^{-1}(\mathbb{R}^3))$ such that

$$1 = \|v_n(0)\|^2_{H^{-1}} \geq n\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\, dt,\ \forall n\geq 0. \tag{6.22}$$

Extracting a subsequence, still denoted by the same indexes, we have that

$$v_n \rightharpoonup v \text{ in } L^\infty((0,T);H^{-1}(\mathbb{R}^3)) \text{ weak* as } n\to\infty$$

and

$$v_n(0) \rightharpoonup v(0) \text{ in } H^{-1}(\mathbb{R}^3) \text{ weak as } n\to\infty,$$

for some solution $v\in C([0,T];H^{-1}(\mathbb{R}^3))$ of the system (6.4). Note that

$$\varphi v_n \rightharpoonup \varphi v \text{ in } L^\infty(0,T;H^{-1}(\mathbb{R}^3)) \text{ weak* as } n\to\infty$$

and this, combined with (6.22) ($\varphi v_n \to 0$ in $L^2\big((0,T);H^{-1}(\mathbb{R}^3)\big)$), yields $\varphi v\equiv 0$ and, hence, $v\equiv 0$ for $|x|>R+1$, $t\in(0,T)$. So, by the unique continuation property as in Step 2, we deduce that $v\equiv 0$ on $\mathbb{R}^3\times(0,T)$. On the other hand, the sequence $(1-\varphi(x/2))v_n(0)$ is bounded in $H^{-1}(\mathbb{R}^3)$ and has compact support contained in $B_{2R+2}(0)$. Therefore, extracting a subsequence, we may assume that it converges strongly in $H^{-2}(\mathbb{R}^3)$. Moreover, its limit is necessarily 0 since

$$(1-\varphi(x/2))v_n(0) \rightharpoonup 0 \text{ in } H^{-2}(\mathbb{R}^3) \text{ as } n\to\infty.$$

Using (6.13), we conclude that $\|v_n(0)\|_{H^{-1}} \to 0$ as $n\to\infty$, which contradicts (6.22). This proves the desired observability inequality (6.3) and finishes the proof of Proposition 6.1.1. □

Finally, we prove Theorem 6.1.

Proof of Theorem 6.1. We use Hilbert's uniqueness method. First, note that, since the Schrödinger equation (6.1) is backward well-posed, we may assume that $u(T)=0$ without loss of generality. Now, consider the two systems

$$\begin{cases} i\partial_t u + \Delta u = \varphi(x)h(x,t) \ on \ [0,T]\times\mathbb{R}^3, \\ u(T)=0, \end{cases} \tag{6.23}$$

with $\varphi(x)$ given by (6.2) and

$$\begin{cases} i\partial_t v + \Delta v = 0 \ on \ [0,T]\times\mathbb{R}^3, \\ v(0)=v_0\in H^{-1}(\mathbb{R}^3). \end{cases}$$

Multiplying the first equation of the first system by $\overline{v}$ and integrating by parts, we obtain

$$i\int_{\mathbb{R}^3}\left[\overline{v}(T)u(T)-\overline{v_0}u(0)\right]\,dx = \int_0^T\int_{\mathbb{R}^3}\varphi(x)h(x,t)\overline{v(x,t)}\,dxdt.$$

Hence, taking $L^2(\mathbb{R}^3)$ as pivot space, one has

$$\langle v_0,-iu_0\rangle = \int_0^T\langle\varphi(x)v,h(t)\rangle\,dt, \tag{6.24}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$. Consider the continuous map $\Lambda : H^{-1}(\mathbb{R}^3)\to H^1(\mathbb{R}^3)$ defined by $\Lambda v = \langle v,\cdot\rangle_1$. Given any $v_0\in H^{-1}(\mathbb{R}^3)$, let $h(t)=\Lambda^{-1}(\varphi v(t))\,(h\in C([0,T];H^1(\mathbb{R}^3)))$ and let u be the corresponding solution of system (6.23). Finally, set $\Gamma(v_0)=-iu(\cdot,0)$. Then, we have

$$\langle v_0,\Gamma(v_0)\rangle = \int_0^T\|\varphi v(t)\|^2_{H^{-1}(\mathbb{R}^3)}dt \geqslant c\,\|v_0\|^2_{H^{-1}(\mathbb{R}^3)},$$

in view of the observability inequality (6.3) and (6.24). It follows from the Lax-Milgram theorem that Γ defines an isomorphism, and this concludes the proof of Theorem 6.1. □

6.2 NONLINEAR SYSTEM: PROOF OF THEOREM 2.4

We have gathered all the necessary information to demonstrate the Theorem 2.4. The proof is based on a perturbation argument due to Zuazua (ZUAZUA, 1990). To use it, consider the following two Schrödinger systems with initial data in H^{-1} and null initial data, namely

$$\begin{cases} i\partial_t \Phi + \Delta\Phi = 0 \text{ on } [0,T]\times\mathbb{R}^3, \\ \Phi(0) = \Phi_0 \in H^{-1}(\mathbb{R}^3) \end{cases}$$

and

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = A\Phi \text{ on } [0,T]\times\mathbb{R}^3, \\ u(T) = 0, \end{cases} \tag{6.25}$$

where A is defined as in Theorem 6.1 by

$$A\Phi := \Lambda^{-1}(\varphi(x)\Phi).$$

Now, define the operator

$$\mathcal{L} : H^{-1}(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$$
$$\Phi_0 \mapsto \mathcal{L}\Phi_0 = u_0 = u(0).$$

The purpose is to show that $\mathcal{L}$ is onto in a small neighborhood of the origin of $H^1(\mathbb{R}^3)$. To this end, split u as $u = v + \Psi$, where is Ψ a solution of

$$\begin{cases} i\partial_t \Psi + \Delta\Psi = A\Phi \text{ on } [0,T]\times\mathbb{R}^3, \\ \Psi(T) = 0 \end{cases}$$

and v is a solution of

$$\begin{cases} i\partial_t v + \Delta v = |u|^4 u \quad \text{on} \quad [0,T]\times\mathbb{R}^3, \\ v(T) = 0. \end{cases} \tag{6.26}$$

Clearly u, v and Ψ belong to $C([0,T], H^1(\mathbb{R}^3)) \cap L^{10}([0,T]; L^{10}(\mathbb{R}^3))$ and $u(0) = v(0) + \Psi(0)$. We write

$$\mathcal{L}\Phi_0 = \mathcal{J}\Phi_0 + \Gamma\Phi_0,$$

where $\mathcal{J}\Phi_0 = v_0$. Observe that $\mathcal{L}\Phi_0 = u_0$, or equivalently, $\Phi_0 = -\Gamma^{-1}\mathcal{J}\Phi_0 + \Gamma^{-1}u_0$. Now, define the operator

$$\mathcal{B} : H^{-1}(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3)$$
$$\Phi_0 \to \mathcal{B}\Phi_0 = \Gamma^{-1}\mathcal{J}\Phi_0 + \Gamma^{-1}u_0,$$

where we are taking into account that Γ is the linear control isomorphism between H^{-1} and H^1, due to Theorem 6.1. The goal is to prove that $\mathcal{B}$ has a fixed point near the origin of $H^{-1}(\mathbb{R}^3)$. More precisely, let us prove that if $\|u_0\|_{H^1}$ is small enough, then $\mathcal{B}$ is a contraction on a small ball B_R of $H^{-1}(\mathbb{R}^3)$. We may assume $T<1$ and we denote by $C>0$ any constant that may have its numerical value changed line by line. Since Γ is an isomorphism, we have

$$\begin{aligned}\|\mathcal{B}\Phi_0\|_{H^{-1}} \leq & \|\Gamma^{-1}\mathcal{J}\Phi_0\|_{H^{-1}} + \|\Gamma^{-1}u_0\|_{H^{-1}} \\ \leq & C\left(\|\mathcal{J}\Phi_0\|_{H^1} + \|u_0\|_{H^1}\right) \\ \leq & C\left(\|v(0)\|_{H^1} + \|u_0\|_{H^1}\right).\end{aligned} \tag{6.27}$$

Claim 1: There exists $C>0$ such that

$$\|v(0)\|_{H^1} \leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x}. \tag{6.28}$$

Indeed, note that due to the classical energy estimate for system (6.26), Strichartz estimates (see Lemma 3.2.1) and a Sobolev embedding (see Lemma 3.2.2), we have

$$\begin{aligned}\|v(0)\|_{L^2} &\leq \|v(T)\|_{L^2} + \left\|\int_0^t e^{i(t-\tau)\Delta}|u|^4 u \, d\tau\right\|_{L^2_x} \\ &\leq C\|u^5\|_{L^1_t L^2_x} \\ &\leq C\|u\|^5_{L^{10}_t L^{10}_x} \\ &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x}\end{aligned}$$

and

$$\begin{aligned}\|\nabla v(0)\|_{L^2} &\leq \|\nabla v(T)\|_{L^2} + \left\|\int_0^t e^{i(t-\tau)\Delta}\nabla|u|^4 u \, d\tau\right\|_{L^2_x} \\ &\leq C\|\nabla u\|_{L^{10}_t L^{\frac{30}{13}}_x}\|u\|^4_{L^{10}_t L^{10}_x} \\ &\leq \|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x}.\end{aligned}$$

Thus,

$$\|v(0)\|^2_{L^2} \leq C\|\nabla u\|^{10}_{L^{10}_t L^{\frac{30}{13}}_x}$$

and

$$\|\nabla v(0)\|^2_{L^2} \leq C\|\nabla u\|^{10}_{L^{10}_t L^{\frac{30}{13}}_x}.$$

Adding up, we have (6.28), showing Claim 1.

Claim 2: There exists $C>0$ such that

$$\|\nabla u\|_{L^{10}_t L^{\frac{30}{13}}_x} \leq C\|\Phi_0\|_{H^{-1}}. \tag{6.29}$$

In fact, an application of Lemma 3.2.1 to system (6.25) ensures that

$$\begin{aligned}\|\nabla u\|_{L^{10}_t L^{\frac{30}{13}}_x} &\leq \|\nabla u(T)\|_{L^2} + C\|\nabla u\|_{L^{10}_t L^{\frac{30}{13}}_x}\|u\|^4_{L^{10}_t L^{10}_x} + C\|\nabla A\Phi\|_{L^1_t L^2_x} \\ &\leq C\left(\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + \|A\Phi\|_{L^2_t H^1_x}\right).\end{aligned}$$

Note that, using the fact that Λ is an isomorphism, we get

$$\|A\Phi\|_{H^1} = \|\Lambda^{-1}(\varphi\Phi)\|_{H^1} \leq C\|\varphi\Phi\|_{H^{-1}}$$

or, equivalently,

$$\|A\Phi\|_{L^2H^1} \leq \left(\int_0^T \|\varphi\Phi\|^2_{H^{-1}}\, dt\right)^{\frac{1}{2}}.$$

Then, the duality (6.24) yields

$$\begin{aligned}\|\nabla u\|_{L^{10}_t L^{\frac{30}{13}}_x} &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + C\left(\int_0^T \|\varphi\Phi\|^2_{H^{-1}}\, dt\right)^{\frac{1}{2}} \\ &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + C\Big(\langle\Gamma\Phi_0, \Phi_0\rangle\Big)^{\frac{1}{2}} \\ &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + C\Big(\|\Gamma\Phi_0\|_{H^1}\|\Phi_0\|_{H^{-1}}\Big)^{\frac{1}{2}} \\ &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + C\Big(\|\Phi_0\|^2_{H^{-1}}\Big)^{\frac{1}{2}} \\ &\leq C\|\nabla u\|^5_{L^{10}_t L^{\frac{30}{13}}_x} + C\|\Phi_0\|_{H^{-1}}.\end{aligned}$$

Using a bootstrap argument, taking $\|\Phi_0\|_{H^{-1}} \leq R$ with R small enough, we get (6.29), showing Claim 2.

Observe that, putting together (6.28) and (6.29) into (6.27), we conclude that

$$\begin{aligned}\|B\Phi_0\|_{H^{-1}} &\leq C\Big(\|v(0)\|_{H^1} + \|u_0\|_{H^1}\Big) \\ &\leq C\Big(\|\Phi_0\|^5_{H^{-1}} + \|u_0\|_{H^1}\Big).\end{aligned}$$

Then, choosing R small enough and $\|u_0\|_{H^1} \leq \frac{R}{2C}$, we get

$$\|B\Phi_0\|_{H^{-1}} \leq R$$

and, therefore, $\mathcal{B}$ reproduces the ball B_R of $H^{-1}(\mathbb{R})$.

Finally, we prove that $\mathcal{B}$ is a contraction map. To do this, let us study the systems

$$\begin{cases} i\partial_t(u_1-u_2) + \Delta(u_1-u_2) - |u_1|^4u_1 + |u_2|^4u_2 = A(\Phi^1-\Phi^2), \\ (u_1-u_2)(T) = 0 \end{cases}$$

and

$$\begin{cases} i\partial_t(v_1-v_2)+\Delta(v_1-v_2)=|u_1|^4u_1-|u_2|^4u_2, \\ (v_1-v_2)(T)=0. \end{cases} \tag{6.30}$$

As done before, we have

$$\|\mathcal{B}\Phi_0^1-\mathcal{B}\Phi_0^2\|_{H^{-1}}\le C\|v_1(0)-v_2(0)\|_{H^1}. \tag{6.31}$$

We estimate $v_1(0)-v_2(0)$ in the H^1-norm. First, applying the Strichatz estimates (see Lemma 3.2.1) to the system (6.30) yields that

$$\begin{aligned}
\|v_1(0)-v_2(0)\|_{L^2}\le&\left\|\int_0^t e^{i(t-\tau)\Delta}(|u_1|^4u_1-|u_2|^4u_2)\,d\tau\right\|_{L^2_x}\\
\le&C\|u_1^5-u_2^5\|_{L^1_tL^2_x}\\
\le&C\|u_1-u_2\|_{L^5_tL^{10}_x}\Big(\|u_1\|^4_{L^5_tL^{10}_x}+\|u_2\|^4_{L^5_tL^{10}_x}\Big)\\
\le&C\|u_1-u_2\|_{L^{10}_tL^{10}_x}\Big(\|u_1\|^4_{L^{10}_tL^{10}_x}+\|u_2\|^4_{L^{10}_tL^{10}_x}\Big)\\
\le&C\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\Big(\|\nabla u_1\|^4_{L^{10}_tL^{\frac{30}{13}}_x}+\|\nabla u_2\|^4_{L^{10}_tL^{\frac{30}{13}}_x}\Big)\\
\le&CR^4\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla v_1(0)-\nabla v_2(0)\|_{L^2}\le&\left\|\int_0^t \nabla e^{i(t-\tau)\Delta}(|u_1|^4u_1-|u_2|^4u_2)\,d\tau\right\|_{L^2_x}\\
\le&\|\nabla(|u_1|^4u_1-|u_2|^4u_2)\|_{L^2_tL^{\frac{6}{5}}_x}\\
\le&C\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}\|u_1\|^4_{L^{10}_tL^{10}_x}\\
\le&C\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_1\|^4_{L^{10}_tL^{\frac{30}{13}}_x}\\
&+C\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\Big(\|\nabla u_1\|^3_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\\
&+\|\nabla u_2\|^3_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\Big)\\
\le&CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}+CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}\\
\le&CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}.
\end{aligned}$$

Thus,

$$\|v_1(0)-v_2(0)\|^2_{L^2}\le CR^8\|\nabla u_1-\nabla u_2\|^2_{L^{10}_tL^{\frac{30}{13}}_x}$$

and

$$\|\nabla v_1(0)-\nabla v_2(0)\|^2_{L^2}\le CR^8\|\nabla u_1-\nabla u_2\|^2_{L^{10}_tL^{\frac{30}{13}}_x}.$$

These bounds together give us the H^1-estimate

$$\|v_1(0) - v_2(0)\|_{H^1} \leq CR^4 \|\nabla u_1 - \nabla u_2\|_{L_t^{10} L_x^{\frac{30}{13}}}.$$

Now, let us bound the right-hand side of this inequality. To this end, first notice that

$$\begin{aligned} \|\nabla(u_1 - u_2)\|_{L_t^{10} L_x^{\frac{30}{13}}} &\leq \|\nabla(|u_1|^4 u_1 - |u_2|^4 u_2)\|_{L_t^2 L_x^{\frac{6}{5}}} + \|\nabla A(\Phi^1 - \Phi^2)\|_{L_t^1 L_x^2} \\ &\leq CR^4 \|\nabla u_1 - \nabla u_2\|_{L_t^{10} L_x^{\frac{30}{13}}} + C\|A(\Phi^1 - \Phi^2)\|_{L_t^2 H_x^1} \\ &\leq CR^4 \|\nabla u_1 - \nabla u_2\|_{L_t^{10} L_x^{\frac{30}{13}}} + C\|\Phi_0^1 - \Phi_0^2\|_{H^{-1}}. \end{aligned}$$

So, choosing $R > 0$ small enough, we get

$$\|\nabla(u_1 - u_2)\|_{L_t^{10} L_x^{\frac{30}{13}}} \leq C\|\Phi_0^1 - \Phi_0^2\|_{H^{-1}}.$$

Therefore,

$$\begin{aligned} \|v_1(0) - v_2(0)\|_{H^1} &= \left(\|v_1(0) - v_2(0)\|_{L^2}^2 + \|\nabla v_1(0) - \nabla v_2(0)\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &\leq CR^4 \|\Phi_0^1 - \Phi_0^2\|_{H^{-1}}. \end{aligned} \tag{6.32}$$

Finally, we get, by (6.31) and (6.32), that

$$\begin{aligned} \|\mathcal{B}\Phi_0^1 - \mathcal{B}\Phi_0^2\|_{H^{-1}} &\leq C\|v_1(0) - v_2(0)\|_{H^1} \\ &\leq CR^4 \|\Phi_0^1 - \Phi_0^2\|_{H^{-1}}, \end{aligned}$$

concluding that $\mathcal{B}$ is a contraction on a small ball B_R of H^{-1}. This completes the proof of Theorem 2.4. □

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.1 PROPAGATION RESULTS FOR THE LINEAR SCHRÖDINGER EQUATION

We collect some results of propagation for solutions of the linear Schrödinger equation which were used throughout this thesis (see, for instance, (DEHMAN; GÉRARD; LEBEAU, 2006)). The main ingredient is basic pseudodifferential analysis.

Proposition .2. *Let $L = i\partial_t + \Delta + R_0$, where $R_0(t,x,D_x)$ is a tangential pseudodifferential operator of order 0 and $(u_n)_{n\in\mathbb{N}}$ a sequence of functions satisfying, for every $\chi \in C_0^\infty(\mathbb{R}^3)$, with $\chi(x) = 1,\ x \in supp(\chi) = K$,*

$$\sup_{t\in[0,T]} \|\chi u_n(t)\|_{H^1(\mathbb{R}^3)} \leq C, \quad \sup_{t\in[0,T]} \|\chi u_n(t)\|_{L^2(\mathbb{R}^3)} \to 0 \ \text{ and } \int_0^T \|Lu_n(t)\|_{L^2}^2\, dt \to 0. \tag{33}$$

There exist a subsequence $(u_{n'})_{n'\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and a positive measure μ on $(0,T)\times\mathbb{R}^3\times S^2$ such that, for every tangential pseudodifferential operator $A = A(t,x,D_x)$ of order 2, with principal symbol $\sigma(A) = a_2(t,x,\xi)$, one has

$$\langle A(t,x,D_x)\chi u_{n'}, \chi u_{n'}\rangle_{L^2} \longrightarrow \int_{(0,T)\times\mathbb{R}^3\times S^3} a_2(t,x,\xi)\, d\mu(t,x,\xi). \tag{34}$$

Moreover, if G_s denotes the geodesic flow on $\mathbb{R}^3\times S^2$, one has, for every $s\in\mathbb{R}$,

$$G_s(\mu) = \mu. \tag{35}$$

In other words, μ is invariant by the geodesic flow "at fixed t."

Demonstração. The construction of the tangential microlocal defect measure μ satisfying (34) is classical (see e.g. (GÉRARD, 1991)). The first estimate in (33) combined with a separability argument allows to find a subsequence $(u_{n'})_{n'\in\mathbb{N}}$ such that the left-hand side of (34) converges for all A. Then the second estimate in (33) and the Gårding inequality imply the existence of some positive measure μ such that (34) holds.

For the propagation, i.e., property (35), we consider $\varphi = \varphi(t) \in C_0^\infty(0,T)$, $B(x,D_x)$ a pseudodifferential operator of order 1, with principal symbol b_1, $A(t,x,D_x) = \varphi(t)B(x,D_x)$ and for a given $\varepsilon > 0$, we write $A_\varepsilon = \varphi B_\varepsilon = Ae^{\varepsilon\Delta}$. Moreover, denote

$$\alpha_n^\varepsilon = \big(Lu_n, A_\varepsilon^* u_n\big)_{L^2([0,T]\times\mathbb{R}^3)} - \big(A_\varepsilon u_n, Lu_n\big)_{L^2([0,T]\times\mathbb{R}^3)}.$$

By assumption (33), $\sup_\varepsilon \alpha_n^\varepsilon \to 0$ as $n \to \infty$. On the other hand,

$$\begin{aligned}
\alpha_n^\varepsilon &= \Big(i\partial_t u_n + \Delta u_n + R_0 u_n, A_\varepsilon^* u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} - \Big(A_\varepsilon u_n, i\partial_t u_n + \Delta u_n + R_0 u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&= i\Big(\partial_t u_n, A_\varepsilon^* u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big(\Delta u_n, A_\varepsilon^* u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big(R_0 u_n, A_\varepsilon^* u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&\quad - \Big(A_\varepsilon u_n, i\partial_t u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} - \Big(A_\varepsilon u_n, \Delta u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} - \Big(A_\varepsilon u_n, R_0 u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&= i\Big((\partial_t A_\varepsilon) u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big(A_\varepsilon \Delta u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big(A_\varepsilon R_0 u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&\quad - i\Big((\partial_t A_\varepsilon) u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} - \Big(\Delta A_\varepsilon u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} - \Big(R_0^* A_\varepsilon u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&= \Big([A_\varepsilon, \Delta] u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big([A_\varepsilon R_0 - R_0^* A_\varepsilon] u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&= \Big([A_\varepsilon, \Delta] u_n + [A_\varepsilon R_0 - R_0^* A_\varepsilon] u_n, u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

So

$$\sup_\varepsilon \Big(\chi[A_\varepsilon, \Delta] u_n + \chi[A_\varepsilon R_0 - R_0^* A_\varepsilon] u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \longrightarrow 0 \text{ as } n \to \infty.$$

Observe that, taking $\Big(\chi[A_\varepsilon R_0 - R_0^* A_\varepsilon] u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} = \beta_n^\varepsilon$, we have $\sup_\varepsilon \beta_n^\varepsilon \to 0$ as $n \to \infty$. Finally, passing to the limit as $\varepsilon \to 0$, we obtain, for all $\chi \in C_0^\infty(\mathbb{R}^3)$,

$$\Big(\chi\varphi[B, \Delta] u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \to 0 \tag{36}$$

as $n \to \infty$. Denoting $D := \varphi[B, \Delta]$, D is a pseudodifferential operator of order 2 and we have

$$\begin{aligned}
\Big(\varphi[B, \Delta]\chi u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} &= \Big(D\chi u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&= \Big([D, \chi] u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} + \Big(\chi D u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, using (36), and

$$\begin{aligned}
\Big([D, \chi] u_n, \chi u_n\Big)_{L^2([0,T]\times\mathbb{R}^3)} &\leq \|[D, \chi] u_n\|_{L^2} \|\chi u_n\|_{L^2} \\
&\leq C\|u_n\|_{H^1} \|\chi u_n\|_{L^2}.
\end{aligned}$$

In view of (34), one has

$$\int_{(0,T)\times\mathbb{R}^3\times S^3} \varphi\{|\xi|_x^2, b_1\}\, d\mu(t, x, \xi) = 0.$$

This identity expresses property (35) and completes the proof. □

Using this tool we obtain the following important corollary.

Corollary .3. *Assume that* $\omega \subset \mathbb{R}^3$ *satisfies Assumption 5.1. Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence of functions bounded in* $L^\infty([0,T], H^1(\mathbb{R}^3))$*, convergent to 0 in* L^2 *and satisfying*

$$\begin{cases} i\partial_t u_n + \Delta u_n \to 0 \text{ in } L^2([0,T], H^1(\mathbb{R}^3)), \\ u_n \to 0 \quad \text{in } L^2([0,T], H^1_{loc}(\omega)). \end{cases} \tag{37}$$

Then, $(u_n)_{n\in\mathbb{N}}$ *strongly converges to 0 in* $L^\infty([0,T], H^1_{loc}(\mathbb{R}^3))$.

Demonstração. By Proposition .2, we can attach to the sequence $(u_n)_{n\in\mathbb{N}}$ a microlocal defect measure in $L^2((0,T), H^1(\mathbb{R}^3))$ that propagates with infinite speed along the geodesics of $\mathbb{R}^3$. Using the second equation of (37), we can deduce that

$$\mu = 0 \text{ on } (0,T) \times \omega \times S^3,$$

which yields, by the propagation (36) and Assumption 5.1, $\mu = 0$ on $(0,T) \times \mathbb{R}^3 \times S^3$. This means that $u_n \to 0$ in $L^2_{loc}((0,T); H^1_{loc}(\mathbb{R}^3))$ as $n \to \infty$. Finally, solving the first equation of (37) with initial data $u_n(t_0)$, where $t_0 \in (0,T)$ is such that $\|u_n(t_0)\|_{H^1_{loc}} \to 0$ as $n \to \infty$, this implies the strong convergence $u_n(t) \to 0$ in the space $L^\infty([0,T], H^1_{loc}(\mathbb{R}^3))$ as $n \to \infty$. $\square$

.2 SMOOTHING

For the sake of completeness, we discuss the smoothing properties of the linear Schrödinger equation

$$\begin{cases} i\partial_t u - u + \Delta u = 0, & x \in \mathbb{R}^3, t \in \mathbb{R}, \\ u(x,0) = \psi(x), & x \in \mathbb{R}^3. \end{cases} \tag{38}$$

For $j \in \{1,2,3\}$, let P_j be the differential operator on $\mathbb{R}^4$ defined by

$$P_j v(t,x) = (x_j + 2it\partial_j)v(t,x) = x_j v(t,x) + 2it\frac{\partial}{\partial x_j}v(t,x). \tag{39}$$

For a multi-index α, define the differential operator P_α on $\mathbb{R}^4$ by

$$P_\alpha = \prod_{j=1}^{3} P_j^{\alpha_j}.$$

In addition, for $x \in \mathbb{R}^3$, set

$$x^\alpha = \prod_{j=1}^{3} x_j^{\alpha_j}.$$

For a given smooth function $u(t,x)$, one has

$$P_j u(t,x) = 2ite^{i\frac{|x|^2}{4t}} \frac{\partial}{\partial x_j}\Big(e^{-i\frac{|x|^2}{4t}} u(t,x)\Big).$$

Indeed,

$$\begin{aligned} 2ite^{i\frac{|x|^2}{4t}}\frac{\partial}{\partial x_j}\Big(e^{-i\frac{|x|^2}{4t}}u\Big) &= -2ite^{i\frac{|x|^2}{4t}}\frac{2ix_j}{4t}e^{-i\frac{|x|^2}{4t}}u(t,x)+2ite^{i\frac{|x|^2}{4t}}e^{-i\frac{|x|^2}{4t}}\frac{\partial}{\partial x_j}u(t,x)\\ &= x_ju(t,x)+2it\frac{\partial}{\partial x_j}u(t,x). \end{aligned}$$

Hence,

$$P_\alpha u(t,x)=(2it)^{|\alpha|}e^{i\frac{|x|^2}{4t}}D^\alpha\Big(e^{-i\frac{|x|^2}{4t}}u(t,x)\Big).$$

On the other hand, a calculation gives

$$[P_j, i\partial_t+\Delta]=0.$$

Therefore, if $u\in C(\mathbb{R},H^1(\mathbb{R}^3))$ is any solution of the linear Schrödinger equation (38), then so is P_ju and $P_\alpha u$.

Proposition .4. *Let α be a multi-index and $T>0$ be given. Let $\psi\in H^1(\mathbb{R}^3)$ be such that $x^\alpha\psi\in H^1(\mathbb{R}^3)$. The corresponding solution u of the IVP*

$$\begin{cases} i\partial_t u+\Delta u-u=0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3,\\ u(x,0)=\psi, & x\in\mathbb{R}^3, \end{cases} \tag{40}$$

satisfies

$$P_\alpha u\in C(\mathbb{R};H^1(\mathbb{R}^3)),$$

and there exists a constant C depending only on T and α such that

$$\|P_\alpha u\|_{H^1(\mathbb{R}^3)}\le C\|x^\alpha\psi\|_{H^1(\mathbb{R}^3)}$$

holds for any $t\in[-T,T]$. In particular, if $\psi\in H^1(\mathbb{R}^3)$ has compact support, then u is infinitely smooth everywhere except at $t=0$.

Demonstração. Using a standard density argument, it is sufficient to prove the result for $\psi\in\mathcal{S}(\mathbb{R}^3)$. Assume first that $|\alpha|=1$, so that $P_\alpha=P_j$ for some $j\in\{1,2,3\}$. First, note that

$$\|u(t)\|_{H^1(\mathbb{R}^3)}=\|\psi\|_{H^1(\mathbb{R}^3)},$$

for any $t\in[-T,T]$. Set $u^j(t,x)=P_ju(t,x)$. Applying the operator P_j to (40) yields

$$\begin{cases} i\partial_t u^j+\Delta u^j-u^j=0,\\ u^j(0,x)=x_j\psi, \end{cases}$$

since $P_j u(0,x) = x_j u(0,x)$. Thus,

$$\|u^j(t)\|_{H^1(\mathbb{R}^3)} = \|x_j\psi\|_{H^1(\mathbb{R}^3)}.$$

The general case $(|\alpha| > 1)$ is obtained by induction. □

Remark .5. The Proposition .4 also holds for the traditional Schrödinger equation without the perturbation term as we can see in (ROSIER; ZHANG, 2009).

.3 PROOF OF THEOREM 2.3

Demonstração. We follow the ideas from (KENIG; MERLE, 2006). Define $I = [0,T]$ and observe that the Cauchy problem (2.5) is equivalent to the integral equation (by Duhamel's formula)

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-\tau)\Delta}[|u|^4u + f]\; d\tau.$$

Define

$$|||u||| = \sup_{t\in I}\|u(t)\|_{L^2} + \sup_{t\in I}\|\nabla u(t)\|_{L^2} + \|u\|_{S(I)} + \|\nabla u\|_{W(I)} + \|\nabla u\|_{Z(I)}.$$

For $R > 0$ to be conveniently chosen later on, consider the set

$$B_R := \Big\{u(t,x) \text{ on } I\times\mathbb{R}^3 : |||u||| \le R\Big\}.$$

We want to show that the operator $\Phi_{u_0} : B_R \longrightarrow B_R$ defined by

$$\Phi_{u_0}(u) = e^{it\Delta}u_0 - \int_0^t e^{i(t-\tau)\Delta}[|u|^4u + f]\; d\tau$$

has a fixed point if R small enough. To this end, first, observe that

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{L^2_x} &\le \|e^{it\Delta}u_0\|_{L^2_x} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}[|u|^4u + f]\; d\tau\right\|_{L^2_x}\\
&\le \|u_0\|_{L^2} + C\||u|^4u\|_{L^1_tL^2_x} + \|f\|_{L^1_tL^2_x}\\
&\le C\|u_0\|_{H^1} + C\|u\|^5_{S(I)} + C_I\|f\|_{L^\infty_tH^1_x}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u)\|_{L^2_x} &\le \|\nabla e^{it\Delta}u_0\|_{L^2_x} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}[u^5 + f]\; d\tau\right\|_{L^2_x}\\
&\le \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \|\nabla f\|_{L^1_tL^2_x}\\
&\le C\|u_0\|_{H^1} + C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + C_I\|f\|_{L^\infty_tH^1_x}.
\end{aligned}$$

Choosing the length of I small enough such that $C_I\|f\|_{L_t^\infty H_x^1} \leq C\|u_0\|_{H^1}$, we have

$$\|\Phi_{u_0}(u)\|_{L_x^2} + \|\nabla\Phi_{u_0}(u)\|_{L_x^2} \leq 2C\|u_0\|_{H^1} + CR^5.$$

Secondly, notice that

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\leq \|\nabla e^{it\Delta}u_0\|_{W(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+f]\, d\tau\right\|_{W(I)} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \|\nabla f\|_{L_t^1L_x^2}.\end{aligned}$$

So, due to Hölder's inequality with $p = \frac{7}{4}$ and $q = \frac{7}{3}$, we get

$$\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} \leq C\|u\|_{S(I)}^4\|\nabla u\|_{W(I)}.$$

Thus,

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\leq C\Big(\|\nabla u_0\|_{L^2} + \|u\|_{S(I)}^4\|\nabla u\|_{W(I)} + \|\nabla g\|_{L_t^1L_x^2}\Big) \\ &\leq C\|u_0\|_{H^1} + CR^5 + C_I\|f\|_{L_t^\infty H_x^1}.\end{aligned}$$

Choosing the length of I small enough such that $C_I\|f\|_{L_t^\infty H_x^1} \leq C\|u_0\|_{H^1}$, one gets

$$\|\nabla\Phi_{u_0}(u)\|_{W(I)} \leq 2C\|u_0\|_{H^1} + CR^5.$$

On the other hand, using Lemma 3.2.1 with $q = 10$ and $r = 30/13$, inequality (3.11) with $q = 10$ and $r = 30/13$ and $m' = 2$ and $n' = \frac{6}{5}$ and Hölder's inequality, one has

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{Z(I)} &\leq \|\nabla e^{it\Delta}u_0\|_{Z(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+f]\, d\tau\right\|_{Z(I)} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^2L_x^{\frac{6}{5}}} + C\|\nabla f\|_{L_t^1L_x^2} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla u\|_{Z(I)}\|u\|_{S(I)}^4 + C\|\nabla f\|_{L_t^1L_x^2} \\ &\leq C\|u_0\|_{H^1} + CR^5 + C_I\|f\|_{L_t^\infty H_x^1} \\ &\leq 2C\|u_0\|_{H^1} + CR^5,\end{aligned}$$

since $C_I\|f\|_{L_t^\infty H_x^1} \leq C\|u_0\|_{H^1}$. Finally,

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{S(I)} &\leq \|\nabla\Phi_{u_0}(u)\|_{Z(I)} \\ &\leq \|\nabla e^{it\Delta}u_0\|_{Z(I)} + \left\|\int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+f]\, d\tau\right\|_{Z(I)} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla u\|_{Z(I)}\|u\|_{S(I)}^4 + C\|\nabla f\|_{L_t^1L_x^2} \\ &\leq C\|u_0\|_{H^1} + CR^5 + C\|\nabla f\|_{L_t^1L_x^2} \\ &\leq C\|u_0\|_{H^1} + CR^5 + C_I\|f\|_{L_t^\infty H_x^1} \\ &\leq 2C\|u_0\|_{H^1} + CR^5,\end{aligned}$$

since $C_I\|f\|_{L_t^\infty H_x^1} \leq C\|u_0\|_{H^1}$. Adding up, we get

$$|||\Phi_{u_0}(u)||| \leq 2C\|u_0\|_{H^1} + CR^5 \leq R,$$

as long as $\|u_0\|_{H^1} \leq \frac{R}{2C} - \frac{R^5}{2}$. Next, denoting $g(u) = |u|^4 u$, we get

$$\begin{aligned}\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L_x^2} &\leq C\|g(u) - g(v)\|_{L_t^1 L_x^2} \\ &\leq C\|u - v\|_{S(I)}\Big(\|u\|_{S(I)}^4 + \|v\|_{S(I)}^4\Big),\end{aligned}$$

$$\begin{aligned}\|\nabla\Phi_{u_0}(u) - \nabla\Phi_{u_0}(v)\|_{L_x^2} &\leq C\|\nabla g(u) - \nabla g(v)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} \\ &\leq C\Bigg(\Big\||u|^4|\nabla u - \nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + \Big\||u - v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} \\ &\quad + \Big\||u - v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}}\Bigg) \\ &\leq C\Bigg(\|u\|_{S(I)}^4\|\nabla u - \nabla v\|_{W(I)} + \|u - v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|_{S(I)}^3 + \\ &\quad + \|u - v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|_{S(I)}^3\Bigg) \\ &\leq CR^4\|u - v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Phi_{u_0}(u) - \nabla\Phi_{u_0}(v)\|_{W(I)} &\leq C\|\nabla g(u) - \nabla g(v)\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} \\ &\leq C\Bigg(\Big\||u|^4|\nabla u - \nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} + \Big\||u - v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}} \\ &\quad + \Big\||u - v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}}\Bigg) \\ &\leq C\Bigg(\|u\|_{S(I)}^4\|\nabla u - \nabla v\|_{W(I)} + \|u - v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|_{S(I)}^3 + \\ &\quad + \|u - v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|_{S(I)}^3\Bigg) \\ &\leq CR^4\|u - v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)}.\end{aligned}$$

Following the same reasoning,

$$\begin{aligned}\|\nabla\Phi_{u_0}(u) - \nabla\Phi_{u_0}(v)\|_{Z(I)} &\leq C\Bigg(\|u\|_{S(I)}^4\|\nabla u - \nabla v\|_{Z(I)} + \|u - v\|_{S(I)}\|\nabla v\|_{Z(I)}\|u\|_{S(I)}^3 + \\ &\quad + \|u - v\|_{S(I)}\|\nabla v\|_{Z(I)}\|v\|_{S(I)}^3\Bigg) \\ &\leq CR^4\|\nabla u - \nabla v\|_{Z(I)} + CR^4\|u - v\|_{S(I)}.\end{aligned}$$

Moreover, by Sobolev's embedding,

$$\|\Phi_{u_0}(u)-\Phi_{u_0}(v)\|_{S(I)} \leq \|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{Z(I)} \leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}.$$

Adding up, we get

$$\begin{aligned}
|||\Phi_{u_0}(u) - \Phi_{u_0}(v)||| &\leq CR^4\|\nabla u - \nabla v\|_{Z(I)} + CR^4\|u - v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)} \\
&\leq CR^4\|\nabla u - \nabla v\|_{Z(I)} + CR^4\|u - v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)} \\
&\quad + CR^4 \sup_{t\in I} \|\nabla u(t) - \nabla v(t)\|_{L^2_x} + CR^4 \sup_{t\in I} \|u(t) - v(t)\|_{L^2} \\
&\leq CR^4|||u - v|||.
\end{aligned}$$

Thus, if $R > 0$ is such that $CR^4 < 1$, then Φ_{u_0} is a contraction in B_R and, therefore, has a unique fixed point, i.e., problem (2.5) has a local solution defined on a maximal interval $[0, T]$. □

Remark .6. Observe that it is possible to use the energy estimates to get global existence, that is, the solution $u = u(x,t)$ of (2.5) is globally well-defined in time. To verify this, first consider the energy defined by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{6}\int_{\mathbb{R}^3} |u|^6,$$

which is conserved if $f = 0$. Multiplying equation (2.5) by $\partial_t \bar{u}$, we have

$$\begin{aligned}
E(t) \leq& E(0) - Re\int_0^t \int_{\mathbb{R}^3} f\partial_t\overline{u}\ dxdt \\
\leq& E(0) - Re\int_0^t \int_{\mathbb{R}^3} f(i\Delta u - i|u|^4u - if)\ dxdt \\
\leq& E(0) + C\int_0^t \|\nabla f(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\ d\tau \\
&+ C\int_0^t \|f(\tau)\|_{L^6}\|u(\tau)^5\|_{L^{\frac{6}{5}}}\ d\tau + \int_0^t \|f(\tau)\|^2_{L^2}\ d\tau \\
\leq& E(0) + C\int_0^t \|\nabla f(\tau)\|_{L^2}\sqrt{E(\tau)}\ d\tau + C\int_0^t \|f(\tau)\|_{L^6}(E(\tau))^{\frac{5}{6}}\ d\tau + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0) + C\int_0^t \|f(\tau)\|_{H^1}\sqrt{E(\tau)}\ d\tau + C\int_0^t \|f(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\ d\tau + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

Then,

$$\begin{aligned}
E(t) \leq& E(0) + C\int_0^t \|f(\tau)\|_{H^1}(E(\tau))^{-\frac{1}{3}}(E(\tau))^{\frac{5}{6}}\ d\tau \\
&+ C\int_0^t \|f(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\ d\tau + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0) + C\int_0^t \|f(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\ d\tau + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0) + C\int_0^t \|f(\tau)\|_{H^1}\Big(1 + (E(\tau))^{\frac{5}{6}}\Big)\ d\tau + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}$$

Therefore,

$$\max_{0\leq t\leq T} E(\tau) \;\; \leq \;\; E(0) + C\Big(1 + \max_{0\leq\tau\leq t}(E(\tau))^{\frac{5}{6}}\Big)\|f\|_{L^1([0,T];H^1(\mathbb{R}^3))} + \|f\|^2_{L^2([0,T]\times\mathbb{R}^3)}.$$

So, finally, it follows that

$$E(t) \leq C\Big(1 + E(0)^6 + \|f\|^{12}_{L^2([0,T]\times\mathbb{R}^3)} + \|f\|^6_{L^1([0,T];H^1(\mathbb{R}^3))}\Big).$$

So, if $f \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$, then the energy is bounded. Now, for the L^2–energy (or mass), define the following quantity

$$\overline{E}(t) = \frac{1}{2}\|u(t)\|^2_{L^2}.$$

Multiplying equation (2.5) by $\overline{u}$, taking its imaginary part and integrating by parts yields

$$\begin{aligned}
\frac{1}{2}\|u(t)\|^2_{L^2} \;\; &\leq \;\; \frac{1}{2}\|u(0)\|^2_{L^2} + Im\int_0^t\int_{\mathbb{R}^3} f\cdot\overline{u}\; dxdt \\
&\leq \;\; \frac{1}{2}\|u(0)\|^2_{L^2} + \int_0^t\int_{\mathbb{R}^3} |f\cdot\overline{u}|\; dxdt \\
&\leq \;\; \frac{1}{2}\|u(0)\|^2_{L^2} + \int_0^t \|f(\tau)\|_{L^2}\|u(\tau)\|_{L^2}dt.
\end{aligned}$$

So,

$$\begin{aligned}
\overline{E}(t) \;\; &\leq \;\; \overline{E}(0) + \int_0^t \|f(\tau)\|_{L^2}\|u(\tau)\|_{L^2}dt \\
&\leq \;\; \overline{E}(0) + C\int_0^t \|f(\tau)\|_{L^2}\sqrt{\overline{E}(\tau)}dt.
\end{aligned}$$

This implies that the L^2–energy is bounded if $f \in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$.