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ON THE STABILITY OF THE KAWAHARA EQUATION WITH A DISTRIBUTED INFINITE MEMORY

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Abstract. This article deals with the stability problem for a higher-order dispersive model governed by the so-called Kawahara equation. To do so, a damping mechanism is introduced, which contains a distributed memory term, and then proves that the solutions of the system are exponentially stable, provided that specific assumptions on the memory kernel are fulfilled. This is possible thanks to the energy method that permits us to obtain a decay rate estimate of the energy of the problem.

1. Introduction.

1.1. Model under consideration and objective.

The fifth-order nonlinear dispersive equation

$$\pm 2\partial_t u + 3u\partial_x u - \nu\partial_x^3 u + \frac{1}{45}\partial_x^5 u = 0, \tag{1}$$

models numerous physical phenomena. Considering suitable assumptions on the amplitude, wavelength, wave steepness, and so on, the properties of the asymptotic models for water waves have been extensively studied in the last years, through (1), to understand the full water wave system. For a rigorous justification of various asymptotic surface and internal waves models, we suggest that the reader consult the following references [1, 5, 29].

On the other hand, we can formulate the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form with at least two parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, non-dimensional, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. In turn, if we introduce another non-dimensional parameter μ, so-called the Bond number, which measures the importance of gravitational forces

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compared to surface tension forces, then the physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves. On the other hand, there are several long-wave approximations depending on the relations between ε and δ. For instance, if we consider $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation, represented by (1), and derived by Hasimoto and Kawahara in [23, 27].

The main concern of this article is to deal with the well-posedness and stability of an initial-boundary-value problem related to (1). Specifically, we are concerned with a fifth-order dispersive partial differential equation with a distributed memory term

$$\begin{cases} \partial_t u(x,t) + \partial_x^3 u(x,t) - a_0 \partial_x^5 u(x,t) + u(x,t)\partial_x u(x,t) & \\ +a_1 \partial_x u(x,t) + (-1)^k \displaystyle\int_0^\infty f(s)\partial_x^{2k} u(x,t-s)ds = 0, & (x,t) \in I \times (0,\infty), \\ u(0,t) = u(L,t) = 0, & t > 0, \\ \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0 & t > 0, \\ u(x,-t) = u_0(x,t), & x \in I,\ t \geq 0. \end{cases} \tag{2}$$

Here u represents the amplitude of the dispersive wave, $k \in \{0,1,2\}$, $L > 0$, $I = (0,L)$, while $a_1 \in \mathbb{R}$ and $a_0 > 0$ are physical parameters of the dispersive equation. Moreover, u_0 is the initial condition and f is the memory kernel satisfying $f : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}$ so as there exists a positive constant c_0 such that:

$$f \in C^2(\mathbb{R}_+), \quad f' < 0, \quad 0 \leq f'' \leq -c_0 f', \quad f(0) > 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{3}$$

After that, the energy associated with the system (2) is

$$E(t) = \frac{1}{2}\left(\|u(t)\|^2 + \int_0^\infty g(s)\|\partial_x^k \eta^t(\cdot,s)\|^2 ds \right),\ t \in \mathbb{R}_+. \tag{4}$$

Observe that $E' < 0$ and hence the energy of our system is decreasing (see Lemma 2.1). This means that the localized damping mechanism and the memory term constitute a damping mechanism and consequently one has to study the decay of the solutions of (2). Notwithstanding, it has been noticed that the stability property of solutions of many physical systems may be lost when a memory effect occurs [32]. Thus, our concern is to provide an answer to the following questions:

Problem $\mathcal{P}$: *Does the energy $E(t)$ decay to 0 as t is sufficiently large? If so, can we provide a decay rate estimate?*

1.2. **Historical background.** Let us present a review of the main results available in the literature.

First, we shall focus on the third-order Korteweg-de Vries (KdV) equation. In the case when a memory term occurs, numerous stability results were obtained in [12, 13] (see also the reference therein). Chentouf [12] considered the KdV equation with a boundary finite memory term in a bounded interval. Additionally, Chentouf and Guesmia [13] studied the stability problem for the KdV equation subject to the effect of a distributed infinite memory term. Recently, in [34], Parada *et al.* studied the stabilization problem of the KdV equation with either a boundary or distributed time-dependent delay. Note that this outcome extends those obtained in [4, 36].

Concerning the analysis of the Kawahara equation in a bounded interval, a pioneer work is due to Silva and Vasconcellos [37, 38], where the authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second endeavor, in this line, was completed by Capistrano-Filho *et al.* [3], where a generalized Kawahara equation in a bounded domain $Q_T = (0,T) \times (0,L)$ is considered:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + a(x)u = 0, & \text{in } Q_T, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = \partial_x^2 u(t,L) = 0, & \text{on } [0,T], \\ u(0,x) = u_0(x), & \text{in } [0,L], \end{cases} \tag{5}$$

with $p \in [1,4)$ and $a(x)$ is a nonnegative function and positive only on an open subset of $(0,L)$. It is proven that the energy of the above system decays exponentially.

The internal controllability problem has been tackled by Chen [10] for the Kawahara equation with homogeneous boundary conditions. Using Carleman estimates associated with the linear operator of the Kawahara equation with an internal observation, a null controllable result was shown when the internal control is effective in a subdomain $\omega \subset (0,L)$. In [8], considering the system (5) with an internal control $f(t,x)$ and homogeneous boundary conditions, the equation is shown to be exactly controllable in L^2-weighted Sobolev spaces and, additionally, controllable by regions in L^2-Sobolev space.

Recently, a new tool for the control properties of the Kawahara operator was proposed. In [7], the authors showed a new type of controllability for the Kawahara equation, what they called *overdetermination control problem*. A boundary control was designed so that the solution to the problem under consideration satisfies an integral condition.

The last studies on the stabilization of the Kawahara equation deal with a localized time-delayed interior control. In [9, 11], under suitable assumptions on the time delay coefficients, the authors were able to prove that solutions of the Kawahara system are exponentially stable. The results were obtained using the Lyapunov approach and a compactness-uniqueness argument. More recently, the authors in [6] gave an analysis to better understand the stabilization issue for the Kawahara equation. Indeed, it is shown that the Kawahara equation under the action of a time-delayed boundary control remains exponentially stable under a condition on the length of the spatial domain. Such a desirable property is proved using two different approaches. It is also worth mentioning that the stability of the solutions to the Kawahara equation has been extensively studied in the context of periodic or non-periodic bounded domain [20, 21, 24, 26] and also in the case when the spacial variable lies in $(-\infty, \infty)$ or $[0, \infty)$ [15, 16, 17, 19, 25, 30].

We end the literature review by mentioning that the occurrence of a memory phenomenon in the Kawahara problem (2) could be explained in practice by the fact that numerous compressible and incompressible fluids are intrinsically viscoelastic and therefore the influence of the past values of the amplitude of the dispersive wave of the fluid is unavoidable [2, 13, 18, 33].

Regarding the main contribution of this paper, we can claim that we go one step further in the study of the stabilization problem for the fifth-order Korteweg-de-Vries type system. Compared to the recent works [3, 6, 9, 11], where damping mechanisms and delay controls are used, this paper closes the gap since it is the first work to treat exponential stability using only infinite memory. It is also noteworthy that the current paper shows that a memory term plays a role of a damping control in the sense that it leads to the stability of the system without any additional damping such as $a(x)u$ used in [3, 11, 39] to get the stability property of the system.

Finally, note that our results remain valid if $a_1 = 0$ and hence the drift term $\partial_x u(x,t)$ can be omitted.

1.3. **Notations and main result.** Throughout this article, C denotes a constant that can be different from one step to another in the demonstrations presented in the sequel. Let us use $\langle\ ,\ \rangle$ and $\|\cdot\|$ to denote the standard real inner product in $L^2(I)$ and its corresponding norm given by

$$\langle u,v\rangle=\int_0^L u(x)v(x)dx \quad \text{and} \quad \|u\|=\left(\int_0^L |u(x)|^2dx\right)^{\frac{1}{2}}.$$

Consider two functions $u(x,\cdot)$ and $u_0(x,\cdot)$ belonging to $C(\mathbb{R}_+,L^2(I))$ and satisfying the boundary conditions of (2). As in [18], introduce the following approximation

$$\eta^t(x,s)=\int_{t-s}^t u(x,\tau)d\tau \text{ and } \eta^0(x,s)=\int_0^s u_0(x,\tau)d\tau,\ x\in I, s,t\in\mathbb{R}_+,$$

for $u(x,\cdot)$ and $u_0(x,\cdot)$. Observe that the functional η^t satisfies

$$\begin{cases}\partial_t\eta^t(x,s)+\partial_s\eta^t(x,s)=u(x,t), & x\in I,\ s,t\in\mathbb{R}_+,\\ \eta^t(0,s)=\eta^t(L,s)=0, & s,t\in\mathbb{R}_+,\\ \eta^t(x,0)=0, & x\in I,\ t\in\mathbb{R}_+.\end{cases} \tag{6}$$

Indeed, differentiating formally η^t with respect to t, we obtain

$$\partial_t\eta^t=-u(x,t-s)+u(x,t).$$

Analogously, we have

$$\partial_s\eta^t=u(x,t-s).$$

Therefore,

$$\partial_t\eta^t+\partial_s\eta^t=u(x,t),\quad \forall x\in I,\ \forall s,t\in\mathbb{R}_+.$$

In addition, note that thanks to the boundary conditions of (2), we ensure that

$$\eta^t(0,s)=\int_{t-s}^t \underbrace{u(0,\tau)}_{=0}d\tau=0,\quad \eta^t(L,s)=\int_{t-s}^t \underbrace{u(L,\tau)}_{=0}d\tau=0,$$

while

$$\eta^t(x,0)=\int_t^t u(x,\tau)d\tau=0,$$

which gives us (6).

Now, in order to express the memory term

$$(-1)^k\int_0^\infty f(s)\partial_x^{2k}u(x,t-s)ds$$

in terms of η^t, pick a function $g:=-f'$, with f satisfying (3). Thus, thanks to the properties of the function f, we get

$$\begin{cases}g\in C^1(\mathbb{R}_+),\ g>0,\ 0\le -g'\le c_0 g,\\ g_0=\displaystyle\int_0^\infty g(s)ds=f(0)>0 \text{ and } \lim_{s\to\infty}g(s)=0.\end{cases} \tag{7}$$

On the other hand, integrating by parts with respect to s and using that $\eta^t(x,0)=0$ and the limit (3), we have that

$$\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds=\int_0^\infty -f'(s)\partial_x^{2k}\eta^t(x,s)ds=\int_0^\infty f(s)\partial_x^{2k}u(x,t-s)ds. \quad (8)$$

Next, with the approximation (8) in hands, we can rewrite our system (2) as

$$\partial_t u+\partial_x^3u-a_0\partial_x^5u+u\partial_xu+a_1\partial_xu+(-1)^k\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds=0. \quad (9)$$

Thereafter, we introduce a state variable U and its initial data U_0 given by

$$U=(u,\eta^t) \quad \text{and} \quad U_0(x,s)=(u_0(x),\eta^0(x,s)),$$

where

$$u\in L^2(I) \ \text{ and } \ \eta^t\in L_g:=\left\{v:\mathbb{R}_+\longrightarrow H_k;\int_0^\infty g(s)\|\partial_x^kv(s)\|^2ds<+\infty\right\},$$

and the space H_k is defined as

$$H_k=\begin{cases}L^2(I), & \text{if } k=0,\\ H_0^1(I), & \text{if } k=1,\\ H_0^2(I), & \text{if } k=2.\end{cases}$$

Furthermore, we will consider in the set L_g, defined above, the inner product and norm given by

$$\langle v,w\rangle_{L_g}=\int_0^\infty g(s)\langle\partial_x^kv(s),\partial_x^kw(s)\rangle ds$$

and

$$\|v(s)\|_{L_g}=\left(\int_0^\infty g(s)\|\partial_x^kv(s)\|^2ds\right)^{\frac{1}{2}},$$

respectively and we define the energy space as $\mathcal{H}=L^2(I)\times L_g$, which will be equipped with the following inner product and its corresponding norm

$$\langle(v_1,v_2),(w_1,w_2)\rangle_{\mathcal{H}}=\langle v_1,w_1\rangle+\langle v_2,w_2\rangle_{L_g}$$

and

$$\|(v(s),w(s))\|_{\mathcal{H}}=\left(\|v(s)\|^2+\|w(s)\|^2_{L_g}\right)^{\frac{1}{2}}.$$

Additionally, to get our stability results we assume the following additional hypothesis on g: There exists a function $\xi:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\xi\in C^1(\mathbb{R}^+),\ \xi'\leq 0,\ \int_0^\infty \xi(s)ds=\infty \text{ and } g'\leq-\xi g. \quad (10)$$

Remark 1.1.

(i) The set of functions g satisfying (7) and (10) is very wide and contains, for example, the ones that converge to zero exponentially like

$$g(s)=d_1e^{-q_1s},$$

where $\xi(s)=q_1=\xi_0$ with $d_1>0$ and $q_1>0$, or polynomially like

$$g(s)=d_1(1+s)^{-q_1},$$

where $\xi(s)=\dfrac{q_1}{s+1}$, $\xi_0=q_1$ with $d_1>0$ and $q_1>1$, or between them like the following one

$$g(s)=d_1e^{-q_1(s+1)^{p_1}},$$

where $\xi(s) = q_1 p_1 (s+1)^{p_1-1}$, $\xi_0 = q_1 p_1$, with $d_1 > 0$, $q_1 > 0$ and $p_1 \in (0,1)$.

(ii) The assumptions on g are classical and used in numerous papers where other types of models and problems are treated (see for instance [13] and the references therein). However, one could relax some of the conditions on g but at the expense of technical complexities. The reader is referred to [14, 22] for further details about this point.

In the sequel, M_P is the smallest positive constant satisfying the Poincaré's Inequality

$$\|v\|^2 \le M_P \|\partial_x v\|^2,$$

for all $v \in H_0^1(I)$. Furthermore, let us denote by M_S the positive constant of the Sobolev embedding $H^1(I) \hookrightarrow L^\infty(I)$

$$\|v\|^2_{L^\infty(I)} \le M_S \|v\|^2_{H^1(I)}, \quad v \in H^1(I).$$

Now, we can announce the main result of this article, precisely, the stability of the solutions of (2).

Theorem 1.2. *Assume that* (3) *and* (10) *are verified. If* $U_0 \in \mathcal{H}$ *satisfies*

$$|a_1| M_P^2 + \frac{2}{3} M_P (M_P + 1) \sqrt{L} M_S \|U_0\| < 5 a_0, \tag{11}$$

then there exist positive constants c *and* $\tilde{c}$ *such that the solution* U *of* (12) *satisfies the following stability estimates*

(*i*) *If* ξ *is a constant function, we have*

$$E(t) \le \tilde{c} e^{-ct}, \ t \in \mathbb{R}_+.$$

(*ii*) *If* ξ *is not a constant function, then it holds*

$$E(t) \le \tilde{c} e^{-c\int_0^t \xi(\tau)d\tau} \left(1 + \int_0^t e^{c\int_0^\sigma \xi(\tau)d\tau} \xi(\sigma) \int_\sigma^\infty g(s) h(\sigma, s) ds d\sigma\right), t \in \mathbb{R}_+,$$

where

$$h(t,s) = t^2 + t + \left\| \int_0^{t-s} \partial_x^k u_0(\cdot, \tau) d\tau \right\|,$$

for $0 \le t \le s$.

The previous theorem permits to solve the Problem $\mathcal{P}$ stated before. To prove this result, we use a classical approach that combines the multiplier method with the energy technique (for further details about this approach, the authors strongly suggest that the reader consult the references [28, 31, 40]).

Our work is outlined as follows: Section 2 is devoted to presenting preliminary results which are essential for the rest of the article. In Section 3, we proved the well-posedness of the damping-memory problem (2). After that, the main result of the article, namely, Theorem 1.2 is shown in Section 4. Finally, we presented further comments in Section 5.

2. **Preliminaries.**

We shall reformulate our problem (2) (see also (9)) as an abstract initial value problem, namely,

$$\begin{cases} \partial_t U(t) = \mathcal{A} U(t), \\ U(0) = U_0, \end{cases} \tag{12}$$

where the operator $\mathcal{A}$ is given by

$$\mathcal{A}U=\left(-\partial_x^3u+a_0\partial_x^5u-u\partial_xu-a_1\partial_xu-(-1)^k\int_0^\infty g(s)\partial_x^{2k}\eta^t(\cdot,s)ds,\ u-\partial_s\eta^t\right),$$

with domain

$$D(\mathcal{A})=\{U\in\mathcal{H};\mathcal{A}U\in\mathcal{H},u\in H_0^2(I),\partial_x^2u(L)=0,\eta^t(x,0)=0\}.$$

Additionally, for $T>0$, we introduce the space

$$\mathcal{B}=C([0,T];L^2(I))\cap L^2(0,T;H^2(I))$$

equipped with the norm

$$\|\cdot\|_{\mathcal{B}}=\|\cdot\|_{C([0,T];L^2(I))}+\|\cdot\|_{L^2(0,T;H^2(I))}.$$

The next lemma gives us a formal calculation of the derivative of $E(t)=\dfrac{1}{2}\|U(t)\|^2_{\mathcal{H}}$, defined by (4), which will be important in the work (the computations will be rigorously justified later).

Lemma 2.1. *Assume that (3) holds. Then the derivative of the energy functional E satisfies*

$$E'(t)=-\frac{1}{2}a_0\left[(\partial_x^2u)(0)\right]^2+\frac{1}{2}\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2ds. \tag{13}$$

Proof. Observe that it follows from (3) that

$$E'(t)=\langle\partial_tu,u\rangle+\frac{1}{2}\partial_t\left(\int_0^\infty g(s)\left\|\partial_x^k\eta^t(\cdot,s)\right\|^2ds\right). \tag{14}$$

We will analyze each part of the $E'(t)$ separately. First, note that by multiplying (9) by u, integrating by parts in and using the boundary condition of (2), we have

$$\int_0^L u\partial_tudx=-a_0\frac{1}{2}\left(\partial_x^2u(0)\right)^2-(-1)^k\int_0^L u\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx. \tag{15}$$

Now, multiplying (6) by $(-1)^k\partial_x^{2k}g(s)\eta^t$ and again, integrating by parts in $I\times\mathbb{R}_+$, we get

$$\begin{aligned}\int_0^L\int_0^\infty(-1)^kg(s)\partial_x^{2k}\eta^t\partial_t\eta^tdsdx+\int_0^L\int_0^\infty(-1)^kg(s)\partial_x^{2k}\eta^t\partial_s\eta^tdsdx\\=\int_0^L\int_0^\infty(-1)^kug(s)\partial_x^{2k}\eta^tdsdx\end{aligned} \tag{16}$$

thanks to the boundary conditions of (6). When $k=0$, (13) holds directly from (14), (15) and (16). For the case $k=1$ or $k=2$, note that integrating by parts $k-$times with respect to the variable x, in the two terms in the left-hand side of (16) and, only once with respect to the variable s in the second term of the left-hand side of (16), we find that

$$\begin{aligned}\frac{1}{2}\partial_t\left(\int_0^\infty g(s)\left\|\partial_x^k\eta^t\right\|^2ds\right)=&\frac{1}{2}\int_0^\infty g'(s)\left\|\partial_x^k\eta^t\right\|^2ds\\&+(-1)^k\int_0^L u\int_0^\infty g(s)\partial_x^{2k}\eta^tdsdx,\end{aligned} \tag{17}$$

since we have that $\eta^t(x,0)=0$ and that the limit (7) holds. Hence, in this case, to get (13) just add (15) and (17). □

Remark 2.2. Let us give some comments.

(i) Since $a_0 > 0$ and due to the assumptions on g' (see (7)), it follows from (13) that $E'(t) \leq 0$. Hence the memory acts as a mechanism of damping feedback.

(ii) Note that the integral term of (13) is well-defined. In fact, observe that since $0 \leq -g' \leq c_0 g$, we have

$$\begin{aligned}\left|\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2 ds\right| =& -\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2 ds\\ \leq& c_0\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds\\ =& c_0\|\partial_x^k\eta^t\|^2_{L_g} < \infty,\end{aligned}$$

for any $\eta^t \in L_g$, showing our claim.

3. **Well-posedness of the memory problem.** In this section, we will study the well-posedness of the system (2). Precisely, we will initially study the well-posedness of the linearized system associated with (2). Then, we will show that the system with source term is well-posed and, finally, we prove that the original nonlinear system (2) is well-posed.

3.1. **Well-posedness: The linearized problem.** In this subsection, we give the details about the well-posedness of the linearized system associated with (2), namely

$$\begin{cases}\partial_t u + \partial_x^3 u - a_0\partial_x^5 u + a_1\partial_x u & \\ \qquad +(-1)^k\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds = 0, & (x,t)\in I\times(0,\infty),\\ \partial_t\eta^t(x,s) + \partial_s\eta^t(x,s) - u(x,t) = 0, & x\in I,\ s,t\in\mathbb{R}_+,\\ \eta^t(0,s) = \eta^t(L,s) = \eta^t(x,0) = 0, & x\in I,\ s,t\in\mathbb{R}_+,\\ u(0,t) = u(L,t) = 0, & t>0,\\ \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t>0,\\ u(x,0) = u_0(x), & x\in I,\end{cases} \tag{18}$$

with some initial data (u_0, η^0). Note that the system (18) can be written in an abstract form in $\mathcal{H}$ as follows

$$\begin{cases}\partial_t\Phi(t) = A\Phi(t),\ t>0\\ \Phi(0) = \Phi_0,\end{cases} \tag{19}$$

with $\Phi = (u,\eta^t),\ \Phi_0 = (u_0,\eta^0)$ and A is a linear operator giving by

$$A\Phi = \left(-\partial_x^3 u + a_0\partial_x^5 u - a_1\partial_x u - (-1)^k\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds,\ u - \partial_s\eta^t\right), \tag{20}$$

with domain

$$D(A) = \{\Phi\in\mathcal{H}; A(\Phi)\in\mathcal{H}, u\in H_0^2(I), \partial_x^2 u(L) = 0, \eta^t(x,0) = 0\}.$$

In turn, recall that, in this section, the generic positive constant C is independent of the initial data Φ_0 but may depend on T, g_0 and the system's parameters a_i, $i = 0,1$. The following result ensures the well-posedness of the linearized system.

Theorem 3.1. *If the condition* (3) *is satisfied, then the following assertions are valid:*

(i) The linear operator A defined by (20) generates a C_0-semigroup of contractions $S(t)$. Moreover, given an initial data $\Phi_0 \in D(A)$, the problem (19) admits a unique classical solution

$$\Phi \in C(\mathbb{R}_+; D(A)) \cup C^1(\mathbb{R}_+, \mathcal{H}). \tag{21}$$

In turn, if $\Phi_0 \in \mathcal{H}$, then (19) has a unique mild solution

$$\Phi \in C(\mathbb{R}_+; \mathcal{H}). \tag{22}$$

(ii) For any $\Phi_0 \in \mathcal{H}$ and $T > 0$, the following estimates holds

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leq C\|(u_0, \eta^0)\|^2_{\mathcal{H}}, \tag{23}$$

for some positive constant C. Additionally, the mapping

$$\Delta : \Phi_0 = (u_0, \eta^0)^T \in \mathcal{H} \to \Phi(\cdot) := S(\cdot)\Phi_0 \in \mathcal{B} \times C([0,T]; L_g)$$

is continuous.

Proof. In order to show (i), consider $\Phi = (u, \eta^t) \in D(A)$. Owing to (13) and (19), we find

$$\langle A(\Phi), \Phi\rangle_{\mathcal{H}} = \langle \partial_t \Phi, \Phi\rangle_{\mathcal{H}} = \left(\frac{1}{2}\|\Phi\|^2_{\mathcal{H}}\right)' = E'(t) < 0.$$

Thus, A is dissipative thanks to Remark 2.2. On the other hand, we can check that the adjoint operator of A is defined by

$$A^*\Psi = \left(\partial_x^3 v - a_0\partial_x^5 v + a_1\partial_x v + (-1)^k \int_0^\infty g(s)\partial_x^{2k}\zeta^t(x,s)ds, -v + \frac{g'(s)}{g(s)}\zeta^t + \partial_s\zeta^t\right),$$

with domain

$$D(A^*) = \{\Psi \in \mathcal{H}; A^*(\Psi) \in \mathcal{H}, v \in H_0^2(I), \partial_x^2 v(0) = 0, \zeta^t(x,0) = 0\}.$$

The same line of thought may be applied to obtain

$$\langle A^*(\Psi), \Psi\rangle_{\mathcal{H}} = -a_0\frac{(\partial_x^2 v(L))^2}{2} + \frac{1}{2}\int_0^\infty g'(s)\|\partial_x^k\zeta^t\|^2 ds \leq 0, \forall \Psi \in D(A^*),$$

and hence A^* is also dissipative. Now, since A is densely defined and closed, the assertion in (i) is a direct consequence of the semigroups theory of linear operators, see for instance [35].

Now, we will show (ii). Let $\Phi_0 = (u_0, \eta^0) \in \mathcal{H}$. As we know that $S(t)$ is a C_0-semigroup of contractions, we have

$$\|S(t)\Phi_0\|_{\mathcal{H}} = \|(u, \eta^t)\|_{\mathcal{H}} \leq \|\Phi_0\|_{\mathcal{H}} = \|(u_0, \eta^0)\|_{\mathcal{H}}, \quad \forall t \in [0,T]. \tag{24}$$

Next, consider the function $p(x,t)$, to be chosen later, and consider a classical solution $\Phi = (u, \eta^t)$ of (19) with initial data $\Phi_0 \in D(A)$. In this case, Φ has the regularity (21). Then, multiplying the equation (18) by $2xu$, integrating by parts over $[0,T]\times I$ and using the boundary conditions, we have:

$$\begin{aligned}
&4\int_0^T \|\partial_x u\|^2 dt + 5a_0\int_0^T \|\partial_x^2 u\|^2 dt\\
=&\int_0^L x u_0^2 dx - \int_0^L x u^2(x,T)dx\\
&-(-1)^k\int_0^T\int_0^L 2xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdxdt + a_1\int_0^T \|u\|^2 dt\\
\le& L\|u_0\|^2 + a_1\int_0^T \|u\|^2 dt\\
&-(-1)^k\int_0^T\int_0^L 2xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdxdt.
\end{aligned}\tag{25}$$

Let us treat the case $k=0$ and $k\in\{1,2\}$ separately.

Case 1: $k=0$.

First, note that

$$\begin{aligned}
-\int_0^T\int_0^L\int_0^\infty 2g(s)xu\eta^t(x,s)dsdxdt \le& L^2\int_0^T\int_0^\infty g(s)\|u\|^2 dsdt\\
&+\int_0^T \|\eta^t(\cdot,s)\|^2_{L_g}dt\\
=&L^2 g_0\int_0^T \|u\|^2 dt + \int_0^T \|\eta^t(\cdot,s)\|^2_{L_g}dt.
\end{aligned}\tag{26}$$

Thus, amalgamating (25) and (26), we deduce that

$$\int_0^T \left(\|u\|^2 + \|\partial_x u\|^2 + \|\partial_x^2 u\|^2\right)dt \le C\|(u_0,\eta^0)\|^2_{\mathcal{H}},$$

where $C=C(T,L,a_0,|a_1|,g_0)>0$, showing (23) for $\Phi\in D(A)$. Finally, the result for $\Phi\in\mathcal{H}$ follows by a density argument. This, together with (24) implies the continuity of Δ.

Case 2: $k=1$.

Now, considering $k=1$, integrating the last term of (25) by parts and using Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
(4-\epsilon g_0 L^2)\int_0^T \|\partial_x u\|^2 dt + 5a_0\int_0^T \|\partial_x^2 u\|^2 dt \le& L\|u_0\|^2 + (a_1+g_0)\int_0^T \|u\|^2 dt\\
&+\left(1+\frac{1}{\epsilon}\right)\int_0^T \|\eta^t\|^2_{L_g}dt.
\end{aligned}$$

Taking $\epsilon = \dfrac{3}{g_0L^2} > 0$, we have

$$\int_0^T \left(\|u\|^2 + \|\partial_x u\|^2 + \|\partial_x^2 u\|^2\right)dt \le C\|(u_0,\eta^0)\|^2_{\mathcal{H}},$$

where $C=C(T,L,a_0,|a_1|,g_0)>0$ showing (23) for $\Phi\in D(A)$. Again, a density argument permits to claim that (23) holds for $\Phi\in\mathcal{H}$ and also the continuity of the mapping Δ is verified.

Case 3: $k=2$.

One has merely to argue as in the previous case. The only difference is that we need to handle the term involving $\|\partial_x^2 u\|^2$ in addition to $\|\partial_x u\|^2$. $\square$

3.2. **Well-posedness: The equation with source term.** The goal of this part is to deal with the well-posedness of the system (18) with a source term $\varphi(x,t)$

$$\begin{cases} \partial_t u + \partial_x^3 u - a_0 \partial_x^5 u + a_1 \partial_x u & \\ \qquad +(-1)^k \displaystyle\int_0^\infty g(s) \partial_x^{2k} \eta^t(x,s) ds = \varphi(x,t), & (x,t) \in I \times (0,\infty), \\ \partial_t \eta^t(x,s) + \partial_s \eta^t(x,s) - u(x,t) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ \eta^t(0,s) = \eta^t(L,s) = \eta^t(x,0) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{27}$$

We have the following result:

Theorem 3.2. *Let us consider $T > 0$. If (3) is verified, then we have:*

(i) If $\Phi_0 = (u_0, \eta^0)^T \in \mathcal{H}$ and $\varphi \in L^1(0,T; L^2(I))$, then there exists a unique mild solution $\Phi = (u, \eta^t)^T$ of (27) such that $\Phi \in \mathcal{B} \times C([0,T]; L_g)$,

$$\|(u,\eta^t)\|^2_{C([0,T];\mathcal{H})} \le C_0 \left(\|(u_0,\eta^0)\|^2_{\mathcal{H}} + \|\varphi\|^2_{L^1(0,T;L^2(I))} \right) \tag{28}$$

and

$$\|u\|^2_{\mathcal{B}} \le C_1 \left(\|(u_0,\eta^0)\|^2_{\mathcal{H}} + \|\varphi\|^2_{L^1(0,T;L^2(I))} \right), \tag{29}$$

for some positive constants C_0, C_1 independent of Φ_0 and φ.

(ii) Given $u \in L^2(0,T; H^2(I))$, we have $u\partial_x u \in L^1(0,T; L^2(I))$ and the map

$$\Theta : u \in L^2(0,T; H^2(I)) \to u\partial_x u \in L^1(0,T; L^2(I))$$

is continuous.

Proof. (i) Since A generates a C_0-semigroup of contractions $S(t)$,

$$\varphi \in L^1(0,T; L^2(I))$$

and to ensure the validity of the computations, we shall work with a regular solution Φ of (27) stemmed from an initial data $\Phi_0 = (u_0, \eta^0)^T \in D(A)$. It is well-known from the semigroups theory [35] that the solution of (27) satisfies

$$\begin{aligned} \|(u,\eta^t)\|_{\mathcal{H}} \le & C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \int_0^t \|\varphi\| dt \right) \\ \le & C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \|\varphi\|_{L^1(0,T;L^2(I))} \right), \end{aligned} \tag{30}$$

and consequently (28) holds. We also have, thanks to (30), that

$$\|u\|_{C([0,T];L^2(I))} \le C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \|\varphi\|_{L^1(0,T;L^2(I))} \right).$$

Therefore, to obtain the H^2-norm of the solution, that is, (29), we use an analogous argument as in the proof of (23), and hence we will omit it. On the other hand, a density argument allows us to extend the results to any initial condition $\Phi_0 \in \mathcal{H}$.

(ii) First, consider $y, z \in L^2(0,T;H^2(I))$. We have

$$\begin{aligned}\|y\partial_x y\|_{L^1(0,T;L^2(I))} \leq & K\int_0^T \|y\|_{H^2(I)}\|\partial_x y\|dt \\ \leq & K\int_0^T \|y\|^2_{H^2(I)}dt \\ = & K\|y\|^2_{L^2(0,T;H^2(I))},\end{aligned} \tag{31}$$

where K is the positive constant of the Sobolev embedding $H^2 \hookrightarrow L^\infty(I)$. Thus,

$$y\partial_x y \in L^1(0,T;L^2(I)),$$

for each $y \in L^2(0,T;H^2(I))$.

In turn, using triangle inequality together with Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|\Theta(y)-\Theta(z)\|_{L^1(0,T;L^2(I))} \leq & K\int_0^T \|y-z\|_{H^2(I)}\|y\|_{H^2(I)}dt \\ & + K\int_0^T \|z\|_{H^2(I)}\|y-z\|_{H^2(I)}dt \\ \leq & K\|y-z\|_{L^2(0,T;H^2(I))}\|y\|_{L^2(0,T;H^2(I))} \\ & + K\|z\|_{L^2(0,T;H^2(I))}\|y-z\|_{L^2(0,T;H^2(I))} \\ = & K\|y-z\|_{L^2(0,T;H^2(I))}\|y\|_{L^2(0,T;H^2(I))} \\ & + K\|y-z\|_{L^2(0,T;H^2(I))}\|z\|_{L^2(0,T;H^2(I))}.\end{aligned}$$

Thus, the mapping Θ is continuous with respect to the corresponding topologies. □

3.3. **Well-posedness: The nonlinear problem.** The next result ensures the well-posedness of the system (2), which is represented by the problem (12).

Theorem 3.3. *Let us consider $T > 0$ and $a_0 > 0$. If (3) is verified, then there exists a positive constant C such that, for every $U_0 \in \mathcal{H}$ with*

$$\|U_0\|^2 < \frac{1}{16C_1^2K^2}, \tag{32}$$

where C_1 is as in (29) and K is the positive constant of the Sobolev embedding $H^2 \hookrightarrow L^\infty(I)$, the problem (12) has a unique global solution U satisfying the regularity (22) and consequently, the problem (2) admits a unique global solution $u \in \mathcal{B}$.

Proof. First, consider $U_0 = (u_0, \eta^0) \in \mathcal{H}$ satisfying (32). Next, define the map $\Gamma : \mathcal{B} \to \mathcal{B}$ by $\Gamma(z) = u$, where u is a solution of (27) with source term $\varphi(x,t) = -z(x,t)\partial_x z(x,t)$ and initial data U_0.

Claim 1: Γ *is well-defined.*

In fact, take $\alpha > 0$ such that

$$\|U_0\|^2_{\mathcal{B}} \leq \alpha < \frac{1}{16C_1^2K^2}.$$

Theorem 3.2 ensures that for each initial data U_0, there exists a unique solution $U = (u, \eta^t)$ of (27) satisfying, thanks to (32), the estimate

$$\|\Gamma(z)\|_{\mathcal{B}} \leq C_1\left(\alpha + \|z\partial_x z\|^2_{L^1(0,T;L^2(I))}\right). \tag{33}$$

Moreover, by using (31), we get

$$\begin{aligned}\|\Gamma(z)\|^2_{\mathcal{B}} \le & C_1\left(\|(u_0,\eta^0)\|^2_{\mathcal{H}}+\|z\partial_x z\|^2_{L^1(0,T;L^2(I))}\right)\\ \le & C_1\left(\alpha+K^2\|z\|^4_{L^2(0,T;H^2(I))}\right)\\ \le & C_1\left(\alpha+K^2\|z\|^4_{\mathcal{B}}\right),\end{aligned} \tag{34}$$

for all $z\in\mathcal{B}$, showing the claim 1.

Claim 2: Γ *is a contraction.*

Indeed, we have

$$\begin{aligned}\|\Gamma(y)-\Gamma(z)\|^2_{\mathcal{B}} \le & 2K^2\|y-z\|^2_{L^2(0,T;H^2(I))}\left(\|y\|^2_{L^2(0,T;H^2(I))}+\|z\|^2_{L^2(0,T;H^2(I))}\right)\\ \le & 2K^2\|y-z\|^2_{\mathcal{B}}\left(\|y\|^2_{\mathcal{B}}+\|z\|^2_{\mathcal{B}}\right).\end{aligned}$$

Then, consider the restriction of Γ to the closed ball $B=\left\{z\in\mathcal{B}; \|z\|^2_{\mathcal{B}}\le r\right\}$, with $r=\dfrac{\sqrt{\alpha}}{2K}$. Thus, (33) and (34) yields that

$$\|\Gamma(z)\|^2_{\mathcal{B}}\le C_1(\alpha+K^2\|z\|^4_{\mathcal{B}})\le C_1(\alpha+K^2r^2)<2C_1\alpha<r$$

and

$$\|\Gamma(y)-\Gamma(z)\|^2_{\mathcal{B}}\le 4rK^2\|y-z\|^2_{\mathcal{B}}\le\frac{1}{2}\|y-z\|^2_{\mathcal{B}}.$$

The mapping Γ is well-defined and contractive on the ball B according to the choice (32), showing claim 2.

Therefore, using the Banach Fixed Point Theorem, we deduce that Γ has a unique fixed element $u\in B$, which turns out to be the unique solution to our problem (2). Lastly, the system (2) being dissipative as its energy is decreasing, the solution is global. □

4. **Proof of the main result.** This section is devoted to the proof of the main result, namely, Theorem 1.2. The principal ingredient of the proof is the utilization of the energy method.

Proof of Theorem 1.2. First, multiplying (9) by xu, integrating by parts several times, observing that $\partial_x(xu)=u+x\partial_x u$, and thanks to the boundary conditions of (2), we have that

$$\begin{aligned}\frac{5a_0}{2}\|\partial_x^2 u\|^2 = & -\partial_t\left(\frac{1}{2}\int_0^L xu^2dx\right)-\frac{3}{2}\|\partial_x u\|^2+\frac{1}{3}\int_0^L u^3dx+\frac{a_1}{2}\|u\|^2\\ & -(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx.\end{aligned} \tag{35}$$

We are now in a position to estimate the terms of the right-hand side of (35).

Estimate 1: First, the Sobolev embedding yields that

$$\left|\int_0^L u^3dx\right|\le\|u\|^2_{L^\infty(I)}\int_0^L|u|dx\le M_S\|u\|^2_{H_1(I)}\int_0^L|u|dx.$$

Thus, the previous inequality together with Hölder's and Poincaré's inequalities give us

$$\begin{aligned}\left|\int_0^L u^3dx\right| \leq & M_S\|u\|^2_{H_1(I)}\left(\int_0^L 1^2dx\right)^{\frac{1}{2}}\left(\int_0^L |u|^2dx\right)^{\frac{1}{2}} \\ \leq & M_S L^{\frac{1}{2}}\left(\|u\|^2+\|\partial_x u\|^2\right)\|u\| \\ \leq & M_S L^{\frac{1}{2}}\left(M_P+1\right)\|\partial_x u\|^2(2E(t))^{\frac{1}{2}} \\ \leq & M_S L^{\frac{1}{2}}\left(M_P+1\right)(2E(0))^{\frac{1}{2}}\|\partial_x u\|^2.\end{aligned}$$

Estimate 2: We claim that for each $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx\right| \leq \epsilon\|\partial_x^2 u\|^2 + C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds.$$

We split this estimate into two parts, namely, $k = 0$ and $k \in \{1, 2\}$.

Indeed, for the case $k = 0$, using the Young's and Poincaré's inequalities we have

$$\begin{aligned}\left|\int_0^L xu\int_0^\infty g(s)\eta^t dsdx\right| \leq & \int_0^\infty g(s)\int_0^L |xu|\left|\eta^t\right|dxds \\ \leq & L\int_0^\infty g(s)\int_0^L\left(\delta|u|^2+\frac{1}{4\delta}\left|\eta^t\right|^2\right)dxds \\ \leq & L\delta\underbrace{\int_0^\infty g(s)ds}_{=g_0}\|u\|^2+L\frac{1}{4\delta}\int_0^\infty g(s)\|\eta^t\|^2ds \\ \leq & \underbrace{LM_P^2g_0\delta}_{\epsilon}\|\partial_x^2u\|^2+\underbrace{L\frac{1}{4\delta}}_{C_\epsilon}\int_0^\infty g(s)\|\eta^t\|^2ds \\ \leq & \epsilon\|\partial_x^2u\|^2+C_\epsilon\int_0^\infty g(s)\|\eta^t\|^2ds,\end{aligned}$$

where $\delta = \dfrac{\epsilon}{Lc_p^2g_0} > 0$, showing the estimate 2.

Now, we turn to the case $k = 1$. Applying once again Young's and Poincaré's inequalities gives the following

$$\begin{aligned}\left|-\int_0^\infty g(s)\int_0^L u\partial_x\eta^t(x,s)dxds\right| \leq & \int_0^\infty\int_0^L\left|g(s)u\partial_x\eta^t(x,s)\right|dxds \\ \leq & \delta\int_0^\infty g(s)\|u\|^2ds \\ & +\frac{1}{4\delta}\int_0^\infty g(s)\|\partial_x\eta^t\|^2ds \\ \leq & \delta\int_0^\infty g(s)\|u\|^2ds \\ & +\frac{1}{4\delta}\|\eta^t\|^2_{L_g} \\ \leq & \delta g_0M_P^2\|\partial_x^2u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_g}.\end{aligned} \tag{36}$$

Similarly, we also have the following estimate

$$
\begin{aligned}
\left|-\int_0^\infty g(s)\int_0^L x\partial_x u\partial_x\eta^t(x,s)dxds\right| \leq & \delta\int_0^\infty g(s)\|x\partial_x u\|^2ds \\
& +\frac{1}{4\delta}\int_0^\infty g(s)\|\partial_x\eta^t\|^2ds \\
\leq & \delta L^2\int_0^\infty g(s)\|\partial_x u\|^2ds \\
& +\frac{1}{4\delta}\|\eta^t\|^2_{L_g} \\
= & \delta g_0L^2\|\partial_x u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_g} \\
\leq & \delta g_0M_PL^2\|\partial_x^2 u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_g}.
\end{aligned}
\tag{37}
$$

Observe that

$$
\begin{aligned}
\left|(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^2\eta^tdsdx\right| \leq & \left|-\int_0^\infty g(s)\int_0^L u\partial_x\eta^t(x,s)dxds\right| \\
& +\left|-\int_0^\infty g(s)\int_0^L x\partial_x u\partial_x\eta^t(x,s)dxds\right|.
\end{aligned}
$$

Thus, thanks to the inequalities (36) and (37) applied on the right-hand side of the previous inequality we have

$$
\begin{aligned}
\left|(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^2\eta^tdsdx\right| \leq & \delta g_0M_P^2\|\partial_x^2u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_g} \\
& +\delta g_0M_PL^2\|\partial_x^2u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_g} \\
= & \underbrace{\delta g_0(M_P+M_P^2)}_{\epsilon}\|\partial_x^2u\|^2+\underbrace{\frac{1}{2\delta}}_{C_\epsilon}\|\eta^t\|^2_{L_g},
\end{aligned}
$$

where $\delta=\dfrac{\epsilon}{g_0(M_P+M_P^2)}>0$, showing the estimate 2.

Now, we turn to the case $k=2$. Thanks to Young's and Poincaré's inequalities, we have

$$
\left|2\int_0^\infty g(s)\int_0^L \partial_x u\partial_x^2\eta^t(x,s)dxds\right| \leq \delta g_0M_P\|\partial_x^2u\|^2+\frac{1}{\delta}\|\eta^t\|^2_{L_g}. \tag{38}
$$

Using the same arguments as for the case $k=1$, we have

$$
\left|-\int_0^\infty g(s)\int_0^L x\partial_x^2 u\partial_x^2\eta^t(x,s)dxds\right| \leq \delta g_0L^2\|\partial_x^2u\|^2+\frac{1}{\delta}\|\eta^t\|^2_{L_g}. \tag{39}
$$

Moreover, since

$$
\begin{aligned}
\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^4\eta^tdsdx\right| \leq & \left|2\int_0^\infty g(s)\int_0^L \partial_x u\partial_x^2\eta^t(x,s)dxds\right| \\
& +\left|\int_0^\infty g(s)\int_0^L x\partial_x^2u\partial_x^2\eta^t(x,s)dxds\right|,
\end{aligned}
$$

and thanks to (38) and (39), we reach

$$\begin{aligned}\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^4\eta^t dsdx\right| \leq& \delta g_0 M_P\|\partial_x^2 u\|^2+\delta g_0 L^2\|\partial_x^2 u\|^2+\frac{2}{\delta}\|\eta^t\|^2_{L_g}\\ =& \underbrace{\delta g_0(M_P+L^2)}_{\epsilon}\|\partial_x^2 u\|^2+\underbrace{\frac{2}{\delta}}_{C_\epsilon}\|\eta^t\|^2_{L_g},\end{aligned}$$

where $\delta=\dfrac{\epsilon}{g_0(M_P+L^2)}>0$, showing the estimate 2.

Estimate 3: There are two constants $C_1>0$ and $C_2>0$ such that

$$\|\partial_x^2 u\|^2 \quad\leq\quad -C_1\partial_t\left(\int_0^L xu^2dx\right)+C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds. \tag{40}$$

Indeed, Estimates 1 and 2 together with the Poincaré inequality yield

$$\begin{aligned}\frac{5a_0}{2}\|\partial_x^2 u\|^2 \quad\leq\quad& -\partial_t\left(\frac{1}{2}\int_0^L xu^2dx\right)+\frac{1}{3}\left|\int_0^L u^3dx\right|\\ &+\frac{a_1}{2}\|u\|^2+\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx\right|\\ \leq\quad& -\partial_t\left(\frac{1}{2}\int_0^L xu^2dx\right)+\frac{1}{3}M_SL^{\frac{1}{2}}(M_P+1)(E(0))^{\frac{1}{2}}\|\partial_x u\|^2\\ &+\frac{a_1}{2}\|u\|^2+\epsilon\|\partial_x^2 u\|^2+C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds\\ \leq\quad& -\partial_t\left(\frac{1}{2}\int_0^L xu^2dx\right)+\frac{1}{3}M_SL^{\frac{1}{2}}M_P(M_P+1)(E(0))^{\frac{1}{2}}\|\partial_x^2 u\|^2\\ &+\frac{|a_1|M_P^2}{2}\|\partial_x^2 u\|^2+\epsilon\|\partial_x^2 u\|^2+C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds\end{aligned}$$

which ensures

$$\begin{aligned}&\underbrace{\left(5a_0-\frac{2}{3}M_SL^{\frac{1}{2}}M_P(M_P+1)(E(0))^{\frac{1}{2}}-|a_1|M_P^2-2\epsilon\right)}_{:=D}\|\partial_x^2 u\|^2\\ &\leq -\partial_t\left(\int_0^L xu^2dx\right)+2C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds.\end{aligned}$$

Thus, taking $\epsilon>0$ small enough, it follows from (11) that $D>0$ and hence (40) holds true for $C_1=\dfrac{1}{D}>0$ and $C_2=\dfrac{2C_\epsilon}{D}>0$, showing the estimate 3.

To conclude the proof, consider the following function

$$F(t)=\mu E(t)+C_1\xi(t)\left(\int_0^L xu^2dx\right),$$

where $\mu = 2\left(C_2 + \frac{1}{M_P^2}\right)$. As $\xi' \le 0$, we have

$$0 \le \xi(t)\left(\int_0^L xu^2 dx\right) \le \xi(0)\left(\int_0^L xu^2 dx\right) \le \xi(0)L\|u\|^2 \le 2L\xi(0)2E(t).$$

Consequently, owing to the previous inequality, we get

$$\begin{aligned}\mu E(t) \le & F(t) \le \mu E(t) + C_1\xi(t)\left(\int_0^L xu^2 dx\right) \\ & \le \mu E(t) + 2LC_1\xi(0)E(t) = \left(\mu + 2LC_1\xi(0)\right)E(t).\end{aligned} \tag{41}$$

Observe that $\xi' \le 0$ ensures that

$$\begin{aligned}F'(t) = & \mu E'(t) + C_1\xi'(t)\left(\int_0^L xu^2 dx\right) + C_1\xi(t)\partial_t\left(\int_0^L xu^2 dx\right) \\ \le & \mu E'(t) + C_1\xi(t)\partial_t\left(\int_0^L xu^2 dx\right).\end{aligned} \tag{42}$$

Now, putting (40) into (42) and using the Poincaré's inequality, we get

$$\begin{aligned}F'(t) \le & \mu E'(t) + \xi(t)\left(C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \|\partial_x^2 u\|^2\right) \\ \le & \mu E'(t) + \xi(t)\left(C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \frac{1}{M_P^2}\|u\|^2\right) \\ = & \mu E'(t) + \xi(t)\left(C_2 + \frac{1}{M_P^2}\right)\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \frac{2}{M_P^2}\xi(t)E(t) \\ \le & \mu E'(t) + \xi(t)\left(C_2 + \frac{1}{M_P^2}\right)\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \lambda_0\xi(t)F(t),\end{aligned} \tag{43}$$

where in the last inequality we have used (41). Here,

$$\lambda_0 = \frac{2}{M_P^2\left[\mu + 2LC_1\xi(0)\right]}.$$

Now on, we shall distinguish two cases.

Case 1: ξ is a constant function.

In this case, taking into account (10) and (13), we have

$$\xi\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds \le -\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2 ds \le -2E'(t)$$

and hence (43) becomes

$$F'(t) \le \mu E'(t) - 2\left(C_2 + \frac{1}{M_P^2}\right)E'(t) - \lambda_0\xi F(t) = -\lambda_0\xi F(t)$$

implying that $F(t) = e^{-ct}F(0)$, with $c = \lambda_0\xi$. Finally, (41) yields

$$E(t) \le \frac{1}{\mu}F(t) = \frac{F(0)}{\mu}e^{-ct} \le \frac{2\xi(0)E(0)}{\mu}e^{-ct},$$

which ensures item (i) of Theorem 1.2.

Case 2: ξ is not a constant function.

First, observe that integrating (40) over $[0,t]$ and using the definition of $E(t)$, since E is decreasing, we get

$$
\begin{aligned}
\int_0^t \|\partial_x^2 u\|^2 ds \leq & -C_1 \int_0^t \partial_\tau \left(\int_0^L xu^2dx\right) d\tau + C_2 \int_0^t \int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds d\tau \\
\leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t \int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds d\tau \\
\leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t 2E(\tau) d\tau \\
\leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t 2E(0) d\tau := C_3(1+t),
\end{aligned}
\tag{44}
$$

where $C_3 = \max\left\{C_1\left(\int_0^L xu_0^2dx\right), 2C_2E(0)\right\}$. Now, Young's and Hölder's inequalities together with (44) , ensures that

$$
\begin{aligned}
\left\|\int_{t-s}^t \partial_x^k u(\cdot,\tau)d\tau\right\|^2 \leq & 2\left\|\int_{t-s}^0 \partial_x^k u(\cdot,\tau)d\tau\right\|^2 + 2\left\|\int_0^t \partial_x^k u(\cdot,\tau)d\tau\right\|^2 \\
\leq & 2\left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau)d\tau\right\|^2 + 2t\int_0^L\int_0^t (\partial_x^k u)^2(\cdot,\tau)d\tau dx \\
\leq & 2\left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau)d\tau\right\|^2 + 2t\int_0^t \|\partial_x^k u\|^2 d\tau := c_1 h(t,s),
\end{aligned}
$$

for $0 \leq t \leq s$. Here $c_1 = 2\max\{1, M_P^{2-k}C_3\}$ and

$$
h(t,s) = t^2 + t + \left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau)d\tau\right\|^2.
$$

On the other hand, thanks to (13)

$$
\begin{aligned}
\xi(t)\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds = & \xi(t)\int_0^t g(s)\|\partial_x^k\eta^t\|^2 ds + \xi(t)\int_t^\infty g(s)\|\partial_x^k\eta^t\|^2 ds \\
\leq & -\int_0^t g'(s)\|\partial_x^k\eta^t\|^2 ds + \xi(t)\int_t^\infty g(s)\|\partial_x^k\eta^t\|^2 ds \\
\leq & -\int_0^t g'(s)\|\partial_x^k\eta^t\|^2 ds + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds \\
\leq & -\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2 ds + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds \\
\leq & -2E'(t) + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds.
\end{aligned}
$$

Recall that

$$
F(t) = \mu E(t) + C_1\xi(t)\left(\int_0^L xu^2dx\right).
$$

Analogously to the previous case, we have

$$\begin{aligned} F'(t) \leq & -\lambda_0 \xi(t) F(t) + \mu E'(t) + \xi(t) \frac{\mu}{2} \int_0^\infty g(s) \|\partial_x^k \eta^t\|^2 ds \\ \leq & -\lambda_0 \xi(t) F(t) + \mu E'(t) - \mu E'(t) + c_1 \frac{\mu}{2} \xi(t) \int_t^\infty g(s) h(t,s) ds. \end{aligned} \tag{45}$$

Setting $c := \lambda_0$, (45) ensures that

$$\begin{aligned} F'(t) + c\xi(t)F(t) \leq & \mu E'(t) - \mu E'(t) + \frac{c_1 \mu}{2} \xi(t) \int_t^\infty g(s) h(t,s) ds \\ = & \frac{c_1 \mu}{2} \xi(t) \int_t^\infty g(s) h(t,s) ds, \end{aligned}$$

or equivalently,

$$\left(e^{c \int_0^t \xi(\tau) d\tau} F(t) \right)' \leq \frac{c_1 \mu}{2} e^{c \int_0^t \xi(\tau) d\tau} \xi(t) \int_t^\infty g(s) h(t,s) ds.$$

Finally, the previous inequality and $E(t) \leq \dfrac{1}{\mu} F(t)$, gives us

$$E(t) \leq \tilde{c} e^{-c \int_0^t \xi(\tau) d\tau} \left(1 + \int_0^t e^{c \int_0^\sigma \xi(\tau) d\tau} \xi(\sigma) \int_\sigma^\infty g(s) h(\sigma, s) ds d\sigma \right),$$

where $\tilde{c} = \dfrac{\max \left\{ F(0), \dfrac{c_1 \mu}{2} \right\}}{\mu}$. Thereby, the proof of the second part (ii) of Theorem 1.2 is complete. □

5. **Conclusion.** In this paper, we considered the well-known Kawahara equation under the presence of only an internal infinite memory term. Then, it is shown that the energy of the system decays under some assumptions of the memory kernel. Moreover, an estimate of the energy decay is provided depending on the property of the kernel. The main ingredient of the proof is the utilization of the Fixed Point Theorem and the energy method. Based on this outcome, one can conclude that the distributed memory term creates enough dissipation for the energy of the system so that the exponential stability holds. On the other hand, we believe that our results remain valid if the memory term occurs in a boundary condition. Of course, this could be the subject of a future work to ascertain the claim.

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REFERENCES

[1] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, *Invent. Math.*, **171** (2008), 485-541.

[2] G. Amendola, M. Fabrizio and J. M. Golden, *Thermodynamics of Materials with Memory: Theory and Applications*, Springer, 2012.

[3] F. D. Araruna, R. A. Capistrano–Filho and G. G. Doronin, Energy decay for the modified Kawahara equation posed in a bounded domain, *J. Math. Anal. Appl.*, **385** (2012), 743-756.

[4] L. Baudouin, E. Crépeau and J. Valein, Two approaches for the stabilization of nonlinear KdV equation with boundary time-delay feedback, *IEEE TAC*, **64** (2019), 1403-1414.

[5] J. L. Bona, D. Lannes and J.-C. Saut, Asymptotic models for internal waves, *J. Math. Pures Appl.*, **89** (2008), 538-566.

[6] R. de A. Capistrano–Filho, B. Chentouf, L. S. de Sousa and V. H. Gonzalez Martinez, Two stability results for the Kawahara equation with a time-delayed boundary control, *Zeitschrift für Angewandte Mathematik und Physik*, **74** (2023), Paper No. 16, 26 pp.

[7] R. de A. Capistrano–Filho and L. S. de Sousa, Control results with overdetermination condition for higher order dispersive system, *Journal of Mathematical Analysis and Applications*, **506** (2022), 1-22.

[8] R. de A. Capistrano–Filho and M. M. de S. Gomes, Well-posedness and controllability of Kawahara equation in weighted Sobolev spaces, *Nonlinear Analysis*, **207** (2021), 1-24.

[9] R. de A. Capistrano–Filho and V. H. Gonzalez Martinez, Stabilization results for delayed fifth-order KdV-type equation in a bounded domain, *Math. Control Relat. Fields*, (2023).

[10] M. Chen, Internal controllability of the Kawahara equation on a bounded domain, *Nonlinear Analysis*, **185** (2019), 356-373.

[11] B. Chentouf, Well-posedness and exponential stability of the Kawahara equation with a time-delayed localized damping, *Mathematical Methods in the Applied Sciences,* **45** (2022), 10312-10330.

[12] B. Chentouf, Qualitative analysis of the dynamic for the nonlinear Korteweg-de Vries equation with a boundary memory, *Qual. Theory Dyn. Syst.*, **20** (2021), Paper No. 36, 29 pp.

[13] B. Chentouf and A. Guesmia, Well-posedness and stability results for the Korteweg-de Vries-Burgers and Kuramoto-Sivashinsky equations with infinite memory: A history approach, *Nonlinear Analysis: Real World Applications*, **65** (2022), 103508.

[14] V. V. Chepyzhov and V. Pata, Some remarks on stability of semigroups arising from linear viscoelasticity, *Asymptot. Anal.*, **46** (2006), 251-273.

[15] G. M. Coclite and L. di Ruvo, Wellposedness of the classical solutions for a Kawahara–Korteweg–de Vries type equation, *J. Evolution Equations*, 21 (2021), 625-651.

[16] G. M. Coclite and L. di Ruvo, Convergence results related to the modified Kawahara equation, *Boll. Unione Mat. Ital.*, **8** (2016), 265-286.

[17] S. Cui and S. Tao, Strichartz estimates for dispersive equations and solvability of the Kawahara equation, *J. Math. Anal. Appl.*, **304** (2005), 683-702.

[18] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, **37** (1970) 297-308.

[19] G. G. Doronin and N. A. Larkin, Kawahara equation in a quarter-plane and in a finite domain, *Bol. Soc. Parana. Mat.*, **25** (2007) 9-16.

[20] G. G. Doronin and N. A. Larkin, Boundary value problems for the stationary Kawahara equation, *Nonlinear Analysis*, **69** (2008), 1655-1665.

[21] G. G. Doronin and N. A. Larkin, Kawahara equation in a bounded domain, *Discrete Contin. Dyn. Syst. Ser. B*, **10** (2008) 783-799.

[22] M. Fabrizio, C. Giorgi and V. Pata, A new approach to equations with memory, *Arch. Rational Mech. Anal.*, **198** (2010), 189-232.

[23] H. Hasimoto, Water waves, *Kagaku*, **40** (1970), 401-408.

[24] H. Hirayama, Local well-posedness for the periodic higher order KdV type equations, *NoDEA Nonlinear Differential Equations Appl.*, **19** (2012) 677-693.

[25] P. Isaza, F. Linares and G. Ponce, Decay properties for solutions of fifth order nonlinear dispersive equations, *J. Differential Equations*, **258** (2015), 764-795.

[26] T. Kato, Low regularity well-posedness for the periodic Kawahara equation, *Differential Integral Equations*, **25** (2012), 1011-1036.

[27] T. Kawahara, Oscillatory solitary waves in dispersive media, *J. Phys. Soc. Japan*, **33** (1972), 260-264.

[28] V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method*, Wiley, 1994.

[29] D. Lannes, *The Water Waves Problem. Mathematical Analysis and Asymptotics*, Mathematical Surveys and Monographs, 188. American Mathematical Society, Providence, RI, 2013. xx+321 pp.

[30] N. A. Larkin and M. H. Simões, The Kawahara equation on bounded intervals and on a half-line, *Nonlinear Anal. TMA*, **127** (2015), 397-412.

[31] J.-L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Syst´emes Distribués*, Tome 1: Contrôlabilité Exacte, RMA 8, Masson, Paris, 1988.

[32] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Diff. Integral Equations*, **21** (2008), 935-958.

[33] L. Pandolfi, *Distributed Systems with Persistent Memory. Control and Moment Problems*, Springer-Verlag, New York, 1983.

[34] H. Parada, C. Timimoun and J. Valein, Stability results for the KdV equation with time-varying delay, *Systems* & *Control Letters*, **177** (2023), 105547.

[35] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[36] J. Valein, On the asymptotic stability of the Korteweg-de Vries equation with time-delayed internal feedback, *Mathematical Control* & *Related Fields*, **12** (2022), 667-694.

[37] C. F. Vasconcellos and P. N. da Silva, Stabilization of the linear Kawahara equation with localized damping, *Asymptotic Analysis*, **58** (2008), 229-252.

[38] C. F. Vasconcellos and P. N. da Silva, Erratum: Stabilization of the linear Kawahara equation with localized damping, *Asymptotic Analysis*, **66** (2010), 119-124.

[39] C. F. Vasconcellos and P. N. da Silva, Stabilization of the Kawahara equation with localized damping, *ESAIM Control Optim. Calc. Var.*, **17** (2011), 102-116.

[40] E. Zuazua, Exact controllability for the semilinear wave equation in one space dimension, *Ann. IHP. Analyse Non Linéaire*, **10** (1993), 109-129.

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