BOUNDARY CONTROLLABILITY OF THE KORTEWEG-DE VRIES EQUATION: THE NEUMANN CASE

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Abstract. This article gives a necessary first step to understanding the critical set phenomenon for the Korteweg-de Vries (KdV) equation posed on interval $[0,L]$ considering the Neumann boundary conditions with only one control input. We showed that the KdV equation is controllable in the critical case, i.e., when the spatial domain L belongs to the set $\mathcal{R}_c$, where $c\neq -1$ and

$$\mathcal{R}_c := \left\{ \frac{2\pi}{\sqrt{3(c+1)}}\sqrt{m^2+ml+m^2};\ m,l\in\mathbb{N}^* \right\} \cup \left\{ \frac{m\pi}{\sqrt{c+1}};\ m\in\mathbb{N}^* \right\},$$

the KdV equation is exactly controllable in $L^2(0,L)$. The result is achieved using the *return method* together with a fixed point argument.

Keywords. Korteweg–de Vries equation, exact boundary controllability, Neumann boundary conditions, Dirichlet boundary conditions, critical set

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1. Introduction

We had known when we formulated the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, there exist two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. See, for instance, [1–4, 23, 26] and references therein for a rigorous justification. Moreover, another non-dimensional parameter μ appears, the Bond number, to measure the importance of gravitational forces compared to surface tension forces also appears. Considering the physical condition $\delta \ll 1$ we can characterize the waves, called long waves or shallow water waves. In particular, considering the relations between ε and δ, we can have the following regime:

- Korteweg-de Vries (KdV): $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$. Under this regime, Korteweg and de Vries [20][1] derived the following well-known equation as a central equation among other dispersive or shallow water wave models called the KdV equation from the equations for capillary-gravity waves:

$$\pm 2\eta_t + 3\eta\eta_x + \left(\frac{1}{3} - \mu\right)\eta_{xxx} = 0.$$

Today, it is well known that this equation has an important phenomenon that directly affects the control problem related to them, the so-called *critical length phenomenon*. Let us briefly present the control problem, which makes the phenomenon of critical lengths emerge. The control problem was presented in a pioneering work of Rosier [24] that studied the following system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ u(0,t) = 0,\ u(L,t) = 0,\ u_x(L,t) = g(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{1.1}$$

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[1] This equation was first introduced by Boussinesq [6], and Korteweg and de Vries rediscovered it twenty years later.

where the boundary value function $g(t)$ is considered as a control input. Precisely, the author showed the following control problem for the system (1.1), giving the origin of the critical length phenomenon for the KdV equation.

Question $\mathcal{A}$: *Given $T>0$ and $u_0,u_T\in L^2(0,L)$, can one find an appropriate control input $g(t)\in L^2(0,T)$ such that the corresponding solution $u(x,t)$ of the system (1.1) satisfies*

$$u(x,0)=u_0(x) \quad \text{and} \quad u(x,T)=u_T(x)? \tag{1.2}$$

Rosier answered the previous question in [24]. He proved that considering $L\notin\mathcal{N}$, where

$$\mathcal{N}:=\left\{\frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}:k,l\in\mathbb{N}^*\right\},$$

the associated linear system (1.1)

$$\begin{cases} u_t+u_x+u_{xxx}=0, & \text{in } (0,L)\times(0,T),\\ u(0,t)=0,\ u(L,t)=0,\ u_x(L,t)=g(t), & \text{in } (0,T),\\ u(x,0)=u_0(x) & \text{in } (0,L),\end{cases} \tag{1.3}$$

is controllable; roughly speaking, if $L\in\mathcal{N}$ system (1.3) is not controllable, that is, there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M}=\mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi\in\mathcal{M}$, $g\in L^2(0,T)$ and $u\in C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ satisfying (1.3) and $u(\cdot,0)=0$, one has $u(\cdot,T)\neq\psi$.

Definition 1.1. *A spatial domain $(0,L)$ is called critical for the system (1.3) if its domain length L belongs to $\mathcal{N}$.*

Following the work of Rosier [24], the boundary control system of the KdV equation posed on the finite interval $(0,L)$ with various control inputs has been intensively studied (cf. [7, 8, 10, 11, 15, 17–19] and see [9, 25] for more complete reviews). Thus, this work gives a necessary step to understanding this phenomenon for the KdV equation with Neumann boundary conditions, completing, in some sense, the results shown in [7].

1.1. Problem set

In this article, we study a class of distributed parameter control systems described by the Korteweg–de Vries (KdV) equation posed on a bounded domain $(0,L)$ with the Neumann boundary conditions

$$\begin{cases} u_t+u_x+u_{xxx}+uu_x=0, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t)=u_{xx}(L,t)=0, & \text{in } (0,T),\\ u_x(L,t)=h(t), & \text{in } (0,T),\\ u(x,0)=u_0(x), & \text{in } (0,L),\end{cases} \tag{1.4}$$

where $h(t)$ will be considered as a control input. Recently, the first author dealt with the control problem related to the system (1.4). Precisely, was proved that their solutions are exactly controllable in a neighborhood of c if the length L of the spatial domain $(0,L)$ does not belong to the set

$$\mathcal{R}_c:=\left\{\frac{2\pi}{\sqrt{3(c+1)}}\sqrt{m^2+ml+m^2};\ m,l\in\mathbb{N}^*\right\}\cup\left\{\frac{m\pi}{\sqrt{c+1}};\ m\in\mathbb{N}^*\right\},$$

that is, the relation (1.2) holds for the solution of the system (1.4). The result can be read as follows.

THEOREM 1.1 (Caicedo, Capistrano–Filho, Zhang [7]). *Let $T>0$, $c\neq -1$ and $L\notin \mathcal{R}_c$. There exists $\delta>0$ such that for any $u_0,u_T\in L^2(0,L)$ with*

$$\|u_0-c\|_{L^2(0,L)}<\delta \quad \textit{and} \quad \|u_T-c\|_{L^2(0,L)}<\delta$$

one can find $h\in L^2(0,T)$ such that the system (1.4) admits a unique solution

$$u\in \mathcal{Z}_T=C\left([0,T];L^2(0,L)\right)\cap L^2\left(0,T;H^1(0,L)\right)$$

satisfying (1.2).

As in [24], the first step is to obtain a control result for the linear system, namely,

$$\begin{cases} v_t+(1+c)v_x+v_{xxx}=0, & \text{in } (0,L)\times(0,T),\\ v_{xx}(0,t)=v_{xx}(L,t)=0, & \text{in } (0,T),\\ v_x(L,t)=h(t), & \text{in } (0,T),\\ v(x,0)=v_0(x), & \text{in } (0,L). \end{cases} \tag{1.5}$$

Precisely the authors in [7] proved the following result.

THEOREM 1.2 (Caicedo, Capistrano–Filho, Zhang [7]). *For $c\neq -1$, the linear system (1.5) is exactly controllable in the space $L^2(0,L)$ if and only if $L\notin\mathcal{R}_c$. Otherwise, that is, if $c=-1$, the system (1.5) is not exactly controllable in the space $L^2(0,L)$ for any $L>0$.*

With this in hand, they extend the result for the nonlinear system (1.4) using fixed point argument achieving Theorem 1.1 whenever $L\notin\mathcal{R}_c$. It is important to point out that fixing $k\in\mathbb{N}^*$ and considering $m=l=k$ we have $L=2k\pi$, when $c=0$, and, from Theorem 1.2, it follows that (1.5) are not exactly controllable. Additionally, as before mentioned, we do not know if the system (1.4) is exactly controllable. So, in this context, the natural questions appear:

Question $\mathcal{B}$: Given $T>0$, $L\in\mathcal{R}_c$ and $y_0,y_T\in L^2(0,L)$ close enough to c, can one find an appropriate control input $h\in L^2(0,T)$ such that the solution y of the system (1.6), corresponding to h and y_0, satisfies $y(\cdot,T)=y_T$?

Question $\mathcal{C}$: Given $T>0$, $L\in\mathcal{R}_c$ and $y_0,y_T\in L^2(0,L)$. The system (1.6) is exactly controllable in the critical length L, that is when $L\in\mathcal{R}_c$?

1.2. Main results

Let us consider the following nonlinear control system

$$\begin{cases} y_t+(1+c)y_x+y_{xxx}+yy_x=0, & \text{in } (0,L)\times(0,T),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h(t),\ y_{xx}(L,t)=0, & \text{in } (0,T),\\ y(x,0)=y_0(x), & \text{in } (0,L), \end{cases} \tag{1.6}$$

where h will be as a control input and y is the state. Theorem 1.1 says that when $L\notin\mathcal{R}_c$, system (1.6) is locally controllable around c but we do not know if the same holds when $L\in\mathcal{R}_c$. The main result in this work provides an affirmative answer to the Question $\mathcal{C}$. Precisely, we have the following:

THEOREM 1.3. *Let $T>0$, $c=0$ and $L\in\mathcal{R}_0$. Then, system (1.6) is exactly controllable around the origin 0 in $L^2(0,L)$, that is, there exists $\delta>0$ such that, for very $y_0,y_T\in L^2(0,L)$ with*

$$\|y_0\|_{L^2(0,L)},\|y_T\|_{L^2(0,L)}<\delta$$

it is possible to find $h\in L^2(0,T)$ such that the corresponding solution of (1.6) satisfying $y(\cdot,0)=y_0$ and $y(\cdot,T)=y_T$.

This previous result can be generalized for any c as follows:

THEOREM 1.4. *Let $T>0$ and $L\in\mathcal{R}_c$. The system (1.6) is exactly controllable around c in $L^2(0,L)$ in the sense of Theorem 1.3.*

To prove the previous results we need an auxiliary property that ensures that for c near enough to 0 (small perturbations of 0), the system (1.4) is exactly controllable in a neighborhood of c in $\mathcal{R}_c$), or precisely, for d close enough to c one has $L\notin\mathcal{R}_d$ so that, the system (1.5) corresponding to d is exactly controllable, answering the Question $\mathcal{B}$. In other words, the set of critical lengths is sensitive to small disturbances in equilibrium c, and the result can be read as follows.

THEOREM 1.5. *Let $T>0$, $c\neq-1$ and $L\in\mathcal{R}_c$. There exists $\varepsilon_c>0$ such that, for every $d\in(c-\varepsilon_c,c+\varepsilon_c)\backslash\{c\}$, $d\neq-1$, we have $L\notin\mathcal{R}_d$. Consequently, the linear system (1.5), with $c=d$, is exactly controllable; and the nonlinear system (1.6) is exactly controllable around the steady state d in $L^2(0,L)$, that is, there exists $\delta_d>0$ such that, for any $y_0,y_T\in L^2(0,L)$ with*

$$\|y_0-d\|_{L^2(0,L)}<\delta_d \quad \text{and} \quad \|y_T-d\|_{L^2(0,L)}<\delta_d,$$

one can find $h\in L^2(0,T)$ such that the system (1.6) admits a unique solution $y\in\mathcal{Z}_T$ satisfying $y(\cdot,0)=y_0$ and $y(\cdot,T)=y_T$.

1.3. Heuristic and paper's outline

The proof of the Theorem 1.5 is based on the topological properties of real numbers together with the Theorem 1.1. Moreover, with this in hand, both results stated in the previous paragraph (Theorems 1.3 and 1.4) rely on the so-called *return method* together with the fixed point argument.

It is important to point out that the return method was introduced by J.M. Coron in [12] (see also [13]) and has been used by several authors to prove control results in the critical lengths for the KdV-type equation (see, for instance, [8,10,11,15]). This method consists of building particular trajectories of the system (1.6) starting and ending at some equilibrium such that the linearization of the system around these trajectories has good properties. Here, we use a combination of this method with a fixed point argument, successfully applied in [16]. We mention that this method can be applied together with quasi-static deformations and power series expansion, we infer to the reader the nice book of Coron [14] about more details of the method.

Concerning the construction of solutions to the Theorems 1.3 and 1.4 we follow the following procedure: In the first time, we construct a solution that starts from y_0 and reaches at time $T/3$ a state which is in some sense close to d (which is yet to be defined). Then we construct a solution (close to the state solution d), which starts at time $2T/3$ from the previous state. In the last step, we bring the latter state to 0 *via* a function y_2, as we can see in Figure 1.1 below. For details of this construction see the characterization of the function y in (3.12).

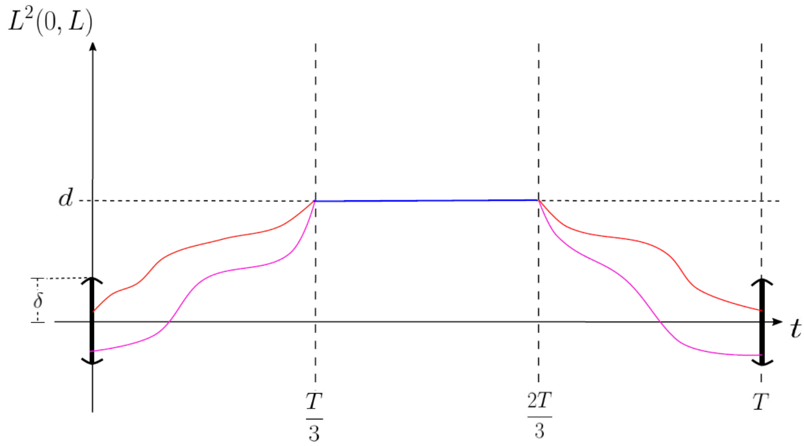

FIG. 1.1. *Solutions driving states close to 0 to constants and vice versa.*

We finish this introduction with an outline of this work which consists of three parts, including the introduction. Section 2 gives an overview of the well-posedness of the system (1.6). Section 3 is devoted to proving carefully the controllability of the system (1.6) when $L\in\mathcal{R}_c$. Precisely, in the first part of Section 3, we deal with the proof of Theorem 1.5. In the second part, we prove the construction of the function y mentioned before, and finally, in the third part of Section 3, we use these previous results to achieve Theorem 1.3.

2. Overview of the well-posedness theory

In this section, we review the well-posedness theory for the KdV equation. The results presented here can be found in [5, 7, 21]. For that, consider $L>0$ and $T_0, T_1\in\mathbb{R}$ with $T_0<T_1$. We define the space

$$\mathcal{Z}_{T_0,T_1} := C\left([T_0,T_1];L^2(0,L)\right)\cap L^2\left([T_0,T_1];H^1(0,L)\right)$$

which is a Banach space with the following norm

$$\|y\|_{\mathcal{Z}_{T_0,T_1}} := \max_{t\in[T_0,T_1]}\|y(\cdot,t)\|_{L^2(0,L)} + \left(\int_{T_0}^{T_1}\|y(\cdot,t)\|^2_{H^1(0,L)}dt\right)^{1/2}.$$

For any $T>0$ we denote $\mathcal{Z}_{0,T}$ simply by $\mathcal{Z}_T$.

Additionally, let $T>0$ be given and consider the space

$$\mathcal{H}_T := H^{-\frac{1}{3}}(0,T)\times L^2(0,T)\times H^{-\frac{1}{3}}(0,T)$$

with a norm

$$\|(h_1,h_2,h_3)\|_{\mathcal{H}_T} := \|h_1\|_{H^{-\frac{1}{3}}(0,T)} + \|h_2\|_{L^2(0,T)} + \|h_3\|_{H^{-\frac{1}{3}}(0,T)}.$$

The next proposition, showed in [7, Proposition 2.5], provides the well-posedness to the following system

$$\begin{cases} u_t+u_{xxx}=f, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t)=h_1(t),\ u_x(L,t)=h_2(t),\ u_{xx}(L,t)=h_3(t), & \text{in } (0,T),\\ u(x,0)=u_0, & \text{in } (0,L). \end{cases} \tag{2.1}$$

PROPOSITION 2.1 (Caicedo, Capistrano-Filho, Zhang [7]). *For any* $v_0 \in L^2(0,L)$, $\mathbf{h} = (h_1,h_2,h_3) \in \mathcal{H}_T$ *and* $f \in L^1(0,T,L^2(0,L))$, *the IBVP* (2.1) *admits a unique mild solution* $u \in \mathcal{Z}_T$, *which satisfies*

$$\partial_x^j u \in L_x^\infty(0,L;H^{(1-j)/3}(0,T)),\ j=0,1,2.$$

Moreover, there exists $C_1>0$ *such that*

$$\|u\|_{\mathcal{Z}_T} + \sum_{j=0}^{2} \left\|\partial_x^j u\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))} \le C_1 \left(\|u_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right).$$

REMARK 2.1. *We highlight that the constant* C_1 *in the above result depends on* T. *However, if* $\theta \in (0,T]$ *then, the estimates in the Proposition* 2.1 *hold with the same constant* C_1 *corresponding to* T.

In fact, let $u_0 \in L^2(0,L)$, $\mathbf{h} \in \mathcal{H}_\theta$, $f \in L^1(0,\theta;L^2(0,L))$ *and* $u \in \mathcal{Z}_\theta$ *the solution of* (2.1) *corresponding to these datas. We extend* $\mathbf{h}$ *and* f *to* $[0,T]$ *(we will also denote these extensions by* $\mathbf{h}$ *and* f*) putting*

$$\mathbf{h}=0 \ \textit{in}\ (\theta,T] \quad \textit{and} \quad f=0 \ \textit{in}\ (\theta,T].$$

Now, denote by $\tilde{u} \in \mathcal{Z}_T$ *the corresponding solution of* (2.1). *Then, from Proposition* 2.1, *we have* $\tilde{u}\big|_{[0,\theta]} = u$ *and*

$$\begin{aligned}\|u\|_{\mathcal{Z}_\theta} \le \|\tilde{u}\|_{\mathcal{Z}_T} &\le C_1 \left(\|u_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right) \\ &= C_1 \left(\|u_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right).\end{aligned}$$

Using the Proposition 2.1 we can get properties for the linear problem

$$\begin{cases} y_t + (ay)_x + y_{xxx} = f, & \text{in } (0,L)\times(0,T), \\ y_{xx}(0,t) = h_1(t),\ y_x(L,t) = h_2(t),\ y_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L), \end{cases} \tag{2.2}$$

where $a \in \mathcal{Z}_T$ is given. To do this, the following lemma will be very useful and was proved in [22, Lemma 3].

LEMMA 2.1 (Kramer, Zhang [22]). *There exists a constant* $C>0$ *such that*

$$\int_0^T \|uv_x(\cdot,t)\|_{L^2(0,L)} dt \le C\left(T^{\frac{1}{2}} + T^{\frac{1}{3}}\right)\|u\|_{\mathcal{Z}_T}\|v\|_{\mathcal{Z}_T},$$

for every $u,v \in \mathcal{Z}_T$.

With these previous results in hand, the following proposition gives us the well-posedness to the general system (2.2), which will be used several times so, for the sake of completeness, we will give the proof.

PROPOSITION 2.2. *For any* $y_0 \in L^2(0,L)$, $\mathbf{h} = (h_1,h_2,h_3) \in \mathcal{H}_T$ *and* $f \in L^1(0,T,L^2(0,L))$, *the IBVP* (2.2) *admits a unique mild solution* $y \in \mathcal{Z}_T$, *which satisfies*

$$\partial_x^j y \in L_x^\infty(0,L;H^{(1-j)/3}(0,T)),\ j=0,1,2,$$

and

$$\|y\|_{\mathcal{Z}_T} \leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right),$$

for some positive constant C_2 which depends only on T and $\|a\|_{\mathcal{Z}_T}$. In addition, the solution y possesses the following sharp trace estimates

$$\sum_{j=0}^{2} \left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))} \leq C_2 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0,T;L^2(0,L))}\right).$$

In particular, the map $(y_0, \mathbf{h}, f) \mapsto y$ is Lipschitz continuous.

Proof. Let $y_0 \in L^2(0,L)$, $\mathbf{h} \in \mathcal{H}_T$ and $f \in L^1(0,T;L^2(0,L))$ be given. Consider θ satisfying $0 < \theta \leq T$ and define the map $\Gamma : \mathcal{Z}_\theta \to \mathcal{Z}_\theta$ in the next way: for $y \in \mathcal{Z}_\theta$, put Γy being the solution of

$$\begin{cases} u_t + u_{xxx} = f - (ay)_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,\theta), \\ u(x,0) = y_0(x), & \text{in } (0,L). \end{cases}$$

Consider the set

$$B = \{y \in \mathcal{Z}_\theta;\ \|y\|_{\mathcal{Z}_\theta} \leq r\},$$

with $r > 0$ to be determined later. From Proposition 2.1 and Lemma 2.1 we have for any $y \in B$ the following estimate

$$\begin{aligned} \|\Gamma y\|_{\mathcal{Z}_\theta} \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))} + \|(ay)_x\|_{L^1(0,\theta;L^2(0,L))}\right) \\ \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) \\ & + 2C_1 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|a\|_{\mathcal{Z}_\theta} \|y\|_{\mathcal{Z}_\theta} \\ \leq & C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right) \\ & + 2C_1 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|a\|_{\mathcal{Z}_T} \|y\|_{\mathcal{Z}_\theta}. \end{aligned} \tag{2.3}$$

Choosing

$$r = 2C_1 \left(\|y_0\|_{L^2(0,L)} + \|\mathbf{h}\|_{\mathcal{H}_\theta} + \|f\|_{L^1(0,\theta;L^2(0,L))}\right)$$

and θ satisfying

$$2C_1 C \left(\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}}\right) \|a\|_{\mathcal{Z}_T} < \frac{1}{2}, \tag{2.4}$$

the inequality (2.3) give us

$$\|\Gamma y\|_{\mathcal{Z}_\theta} \leq \frac{r}{2} + \frac{r}{2} = r,$$

that is, $\Gamma(B) \subset B$. Furthermore, $\Gamma y - \Gamma w$ solves

$$\begin{cases} u_t + u_{xxx} = [a(-y+w)]_x, & \text{in } (0,L)\times(0,\theta), \\ u_{xx}(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & \text{in } (0,\theta), \\ u(x,0) = 0, & \text{in } (0,L). \end{cases}$$

So, from Proposition 2.1, Lemma 2.1 and inequality (2.4), we have

$$\begin{aligned}\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\theta} &\leq C_1\|[a(y-w)]_x\|_{L^1(0,\theta;L^2(0,L))}\\ &\leq 2C_1C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta}\\ &\leq 2C_1C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|a\|_{\mathcal{Z}_T}\|y-w\|_{\mathcal{Z}_\theta}\\ &<\frac{1}{2}\|y-w\|_{\mathcal{Z}_\theta}.\end{aligned}$$

Thus, $\Gamma: B\to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y\in B$ which is a solution to the problem (2.2) in $[0,\theta]$, corresponding to data $(y_0,\mathbf{h},f)$. Additionally, inequalities (2.3) and (2.4) yields that

$$\|y\|_{\mathcal{Z}_\theta}\leq C_1\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right)+\frac{1}{2}\|y\|_{\mathcal{Z}_\theta}$$

and, therefore,

$$\|y\|_{\mathcal{Z}_\theta}\leq 2C_1\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right).$$

Since θ depends only on a, with a standard continuation extension argument, the solution y can be extended to interval $[0,T]$ and the following estimate holds

$$\|y\|_{\mathcal{Z}_T}\leq C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}\right), \tag{2.5}$$

for some suitable constant $C_2>0$ which only depends on T and $\|a\|_{\mathcal{Z}_T}$. Therefore, it follows from (2.5) that the map $(y_0,\mathbf{h},f)\mapsto y$ is Lipschitz continuous and, as a consequence of this, we have the uniqueness of the solution y in $\mathcal{Z}_T$. The sharp trace estimates follow as a consequence of the Proposition 2.1, showing the proposition. □

Now, we will study the well-posedness of the nonlinear problem, namely:

$$\begin{cases} y_t+(ay)_x+y_{xxx}+yy_x=f, & \text{in } (0,L)\times(0,T),\\ y_{xx}(0,t)=h_1(t),\ y_x(L,t)=h_2(t),\ y_{xx}(L,t)=h_3(t), & \text{in } (0,T),\\ y(x,0)=y_0(x), & \text{in } (0,L).\end{cases} \tag{2.6}$$

The result can be read as follows.

PROPOSITION 2.3. *Let $a\in\mathcal{Z}_T$ be given. For every $y_0\in L^2(0,L)$, $\mathbf{h}=(h_1,h_2,h_3)\in\mathcal{H}_T$ and $f\in L^1(0,T,L^2(0,L))$ there exists a unique mild solution $y\in\mathcal{Z}_T$ of (2.6) which satisfies*

$$\|y\|_{\mathcal{Z}_T}\leq C_3\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{L^2(0,T)}+\|f\|_{L^1(0,T;L^2(0,L))}\right)$$

and

$$\begin{aligned}\sum_{j=0}^{2}\left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))}\leq & C_3\left(\|y_0\|_{L^2(0,L)}+\|h\|_{L^2(0,T)}+\|f\|_{L^1(0,T;L^2(0,L))}\right)\\ &+C_3\left(\|y_0\|_{L^2(0,L)}+\|h\|_{L^2(0,T)}+\|f\|_{L^1(0,T;L^2(0,L))}\right)^2,\end{aligned}$$

for some constant $C_3>0$. Furthermore, the corresponding solution map S is locally Lipschitz continuous, that is, given $\lambda>0$ there exist $L_\lambda>0$ such that, for every

$$(y_0,\mathbf{h},f_0),(y_1,\mathbf{g},f_1)\in L^2(0,L)\times\mathcal{H}_T\times L^1(0,T;L^2(0,L))$$

with

$$\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f_0\|_{L^1(0,T;L^2(0,L))}<\lambda,$$
$$\|y_1\|_{L^2(0,L)}+\|\mathbf{g}\|_{\mathcal{H}_T}+\|f_1\|_{L^1(0,T;L^2(0,L))}<\lambda,$$

we have

$$\begin{aligned}\|S(y_0,\mathbf{h},f_0)-S(y_1,\mathbf{g},f_1)\|_{\mathcal{Z}_T}\leq L_\lambda\big(&\|y_0-y_1\|_{L^2(0,L)}\\ &+\|\mathbf{h}-\mathbf{g}\|_{\mathcal{H}_T}+\|f_0-f_1\|_{L^1(0,T;L^2(0,L))}\big).\end{aligned}$$

Proof. We will proceed as follows: First, we will show the existence of a solution and obtain the desired estimates. Secondly, we will get an estimate that provides the uniqueness of the solution and guarantees that S is locally Lipschitz continuous.

To do that, consider $y_0\in L^2(0,L)$, $\mathbf{h}\in\mathcal{H}_T$ and $f\in L^1(0,T;L^2(0,L))$. Let θ satisfying $0<\theta\leq T$ and define the map $\Gamma:\mathcal{Z}_\theta\to\mathcal{Z}_\theta$ in the following way: For $y\in\mathcal{Z}_\theta$, pick Γy as solution of the following problem

$$\begin{cases} u_t+(au)_x+u_{xxx}=f-yy_x, & \text{in } (0,L)\times(0,\theta),\\ u_{xx}(0,t)=h_1(t),\ u_x(L,t)=h_2(t),\ u_{xx}(L,t)=h_3(t), & \text{in } (0,\theta),\\ u(x,0)=y_0(x), & \text{in } (0,L).\end{cases}$$

Consider the set

$$B=\{y\in\mathcal{Z}_\theta;\ \|y\|_{\mathcal{Z}_\theta}\leq r\},$$

with $r>0$ to be determined later. Using Proposition 2.2 and Lemma 2.1 we obtain, for any $y\in B$,

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\theta}&\leq C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}+\|yy_x\|_{L^1(0,\theta;L^2(0,L))}\right)\\ &\leq C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right)+C_2C\left(\theta^{\frac12}+\theta^{\frac13}\right)\|y\|^2_{\mathcal{Z}_T}.\end{aligned}\tag{2.7}$$

Choosing

$$r=4C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right)$$

and θ satisfying

$$C_2C\left(\theta^{\frac12}+\theta^{\frac13}\right)r<\frac14,\tag{2.8}$$

inequality (2.7) give us

$$\|\Gamma\|_{\mathcal{Z}_T}\leq\frac r4+\frac r4=\frac r2<r,$$

that is, $\Gamma(B)\subset B$. Moreover given $y,w\in B$, $\Gamma y-\Gamma w$ solves

$$\begin{cases} u_t+(au)_x+u_{xxx}=-yy_x+ww_x, & \text{in } (0,L)\times(0,\theta),\\ u_{xx}(0,t)=0,\ u_x(L,t)=0,\ u_{xx}(L,t)=0, & \text{in } (0,\theta),\\ u(x,0)=0, & \text{in } (0,L).\end{cases}$$

Therefore, from Proposition 2.2,

$$\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\theta}\leq C_2\|yy_x-ww_x\|_{L^1(0,\theta;L^2(0,L))}.$$

Note that

$$yy_x - ww_x = \frac{1}{2}\left[(y+w)_x(y-w)+(y+w)(y-w)_x\right].$$

Then, thanks to the Lemma 2.1, we get that

$$\begin{aligned}
&\|yy_x - ww_x\|_{L^1(0,\theta;L^2(0,L))}\\
&\le \frac{1}{2}\left[\|(y+w)_x(y-w)\|_{L^1(0,\theta;L^2(0,L))}+\|(y+w)(y-w)_x\|_{L^1(0,\theta;L^2(0,L))}\right]\\
&\le \frac{1}{2}\left[C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta}+C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta}\right]\\
&= C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|y+w\|_{\mathcal{Z}_\theta}\|y-w\|_{\mathcal{Z}_\theta}\\
&\le 2C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)r\|y-w\|_{\mathcal{Z}_\theta}
\end{aligned}$$

and, using (2.8) yields

$$\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\theta}\le 2C_2C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)r\|y-w\|_{\mathcal{Z}_\theta}<\frac{1}{2}\|y-w\|_{\mathcal{Z}_\theta}.$$

Hence, $\Gamma: B\to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y\in B$ which is a solution to the problem (2.6) in $[0,\theta]$, corresponding to data $(y_0,\mathbf{h},f)$.

Furthermore, due to the inequalities (2.7) and (2.8), we have

$$\|y\|_{\mathcal{Z}_\theta}\le C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right)+\frac{1}{4}\|y\|_{\mathcal{Z}_\theta},$$

and, therefore

$$\|y\|_{\mathcal{Z}_\theta}\le \frac{4}{3}C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_\theta}+\|f\|_{L^1(0,\theta;L^2(0,L))}\right).$$

Since θ depends only on a, with a standard continuation extension argument, the solution y can be extended to interval $[0,T]$ and the following estimate holds

$$\|y\|_{\mathcal{Z}_T}\le \tilde{C}_3\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}\right), \tag{2.9}$$

for some suitable constant $\tilde{C}_3>0$ which only depends on T and a. Now, using one more time the Proposition 2.3 together with Lemma 2.1 we obtain

$$\begin{aligned}
&\sum_{j=0}^{2}\left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))}\\
&\le C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}+\|yy_x\|_{L^1(0,T;L^2(0,L))}\right)\\
&\le C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}\right)+C_2C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\|y\|^2_{\mathcal{Z}_T}.
\end{aligned}$$

By (2.9) it follows that

$$\sum_{j=0}^{2}\left\|\partial_x^j y\right\|_{L_x^\infty(0,L;H^{(1-j)/3}(0,T))}$$

$$
\begin{aligned}
&\leq C_2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}\right)\\
&+C_2C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\tilde{C}^2\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f\|_{L^1(0,T;L^2(0,L))}\right)^2.
\end{aligned}
$$

Choosing

$$
C_3:=\max\left\{\tilde{C}_3,C_2,C_2C\left(\theta^{\frac{1}{2}}+\theta^{\frac{1}{3}}\right)\tilde{C}^2\right\}
$$

we obtain the desired estimates.

Now, consider

$$
(y_0,\mathbf{h},f_0),(y_1,\mathbf{g},f_1)\in L^2(0,L)\times\mathcal{H}_T\times L^1(0,T;L^2(0,L))
$$

with

$$
\begin{aligned}
&\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f_0\|_{L^1(0,T;L^2(0,L))}<\lambda,\\
&\|y_1\|_{L^2(0,L)}+\|\mathbf{g}\|_{\mathcal{H}_T}+\|f_1\|_{L^1(0,T;L^2(0,L))}<\lambda.
\end{aligned}
$$

Write $\mathbf{h}=(h_1,h_2,h_3)$ and $\mathbf{g}=(g_1,g_2,g_3)$. Let y and u be solutions of

$$
\begin{cases}
y_t+(ay)_x+y_{xxx}+yy_x=f_0, & \text{in } (0,L)\times(0,T),\\
y_{xx}(0,t)=h_1(t),\ y_x(L,t)=h_2(t),\ y_{xx}(L,t)=h_3(t), & \text{in } (0,T),\\
y(x,0)=y_0(x) & \text{in } (0,L),
\end{cases}
$$

and

$$
\begin{cases}
u_t+(au)_x+u_{xxx}+uu_x=f_1, & \text{in } (0,L)\times(0,T),\\
u_{xx}(0,t)=g_1(t),\ u_x(L,t)=g_2(t),\ u_{xx}(L,t)=g_3(t), & \text{in } (0,T),\\
u(x,0)=y_1(x) & \text{in } (0,L),
\end{cases}
$$

respectively. Then, $w=y-u$ solves the problem

$$
\begin{cases}
w_t+\left[\left(a+\frac{1}{2}(y+u)\right)w\right]_x+w_{xxx}=f_0-f_1, & \text{in } (0,L)\times(0,T,)\\
w_{xx}(0,\cdot)=h_1-g_1,\ w_x(L,\cdot)=h_2-g_2,\ w_{xx}(L,\cdot)=h_3-g_3, & \text{in } (0,T),\\
w(x,0)=y_0(x)-y_1(x), & \text{in } (0,L).
\end{cases}
$$

From Proposition 2.2 it follows that

$$
\|y-u\|_{\mathcal{Z}_T}\leq D\left(\|y_0-y_1\|_{L^2(0,L)}+\|\mathbf{h}-\mathbf{g}\|_{\mathcal{H}_T}+\|f_0-f_1\|_{L^1(0,T;L^2(0,L))}\right), \tag{2.10}
$$

where D is a constant which depends on T and $\|a+\frac{1}{2}(y+u)\|_{\mathcal{Z}_T}$. But, using (2.9) we have

$$
\begin{aligned}
\|a+\frac{1}{2}(y+u)\|_{\mathcal{Z}_T}\leq&\|a\|_{\mathcal{Z}_T}+\frac{1}{2}\left(\|y\|_{\mathcal{Z}_T}+\|u\|_{\mathcal{Z}_T}\right)\\
\leq&\|a\|_{\mathcal{Z}_T}+\frac{\tilde{C}_3}{2}\left(\|y_0\|_{L^2(0,L)}+\|\mathbf{h}\|_{\mathcal{H}_T}+\|f_0\|_{L^1(0,T;L^2(0,L))}\right)\\
&+\frac{\tilde{C}_3}{2}\left(\|y_1\|_{L^2(0,L)}+\|\mathbf{g}\|_{\mathcal{H}_T}+\|f_1\|_{L^1(0,T;L^2(0,L))}\right)
\end{aligned}
$$

and, consequently,

$$
\|a+\frac{1}{2}(y+u)\|_{\mathcal{Z}_T}\leq\|a\|_{\mathcal{Z}_T}+\tilde{C}_3\lambda.
$$

Hence, the constant D can be chosen depending only on T and $\|a\|_{\mathcal{Z}_T}$ (and also on λ), so that, (2.10) gives us the uniqueness of the solution which turns the solution map S well-defined. Moreover, from (2.10)

$$\begin{aligned}\|S(y_0,\mathbf{h},f_0)-S(y_1,\mathbf{g},f_1)\|_{\mathcal{Z}_T}\leq D\big(&\|y_0-y_1\|_{L^2(0,L)}\\&+\|\mathbf{h}-\mathbf{g}\|_{\mathcal{H}_T}+\|f_0-f_1\|_{L^1(0,T;L^2(0,L))}\big),\end{aligned}$$

which concludes the proof. □

Finally, the following Lemma, whose proof can be found in [24], will be useful in the next section.

LEMMA 2.2 (Rosier [24]). *If $y\in\mathcal{Z}_T$ then $yy_x\in L^1(0,T;L^2(0,L))$ and the map*

$$\begin{aligned}\mathcal{Z}_T&\longrightarrow L^1(0,T;L^2(0,L))\\ y&\longmapsto yy_x,\end{aligned}$$

is continuous. More precisely, for every $y,z\in\mathcal{Z}_T$ we have that

$$\begin{aligned}\|yy_x-zz_x\|_{L^1(0,T;L^2(0,L))}\leq C_4\big(&\|y\|_{L^2(0,T;H^1(0,L))}\\&+\|z\|_{L^2(0,T;H^1(0,L))}\big)\|y-z\|_{L^2(0,T;H^1(0,L))},\end{aligned}$$

where C_4 is a positive constant that depends only on L.

REMARK 2.2. *We end this subsection with the following remarks.*

1. *For every $a\in\mathcal{Z}_T$, the Proposition 2.2 give us the well-definition for the solution operator*

$$\Lambda_a: L^2(0,L)\times\mathcal{H}_T\times L^1(0,T;L^2(0,L))\to\mathcal{Z}_T$$

where, for each $(y_0,\mathbf{h},f)\in L^2(0,L)\times\mathcal{H}_T\times L^1(0,T;L^2(0,L))$, $\Lambda_a(y_0,\mathbf{h},f)$ is the corresponding solution to the problem (2.2). Furthermore, the Proposition 2.3 guarantees that Λ_a is a bounded linear operator.

2. *Of course, the constant $C_2>0$ in the Proposition 2.2 depends on $\|a\|_{\mathcal{Z}_T}$. However, given $M>0$, the same constant C_2 can be used for every $a\in\mathcal{Z}_T$ with $\|a\|_{\mathcal{Z}_T}\leq M$.*

3. Boundary controllability in the critical lenght

In this section, we study the controllability of the system

$$\begin{cases} y_t+y_x+y_{xxx}+yy_x=0, & \text{in } (0,L)\times(0,T),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h(t),\ y_{xx}(L,t)=0, & \text{in } (0,T),\\ y(x,0)=y_0(x) & \text{in } (0,L),\end{cases}\tag{3.1}$$

when L is a critical length. We will use the return method together with the fixed point argument to ensure the controllability of the system (3.1) when $L\in\mathcal{R}_c$. Before presenting the proof of the main result, let us give a preliminary result which is important in our analysis.

3.1. An auxiliary result

Given $c\neq-1$, the set of critical lengths for the linearization of (3.1) around c is

$$\mathcal{R}_c:=\left\{\frac{2\pi}{\sqrt{3(c+1)}}\sqrt{m^2+ml+l^2}\ ;\ m,l\in\mathbb{N}^*\right\}\cup\left\{\frac{m\pi}{\sqrt{c+1}};\ m\in\mathbb{N}^*\right\}.$$

As mentioned before, when $L\in\mathcal{R}_c$, the linear system

$$\begin{cases} y_t+(1+c)y_x+y_{xxx}=0, & \text{in } (0,L)\times(0,T),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h(t),\ y_{xx}(L,t)=0, & \text{in } (0,T),\\ y(x,0)=y_0(x), & \text{in } (0,L), \end{cases} \tag{3.2}$$

is not exactly controllable. So, in this section, we first give the proof of Theorem 1.5.

Proof. **(Proof of Theorem 1.5.)** We will split the proof into two cases.

First case: $L=2\pi\frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}}$ for some $(j,k)\in\mathbb{N}^*\times\mathbb{N}^*$.

Let $d\neq -1$ and suppose that $L\in\mathcal{R}_d$. Then

$$L=2\pi\frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}};\ m,l\in\mathbb{N}^* \tag{3.3}$$

or

$$L=\frac{m\pi}{\sqrt{d+1}};\ m\in\mathbb{N}^*. \tag{3.4}$$

If (3.3) is the case, we have that

$$2\pi\frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}}=2\pi\frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}},$$

which implies

$$d=\frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2}-1$$

with $m,l\in\mathbb{N}^*$.

Otherwise, if (3.4) holds, so

$$2\pi\frac{\sqrt{j^2+jk+k^2}}{\sqrt{3(c+1)}}=\frac{m\pi}{\sqrt{d+1}}$$

giving that

$$d=\frac{3m^2(c+1)}{4(j^2+jk+k^2)}-1$$

where $m\in\mathbb{N}^*$. Therefore, if $L\in\mathcal{R}_d$ then we necessarily have $d\in\mathcal{A}_1\cup\mathcal{B}_1$. Here,

$$\mathcal{A}_1:=\left\{\frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2}-1,\ m,l\in\mathbb{N}^*\right\}$$

and

$$\mathcal{B}_1:=\left\{\frac{3m^2(c+1)}{4(j^2+jk+k^2)}-1;\ m\in\mathbb{N}^*\right\}.$$

We are now in a position to prove that $\mathcal{A}_1\cup\mathcal{B}_1$ is discrete. To do that, consider $x,y\in\mathcal{A}_1$ such that $x\neq y$ in the form

$$x=\frac{(c+1)(m_1^2+m_1l_1+l_1^2)}{j^2+jk+k^2}-1$$

and

$$y=\frac{(c+1)(m_2^2+m_2l_2+l_2^2)}{j^2+jk+k^2}-1,$$

with $m_1,l_1,m_2,l_2\in\mathbb{N}^*$. Note that

$$x-y=\frac{c+1}{j^2+jk+k^2}\left[(m_1^2+m_1l_1+l_1^2)-(m_2^2+m_2l_2+l_2^2)\right].$$

Since $x\neq y$ we have

$$(m_1^2+m_1l_1+l_1^2)\neq(m_2^2+m_2l_2+l_2^2).$$

Thus

$$\left|(m_1^2+m_1l_1+l_1^2)-(m_2^2+m_2l_2+l_2^2)\right|\geq 1$$

so that

$$d(x,y)\geq\frac{c+1}{j^2+jk+k^2},\ \forall x,y\in\mathcal{A}_1,\ x\neq y.$$

Analogously, we get

$$d(x,y)\geq\frac{3(c+1)}{4(j^2+jk+k^2)},\ \forall\ x,y\in\mathcal{B}_1,x\neq y.$$

Now, let $x\in\mathcal{A}_1$ and $y\in\mathcal{B}_1$ with $x\neq y$, as follow:

$$x=\frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2}-1$$

and

$$y=\frac{3p^2(c+1)}{4(j^2+jk+k^2)}-1,$$

where $m,l,p\in\mathbb{N}^*$. Observe that

$$\begin{aligned}x-y&=\frac{(c+1)(m^2+ml+l^2)}{j^2+jk+k^2}-\frac{3p^2(c+1)}{4(j^2+jk+k^2)}\\&=\frac{4(c+1)(m^2+ml+l^2)}{4(j^2+jk+k^2)}-\frac{3p^2(c+1)}{4(j^2+jk+k^2)}\\&=\frac{c+1}{4(j^2+jk+k^2)}\left[4(m^2+ml+l^2)-3p^2\right].\end{aligned}$$

Since $x\neq y$ we have that $4(m^2+ml+l^2)$ and $3p^2$ are distinct natural numbers so

$$\left|4(m^2+ml+l^2)-3p^2\right|\geq 1$$

and, consequently, we have

$$d(x,y)\geq\frac{c+1}{4(j^2+jk+k^2)}.$$

From all the above we conclude that

$$d(x,y) \geq \frac{c+1}{4(j^2+jk+k^2)}, \ \forall \ x,y \in \mathcal{A}_1 \cup \mathcal{B}_1, \ x \neq y,$$

which implies that $\mathcal{A}_1 \cup \mathcal{B}_1$ is discrete.

Note that $c \in \mathcal{A}_1 \cup \mathcal{B}_1$ since

$$c = \left[\frac{(c+1)(j^2+jk+k^2)}{j^2+jk+k^2} - 1 \right] \in \mathcal{A}_1.$$

As any point in $\mathcal{A}_1 \cup \mathcal{B}_1$ is isolated, there exists $\epsilon_1 > 0$ such that

$$(c-\epsilon_1, c+\epsilon_1) \cap [\mathcal{A}_1 \cup \mathcal{B}_1] = \{c\}.$$

Therefore for $d \in (c-\epsilon_1, c+\epsilon_1) \backslash \{c\}$ we have that $d \notin \mathcal{A}_1 \cup \mathcal{B}_1$ and, therefore, $L \notin \mathcal{R}_d$.

Second case: $L = \frac{k\pi}{\sqrt{c+1}}$ for some $k \in \mathbb{N}^*$.

Let $d \neq -1$ and suppose that $L \in \mathcal{R}_d$, that is, (3.3) or (3.4) holds. If (3.3) holds, then

$$\frac{k\pi}{\sqrt{c+1}} = 2\pi \frac{\sqrt{m^2+ml+l^2}}{\sqrt{3(d+1)}}$$

giving that

$$d = \frac{(c+1)(m^2+ml+l^2)}{3k^2} - 1,$$

with $m, l \in \mathbb{N}^*$. If (3.4) is the case, thus

$$\frac{k\pi}{\sqrt{c+1}} = \frac{m\pi}{\sqrt{d+1}}$$

so that

$$d = \frac{(c+1)m^2}{k^2} - 1.$$

Therefore, if $L \in \mathcal{R}_d$ then we have necessarily $d \in \mathcal{A}_2 \cup \mathcal{B}_2$ where

$$\mathcal{A}_2 := \left\{ \frac{(c+1)(m^2+ml+l^2)}{3k^2} - 1; \ m, l \in \mathbb{N}^* \right\}$$

and

$$\mathcal{B}_2 := \left\{ \frac{(c+1)m^2}{k^2} - 1; \ m \in \mathbb{N}^* \right\}.$$

As done before, taking $x, y \in \mathcal{A}_2$ with $x \neq y$, such that

$$x = \frac{(c+1)(m_1^2+m_1l_1+l_1^2)}{3k^2} - 1$$

and

$$y = \frac{(c+1)(m_2^2+m_2l_2+l_2^2)}{3k^2} - 1,$$

where $m_1, l_1, m_2, l_2 \in \mathbb{N}^*$,yields that

$$x-y=\frac{c+1}{3k^2}\left[(m_1^2+m_1l_1+l_1^2)-(m_2^2+m_2l_2+l_2^2)\right]$$

and, as $x\neq y$ we have

$$\left|(m_1^2+m_1l_1+l_1^2)-(m_2^2+m_2l_2+l_2^2)\right|\geq 1.$$

Consequently,

$$d(x,y)\geq\frac{c+1}{3k^2}, \forall x,y\in\mathcal{A}_2,\ x\neq y.$$

In an analogous way,

$$d(x,y)\geq\frac{c+1}{k^2},\ \forall x,y\in\mathcal{B}_2,\ x\neq y$$

and

$$d(x,y)\geq\frac{c+1}{3k^2},\ \forall x\in\mathcal{A}_2,\ y\in\mathcal{B}_2,\ x\neq y.$$

Therefore,

$$d(x,y)\geq\frac{c+1}{3k^2},\ \forall x,y\in\mathcal{A}_2\cup\mathcal{B}_2,\ x\neq y.$$

Proceeding as in the first case, we conclude that there exists $\epsilon_2>0$ such that, for every $d\in(c-\epsilon_2,c+\epsilon_2)\setminus\{c\}$ we have $d\notin\mathcal{A}_2\cup\mathcal{B}_2$ so that $L\notin\mathcal{R}_d$. Considering $\epsilon_c:=\min\{\epsilon_1,\epsilon_2\}$, thanks thanks to the [7, Proposition 3.6] and Theorem 1.1, the proof is completed. □

3.2. Construction of the trajectories

In this subsection, for simplicity, we will consider $c=0$ in $\mathcal{R}_c$. Let $T>0$ and note that by Theorem 1.5 there exist $\epsilon_0>0$ such that for every $d\in(0,\epsilon_0)$, the system (3.2) is exactly controllable around d. Henceforth, we use C, C_1, C_2 and C_3 to denote the positive constants given in Proposition 2.3, corresponding to T. One can see that, for every $\tau\in[0,T]$, the same constants can be used to apply the corresponding results for $\|\cdot\|_{\mathcal{Z}_\tau}$ so, for simplicity, we will consider now on $\tau=\frac{T}{3}$.

The first result of this section ensures that we can construct solutions for the system (3.1) which starts close to 0 (left-hand side of the spatial domain) and achieves some non-null equilibrium in a certain time $T/3$. The result is shown by a fixed-point argument.

PROPOSITION 3.1. *There exist $\delta_1>0$ such that, for every $d\in(0,\delta_1)$ and $y_0\in L^2(0,L)$ with $\|y_0\|_{L^2(0,L)}<\delta_1$, there exists $h_1\in L^2(0,T/3)$ such that, the solution of (3.1) for $t\in[0,T/3]$, satisfies*

$$y(\cdot,T/3)=d.$$

Proof. Let $\delta_1\in(0,\epsilon_0)$ be a number to be chosen later. Consider $d\in(0,\delta_1)$ and $y_0\in L^2(0,L)$ satisfying

$$\|y_0\|_{L^2(0,L)}<\delta_1.$$

For $\varepsilon\in(0,\epsilon_0)$ such that

$$C\left(\tau^{\frac{1}{2}}+\tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_T}<\delta_1 \quad \text{and} \quad C\left(\tau^{\frac{1}{2}}+\tau^{\frac{1}{3}}\right)\|\varepsilon\|^2_{\mathcal{Z}_T}<\delta_1, \tag{3.5}$$

where $C>0$ is the positive constant given in Lemma 2.1, [7, Proposition 3.6] guarantees the existence of a bounded linear operator

$$\Psi^{\varepsilon}:L^2(0,L)\times L^2(0,L)\to L^2(0,\tau)$$

such that, for any $u_0,u_\tau\in L^2(0,L)$, the solution u of

$$\begin{cases} u_t+(1+\varepsilon)u_x+u_{xxx}=0, & \text{in } (0,L)\times(0,\tau),\\ u_{xx}(0,t)=0,\ u_x(L,t)=h(t),\ u_{xx}(L,t)=0, & \text{in } (0,\tau),\\ u(x,0)=u_0(x), & \text{in } (0,L),\end{cases}$$

with the control $h=\Psi^{\varepsilon}(u_0,u_\tau)$ satisfies $u(\cdot,\tau)=u_\tau$.

We will denote, for simplicity, the operator $\Lambda_{1+\varepsilon}$ (given in Proposition 2.2 and Remark 2.2 with $a=1+\varepsilon$) by Λ_ε. Observe that, if y is solution for (3.1) for some control h, then y is a solution of

$$\begin{cases} y_t+(1+\varepsilon)y_x+y_{xxx}=-yy_x+\varepsilon y_x, & \text{in } (0,L)\times(0,\tau),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h(t),\ y_{xx}(L,t)=0, & \text{in } (0,\tau),\\ y(x,0)=y_0(x) & \text{in } (0,L),\end{cases}$$

that is,

$$y=\Lambda_\varepsilon(y_0,h,-yy_x+\varepsilon y_x)=\Lambda_\varepsilon(y_0,h,0)+\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x). \tag{3.6}$$

Let $y\in\mathcal{Z}_\tau$ and $h_y\in L^2(0,\tau)$ given by

$$h_y=\Psi^{\varepsilon}\big(y_0,d-\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\big).$$

Define the map $\Gamma:\mathcal{Z}_\tau\to\mathcal{Z}_\tau$ by

$$\Gamma y=\Lambda_\varepsilon(y_0,h_y,0)+\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x).$$

Note that if Γ has a fixed point y, then, from the above construction, it follows that y is a solution of (3.1) with the control h_y. Moreover, from (3.6) we have

$$y=\Lambda_\varepsilon(y_0,h_y,0)+\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)$$

so, by definitions of Λ_ε, h_y and Ψ^ε, we get that

$$\begin{aligned} y(\cdot,\tau)&=\Lambda_\varepsilon(y_0,h_y,0)(\cdot,\tau)+\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\\ &=d-\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)+\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\\ &=d,\end{aligned}$$

and our problem would be solved. So we will focus our efforts on showing that Γ has a fixed point in a suitable metric space.

To do that, let B the set

$$B=\{y\in\mathcal{Z}_\tau;\ \|y\|_{\mathcal{Z}_\tau}\leq r\},$$

with $r>0$ to be chosen later. By (3.6) and Proposition 2.2 (together with Remark 2.2) we have, for $y\in B$, that

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\tau} &\le \|\Lambda_\varepsilon(y_0,h_y,0)\|_{\mathcal{Z}_\tau}+\|\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)\|_{\mathcal{Z}_\tau}\\ &\le C_2\left(\|y_0\|_{L^2(0,L)}+\|h_y\|_{L^2(0,\tau)}+\|-yy_x+\varepsilon y_x\|_{L^1(0,\tau;L^2(0,L))}\right).\end{aligned}$$

From Lemma 2.1 and Young inequality we ensure that

$$\begin{aligned}\|yy_x-\varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))} &= \|(y-\varepsilon)y_x\|_{L^1(0,\tau,L^2(0,L))}\\ &\le C\left(\tau^{\frac12}+\tau^{\frac13}\right)\|y-\varepsilon\|_{\mathcal{Z}_\tau}\|y\|_{\mathcal{Z}_\tau}\\ &\le C\left(\tau^{\frac12}+\tau^{\frac13}\right)\frac12\left(\|y-\varepsilon\|^2_{\mathcal{Z}_\tau}+\|y\|^2_{\mathcal{Z}_\tau}\right)\\ &\le C\left(\tau^{\frac12}+\tau^{\frac13}\right)\frac12\left(\|y\|^2_{\mathcal{Z}_\tau}+\|y\|^2_{\mathcal{Z}_\tau}+\|\varepsilon\|^2_{\mathcal{Z}_\tau}+\|\varepsilon\|^2_{\mathcal{Z}_\tau}+\|y\|^2_{\mathcal{Z}_\tau}\right).\end{aligned}$$

Thus,

$$\|yy_x-\varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))}\le C\left(\tau^{\frac12}+\tau^{\frac13}\right)\frac32\|y\|^2_{\mathcal{Z}_\tau}+C\left(\tau^{\frac12}+\tau^{\frac13}\right)\|\varepsilon\|^2_{\mathcal{Z}_\tau}$$

and, by (3.5) we obtain

$$\|yy_x-\varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))}\le C\left(\tau^{\frac12}+\tau^{\frac13}\right)\frac32\|y\|^2_{\mathcal{Z}_\tau}+\delta_1,$$

that is,

$$\|yy_x-\varepsilon y_x\|_{L^1(0,\tau,L^2(0,L))}\le \overline{C}\|y\|^2_{\mathcal{Z}_\tau}+\delta_1,$$

where

$$\overline{C}:=\frac{3C}{2}\left(\tau^{\frac12}+\tau^{\frac13}\right).$$

In this way

$$\begin{aligned}\|h_y\|_{L^2(0,\tau)} &= \|\Psi^\varepsilon\left(y_0,d-\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\right)\|_{L^2(0,\tau)}\\ &\le \|\Psi^\varepsilon\|\left(\|y_0\|_{L^2(0,L)}+\|d\|_{L^2(0,L)}+\|\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)\|_{L^2(0,L)}\right)\\ &\le \|\Psi^\varepsilon\|\delta_1+\|\Psi^\varepsilon\|d\sqrt{L}+\|\Psi^\varepsilon\|\|\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)\|_{\mathcal{Z}_\tau}\\ &\le \|\Psi^\varepsilon\|\delta_1+\|\Psi^\varepsilon\|\delta_1\sqrt{L}+\|\Psi^\varepsilon\|C_2\|-yy_x+\varepsilon y_x\|_{L^1(0,\tau;L^2(0,L))}\\ &\le \|\Psi^\varepsilon\|\delta_1+\|\Psi^\varepsilon\|\delta_1\sqrt{L}+\|\Psi^\varepsilon\|C_2\overline{C}\|y\|^2_{\mathcal{Z}_\tau}+\|\Psi^\varepsilon\|C_2\delta_1\\ &= \left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_1+C_2\overline{C}\|\Psi^\varepsilon\|r^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma y\|_{\mathcal{Z}_\tau} &\le C_2\delta_1+C_2\left[\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_1+C_2\overline{C}\|\Psi^\varepsilon\|r^2\right]+C_2\left(\overline{C}r^2+\delta_1\right)\\ &= \left[2C_2+C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\right]\delta_1+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\overline{C}r^2.\end{aligned}$$

Choosing

$$r=2\left[2C_2+C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\right]\delta_1$$

and δ_1 small enough such that

$$\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\overline{C}r<\frac{1}{2},\quad 2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r<\frac{1}{2},\quad \left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\delta_1<\frac{1}{2},\tag{3.7}$$

yields that

$$\|\Gamma\|_{\mathcal{Z}_\tau}\leq\frac{r}{2}+\frac{r}{2}=r\Longrightarrow\Gamma(B)\subset B.$$

Additionally, observe that for $y,w\in B$, Proposition 2.2 give us

$$\begin{aligned}\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\tau}=&\|\Lambda_\varepsilon(0,h_y-h_w,0)+\Lambda_\varepsilon(0,0,-yy_x+ww_x+\varepsilon y_x-\varepsilon w_x)\|_{\mathcal{Z}_\tau}\\ \leq&C_2\|h_y-h_w\|_{L^2(0,\tau)}+C_2\|yy_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+C_2\|\varepsilon(y_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Since

$$\begin{aligned}h_y-h_w=&\Psi^\varepsilon\left(0,-\Lambda_\varepsilon(0,0,-yy_x+\varepsilon y_x)(\cdot,\tau)+\Lambda_\varepsilon(0,0,-ww_x+\varepsilon w_x)(\cdot,\tau)\right)\\ =&\Psi^\varepsilon\left(0,\Lambda_\varepsilon(yy_x-\varepsilon y_x-ww_x+\varepsilon w_x)(\cdot,\tau)\right),\end{aligned}$$

we have again from Proposition 2.2 that

$$\begin{aligned}C_2\|h_y-h_w\|_{L^2(0,\tau)}\leq&C_2^2\|\Psi^\varepsilon\|\|yy_x-\varepsilon y_x-ww_x+\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}\\ \leq&C_2^2\|\Psi^\varepsilon\|\|yy_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+C_2^2\|\Psi^\varepsilon\|\|\varepsilon y_x-\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Putting these two previous inequalities together, we find that

$$\begin{aligned}\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\tau}\leq&\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|yy_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|\varepsilon(y_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

From Lemmas 2.1 and 2.2, together with the choices (3.5) and (3.7), it follows that

$$\begin{aligned}\|\Gamma y-\Gamma w\|_{\mathcal{Z}_\tau}\leq&2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r\|y-w\|_{\mathcal{Z}_\tau}\\ &+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)C\left(\tau^{\frac{1}{2}}+\tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_\tau}\|y-w\|_{\mathcal{Z}_\tau}\\ \leq&\left[2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\delta_1\right]\|y-w\|_{\mathcal{Z}_\tau}\\ \leq&\|y-w\|_{\mathcal{Z}_\tau}\end{aligned}$$

Therefore, $\Gamma:B\to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $y\in B$, concluding the proof. □

The second result of this section ensures the construction of solutions for the system (3.1) on $[2T/3,T]$ starting in one non-null equilibrium and ending near 0.

Proposition 3.2. *There exists $\delta_2>0$ such that, for every $d\in(0,\delta_2)$ and $y_T\in L^2(0,L)$ satisfying $\|y_T\|_{L^2(0,L)}<\delta_2$, there exists $h_2\in L^2(2T/3,T)$ such that, the solution of (1.6) for $t\in[2T/3,T]$ satisfies*

$$y(\cdot,2T/3)=d\quad\textit{and}\quad y(\cdot,T/3)=y_T.$$

Proof. Let $\delta_2 \in (0,\epsilon_0)$ be a number to be chosen later. Consider $d \in (0,\delta_2)$ and $y_T \in L^2(0,L)$ satisfying

$$\|y_T\|_{L^2(0,L)} < \delta_2.$$

Let $\varepsilon \in (0,\epsilon_0)$ be such that

$$C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|_{\mathcal{Z}_T} < \delta_2 \quad \text{and} \quad C\left(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}}\right)\|\varepsilon\|^2_{\mathcal{Z}_T} < \delta_2, \tag{3.8}$$

where $C > 0$ is the positive constant given in Lemma 2.1. If z is a solution to the problem

$$\begin{cases} z_t + z_x + z_{xxx} + zz_x = 0, & \text{in } (0,L)\times(0,\tau), \\ z_{xx}(0,t) = 0,\ z_x(L,t) = h(t),\ z_{xx}(L,t) = 0 & \text{in } (0,\tau), \\ z(x,0) = d, & \text{in } (0,L), \end{cases} \tag{3.9}$$

then z is a solution of

$$\begin{cases} z_t + (1+\varepsilon)z_x + z_{xxx} = -zz_x + \varepsilon z_x, & \text{in } (0,L)\times(0,\tau), \\ z_{xx}(0,t) = 0,\ z_x(L,t) = h(t),\ z_{xx}(L,t) = 0, & \text{in } (0,\tau), \\ z(x,0) = d, & \text{in } (0,L), \end{cases}$$

that is,

$$z = \Lambda_\varepsilon(d,h,-zz_x+\varepsilon z_x) = \Lambda_\varepsilon(d,h,0) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x). \tag{3.10}$$

Given $z \in \mathcal{Z}_\tau$, let $h_z \in L^2(0,\tau)$ defined by

$$h_z = \Psi^\varepsilon\big(d, y_T - \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau)\big).$$

Now, consider the map $\Gamma : \mathcal{Z}_\tau \to \mathcal{Z}_\tau$ given by

$$\Gamma z = \Lambda_\varepsilon(d,h_z,0) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x).$$

Once again, if Γ has a fixed point z, from the above construction, it follows that z is a solution of (3.9) with the control h_z. Moreover, from (3.10) we have

$$z = \Lambda_\varepsilon(d,h_z,0) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)$$

so, by definitions of Λ_ε, h_z and Ψ^ε, it follows that

$$z(\cdot,0) = \Lambda_\varepsilon(d,h_z,0)(\cdot,0) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,0) = d + 0 = d$$

and

$$\begin{aligned} z(\cdot,\tau) &= \Lambda_\varepsilon(d,h_z,0)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) \\ &= y_T - \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) + \Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau) \\ &= y_T. \end{aligned}$$

Hence our issue would be solved defining $y : [0,L]\times[2T/3,T] \to \mathbb{R}$ by

$$y(x,t) = z(x,t-2T/3).$$

Now ow, our focus is to show that Γ has a fixed point in a suitable metric space. To do that, consider the set B given by

$$B = \{z \in \mathcal{Z}_\tau;\ \|z\|_{\mathcal{Z}_\tau} \le r\},$$

with $r>0$ to be chosen later. By (3.10) and Proposition 2.2 (together with Remark 2.2) we have, for $z\in B$, that

$$\begin{aligned}\|\Gamma z\|_{\mathcal{Z}_\tau} &\leq \|\Lambda_\varepsilon(d,h_z,0)\|_{\mathcal{Z}_\tau}+\|\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)\|_{\mathcal{Z}_\tau}\\ &\leq C_2\left(\|d\|_{L^2(0,L)}+\|h_z\|_{L^2(0,\tau)}+\|-zz_x+\varepsilon z_x\|_{L^1(0,\tau;L^2(0,L))}\right).\end{aligned}$$

As in the proof of the Proposition 3.1,

$$\|zz_x-\varepsilon z_x\|_{L^1(0,\tau,L^2(0,L))}\leq \overline{C}\|z\|^2_{\mathcal{Z}_\tau}+\delta_2.$$

Moreover,

$$\begin{aligned}\|h_z\|_{L^2(0,\tau)} &= \|\Psi^\varepsilon\left(d,y_T-\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau)\right)\|_{L^2(0,\tau)}\\ &\leq \|\Psi^\varepsilon\|\left(\|d\|_{L^2(0,L)}+\|y_T\|_{L^2(0,L)}+\|\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau)\|_{L^2(0,L)}\right)\\ &\leq \|\Psi^\varepsilon\|d\sqrt{L}+\|\Psi^\varepsilon\|\delta_2+\|\Psi^\varepsilon\|\|\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)\|_{\mathcal{Z}_\tau}\\ &\leq \|\Psi^\varepsilon\|\delta_2+\|\Psi^\varepsilon\|\delta_2\sqrt{L}+\|\Psi^\varepsilon\|C_2\|-zz_x+\varepsilon z_x\|_{L^1(0,\tau;L^2(0,L))}\\ &\leq \|\Psi^\varepsilon\|\delta_2+\|\Psi^\varepsilon\|\delta_2\sqrt{L}+\|\Psi^\varepsilon\|C_2\overline{C}\|z\|^2_{\mathcal{Z}_\tau}+\|\Psi^\varepsilon\|C_2\delta_2\\ &= \left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_2+C_2\overline{C}\|\Psi^\varepsilon\|r^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma z\|_{\mathcal{Z}_\tau} &\leq C_2\delta_2\sqrt{L}+C_2\left[\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|\delta_2+C_2\overline{C}\|\Psi^\varepsilon\|r^2\right]+C_2\left(\overline{C}r^2+\delta_2\right)\\ &= \left[C_2\sqrt{L}+C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|+C_2\right]\delta_2+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\overline{C}r^2.\end{aligned}$$

Choosing

$$r=2\left[C_2\sqrt{L}+C_2\left(1+\sqrt{L}+C_2\right)\|\Psi^\varepsilon\|+C_2\right]\delta_2$$

and δ_2 small enough such that

$$\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\overline{C}r<\frac{1}{2},\quad 2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r<\frac{1}{2},\quad \left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\delta_1<\frac{1}{2}, \tag{3.11}$$

we get that

$$\|\Gamma z\|_{\mathcal{Z}_\tau}\leq \frac{r}{2}+\frac{r}{2}=r\Longrightarrow \Gamma(B)\subset B$$

Furthermore, observe that for $z,w\in B$, Proposition 2.2 give us

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau} =&\|\Lambda_\varepsilon(0,h_z-h_w,0)+\Lambda_\varepsilon(0,0,-zz_x+ww_x+\varepsilon z_x-\varepsilon w_x)\|_{\mathcal{Z}_\tau}\\ \leq& C_2\|h_z-h_w\|_{L^2(0,\tau)}+C_2\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ &+C_2\|\varepsilon(z_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Since

$$\begin{aligned}h_z-h_w &= \Psi^\varepsilon\left(0,-\Lambda_\varepsilon(0,0,-zz_x+\varepsilon z_x)(\cdot,\tau)+\Lambda_\varepsilon(0,0,-ww_x+\varepsilon w_x)(\cdot,\tau)\right)\\ &= \Psi^\varepsilon\left(0,\Lambda_\varepsilon(zz_x-\varepsilon z_x-ww_x+\varepsilon w_x)(\cdot,\tau)\right),\end{aligned}$$

we have, again from Proposition 2.2, that

$$\begin{aligned}C_2\|h_z-h_w\|_{L^2(0,\tau)}\leq & C_2^2\|\Psi^\varepsilon\|\|zz_x-\varepsilon z_x-ww_x+\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}\\ \leq & C_2^2\|\Psi^\varepsilon\|\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ & +C_2^2\|\Psi^\varepsilon\|\|\varepsilon z_x-\varepsilon w_x\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

Then,

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau}\leq & \left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|zz_x-ww_x\|_{L^1(0,\tau;L^2(0,L))}\\ & +\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\|\varepsilon(z_x-w_x)\|_{L^1(0,\tau;L^2(0,L))}.\end{aligned}$$

From Lemmas 2.1 and 2.2, together with (3.8) and (3.11), it follows that

$$\begin{aligned}\|\Gamma z-\Gamma w\|_{\mathcal{Z}_\tau}\leq & 2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r\|z-w\|_{\mathcal{Z}_\tau}\\ & +\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)C\left(\tau^{\frac12}+\tau^{\frac13}\right)\|\varepsilon\|_{\mathcal{Z}_\tau}\|z-w\|_{\mathcal{Z}_\tau}\\ \leq & \left[2\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)r+\left(C_2^2\|\Psi^\varepsilon\|+C_2\right)\delta_2\right]\|z-w\|_{\mathcal{Z}_\tau}\\ \leq & \|z-w\|_{\mathcal{Z}_\tau}.\end{aligned}$$

Therefore, $\Gamma:B\to B$ is a contraction so that, by Banach's fixed point theorem, Γ has a fixed point $z\in B$ which concludes our proof. □

3.3. Controllability on $\mathcal{R}_c$

We are in a position to prove Theorems 1.3 and 1.4. For the sake of simplicity, we will give the proof of the case $L\in\mathcal{R}_0$ (Theorem 1.3), and the case $L\in\mathcal{R}_c$ (Theorem 1.4) follows similarly.

Proof. **(Proof of Theorem 1.3.)** Let δ_1 and δ_2 be positive real numbers given in Propositions 3.1 and 3.2, respectively. Define $\delta:=\min\{\delta_1,\delta_2\}$ and consider $d\in(0,\delta)$ and $y_0,y_T\in L^2(0,L)$ such that

$$\|y_0\|_{L^2(0,L)},\|y_T\|_{L^2(0,L)}<\delta.$$

From Proposition 3.1 there exists $h_1\in L^2(0,T/3)$ such that, the solution $y_1\in\mathcal{Z}_{T/3}$ of

$$\begin{cases}y_t+y_x+y_{xxx}+yy_x=0, & \text{in } (0,L)\times(0,T/3),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h_1(t),\ y_{xx}(L,t)=0, & \text{in } (0,T/3),\\ y(x,0)=y_0(x), & \text{in } (0,L),\end{cases}$$

satisfies

$$y_1(x,T/3)=d.$$

On the other hand, thanks to the Proposition 3.2, there exists $h_2\in L^2(2T/3,T)$ such that, the solution $y_2\in\mathcal{Z}_{2T/3,T}$ of

$$\begin{cases}y_t+y_x+y_{xxx}+yy_x=0, & \text{in } (0,L)\times(2T/3,T),\\ y_{xx}(0,t)=0,\ y_x(L,t)=h_2(t),\ y_{xx}(L,t)=0, & \text{in } (2T/3,T),\\ y(x,2T/3)=d, & \text{in } (0,L),\end{cases}$$

satisfies

$$y_2(x,T)=y_T.$$

Defining $y:[0,L]\times[0,T]\to\mathbb{R}$ by

$$y=\begin{cases} y_1, & \text{in } [0,T/3],\\ d, & \text{in } [T/3,2T/3],\\ y_2, & \text{in } [2T/3,T],\end{cases} \tag{3.12}$$

we have that $y\in\mathcal{Z}_T$ and y is solution of (3.1) driving y_0 to y_T at time T, showing that the system (3.1) is exactly controllable, and the proof is completed. □

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