CONTROL OF THE SCHRÖDINGER EQUATION IN $\mathbb{R}^3$: THE CRITICAL CASE

PABLO BRAZ E SILVA, ROBERTO DE A. CAPISTRANO–FILHO,
JACKELLYNY DASSY DO NASCIMENTO CARVALHO, AND DAVID DOS SANTOS FERREIRA

Abstract. This article deals with the $\dot{H}^1$–level exact controllability for the defocusing critical nonlinear Schrödinger equation in $\mathbb{R}^3$. Firstly, we show the problem under consideration to be well-posed using Strichartz estimates. Moreover, through the Hilbert uniqueness method, we prove the linear Schrödinger equation to be controllable. Finally, we use a perturbation argument and show local exact controllability for the critical nonlinear Schrödinger equation.

1. Introduction

1.1. **Addressed issues and review of the literature.**

We consider the $\dot{H}^1(\mathbb{R}^3)$ exact controllability for the defocusing critical nonlinear Schrödinger equation (C-NLS)

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = f(x,t), \text{ on } \mathbb{R}^3 \times [0,+\infty), \\ u(x,0) = u_0(x) \in \dot{H}^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function of two variables $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, the subscripts denote the corresponding partial derivatives, while the function $f(x,t)$ is a control input. We are mainly concerned with the following exact control problem for system (1.1).

Control problem: Let $T > 0$ be given. For any given $u_0, u_T \in \dot{H}^1(\mathbb{R}^3)$, can one find a control $f(x,t)$ such that system (1.1) admits a solution $u \in C\left([0,T]; \dot{H}^1(\mathbb{R}^3)\right)$ satisfying $u(x,T) = u_T(x)$ in $\mathbb{R}^3$?

Control properties of Schrödinger equations have received a lot of attention in the last decades. For example, regarding control issues, one may see [12, 19, 20, 21] and the references therein. As for Carleman estimates and applications to inverse problems, we cite [2, 4, 5, 16, 18, 27] and the references therein. An excellent review of the contributions up to 2003 is in [29].

Let us detail some recent results. The results due to Illner *et al.* [8, 9] considered internal controllability of the nonlinear Schrödinger equation posed on a finite interval $(-\pi,\pi)$

$$\begin{cases} i\partial_t v + \partial_{xx} v + \lambda |v|^2 v = f(x,t), \quad x \in (-\pi,\pi), \\ v(-\pi,t) = v(\pi,t), \quad \partial_x v(-\pi,t) = \partial_x v(\pi,t), \end{cases} \tag{1.2}$$

where the forcing function $f = f(x,t)$, supported in a sub-interval of $(-\pi,\pi)$, is a control input. They showed that system (1.2) is locally exactly controllable in the space

$$H_p^1(-\pi,\pi) := \{v \in H^1(-\pi,\pi) : v(-\pi) = v(\pi)\}.$$

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*Corresponding author: ddsf@math.cnrs.fr.

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Later, Lange and Teismann [13] considered the internal control of the same nonlinear Schrödinger equation in (1.2) posed on a finite interval but with the homogeneous Dirichlet boundary conditions

$$v(-\pi,t)=v(\pi,t)=0,$$

and showed that this is locally exactly controllable in the space $H_0^1(0,\pi)$ around a special ground state of the equation (see also [14] for internal controllability of the nonlinear Schrödinger equation posed on a finite interval).

In [24], Rosier and Zhang considered the nonlinear Schrödinger equation

$$i\partial_t u+\Delta u+\lambda|u|^2u=0 \tag{1.3}$$

posed on a bounded domain K in $\mathbb{R}^n$ with both Dirichlet boundary conditions and Neumann boundary conditions. They showed that if either

$$s>\frac{n}{2}$$

or

$$0\leq s<\frac{n}{2},\quad \text{with } 1\leq n<2+2s$$

or

$$s=0,1\quad \text{with } n=2,$$

then both systems, with Dirichlet and Neumann conditions, when the control inputs are acting on the whole boundary of K, are locally exactly controllable in the classical Sobolev space $H^s(K)$ around any smooth solution of the Schrödinger equation.

In [25], the authors extend the results of Rosier and Zhang [23]. More precisely, they assume that the spatial variable belongs to the rectangle

$$\mathcal{R}=(0,l_1)\times\cdots\times(0,l_n)$$

and investigate the control properties of the semi-linear Schrödinger equation

$$i\partial_t u+\Delta u+\lambda|u|^\alpha u=ia(x)h(x,t),\quad x\in\mathbb{T}^n\quad t\in(0,T),$$

where $\lambda\in\mathbb{R}$ and $\alpha\in 2\mathbb{N}^*$ by combining new linear controllability results in the spaces $H^s(\mathcal{R})$ with Bourgain analysis. In this case, the geometric control condition is not required (see [25] for more details).

Finally, considering a $2d$-compact Riemann manifold M without boundary, Dehman *et al.* [7] studied internal control and stabilization of nonlinear Schrödinger equations

$$i\partial_t w+\Delta w-|w|^2w=f(x,t),\ x\in M.$$

They showed, in particular, that the system is semi-globally exactly controllable and semi-globally exponentially stabilizable in the space $H^1(M)$ assuming that both the geometric control condition and a unique continuation condition are satisfied (see [7] for more details). The interesting work [3] extended this one: The authors studied global controllability and stabilization properties for the fractional Schrödinger equation on $d-$dimensional compact Riemannian manifolds without boundary (M,g). They used microlocal analysis to show the propagation of regularity which, together with the geometric control condition and a unique continuation property, allowed them to prove global control results.

1.2. **Main results.** As discussed above, most of the available results at the moment are for the classical Schrödinger equation (1.3) in different domains and considering the control inputs acting on the whole boundary. However, in the critical case, namely, system (1.1), the internal control problem remains open.

With the aim of presenting the first answer to the control problem stated at the beginning of the section, let us consider the control system

$$\begin{cases} i\partial_t u+\Delta u-|u|^4u=\varphi(x)h(x,t),\ \text{on } \mathbb{R}^3\times[0,+\infty),\\ u(x,0)=u_0(x)\in\dot{H}^1(\mathbb{R}^3),\end{cases} \tag{1.4}$$

where the function $\varphi \in C^\infty(\mathbb{R}^3, [0,1])$ and satisfies

$$\varphi(x) = \begin{cases} 0, & if\ |x| \le R, \\ 1, & if\ |x| \ge R+1, \end{cases} \tag{1.5}$$

for some $R > 0$ large enough. Our result below gives a first answer in this direction.

Theorem 1.1 (Local exact controllability)**.** *Let $T > 0$ be given. There exists $\delta > 0$ such that for any u_0 and $\tilde{u}_0$ in $H^1(\mathbb{R}^3)$ satisfying $\|u_0\|_{H^1} \le \delta$ and $\|\tilde{u}_0\|_{H^1} \le \delta$, one can find $h(x,t) \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ such that system (1.4) admits a solution $u \in C([0,T]; H^1(\mathbb{R}^3))$ satisfying $u(T) = \tilde{u}_0$.*

In our arguments, Strichartz-type inequalities play a fundamental role giving the well-posedness of system (1.4). In addition, as a first step, we have used a Carleman estimate to obtain the following linear control result.

Theorem 1.2. *For every initial data $u_0 \in H^1(\mathbb{R}^3)$ and every $T > 0$, there exists a control $h(x,t) \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ with support in $\mathbb{R} \times (\mathbb{R}^3 \backslash B_R(0))$, $R > 0$, such that the unique solution of the linear system associated to (1.4) satisfies $u(T, \cdot) = 0$.*

With Theorem 1.2 in hand, a perturbation argument ensures that we can get a local exact controllability result for the critical Schrödinger equation (1.4), so giving Theorem 1.1.

1.3. **Structure of this work.** We finish our introduction by giving an outline of this work. It is divided as follows:

– In Section 2, we give auxiliary results that play an important role in establishing our control result. Precisely, we present a review of the Cauchy problem for the Schrödinger equation.

– In Section 3, we present the proof of our main results. Firstly, we show the exact controllability around the null trajectories for the linear system associated with (1.4), proving Theorem 1.2. Then, through a perturbation argument, we show Theorem 1.1.

– Finally, we discuss some future perspectives in Section 4.

2. A review of the Cauchy problem

2.1. **Smoothing.** For the sake of completeness, we discuss the smoothing properties of the linear Schrödinger equation,

$$\begin{cases} i\partial_t u + \Delta u = 0, & x \in \mathbb{R}^3, t \in \mathbb{R}, \\ u(x,0) = \psi(x), & x \in \mathbb{R}^3, \end{cases} \tag{2.1}$$

which will play an important role in establishing the exact controllability for the defocusing critical nonlinear Schrödinger equation (1.4). To this end, for $j \in \{1,2,3\}$, let P_j be the differential operator on $\mathbb{R}^4$ defined by

$$P_j v(t,x) = (x_j + 2it\partial_j) v(t,x). \tag{2.2}$$

For a multi-index α, define the differential operator P_α on $\mathbb{R}^4$ by

$$P_\alpha = \prod_{j=1}^{3} P_j^{\alpha_j}.$$

Additionally, for $x \in \mathbb{R}^3$, consider

$$x^\alpha = \prod_{j=1}^{3} x_j^{\alpha_j}.$$

For any smooth function $u(t,x)$, one has

$$P_j u(t,x) = 2ite^{i\frac{|x|^2}{4t}} \frac{\partial}{\partial x_j}\left(e^{-i\frac{|x|^2}{4t}} u(t,x)\right).$$

Indeed, note that

$$\begin{aligned} 2ite^{i\frac{|x|^2}{4t}}\frac{\partial}{\partial x_j}\big(e^{-i\frac{|x|^2}{4t}}u\big) &= -2ite^{i\frac{|x|^2}{4t}}\frac{2ix_j}{4t}e^{-i\frac{|x|^2}{4t}}u(t,x)+2ite^{i\frac{|x|^2}{4t}}e^{-i\frac{|x|^2}{4t}}\frac{\partial}{\partial x_j}u(t,x) \\ &= x_ju(t,x)+2it\frac{\partial}{\partial x_j}u(t,x). \end{aligned}$$

Hence,

$$P_\alpha u(t,x)=(2it)^{|\alpha|}e^{i\frac{|x|^2}{4t}}D^\alpha\big(e^{-i\frac{|x|^2}{4t}}u(t,x)\big).$$

On the other hand, we easily obtain

$$[P_j, i\partial_t+\Delta]=0.$$

Thus, considering $u\in C(\mathbb{R},H^1(\mathbb{R}^3))$ to be any solution of the linear Schrödinger equation (2.1), one has that P_ju and $P_\alpha u$ is also a solution.

Now, with the previous analysis in hand and taking into account the relation (2.2), we present the next result, which gives a local smoothing property for the linear Schrödinger equation (2.1).

Proposition 2.1. *Let α be a multi-index and $T>0$ be given. Let $\psi\in H^1(\mathbb{R}^3)$ be such that $x^\alpha\psi\in H^1(\mathbb{R}^3)$. Then, the corresponding solution u of the IVP (2.1) satisfies*

$$P_\alpha u\in C(\mathbb{R};H^1(\mathbb{R}^3))$$

and there exists a constant C depending only on T and α such that

$$\|P_\alpha u\|_{H^1(\mathbb{R}^3)}\le C\|x^\alpha\psi\|_{H^1(\mathbb{R}^3)}$$

holds for any $t\in[-T,T]$. In particular, if $\psi\in H^1(\mathbb{R}^3)$ has compact support, then u is infinitely smooth everywhere except at $t=0$.

Proof. A standard density argument assures that it is sufficient to prove the result for $\psi\in\mathcal{S}(\mathbb{R}^3)$. To this end, assume, first, that $|\alpha|=1$, so that $P_\alpha=P_j$ for some $j\in\{1,2,3\}$. Now, note that

$$\|u(t)\|_{H^1(\mathbb{R}^3)}=\|\psi\|_{H^1(\mathbb{R}^3)},$$

for any $t\in[-T,T]$. Let $u^j(t,x)=P_ju(t,x)$. Applying the operator P_j to (2.1) yields

$$\begin{cases} i\partial_t u^j+\Delta u^j=0,\\ u^j(0,x)=x_j\psi,\end{cases}$$

due to the fact that $P_ju(0,x)=x_ju(0,x)$. Since

$$u^j(t)=e^{it\Delta}(x_j\psi),$$

we get that

$$\|u^j(t)\|_{H(\mathbb{R}^3)}\le C\|x_j\psi\|_{H^1(\mathbb{R}^3)}.$$

The general case ($|\alpha|>1$) follows by induction. ☐

2.2. **Existence.** Here, we are interested in studying the $\dot H^1$ critical, defocusing, Cauchy problem for the nonlinear Schrödinger equation (C-NLS)

$$\begin{cases} i\partial_t u+\Delta u-|u|^4u=g, & (t,x)\in\mathbb{R}\times\mathbb{R}^3,\\ u(0)=u_0\in\dot H^1(\mathbb{R}^3),\end{cases} \tag{2.3}$$

where $g(t,x)=g\in L^\infty_{loc}(\mathbb{R},H^1(\mathbb{R}^3))$. Let us first present some definitions used throughout the paper.

Definition 1. Let s $\in\mathbb{R}$. The homogeneous Sobolev space $\dot H^s(\mathbb{R}^d)$ is the space of tempered distributions u over $\mathbb{R}^d$ which have Fourier transform belonging to $L^1_{loc}(\mathbb{R}^d)$ and satisfying

$$\|u\|^2_{\dot H^s}\overset{\text{def}}{=}\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat u(\xi)|^2d\xi<\infty.$$

We note that the spaces $\dot H^s$ and $\dot H^{s'}$ are not comparable for inclusion.

Definition 2. A pair (q, r) is called L^2-admissible if $r \in [2, 6)$ and q satisfies

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2}.$$

Such a pair is called H^1-admissible if $r \in [6, +\infty)$ and q satisfies

$$\frac{2}{q} + \frac{3}{r} = \frac{1}{2}.$$

With these definitions in hand, we present two results that are paramount to prove that the Cauchy problem (2.3) is well-posed. The first one is *Strichartz estimates* and the second one is a standard Sobolev embedding. These results can be found in [6, 10].

Lemma 2.2. *Let (q, r) be a L^2-admissible pair. We have*

$$\|e^{it\Delta}h\|_{L^q_t L^r_x} \leq c\|h\|_{L^2}, \tag{2.4}$$

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g \, d\tau \right\|_{L^q_t L^r_x} + \left\| \int_0^t e^{i(t-\tau)\Delta} g \, d\tau \right\|_{L^q_t L^r_x} \leq c\|g\|_{L^{q'}_t L^{r'}_x}, \tag{2.5}$$

and

$$\left\| \int_{-\infty}^{+\infty} e^{it\Delta} g(\tau) \, d\tau \right\|_{L^2_x} \leq C\|g\|_{L^{q'}_t L^{r'}_x}.$$

Additionally, we have

$$\left\| \int_{-\infty}^{+\infty} e^{i(t-\tau)\Delta} g(\tau) \, d\tau \right\|_{L^q_t L^r_x} \leq C\|g\|_{L^{m'}_t L^{n'}_x} \tag{2.6}$$

where (q, r), (m, n) are any L^2-admissible pair, wich is an generalization of (2.5).

Lemma 2.3. *For $v \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$, we have*

$$\|v\|_{L^{10}_t L^{10}_x} \leq C\|\nabla v\|_{L^{10}_t L^{\frac{30}{13}}_x}.$$

For an interval I, define the norms $S(I)$, $W(I)$ and $Z(I)$ by

$$\|u\|_{S(I)} = \|u\|_{L^{10}(I;L^{10}(\mathbb{R}^3))}, \quad \|u\|_{Z(I)} = \|u\|_{L^{10}(I;L^{\frac{30}{13}}(\mathbb{R}^3))} \text{ and } \quad \|u\|_{W(I)} = \|u\|_{L^{\frac{10}{3}}(I;L^{\frac{10}{3}}(\mathbb{R}^3))}.$$

The following theorem gives us the solution to problem (2.3).

Theorem 2.4. *Let $u_0 \in \dot{H}^1(\mathbb{R}^3)$. If $\|u_0\|_{\dot{H}^1}$ is small enough, then there exists an interval I and a unique solution $u(t, x)$ of problem* (2.3) *in $I \times \mathbb{R}^3$, with $u \in C(I; \dot{H}^1(\mathbb{R}^3))$, satisfying*

$$\|\nabla u\|_{W(I)} < \infty, \ \|u\|_{S(I)} < \infty \text{ and } \|\nabla u\|_{Z(I)} < \infty.$$

Proof. We follow the ideas from [11]. Assume, without loss of generality, that $t_0 = 0$. Observe that the Cauchy problem (2.3) is equivalent to the integral equation (Duhamel's formula)

$$u(t) = e^{it\Delta}u_0 - \int_0^t e^{i(t-\tau)\Delta}[|u|^4 u + g] \, d\tau.$$

Define

$$|||u||| = \sup_{t \in I} \|u(t)\|_{L^2} + \sup_{t \in I} \|\nabla u(t)\|_{L^2} + \|u\|_{S(I)} + \|\nabla u\|_{W(I)} + \|\nabla u\|_{Z(I)}.$$

For $R > 0$ to be conveniently chosen later on, consider the set

$$B_R := \Big\{ u(t, x) \text{ on } I \times \mathbb{R}^3 : |||u||| \leq R \Big\}.$$

We want to show that the operator $\Phi_{u_0} : B_R \longrightarrow B_R$ defined by

$$\Phi_{u_0}(u) = e^{it\Delta}u_0 - \int_0^t e^{i(t-\tau)\Delta}[|u|^4 u + g] \, d\tau$$

has a fixed point for R small enough. To this end, first, observe that

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{L^2_x} &\leq \|e^{it\Delta}u_0\|_{L^2_x} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[|u|^4u+g]\, d\tau \right\|_{L^2_x} \\ &\leq \|u_0\|_{L^2} + C\||u|^4u\|_{L^1_tL^2_x} + \|g\|_{L^1_tL^2_x} \\ &\leq C\|u_0\|_{\dot{H}^1} + C\|u\|^5_{S(I)} + C_I\|g\|_{L^\infty_t H^1_x}\end{aligned}$$

and

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{L^2_x} &\leq \|\nabla e^{it\Delta}u_0\|_{L^2_x} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+g]\, d\tau \right\|_{L^2_x} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \|\nabla g\|_{L^1_tL^2_x} \\ &\leq C\|u_0\|_{\dot{H}^1} + C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + C_I\|g\|_{L^\infty_t H^1_x}.\end{aligned}$$

Choosing the length of I small enough such that $C_I\|g\|_{L^\infty_t H^1_x} \leq C\|u_0\|_{\dot{H}^1}$, we have

$$\|\Phi_{u_0}(u)\|_{L^2_x} + \|\nabla\Phi_{u_0}(u)\|_{L^2_x} \leq 2C\|u_0\|_{\dot{H}^1} + CR^5.$$

Secondly, notice that

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\leq \|\nabla e^{it\Delta}u_0\|_{W(I)} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+g]\, d\tau \right\|_{W(I)} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \|\nabla g\|_{L^1_tL^2_x}.\end{aligned}$$

Due to Hölder's inequality with $p = \frac{7}{4}$ and $q = \frac{7}{3}$, we get

$$\|\nabla|u|^4u\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} \leq C\|u\|^4_{S(I)}\|\nabla u\|_{W(I)}.$$

Thus

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{W(I)} &\leq C\Big(\|\nabla u_0\|_{L^2} + \|u\|^4_{S(I)}\|\nabla u\|_{W(I)} + \|\nabla g\|_{L^1_tL^2_x}\Big) \\ &\leq C\|u_0\|_{\dot{H}^1} + CR^5 + C_I\|g\|_{L^\infty_t H^1_x}.\end{aligned}$$

Choosing the length of I small enough such that $C_I\|g\|_{L^\infty_t H^1_x} \leq C\|u_0\|_{\dot{H}^1}$, one gets

$$\|\nabla\Phi_{u_0}(u)\|_{W(I)} \leq 2C\|u_0\|_{\dot{H}^1} + CR^5.$$

On other hand, using estimate (2.4) with $q = 10$ and $r = 30/13$, due to the inequality (2.6) with the same q, r and $m' = 2$ and $n' = \frac{6}{5}$, Hölder's inequality gives

$$\begin{aligned}\|\nabla\Phi_{u_0}(u)\|_{Z(I)} &\leq \|\nabla e^{it\Delta}u_0\|_{Z(I)} + \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+g]\, d\tau \right\|_{Z(I)} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla|u|^4u\|_{L^2_tL_x^{\frac{6}{5}}} + C\|\nabla g\|_{L^1_tL^2_x} \\ &\leq \|\nabla u_0\|_{L^2} + C\|\nabla u\|_{Z(I)}\|u\|^4_{S(I)} + C\|\nabla g\|_{L^1_tL^2_x} \\ &\leq C\|u_0\|_{\dot{H}^1} + CR^5 + C_I\|g\|_{L^\infty_t H^1_x} \\ &\leq 2C\|u_0\|_{\dot{H}^1} + CR^5\end{aligned}$$

since $C_I\|g\|_{L^\infty_t H^1_x} \leq C\|u_0\|_{\dot H^1}$ Finally,

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{S(I)} &\leq \|\nabla\Phi_{u_0}(u)\|_{Z(I)}\\
&\leq \|\nabla e^{it\Delta}u_0\|_{Z(I)} + \Big\|\int_0^t \nabla e^{i(t-\tau)\Delta}[u^5+g]\ d\tau\Big\|_{Z(I)}\\
&\leq \|\nabla u_0\|_{L^2} + C\|\nabla u\|_{Z(I)}\|u\|^4_{S(I)} + C\|\nabla g\|_{L^1_tL^2_x}\\
&\leq C\|u_0\|_{\dot H^1} + CR^5 + C\|\nabla g\|_{L^1_tL^2_x}\\
&\leq C\|u_0\|_{\dot H^1} + CR^5 + C_I\|g\|_{L^\infty_t H^1_x}\\
&\leq 2C\|u_0\|_{\dot H^1} + CR^5,
\end{aligned}$$

since $C_I\|g\|_{L^\infty_t H^1_x} \leq C\|u_0\|_{\dot H^1}$. Summing up, we get

$$|||\Phi_{u_0}(u)||| \leq 2C\|u_0\|_{\dot H^1} + CR^5 \leq R$$

as long as $\|u_0\|_{\dot H^1} \leq \frac{R}{2C} - \frac{R^5}{2}$, where $R < (1/C)^{1/4}$.

Next, denoting $f(u) = |u|^4u$, we get

$$\begin{aligned}
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{L^2_x} &\leq C\|f(u)-f(v)\|_{L^1_tL^2_x}\\
&\leq C\|u-v\|_{S(I)}\big(\|u\|^4_{S(I)} + \|v\|^4_{S(I)}\big),
\end{aligned}$$

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u) - \nabla\Phi_{u_0}(v)\|_{L^2_x} &\leq C\|\nabla f(u) - \nabla f(v)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\leq C\Bigg(\Big\||u|^4|\nabla u - \nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \Big\||u-v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\qquad + \Big\||u-v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\Bigg)\\
&\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u - \nabla v\|_{W(I)} + \|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|^3_{S(I)} +\\
&\qquad + \|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|^3_{S(I)}\Bigg)\\
&\leq CR^4\|u-v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\Phi_{u_0}(u) - \nabla\Phi_{u_0}(v)\|_{W(I)} &\leq C\|\nabla f(u) - \nabla f(v)\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\leq C\Bigg(\Big\||u|^4|\nabla u - \nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}} + \Big\||u-v||u|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\\
&\qquad + \Big\||u-v||v|^3|\nabla v|\Big\|_{L_t^{\frac{10}{7}}L_x^{\frac{10}{7}}}\Bigg)\\
&\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u - \nabla v\|_{W(I)} + \|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|u\|^3_{S(I)} +\\
&\qquad + \|u-v\|_{S(I)}\|\nabla v\|_{W(I)}\|v\|^3_{S(I)}\Bigg)\\
&\leq CR^4\|u-v\|_{S(I)} + CR^4\|\nabla u - \nabla v\|_{W(I)}.
\end{aligned}$$

Following the same reasoning as before,

$$
\begin{aligned}
\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{Z(I)} &\leq C\Bigg(\|u\|^4_{S(I)}\|\nabla u-\nabla v\|_{Z(I)}+\|u-v\|_{S(I)}\|\nabla v\|_{Z(I)}\|u\|^3_{S(I)}+ \\
&\qquad +\|u-v\|_{S(I)}\|\nabla v\|_{Z(I)}\|v\|^3_{S(I)}\Bigg) \\
&\leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}.
\end{aligned}
$$

Moreover, by Sobolev's embedding,

$$
\|\Phi_{u_0}(u)-\Phi_{u_0}(v)\|_{S(I)}\leq\|\nabla\Phi_{u_0}(u)-\nabla\Phi_{u_0}(v)\|_{Z(I)}\leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}.
$$

Summing up yields

$$
\begin{aligned}
|||\Phi_{u_0}(u)-\Phi_{u_0}(v)||| &\leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}+CR^4\|\nabla u-\nabla v\|_{W(I)} \\
&\leq CR^4\|\nabla u-\nabla v\|_{Z(I)}+CR^4\|u-v\|_{S(I)}+CR^4\|\nabla u-\nabla v\|_{W(I)} \\
&\quad +CR^4\sup_{t\in I}\|\nabla u(t)-\nabla v(t)\|_{L^2_x}+CR^4\sup_{t\in I}\|u(t)-v(t)\|_{L^2} \\
&\leq CR^4|||u-v|||
\end{aligned}
$$

Thus, if $R>0$ is such that $CR^4<1$, then Φ_{u_0} is a contraction in B_R and, therefore, has a unique fixed point, i.e., problem (2.3) has a local solution defined on a maximal interval $[0,T]$. □

Remark 1. *Observe that it is possible to use the energy estimates to get global existence, that is, the solution $u=u(x,t)$ of problem (2.3) is globally well-defined in time. To verify this, first consider the energy defined by*

$$
E(t)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{1}{6}\int_{\mathbb{R}^3}|u|^6,
$$

which is conserved if $g=0$. Multiplying equation (2.3) by $\partial_t\bar{u}$ and integrating, we have

$$
\begin{aligned}
E(t) \leq& E(0)-\operatorname{Re}\int_0^t\int_{\mathbb{R}^3}g\partial_t\overline{u}\;dxdt \\
\leq& E(0)-\operatorname{Re}\int_0^t\int_{\mathbb{R}^3}g(i\Delta u-i|u|^4u-ig)\;dxdt \\
\leq& E(0)+C\int_0^t\|\nabla g(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}\;d\tau \\
&+C\int_0^t\|g(\tau)\|_{L^6}\|u(\tau)^5\|_{L^{\frac{6}{5}}}\;d\tau+\int_0^t\|g(\tau)\|^2_{L^2}\;d\tau \\
\leq& E(0)+C\int_0^t\|\nabla g(\tau)\|_{L^2}\sqrt{E(\tau)}\;d\tau+C\int_0^t\|g(\tau)\|_{L^6}(E(\tau))^{\frac{5}{6}}\;d\tau+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0)+C\int_0^t\|g(\tau)\|_{H^1}\sqrt{E(\tau)}\;d\tau+C\int_0^t\|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\;d\tau+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}
$$

Choosing τ such that $E(\tau)>k$, we have

$$
\begin{aligned}
E(t) \leq& E(0)+C\int_0^t\|g(\tau)\|_{H^1}(E(\tau))^{-\frac{1}{3}}(E(\tau))^{\frac{5}{6}}\;d\tau \\
&+C\int_0^t\|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\;d\tau+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0)+C\int_0^t\|g(\tau)\|_{H^1}(E(\tau))^{\frac{5}{6}}\;d\tau+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)} \\
\leq& E(0)+C\int_0^t\|g(\tau)\|_{H^1}\left(1+(E(\tau))^{\frac{5}{6}}\right)\;d\tau+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.
\end{aligned}
$$

Therefore,

$$\max_{0\le t\le T} E(\tau) \le E(0)+C\Big(1+\max_{0\le\tau\le t}(E(\tau))^{\frac{5}{6}}\Big)\|g\|_{L^1([0,T];H^1(\mathbb{R}^3))}+\|g\|^2_{L^2([0,T]\times\mathbb{R}^3)}.$$

So, finally, it follows that

$$E(t)\le C\Big(1+E(0)^6+\|g\|^{10}_{L^2([0,T]\times\mathbb{R}^3)}+\|g\|^6_{L^1([0,T];H^1(\mathbb{R}^3))}\Big).$$

So, if $g\in L^\infty_{loc}(\mathbb{R},H^1(\mathbb{R}^3))$, then the energy is bounded, which implies global existence in $\dot H^1(\mathbb{R}^3)$. Now, for the L^2–energy (or mass), define the following quantity

$$\overline{E}(t)=\frac{1}{2}\|u(t)\|^2_{L^2}.$$

Multiplying equation (2.3) by $\overline{u}$, taking the imaginary part and integrating by parts yields

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2} &\le \frac{1}{2}\|u(0)\|^2_{L^2}+\operatorname{Im}\int_0^t\int_{\mathbb{R}^3} g\cdot\overline{u}\ dxdt\\ &\le \frac{1}{2}\|u(0)\|^2_{L^2}+\int_0^t\int_{\mathbb{R}^3}|g\cdot\overline{u}|\ dxdt\\ &\le \frac{1}{2}\|u(0)\|^2_{L^2}+\int_0^t \|g(\tau)\|_{L^2}\|u(\tau)\|_{L^2}dt.\end{aligned}$$

So

$$\begin{aligned}\overline{E}(t) &\le \overline{E}(0)+\int_0^t\|g(\tau)\|_{L^2}\|u(\tau)\|_{L^2}dt\\ &\le \overline{E}(0)+C\int_0^t\|g(\tau)\|_{L^2}\sqrt{\overline{E}(\tau)}dt.\end{aligned}$$

This implies that the L^2–energy is bounded if $g\in L^\infty_{loc}(\mathbb{R},H^1(\mathbb{R}^3))$ and yields global existence in $L^2(\mathbb{R}^3)$.

3. Controllability for the critical nonlinear Schrödinger equation

In this section, we prove Theorem 1.1 using a duality strategy which reduces the controllability problem (1.4) to prove an observability inequality, the so-called "Hilbert Uniqueness Method" [17], for the solutions of the linear system

$$\begin{cases} i\partial_t u+\Delta u=\varphi(x)h(x,t), & x\in\mathbb{R}^3,\ t\in(0,T),\\ u(0)=u_0,\end{cases}\tag{3.1}$$

where $\varphi=\varphi(x)$ is defined by (1.5).

3.1. Linear Schrödinger equation: Exact controllability.

We prove the following:

Theorem 3.1. *For every initial data $u_0\in H^1(\mathbb{R}^3)$ and every $T>0$, there exist $R>0$ and a control $h(x,t)\in C(\mathbb{R},H^1(\mathbb{R}^3))$ with support in $\mathbb{R}\times(\mathbb{R}^3\backslash B_R(0))$ such that the unique solution of system (3.1) satisfies $u(T,\cdot)=0$.*

The exact controllability of system (3.1) follows from the observability inequality, namely,

$$\|v_0\|^2_{H^{-1}}\le C\int_0^T\|\varphi v(t)\|^2_{H^{-1}}\ dt,\tag{3.2}$$

where $v(x,t)$ is a solution to the adjoint system associated to (3.1)

$$\begin{cases} i\partial_t v+\Delta v=0 \text{ in } \mathbb{R}\times\mathbb{R}^3,\\ v(0)=v_0\in H^{-1}(\mathbb{R}^3).\end{cases}\tag{3.3}$$

The observability inequality (3.2) is given by the following result.

Proposition 3.2. *Let φ be a C^∞ real function on $\mathbb{R}^3$ as in (1.5). Then, for every $T > 0$, there exists a constant $C = C(T) > 0$ such that inequality (3.2) holds for every solution $v(t, x)$ of system (3.3).*

Proof. We will split the proof into several steps.

First step: H^1*–observability.*

Lemma 3.3. *Consider the system*

$$\begin{cases} i\partial_t w + \Delta w = 0, & x \in \mathbb{R}^3,\ t \in (0, T), \\ w(0) = w_0 \in H^1(\mathbb{R}^3). \end{cases} \tag{3.4}$$

There exists a constant $C > 0$ such that for each $w_0 \in H^1(\mathbb{R}^3)$, the solution $w(t)$ of (3.4) satisfies

$$\|w_0\|^2_{H^1(\mathbb{R}^3)} \leq C \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\, dt. \tag{3.5}$$

Proof. Let $q \in C_0^\infty(\mathbb{R}^3)$ such that

$$q(x) = \begin{cases} x, & if\ |x| \leq R + 2, \\ 0, & if\ |x| \geq R + 3. \end{cases}$$

Multiplying the equation in (3.4) by $q \cdot \nabla\overline{w} + \frac{1}{2}\overline{w}(div_x q)$, taking the real part and integrating by parts gives

$$\begin{aligned} &\frac{1}{2}\mathrm{Im}\int_{\mathbb{R}^3} (wq \cdot \nabla\overline{w})\, dx \bigg|_0^T + \frac{1}{2}\mathrm{Re}\int_0^T \int_{\mathbb{R}^3} w\nabla(div_x q) \cdot \nabla\overline{w}\, dxdt \\ &\qquad + \mathrm{Re}\int_0^T \int_{\mathbb{R}^3} \sum_{j,k=1}^{3} \left(\frac{\partial q_k}{\partial x_j}\frac{\partial \overline{w}}{\partial x_k}\frac{\partial w}{\partial x_j}\right) dxdt = 0, \end{aligned} \tag{3.6}$$

where we have used the fact that the function $q(x)$ has compact support. Notice that (3.4) is forward and backward well-posed in $H^1(\mathbb{R}^3)$, so for any $t_0 \in [0, T]$, there exists a constant $c > 0$ such that

$$\|w(t_0)\|^2_{H^1(\mathbb{R}^3)} \leq c \int_0^T \|w(t_0)\|^2_{H^1(\mathbb{R}^3)}\, dt = c\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\, dt. \tag{3.7}$$

Thus, it follows from (3.6) and (3.7) that

$$\begin{aligned} \int_0^T \int_{B_{R+2}(0)} |\nabla w|^2\, dxdt \ &\leq\ C_\varepsilon\left(\int_0^T \int_{B_{R+3}(0)\setminus B_{R+2}(0)} |\nabla w|^2\, dxdt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\, dt\right) \\ &\quad + \varepsilon\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\, dt, \end{aligned}$$

for any $\varepsilon > 0$ and some constant $C_\varepsilon > 0$. We also have

$$\|w(t)\|^2_{H^1(\mathbb{R}^3)} \leq C\left(\int_{B_{R+2}(0)} |\nabla w|^2\, dx + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\right).$$

Indeed, observe that

$$\begin{aligned} \|w(t)\|^2_{H^1(\mathbb{R}^3)} &= \|w(t)\|^2_{H^1(B_{R+1}(0))} + \|w(t)\|^2_{H^1(\mathbb{R}^3\setminus B_{R+1}(0))} \\ &= \|w(t)\|^2_{H^1(B_{R+1}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3\setminus B_{R+1}(0))} \\ &\leq \|w(t)\|^2_{H^1(B_{R+2}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)} \\ &\leq \|\nabla w(t)\|^2_{L^2(B_{R+2}(0))} + \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}, \end{aligned}$$

showing the claim. Moreover, if ε is small enough, we obtain

$$\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \le C\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w(t)\|^2_{L^2(\mathbb{R}^3)}\bigg). \tag{3.8}$$

Note that

$$\begin{aligned}\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt &\le C\bigg(\int_0^T \|\nabla w(t)\|^2_{L^2(B_{R+2}(0))}\,dt + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt\bigg)\\ &\le C_\varepsilon\bigg(\int_0^T\int_{B_{R+3}(0)\setminus B_{R+2}(0)} |\nabla w|^2\,dxdt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\,dt\\ &\quad + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt\bigg) + \varepsilon\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt\\ &\le C_\varepsilon\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}dt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\,dt\\ &\quad + \int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt\bigg) + \varepsilon\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt\\ &\le C_\varepsilon\bigg(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}dt + \int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\,dt\bigg)\\ &\quad + \varepsilon\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt.\end{aligned}$$

So, it remains to show

$$\int_0^T \|w(t)\|_{L^2(\mathbb{R}^3)}\,dt \le c\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}dt \tag{3.9}$$

to achieve Lemma 3.3. To this end, let us argue by contradiction, that is, suppose that (3.9) does not hold. If this is the case, there exists a sequence $\{w_{n,0}\}$ in $H^1(\mathbb{R}^3)$ such that the corresponding sequence of solutions $\{w_n\}$ of (3.4) satisfies

$$1 = \int_0^T \|w_n(t)\|_{L^2(\mathbb{R}^3)}\,dt \ge n\int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)}dt, \quad n = 1, 2, \dots \tag{3.10}$$

Due to inequalities (3.8) and (3.10), we get

$$\int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \le C\bigg(\int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w_n(t)\|^2_{L^2(\mathbb{R}^3)}\bigg) \le C,$$

and so the sequence $\{w_n\}$ is bounded in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$. Hence, by inequality (3.7), the sequence $\{w_n(0) = w_{n,0}\}_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Extracting a subsequence and still denoting it by $\{w_{n,0}\}$, we may assume that

$$w_{n,0} \rightharpoonup w_0 \text{ weakly in } H^1(\mathbb{R}^3)$$

and

$$w_n \rightharpoonup w \text{ weakly in } L^2\big((0,T);H^1(\mathbb{R}^3)\big),$$

where $w \in C\big([0,T];H^1(\mathbb{R}^3)\big)$ is a solution of (3.4). By inequality (3.10), $\varphi w_n \to 0$ in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$ strongly. Since $\varphi w_n \rightharpoonup 0$ in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$ weakly, we conclude that $\varphi w \equiv 0$ on $(0,T)\times\mathbb{R}^3$. Therefore,

$$w \equiv 0, \quad |x| > R+1, \ t \in (0,T).$$

According to Proposition 2.1, one has $w \in C^\infty(\mathbb{R}^3\times(0,T))$. Now, we are in a position to use the unique continuation property for the Schrödinger equation showed in [18] to conclude that

$$w \equiv 0 \text{ on } \mathbb{R}^3\times(0,T).$$

Since $\varphi w_n \to 0$ strongly in $L^2\big((0,T);H^1(\mathbb{R}^3)\big)$, we get

$$w_n \to 0 \text{ strongly in } L^2\big((0,T);H^1(\mathbb{R}^3\backslash B_{R+1}(0))\big). \tag{3.11}$$

On the other hand, taking into account (3.4) and (3.8), we have

$$\begin{aligned}
\int_0^T \|w_n(t)\|^2_{H^1(B_{R+1}(0))}\,dt &\leq \int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \\
&\leq C\Big(\int_0^T \|\varphi w_n(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w_n(t)\|^2_{L^2(\mathbb{R}^3)}\,dt\Big),
\end{aligned}$$

$$\begin{aligned}
\int_0^T \|\partial_t w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt &= \int_0^T \|-\Delta w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&= \int_0^T \|(1-\Delta)w_n(t) - w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\leq \int_0^T \|(1-\Delta)w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\quad + 2\int_0^T \|(1-\Delta)w_n(t)\|_{H^{-1}(B_{R+1}(0))}\|w_n(t)\|_{H^{-1}(B_{R+1}(0))}\,dt \\
&\quad + \int_0^T \|w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt \\
&\leq C\int_0^T \|w_n(t)\|^2_{H^1(B_{R+1}(0))}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \|\nabla w_n(t)\|^2_{H^{-1}(B_{R+1}(0))}\,dt &\leq C\int_0^T \|\nabla w_n(t)\|^2_{L^2(B_{R+1}(0))}\,dt \\
&\leq C\int_0^T \|\nabla w_n(t)\|^2_{L^2(\mathbb{R}^3))}\,dt \\
&\leq C\int_0^T \|w_n(t)\|^2_{H^1(\mathbb{R}^3))}\,dt.
\end{aligned}$$

Therefore,

$$w_n \text{ is bounded in } L^2\big((0,T);H^1(B_{R+1}(0))\big)\cap H^1\big((0,T);H^{-1}(B_{R+1}(0))\big).$$

Due to Aubin's lemma (see [26]) and the convergence (3.11), we conclude that for a subsequence, still denoted by $\{w_n\}$,

$$w_n \to w = 0 \text{ strongly in } L^2\big((0,T);L^2(\mathbb{R}^3)\big)$$

which contradicts (3.10). So, the estimate (3.5) follows from (3.7), (3.8) and (3.9) as follows

$$\begin{aligned}
\|w(0)\|^2_{H^1(\mathbb{R}^3)} &\leq C\int_0^T \|w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt \\
&\leq C\Big(\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt + \int_0^T \|w(t)\|^2_{L^2(\mathbb{R}^3)}\,dt\Big) \\
&\leq C\int_0^T \|\varphi w(t)\|^2_{H^1(\mathbb{R}^3)}\,dt,
\end{aligned}$$

showing the lemma. □

Second step: *Weak observability inequality.*

We prove a bound which is weaker than the observability inequality (3.2).

Lemma 3.4. *Let v be the solution of system (3.3) with $v_0 \in H^{-1}(\mathbb{R}^3)$. Then,*

$$\|v_0\|^2_{H^{-1}} \leq C\left(\int_0^T \|\varphi v(t)\|^2_{H^{-1}}\, dt + \|(1-\varphi(x/2))v_0\|^2_{H^{-2}}\right). \tag{3.12}$$

Proof. Again, let us argue by contradiction. If inequality (3.12) is not verified, there exists a sequence $\{v_n\}$ of solutions to problem (3.3) in $C([0,T];H^{-1}(\mathbb{R}^3))$ such that

$$1 = \|v_n(0)\|^2_{H^{-1}} \geq n\left(\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\, dt + \|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}}\right). \tag{3.13}$$

Up to a subsequence, we may assume that

$$v_n \rightharpoonup v \ \text{ in } \ L^\infty((0,T);H^{-1}(\mathbb{R}^3)) \text{ weak*}$$

and

$$v_n(0) \rightharpoonup v(0) \text{ in } H^{-1}(\mathbb{R}^3) \text{ weak,} \tag{3.14}$$

where $v \in C([0,T];H^{-1}(\mathbb{R}^3))$ is a solution of problem (3.3). By inequality (3.13),

$$\varphi v_n \to 0 \text{ (strongly) in } L^2((0,T);H^{-1}(\mathbb{R}^3)).$$

Since

$$\varphi v_n \rightharpoonup \varphi v \text{ in } L^\infty((0,T);H^{-1}(\mathbb{R}^3)) \text{ weak*},$$

we conclude that $\varphi v \equiv 0$. Therefore, $v(t,x) = 0$ for $|x| > R+1$ and $t \in (0,T)$. So, using the unique continuation property as in Step 1, we get that $v \equiv 0$. In particular, $v(0) = 0$.

Now, we claim that

$$\|\varphi(x/2)v_n(0)\|^2_{H^{-2}} \leq C\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\, dt. \tag{3.15}$$

To prove (3.15), introduce the function $\tilde{v}_n(x,t) = \varphi(x/2)v_n(x,t)$ which satisfies

$$i\partial_t \tilde{v}_n + \Delta\tilde{v}_n = f_n$$

where $f_n = [\Delta\varphi(x/2)]v_n + 2\nabla\varphi(x/2)\nabla v_n$. Then, the fact that $\operatorname{supp}[\varphi(x/2)] \subset \{\varphi = 1\}$ yields

$$\begin{aligned}\|\tilde{v}_n(0)\|^2_{H^{-2}(\mathbb{R}^3)} \leq& c\left(\int_0^T \|\tilde{v}_n(t)\|^2_{H^{-2}(\mathbb{R}^3)}\, dt + \int_0^T \|f_n(t)\|^2_{H^{-2}(\mathbb{R}^3)}\, dt\right)\\ \leq& c\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}(\mathbb{R}^3)},\end{aligned}$$

giving (3.15). Now, using (3.13), one has

$$\begin{aligned}\|v_n(0)\|^2_{H^{-2}} &\leq 2\Big(\|\varphi(x/2)v_n(0)\|^2_{H^{-2}} + \|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}}\Big)\\ &\leq c\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\, dt + 2\|(1-\varphi(x/2))v_n(0)\|^2_{H^{-2}} \to 0,\end{aligned}$$

that is,

$$v_n(0) \to 0 \text{ strongly in } H^{-2}(\mathbb{R}^3). \tag{3.16}$$

Let $w_n = (1-\Delta)^{-1}v_n$. Then $w_n \in C([0,T];H^1(\mathbb{R}^3))$ is a solution of the equation (3.3). By the convergences (3.14) and (3.16), we can ensure that

$$w_n(0) \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^3) \text{ weakly}$$

and

$$w_n \to 0 \text{ in } C([0,T];L^2(\mathbb{R}^3)) \text{ strongly.} \tag{3.17}$$

Now, split φw_n as

$$\varphi w_n = (1-\Delta)^{-1}(\varphi v_n) - (1-\Delta)^{-1}[\varphi,(1-\Delta)]w_n.$$

Observe that the operator $[\varphi,(1-\Delta)]$ maps $L^2(\mathbb{R}^3)$ continuously into $H^{-1}(\mathbb{R}^3)$. So, due to the convergence (3.17), we get that

$$(1-\Delta)^{-1}[\varphi,(1-\Delta)]w_n \to 0 \text{ in } C([0,T];H^1(\mathbb{R}^3)). \tag{3.18}$$

On the other hand, by (3.13),

$$(1-\Delta)^{-1}(\varphi v_n) \to 0 \text{ in } L^2((0,T);H^1(\mathbb{R}^3)). \tag{3.19}$$

Therefore, by the convergences (3.18) and (3.19) above, it follows that

$$\varphi w_n \to 0 \text{ in } L^2((0,T);H^1(\mathbb{R}^3)).$$

Since w_n satisfies (3.3), using Lemma 3.3, more precisely, the observability inequality (3.5), we conclude that

$$w_n(0) \to 0 \text{ in } H^1(\mathbb{R}^3) \text{ strongly,}$$

and so

$$v_n(0) \to 0 \text{ in } H^{-1}(\mathbb{R}^3) \text{ strongly,}$$

which is a contradiction to the fact that $\|v_n(0)\|^2_{H^{-1}} = 1$, for all n. This finishes the proof. □

Third step: *Proof of the observability inequality (3.2).*

If (3.2) is false, then there exists a sequence $\{v_n\}$ of solutions to (3.3) in $C([0,T];H^{-1}(\mathbb{R}^3))$ such that

$$1 = \|v_n(0)\|^2_{H^{-1}} \geq n\int_0^T \|\varphi v_n(t)\|^2_{H^{-1}}\, dt, \ \forall n \geq 0. \tag{3.20}$$

Extracting a subsequence, still denoted by the same indexes, we have that

$$v_n \rightharpoonup v \ \text{ in } \ L^\infty((0,T);H^{-1}(\mathbb{R}^3)) \text{ weak}^*$$

and

$$v_n(0) \rightharpoonup v(0) \text{ in } H^{-1}(\mathbb{R}^3) \text{ weak,}$$

for some solution $v \in C([0,T];H^{-1}(\mathbb{R}^3))$ of the system (3.3). Note that

$$\varphi v_n \rightharpoonup \varphi v \text{ in } L^\infty(0,T;H^{-1}(\mathbb{R}^3)) \text{ weak}^*$$

and this, combined with (3.20) ($\varphi v_n \to 0$ in $L^2\big((0,T);H^{-1}(\mathbb{R}^3)\big)$), yields $\varphi v \equiv 0$ and, hence, $v \equiv 0$ for $|x| > R+1$, $t \in (0,T)$. So, by the unique continuation property as in Step 2, we deduce that $v \equiv 0$ on $\mathbb{R}^3 \times (0,T)$. On the other hand, the sequence $(1-\varphi(x/2))v_n(0)$ is bounded in $H^{-1}(\mathbb{R}^3)$ and has compact support contained in $B_{2R+2}(0)$. Therefore, extracting a subsequence, we may assume that it converges strongly in $H^{-2}(\mathbb{R}^3)$. Moreover, its limit is necessarily 0 since

$$(1-\varphi(x/2))v_n(0) \rightharpoonup 0 \text{ in } H^{-2}(\mathbb{R}^3).$$

Using inequality (3.12), we conclude that $\|v_n(0)\|_{H^{-1}} \to 0$, which contradicts (3.20). This proves the desired observability inequality (3.3) and finishes the proof of Proposition 3.2. □

Proof of Theorem 3.1. We prove Theorem 3.1 using Hilbert's uniqueness method. First, note that since the Schrödinger equation (3.1) is backward well-posed, we may assume that $u(T)=0$ without loss of generality.

Now, consider the two systems

$$\begin{cases} i\partial_t u + \Delta u = \varphi(x)h(x,t) \ in \ [0,T]\times\mathbb{R}^3, \\ u(T) = 0, \end{cases} \tag{3.21}$$

with $\varphi(x)$ satisfying (1.5), and

$$\begin{cases} i\partial_t v + \Delta v = 0 \ in \ [0,T]\times\mathbb{R}^3, \\ v(0) = v_0 \in H^{-1}(\mathbb{R}^3). \end{cases}$$

Multiplying the equation of the first system by $\overline{v}$ and integrating by parts, we obtain

$$i\int_{\mathbb{R}^3}\Big[\overline{v}(T)u(T) - \overline{v_0}u(0)\Big]\, dx = \int_0^T\int_{\mathbb{R}^3} \varphi(x)h(x,t)\overline{v(x,t)}\, dxdt.$$

Hence, taking $L^2(\mathbb{R}^3)$ as pivot space, one has

$$\langle v_0, -iu_0\rangle = \int_0^T \langle \varphi(x)v, h(t)\rangle \, dt, \tag{3.22}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}\left(\mathbb{R}^3\right)$ and $H^1\left(\mathbb{R}^3\right)$. Consider the continuous map $\Lambda : H^{-1}(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ defined by $\Lambda v = (v,\cdot)_1$. Given any $v_0 \in H^{-1}\left(\mathbb{R}^3\right)$, let $h(t) = \Lambda^{-1}(\varphi v(t))\left(h \in C\left([0,T]; H^1\left(\mathbb{R}^3\right)\right)\right)$ and let u be the corresponding solution of system (3.21). Finally, set $\Gamma\left(v_0\right) = -iu(\cdot,0)$. Then, we have

$$\langle v_0, \Gamma\left(v_0\right)\rangle = \int_0^T \|\varphi v(t)\|^2_{H^{-1}(\mathbb{R}^3)} dt \geqslant c\left\|v_0\right\|^2_{H^{-1}(\mathbb{R}^3)},$$

in view of the observability inequality (3.2) and identity (3.22). It follows from the Lax-Milgram theorem that Γ defines an isomorphism, and this concludes the proof of Theorem 3.1. □

3.2. **Nonlinear system: Proof of Theorem 1.1.** The proof is based on a perturbation argument due to E. Zuazua [28]. To use it, consider the following two Schrödinger systems with initial data in H^{-1} and null initial data, namely

$$\begin{cases} i\partial_t\Phi + \Delta\Phi = 0 \text{ on } [0,T]\times\mathbb{R}^3, \\ \Phi(0) = \Phi_0 \in H^{-1}(\mathbb{R}^3), \end{cases}$$

and

$$\begin{cases} i\partial_t u + \Delta u - |u|^4 u = A\Phi \text{ on } [0,T]\times\mathbb{R}^3, \\ u(T) = 0, \end{cases} \tag{3.23}$$

respectively, where A is defined as in Theorem 3.1 by

$$A\Phi := \Lambda^{-1}(\varphi(x)\Phi).$$

Now, define the operator

$$\begin{aligned} \mathcal{L} : H^{-1}(\mathbb{R}^3) &\to H^1(\mathbb{R}^3) \\ \Phi_0 &\mapsto \mathcal{L}\Phi_0 = u_0 = u(0). \end{aligned}$$

The goal is then to show that $\mathcal{L}$ is onto in a small neighborhood of the origin of $H^1(\mathbb{R}^3)$. To this end, split u as $u = v + \Psi$, where is Ψ a solution of

$$\begin{cases} i\partial_t\Psi + \Delta\Psi = A\Phi \ in \ [0,T]\times\mathbb{R}^3, \\ \Psi(T) = 0, \end{cases}$$

and v is a solution of

$$\begin{cases} i\partial_t v + \Delta v = |u|^4 u \ in \ [0,T]\times\mathbb{R}^3, \\ v(T) = 0. \end{cases} \tag{3.24}$$

Clearly u, v and Ψ belong to $C([0,T], H^1(\mathbb{R}^3)) \cap L^{10}([0,T]; L^{10}(\mathbb{R}^3))$ and $u(0) = v(0) + \Psi(0)$. We can write

$$\mathcal{L}\Phi_0 = \mathcal{J}\Phi_0 + \Gamma\Phi_0,$$

where $\mathcal{J}\Phi_0 = v_0$. Observe that $\mathcal{L}\Phi_0 = u_0$, or equivalently, $\Phi_0 = -\Gamma^{-1}\mathcal{J}\Phi_0 + \Gamma^{-1}u_0$.

Now, define the operator

$$\begin{aligned} \mathcal{B} : H^{-1}(\mathbb{R}^3) &\to H^{-1}(\mathbb{R}^3) \\ \Phi_0 &\to \mathcal{B}\Phi_0 = \Gamma^{-1}\mathcal{J}\Phi_0 + \Gamma^{-1}u_0, \end{aligned}$$

where we are taking into account that Γ is the linear control isomorphism between H^{-1} and H^1, due to Theorem 3.1.

Now, the goal is to prove that $\mathcal{B}$ has a fixed point near the origin of $H^{-1}(\mathbb{R}^3)$. More precisely, let us prove that if $\|u_0\|_{H^1}$ is small enough, then $\mathcal{B}$ is a contraction on a small ball B_R of $H^{-1}(\mathbb{R}^3)$.

We may assume $T<1$ and we will denote by $C>0$ any constant that may have its numerical value changed line by line. Since Γ is an isomorphism, we have

$$\begin{aligned}\|\mathcal{B}\Phi_0\|_{H^{-1}} \leq&\|\Gamma^{-1}\mathcal{J}\Phi_0\|_{H^{-1}}+\|\Gamma^{-1}u_0\|_{H^{-1}}\\ \leq&C\left(\|\mathcal{J}\Phi_0\|_{H^1}+\|u_0\|_{H^1}\right)\\ \leq&C\left(\|v(0)\|_{H^1}+\|u_0\|_{H^1}\right).\end{aligned} \tag{3.25}$$

Claim 1: There exists $C>0$ such that

$$\|v(0)\|_{H^1}\leq C\|\nabla u\|^5_{L^{10}_tL^{\frac{30}{13}}_x}. \tag{3.26}$$

Indeed, note that due to the classical energy estimate for system (3.24), Strichartz estimates (see Lemma 2.2) and a Sobolev embedding (see Lemma 2.3), we have

$$\begin{aligned}\|v(0)\|_{L^2} &\leq \|v(T)\|_{L^2}+\left\|\int_0^t e^{i(t-\tau)\Delta}|u|^4u\;d\tau\right\|_{L^2_x}\\ &\leq C\|u^5\|_{L^1_tL^2_x}\\ &\leq C\|u\|^5_{L^{10}_tL^{10}_x}\\ &\leq C\|\nabla u\|^5_{L^{10}_tL^{\frac{30}{13}}_x}\end{aligned}$$

and

$$\begin{aligned}\|\nabla v(0)\|_{L^2} &\leq \|\nabla v(T)\|_{L^2}+\left\|\int_0^t e^{i(t-\tau)\Delta}\nabla|u|^4u\;d\tau\right\|_{L^2_x}\\ &\leq C\|\nabla u\|_{L^{10}_tL^{\frac{30}{13}}_x}\|u\|^4_{L^{10}_tL^{10}_x}\\ &\leq \|\nabla u\|^5_{L^{10}_tL^{\frac{30}{13}}_x}.\end{aligned}$$

Thus,

$$\|v(0)\|^2_{L^2}\leq C\|\nabla u\|^{10}_{L^{10}_tL^{\frac{30}{13}}_x}$$

and

$$\|\nabla v(0)\|^2_{L^2}\leq C\|\nabla u\|^{10}_{L^{10}_tL^{\frac{30}{13}}_x}.$$

Summing up, we have (3.26), thus showing Claim 1.

Claim 2: There exists $C>0$ such that

$$\|\nabla u\|_{L^{10}_tL^{\frac{30}{13}}_x}\leq C\|\Phi_0\|_{H^{-1}}. \tag{3.27}$$

In fact, applying Lemma 2.2 to system (3.23), one gets

$$\begin{aligned}\|\nabla u\|_{L^{10}_tL^{\frac{30}{13}}_x} &\leq \|\nabla u(T)\|_{L^2}+C\|\nabla u\|_{L^{10}_tL^{\frac{30}{13}}_x}\|u\|^4_{L^{10}_tL^{10}_x}+C\|\nabla A\Phi\|_{L^1_tL^2_x}\\ &\leq C\left(\|\nabla u\|^5_{L^{10}_tL^{\frac{30}{13}}_x}+\|A\Phi\|_{L^2_tH^1_x}\right).\end{aligned}$$

Note that, using the fact that Λ is an isomorphism, we get

$$\|A\Phi\|_{H^1}=\|\Lambda^{-1}(\varphi\Phi)\|_{H^1}\leq C\|\varphi\Phi\|_{H^{-1}}$$

or, equivalently,

$$\|A\Phi\|_{L^2H^1}\leq\left(\int_0^T\|\varphi\Phi\|^2_{H^{-1}}\;dt\right)^{\frac{1}{2}}.$$

Then, the duality (3.22) yields

$$\begin{aligned}
\|\nabla u\|_{L_t^{10}L_x^{\frac{30}{13}}} &\leq C\|\nabla u\|^5_{L_t^{10}L_x^{\frac{30}{13}}} + C\Big(\int_0^T \|\varphi\Phi\|^2_{H^{-1}}\,dt\Big)^{\frac{1}{2}} \\
&\leq C\|\nabla u\|^5_{L_t^{10}L_x^{\frac{30}{13}}} + C\Big(\langle\Gamma\Phi_0,\Phi_0\rangle\Big)^{\frac{1}{2}} \\
&\leq C\|\nabla u\|^5_{L_t^{10}L_x^{\frac{30}{13}}} + C\Big(\|\Gamma\Phi_0\|_{H^1}\|\Phi_0\|_{H^{-1}}\Big)^{\frac{1}{2}} \\
&\leq C\|\nabla u\|^5_{L_t^{10}L_x^{\frac{30}{13}}} + C\Big(\|\Phi_0\|^2_{H^{-1}}\Big)^{\frac{1}{2}} \\
&\leq C\|\nabla u\|^5_{L_t^{10}L_x^{\frac{30}{13}}} + C\|\Phi_0\|_{H^{-1}}.
\end{aligned}$$

Using a bootstrap argument, taking $\|\Phi_0\|_{H^{-1}} \leq R$ with R small enough, we get (3.27), showing Claim 2.

Note that putting together inequalities (3.26) and (3.27) into (3.25), we conclude

$$\begin{aligned}
\|B\Phi_0\|_{H^{-1}} &\leq C\Big(\|v(0)\|_{H^1} + \|u_0\|_{H^1}\Big) \\
&\leq C\Big(\|\Phi_0\|^5_{H^{-1}} + \|u_0\|_{H^1}\Big).
\end{aligned}$$

Then, choosing R small enough and $\|u_0\|_{H^1} \leq \frac{R}{2C}$, we get

$$\|B\Phi_0\|_{H^{-1}} \leq R$$

and, therefore, $\mathcal{B}$ reproduces the ball B_R of $H^{-1}(\mathbb{R})$.

Finally, we prove that $\mathcal{B}$ is a contraction map. To do this, let us study the systems

$$\begin{cases} i\partial_t(u_1-u_2) + \Delta(u_1-u_2) - |u_1|^4u_1 + |u_2|^4u_2 = A(\Phi^1-\Phi^2), \\ (u_1-u_2)(T) = 0, \end{cases}$$

and

$$\begin{cases} i\partial_t(v_1-v_2) + \Delta(v_1-v_2) = |u_1|^4u_1 - |u_2|^4u_2, \\ (v_1-v_2)(T) = 0. \end{cases} \tag{3.28}$$

As before, we also have

$$\|\mathcal{B}\Phi_0^1 - \mathcal{B}\Phi_0^2\|_{H^{-1}} \leq C\|v_1(0)-v_2(0)\|_{H^1}. \tag{3.29}$$

We now estimate of $v_1(0)-v_2(0)$ in the H^1-norm. First, applying the Strichatz estimates (see Lemma 2.2) to the system (3.28) yields that

$$\begin{aligned}
\|v_1(0)-v_2(0)\|_{L^2} \leq& \Big\|\int_0^t e^{i(t-\tau)\Delta}(|u_1|^4u_1 - |u_2|^4u_2)\,d\tau\Big\|_{L^2_x} \\
\leq& C\|u_1^5 - u_2^5\|_{L_t^1L_x^2} \\
\leq& C\|u_1-u_2\|_{L_t^5L_x^{10}}\Big(\|u_1\|^4_{L_t^5L_x^{10}} + \|u_2\|^4_{L_t^5L_x^{10}}\Big) \\
\leq& C\|u_1-u_2\|_{L_t^{10}L_x^{10}}\Big(\|u_1\|^4_{L_t^{10}L_x^{10}} + \|u_2\|^4_{L_t^{10}L_x^{10}}\Big) \\
\leq& C\|\nabla u_1-\nabla u_2\|_{L_t^{10}L_x^{\frac{30}{13}}}\Big(\|\nabla u_1\|^4_{L_t^{10}L_x^{\frac{30}{13}}} + \|\nabla u_2\|^4_{L_t^{10}L_x^{\frac{30}{13}}}\Big) \\
\leq& CR^4\|\nabla u_1-\nabla u_2\|_{L_t^{10}L_x^{\frac{30}{13}}}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla v_1(0)-\nabla v_2(0)\|_{L^2} \leq & \left\| \int_0^t \nabla e^{i(t-\tau)\Delta}(|u_1|^4u_1-|u_2|^4u_2)\, d\tau \right\|_{L^2_x} \\
\leq & \|\nabla(|u_1|^4u_1-|u_2|^4u_2)\|_{L^2_tL^{\frac{6}{5}}_x} \\
\leq & C\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}\|u_1\|^4_{L^{10}_tL^{10}_x} \\
\leq & C\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_1\|^4_{L^{10}_tL^{\frac{30}{13}}_x} \\
& + C\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\Big(\|\nabla u_1\|^3_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x} \\
& + \|\nabla u_2\|^3_{L^{10}_tL^{\frac{30}{13}}_x}\|\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}\Big) \\
\leq & CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x} + CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x} \\
\leq & CR^4\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x}.
\end{aligned}$$

Thus,

$$\|v_1(0)-v_2(0)\|^2_{L^2} \leq CR^8\|\nabla u_1-\nabla u_2\|^2_{L^{10}_tL^{\frac{30}{13}}_x}$$

and

$$\|\nabla v_1(0)-\nabla v_2(0)\|^2_{L^2} \leq CR^8\|\nabla u_1-\nabla u_2\|^2_{L^{10}_tL^{\frac{30}{13}}_x}.$$

These bounds together give us the H^1-estimate

$$\|v_1(0)-v_2(0)\|_{H^1} \leq CR^4\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x}.$$

Now, let us bound the right-hand side of this inequality. To this end, first notice that

$$\begin{aligned}
\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x} & \leq \|\nabla(|u_1|^4u_1-|u_2|^4u_2)\|_{L^2_tL^{\frac{6}{5}}_x} + \|\nabla A(\Phi^1-\Phi^2)\|_{L^1_tL^2_x} \\
& \leq CR^4\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x} + C\|A(\Phi^1-\Phi^2)\|_{L^2_tH^1_x} \\
& \leq CR^4\|\nabla u_1-\nabla u_2\|_{L^{10}_tL^{\frac{30}{13}}_x} + C\|\Phi^1_0-\Phi^2_0\|_{H^{-1}}.
\end{aligned}$$

So, choosing $R>0$ small enough, we get

$$\|\nabla(u_1-u_2)\|_{L^{10}_tL^{\frac{30}{13}}_x} \leq C\|\Phi^1_0-\Phi^2_0\|_{H^{-1}}.$$

Therefore,

$$\begin{aligned}
\|v_1(0)-v_2(0)\|_{H^1} = & \left(\|v_1(0)-v_2(0)\|^2_{L^2}+\|\nabla v_1(0)-\nabla v_2(0)\|^2_{L^2}\right)^{\frac{1}{2}} \\
\leq & CR^4\|\Phi^1_0-\Phi^2_0\|_{H^{-1}}.
\end{aligned} \tag{3.30}$$

Finally, we get by inequalities (3.29) and (3.30) that

$$\begin{aligned}
\|\mathcal{B}\Phi^1_0-\mathcal{B}\Phi^2_0\|_{H^{-1}} & \leq C\|v_1(0)-v_2(0)\|_{H^1} \\
& \leq CR^4\|\Phi^1_0-\Phi^2_0\|_{H^{-1}},
\end{aligned}$$

concluding that $\mathcal{B}$ is a contraction on a small ball B_R of H^{-1}. This completes the proof of Theorem 1.1. $\square$

4. Comments and open issues

In this work, we have studied the local exact controllability of the critical C-NLS in dimension 3. Considering the defocusing critical nonlinear Schrödinger equation (1.4), we showed this system to be exact controllable in $\dot{H}^1(\mathbb{R}^3)$–level, that is, it satisfies $u(T) = \tilde{u}_0$ for u_0 and $\tilde{u}_0$ in $\dot{H}^1(\mathbb{R}^3)$ satisfying $\|u_0\|_{H^1} \leq \delta$, $\|\tilde{u}_0\|_{H^1} \leq \delta$, and $h(x,t) \in C([0,T]; H^1(\mathbb{R}^3))$. Concerning our main results, Theorems 1.1 and 1.2, the following remarks are worth mentioning.

- The main tools to achieve these results were
  – *Strichartz estimates* for the Schrödinger operator [6, 10], which give the well-posedness theory for system (1.4);
  – *A unique continuation property* for the linear system associated with (1.4), which follows by the Carleman estimate showed in [18] (see also [15]);
  – *A perturbation argument*, as presented in [28], which allows one to extend the result for the nonlinear system (1.4).
- For the $3d$–case, Laurent [15] showed large time global internal controllability for the nonlinear Schrödinger equation on some compact manifolds of dimension 3. The results therein were obtained for the nonlinearity $|u|^2u$ instead of $|u|^4u$. In this sense, our result completes the local control result given by the author for the critical case (1.4) in $\mathbb{R}^3$.
- Since we are working on the whole space $\mathbb{R}^3$ and we have a control taking the form $\varphi(x)h(x,t)$ in the system (1.4) (remember the definition of φ in (1.5)), the geometric control condition (see, for instance, [1, 22]) is easily satisfied.

Thus, our work gives the first step to understanding control problems for the C-NLS in dimension 3, and, therefore, some important open issues appear. Let us present them below.

($\mathcal{A}$) **Stabilization problem:** Can one find a feedback control law $f = Ku$ *so that the resulting closed-loop system*

$$i\partial_t u + \Delta u - |u|^4 u = Ku, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3,$$

is asymptotically stable at an equilibrium point as $t \to +\infty$*?*

($\mathcal{B}$) **Global control problem:** If the answer to the previous question is positive, another natural issue is to obtain a global control result for the system (1.4), i.e., a control result for large data. This would be a consequence of the global stabilization result together with the local control result shown in Theorem 1.1.

We hope to address these problems in a forthcoming paper.

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Departamento de Matemática, Universidade Federal de Pernambuco, S/N Cidade Universitária, 50740-545, Recife (PE), Brazil
Email address: pablo.braz@ufpe.br

Email address: roberto.capistranofilho@ufpe.br

Email address: jackellyny.dassy@ufpe.br

Institut Élie Cartan de Lorraine, UMR CNRS 7502, équipe SPHINX, INRIA, Université de Lorraine, F-54506 Vandoeuvre-les-Nancy Cedex, France.
Email address: ddsf@math.cnrs.fr