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WELL-POSEDNESS AND STABILIZATION THEORY FOR DISPERSIVE SYSTEMS

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Well-posedness and stabilization theory for dispersive systems

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To God, my family, and my friends.

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RESUMO

Este trabalho trata do estudo da boa colocação e estabilização de equações dispersivas não lineares em domínios limitados. Iniciamos provando teoremas do tipo Massera para a equação de Kawahara não linear. Mais precisamente, graças às propriedades do semigrupo do operador linear associado à equação estudada e ao decaimento exponencial das soluções do sistema linear, foi possível mostrar que as soluções da equação de Kawahara são periódicas e quase periódicas. Em um segundo momento, estudamos problemas de estabilização desta mesma equação. Precisamente, introduzindo somente um termo de memória infinita na equação de Kawahara, que desempenhou um papel de mecanismo de amortecimento, garantimos a estabilidade exponencial das soluções do sistema. Além disso, projetando uma lei de *feedback* de fronteira para o sistema de Kawahara, que combina um termo de amortecimento e um termo de memória finita, mostramos que a energia associada a este sistema, com a presença desta lei de *feedback*, decai exponencialmente. Por fim, estudamos uma outra equação, a saber, equação linear de Schrödinger de quarta ordem ou equação biharmônica de Schrödinger. Aqui, acrescentando um termo de memória infinita, provamos que a energia desta equação decai em taxas do tipo polinomial.

Palavras-chave: Equação de Kawahara, Boa colocação, Soluções periódicas, Teoremas do tipo Massera, Problemas de estabilização, Funcional de Lyapunov, Método de energia, Equação biharmônica de Schröndinger, Memória infinita, Mecanismo de amortecimento.

ABSTRACT

This work deals with the study of the well-posedness and stabilization of nonlinear dispersive equations in bounded domains. We start by proving Massera-type theorems for the nonlinear Kawahara equation. More precisely, thanks to the properties of the semigroup of the linear operator associated with the equation studied and the exponential decay of the solutions of the linear system, it was possible to show that the solutions of the Kawahara equation are periodic and quasi-periodic. In a second moment, we study the stabilization problems of this same equation. Precisely, by introducing only one term of infinite memory in the Kawahara equation, which played a role as a damping mechanism, we guarantee the exponential stability of the system solutions. Furthermore, by designing a boundary *feedback* law for the Kawahara system, which combines a damping term and a finite memory term, we show that the energy associated with this system, with the presence of this *feedback* law, decays exponentially. Finally, we study another equation, namely, the fourth-order linear Schrödinger equation or biharmonic Schrödinger equation. Here, adding an infinite memory term, we prove that the energy associated with this equation decays at polynomial-type rates.

Keywords: Kawahara equation, Well-posedness, Periodic solutions, Massera's type theorems, Stabilization problem, Lyapunov function, Energy method, Biharmonic Schrödinger equation, Infinite memory, Damping mechanism.

Contents

1 General Introduction

This thesis deals with the stabilization of dispersive systems governed by partial differential equations. In what concerns the stabilization, we study the asymptotic behavior of solutions, i.e., through an initial analysis of the energy associated with the model, the initial question is: Is it possible to ensure that the solutions are asymptotically stable for arbitrarily large time t? Having a positive response to this question, we are interested in seeking the rate of decay of these solutions.

The dispersive models studied here are the Kawahara equation and the Biharmonic Schrödinger equation posed on a bounded domain. To provide a minimum of the theory used in the course of the following chapters, we will present a small sample of the history of the dispersive systems, as well as some concepts necessary for the development of this thesis. Let us begin with a relevant history.

1.1 The beginning of the dispersive equations

To tell any story about the dispersive equations, we must start with John Scott Russell's vivid description of his discovery of solitary waves in his Report on Waves [84] and the origin of the Korteweg-de Vries equation or the *wave of translation*.

John Scott Russell (1808-1882) was a Scottish civil engineer, naval architect, and shipbuilder. He made the translational wave discovery that gave rise to the modern study of solitons and developed the wave-line system for building ships. In 1834, Russell was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon which he called the translation wave [84], and he said:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel. Such, in August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

After this discovery, he built a 30-foot water tank in his backyard and designed experiments generating long waves to investigate the phenomenon he had observed. He studied the shape of waves, their speed of propagation, and their stability. He then challenged the mathematical community to theoretically prove the possible existence of a stable solitary wave propagating without changing shape.

Russell's experiments, known as the *The Hull Construction Wave Line System,* consisted of raising an area of fluid behind an obstacle and then removing the obstacle so that a long, pile-shaped wave spread through the channel. His developments revolutionized naval architecture in the 19th century, and he was awarded the Gold Medal of the Royal Society of Edinburgh for his work in 1837. Russell's experiments contradicted physical conjectures such as those of G. B. Airy [2] about water wave theory, in which the traveling wave could not exist because it eventually changed its speed or its shape, or G. G. Stokes' theory [88], where waves of finite amplitude and fixed form were possible, but only in deep water and only in periodic form. However, Stokes was aware of the unfinished state of Russell's theory, so Stoke said:

"It is the opinion of Mr. Russell that the solitary wave is a phenomenon sui generis, in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered."

Consequently, to convince the physics community, Scott Russell challenged the mathematical community to theoretically prove the existence of the phenomenon he witnessed:

"Having verified that no one has managed to predict the phenomenon that I ventured to call the translation wave,... to be done... to show how it should have been predicted from the known general equations of fluid motion. In other words, it now remained for the mathematician to predict the discovery after it had taken place, that is, to give an a priori a posteriori proof."

Several researchers took up Russell's challenge. The first mathematician to respond was Joseph Boussinesq, a French mathematician, and physicist who got important results [15] in 1871. In 1876, the English physicist Lord Rayleigh obtained a different result [82], and in 1895 the Dutch mathematicians D. J. Korteweg and his student G. de Vries gave the last significant result of the 19th-century [64].

Precisely, as the first answer to Russell's challenge, Boussinesq considered a model of long, incompressible, and rotation-free waves in a shallow channel with a rectangular cross-section neglecting the friction along the boundaries, and he obtained the equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\Big[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\Big],$$

where (t, x) are the coordinates of a fluid particle at time t, h is the amplitude of the wave, H is the height of the water in equilibrium and g is the gravitational constant.

Rayleigh, independently, considered the same phenomenon and added the assumption of the existence of a stationary wave vanishing at infinity. He considered only spatial dependence and captured the desired behavior in the equation

$$\left(\frac{dh}{dx}\right)^2 + \frac{3}{H^3}h^2(h-h_0) = 0,$$

with h_0 being the crest of the wave and the other parameters defined as before. This equation has an explicit solution given by

$$h(x) = h_0 \operatorname{sech}^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right).$$

In 1876, Rayleigh wrote in his article [82]:

"I have lately seen a memoir by M. Boussinesq, Comptes Rendus, Vol. LXXII, which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to Boussinesq J."

The last proof of the existence of "translation waves" was given by Diederik Johannes Korteweg and Gustav de Vries. They constructed a nonlinear partial differential equation that has a solution describing the phenomenon discovered by Russell, thus giving the Korteweg–de Vries equation its name, often abbreviated as the KdV equation. In 1895, they published an article deriving the equation

$$\frac{\partial \eta}{\partial l} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right),$$

in which η is the surface elevation above the equilibrium level l, α is a small arbitrary constant related to the motion of the liquid, g is the gravitational constant, and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$, with surface capillary tension T and density ρ. Eliminating the physical constants by the change of variables

$$t \to \frac{1}{2}\sqrt{\frac{g}{l\sigma}}t, \quad x \to -\frac{x}{\sqrt{\sigma}} \quad \text{and} \quad u \to -\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

one obtains the standard Korteweg- de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

which is a model describing the propagation of small amplitude, long wavelength waves on an air-sea interface in a canal of the rectangular cross-section.

After that several important dispersive equations appears in the literature, let us now introduce two of them which were the equations studied in this work.

1.2 Dispersive systems

In partial differential equations, the "word" dispersive means that waves of different wavelengths propagate at different phase velocities. There are several examples of dispersive partial differential equations in the literature, however, in this thesis, we will treat two of them, as mentioned before, the Kawahara equation, a fifth-order KdV type equation, and the Biharmonic Schrödinger equation or fourth-order Schrödinger equation, a Schrödinger type equation. Let us give a brief motivation about these equations.

1.2.1 Kawahara equation

After many years, it is well known today that under suitable assumptions on amplitude, wavelength, wave steepness, and so on, the study on asymptotic models for water waves has been extensively investigated to understand the full water wave system, see, for instance, [4, 13, 14] and references therein for a rigorous justification of various asymptotic models for surface and internal waves.

Formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces. The physical condition $\delta \ll 1$ characterizes the waves, called long waves or shallow water waves. In particular, considering the relations between ε and δ, we can have two well-known regimes.

The first one, considering $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$, gives us the famous Korteweg-de Vries (KdV)[1], which was introduced in the previous section. However, considering $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the fifth-order KdV equation which one takes the form

$$\pm 2\eta_t + 3\eta\eta_x - \nu\eta_{xxx} + \frac{1}{45}\eta_{xxxxx} = 0.$$

This equation was introduced by Hasimoto [51] in 1970 and Takuji Kawahara in 1972. Kawahara, in an article titled "Oscillatory Solitary Waves in Dispersive Media" [62], generalized the KdV equation, whose most common denominations are Kawahara equation or generalized KdV equation, which is

$$\eta_t + \frac{3}{2}\eta\eta_x + \alpha\eta_{xxx} - \beta\eta_{xxxxx} = 0,$$

where $\alpha > 0$ and $\beta > 0$. This equation differs from KdV by the fifth-order derivative. Fifth-order dispersive differential equations, the main type of equation analyzed in this thesis,

[1] This equation was first introduced by Boussinesq [15], and Korteweg and de Vries rediscovered it twenty years later.

describe the propagation of waves of small amplitudes in one dimension. In addition, fluid-related problems and plasma physics are generally physical formulations represented by the Kawahara equation.

1.2.2 Schrödinger type equation

Erwin Rudolf Josef Alexander Schrödinger, born in Vienna-Erdberg on August 12, 1887, died in Vienna on January 4, 1961, was an Austrian physicist, known for his contributions to quantum mechanics, especially the Schrödinger equation, for which he received the Nobel Prize in Physics in 1933. Erwin Schrödinger began his work on wave mechanics at the end of 1925, as a consequence of studying De Broglie's thesis in 1924.

Around Christmas time 1925, during Schrödinger's stay at Villa Herwig in the Alps, Schrödinger attempted to design a relativistic wave equation, following De Broglie's approach. This deduction was not published but was found in a manuscript probably written in late 1925. Schrödinger applied this wave equation to the hydrogen atom and got erroneous results for the energy levels. After struggling briefly with a relativistic theory, he turned to a non-relativistic approach. From a variational problem, Schrödinger deduced the wave equation for the hydrogen atom. In this deduction presented in Schrödinger's first paper of 1926 his only "justification" is that the wave equation leads to the correct energy levels for the hydrogen atom.

Schrödinger's theory can be applied to cases such as accelerated motion and rotation, to particles in situations where the wavelength is comparable to the dimensions of the region containing the electron. To access the detailed history of the discovery of the wave equation and the comparison of the theory developed by Schrödinger with the one produced by Broglie, we cite for the reader the following reference [48].

The fourth-order nonlinear Schrödinger equation (4NLS) or biharmonic cubic nonlinear Schrödinger equation, which is a generalization of the famous Schrödinger equation, takes the form as

$$i\partial_t y + \Delta y - \Delta^2 y = \lambda |y|^2 y, \tag{1.1}$$

and was introduced by Karpman [60] and Karpman and Shagalov [59] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The equation (1.1) arises in many scientific fields, such as quantum mechanics, nonlinear optics, and plasma physics, and has been intensively studied with fruitful references see, for example, [8, 60, 77, 78] and references contained therein.

Now on, this thesis will treat these two important systems with several applications in fluid dynamics and physics. Before presenting the results achieved here, let us give a simple list of the main tools used.

1.3 Basic theory

In this section, we will address some definitions, concepts, and methods used in this thesis.

1.3.1 Theory of distributions and Sobolev spaces

The results of this subsection can be found in [1, 17, 73, 89] and in the references therein.

Given $\Omega \subset \mathbb{R}^n$ an open set and $f : \Omega \to \mathbb{R}$ a continuous function, the support of f, denoted by $\mathrm{supp}(f)$, is the closure in Ω of the set $\{x \in \Omega; f(x) \neq 0\}$. Thus, $\mathrm{supp}(f)$ is a closed subset of Ω. A n-upla of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called multi-index and its order is defined by $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let us represents by D^α the order derivation operator $|\alpha|$, that is,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}.$$

For $\alpha = (0, 0, \cdots, 0)$, we define

$$D^0 u := u, \ \forall u.$$

Let $C_0^\infty(\Omega)$ be the vector space of numerical functions defined in Ω, that have compact support and are infinitely differentiable in Ω. A classic example of a function of $C_0^\infty(\Omega)$ is given below.

Exemple 1.3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open such that $B_1(0) = \{x \in \mathbb{R}^n; \|x\| < 1\}$ is compactly contained in Ω. Let us consider $f : \Omega \to \mathbb{R}$ such that*

$$f(x) = \begin{cases} e^{\frac{1}{\|x\|^2 - 1}}, & \text{if } \|x\| < 1, \\ 0, & \text{if } \|x\| \geq 1 \end{cases}$$

where $x = (x_1, \ldots, x_n)$ and $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ is the Euclidean norm of x. We have $f \in C^\infty(\Omega)$ and $supp(f) = \overline{B_1(0)}$ is compact, that is, $f \in C_0^\infty(\Omega)$.

Definition 1.3.1. *One say that a sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C_0^\infty(\Omega)$ converges to φ in $C_0^\infty(\Omega)$ when the following conditions are satisfied:*

(i) There exists a compact K of Ω such that $supp(\varphi) \subset K$ and $supp(\varphi_n) \subset K$, $\forall n \in \mathbb{N}$;

(ii) $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly in K, for all multi-index α.

The space $C_0^\infty(\Omega)$, equipped with the convergence defined above, will be denoted by $D(\Omega)$ and called space of test functions on Ω.

Remark 1.3.1. *Note that it is possible to equip $C_0^\infty(\Omega)$ with a topology so that the notion of convergence in this topology coincides with that given by the previous definition.*

A (scalar) distribution over Ω is every continuous linear functional over $\mathcal{D}(\Omega)$. More precisely, a distribution over Ω is a functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ satisfying the following conditions:

(i) $T(\alpha\varphi + \beta\psi) = \alpha T(\varphi) + \beta T(\psi),\ \forall \alpha, \beta \in \mathbb{R}$ and $\forall \varphi, \psi \in \mathcal{D}(\Omega)$,

(ii) T is continuous, that is, if $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $\mathcal{D}(\Omega)$, then $(T(\varphi_n))_{n\in\mathbb{N}}$ converges to $T(\varphi)$ in $\mathbb{R}$.

It is common to denote the value of the distribution T in φ by $\langle T, \varphi\rangle$. Moreover, the set of all distributions over Ω with the usual operations is a vector space, which is represented by $D'(\Omega)$. The following examples of scalar distributions play a key role in the theory.

Exemple 1.3.2. *Let $u \in L^1_{loc}(\Omega)$. The functional $T_u : D(\Omega) \to \mathbb{R}$, defined by*

$$\langle T_u, \varphi\rangle = \int_\Omega u(x)\varphi(x)dx,$$

is a distribution over Ω uniquely determined by u. For this reason, u is identified to the distribution T_u defined by it and, $L^1_{loc}(\Omega)$ will be identified to a (proper) part of $\mathcal{D}'(\Omega)$.

Definition 1.3.2. *A sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{D}'(\Omega)$ is said to converge to T in $\mathcal{D}'(\Omega)$ when the numeric sequence $(\langle T_n, \varphi\rangle)_{n\in\mathbb{N}}$ converge to $\langle T, \varphi\rangle$ in $\mathbb{R}$, for all $\varphi \in \mathcal{D}(\Omega)$.*

Lemma 1.3.1 (Du Bois Raymond)**.** *Let $u \in L^1_{loc}(\Omega)$. Then*

$$\int_\Omega u(x)\varphi(x)dx = 0,\ \forall \varphi \in \mathcal{D}(\Omega),$$

if and only if $u = 0$ almost everywhere in Ω.

Exemple 1.3.3. *Consider $0 \in \Omega$ and the functional $\delta_0 : \mathcal{D}(\Omega) \to \mathbb{R}$, defined by*

$$\langle \delta_0, \varphi\rangle = \varphi(0),$$

then, it can be shown that δ_0 is a distribution over Ω, called the dirac distribution. Furthermore, it is shown that δ_0 is not defined by a function of $L^1_{loc}(\Omega)$.

Definition 1.3.3. *Let T be a distribution over Ω and α be a multi-index. The derivative $D^\alpha T$ (in the sense of distributions) of order $|\alpha|$ of T is the functional defined in $\mathcal{D}(\Omega)$ by*

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle,\ \forall \varphi \in \mathcal{D}(\Omega).$$

Remark 1.3.2. *It follows from Definition 1.3.3 that each distribution T over Ω has derivatives of all orders.*

Remark 1.3.3. *$D^\alpha T$ is a distribution over Ω, where $T \in \mathcal{D}'(\Omega)$. In fact, it is easily seen that $D^\alpha T$ is linear. Now, for continuity, consider $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ in $\mathcal{D}(\Omega)$. Thus,*

$$|\langle D^\alpha T, \varphi_n\rangle - \langle D^\alpha T, \varphi\rangle| \leq |\langle T, D^\alpha \varphi_n - D^\alpha \varphi\rangle| \to 0$$

when $n \to \infty$.

Remark 1.3.4. *The map $D^\alpha : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$, such that $T \mapsto D^\alpha T$, is linear and continuous in the sense of convergence defined in $\mathcal{D}'(\Omega)$.*

Given an integer $m > 0$, by $W^{m,p}(\Omega),\ 1 \leq p \leq \infty$, represents the Sobolev space of order m, over Ω of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$, for every multi-index α, with $|\alpha| \leq m$. $W^{m,p}(\Omega)$ is a vector space, whatever $1 \leq p < \infty$. Considering the following norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p dx\right)^{\frac{1}{p}}$$

when $1 \leq p < \infty$ and

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \operatorname{supess}_{x\in\Omega} |D^\alpha u(x)|$$

when $p = \infty$, then Sobolev spaces $W^{m,p}(\Omega)$ is a Banach space.

Remark 1.3.5. *When $p = 2$, the space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which equipped with the inner product*

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

is a Hilbert space.

Let us denote by $H_0^m(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$ relative to the norm of the space $H^m(\Omega)$. The set $H_0^m(\Omega)$ equipped with the induced inner product of $H^m(\Omega)$ is a vector subspace. Furthermore, it is possible to prove that the induced inner product of $H^m(\Omega)$ and the induced norm of $H^m(\Omega)$ are equivalent, respectively, to

$$((u, v))_{H^m(\Omega)} = \sum_{|\alpha|=m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

and

$$\|u\|^2_{H^m(\Omega)} = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^2 dx.$$

We then have the following results:

Lemma 1.3.2 (Poincaré-Friedrichs inequality)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$. If $u \in H_0^1(\Omega)$, then there exists a constant $C > 0$, such that*

$$\|u\|^2_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}.$$

Lemma 1.3.3 (Gagliardo–Nirenberg inequality). *Let $I = (0,1)$, $1 \leq q < \infty$ and $1 < r \leq \infty$. Then*

$$\|u\|_{L^\infty(I)} \leq C\|u\|^a_{W^{1,r}(I)}\|u\|^{1-a}_{L^q(I)}, \ \forall u \in W^{1,r}(I)$$

for some constant $C = C(q,r)$, where $0 < a < 1$ is defined by $a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}$.

Lemma 1.3.4 (Sobolev embedding). *Let Ω be a bounded open of $\mathbb{R}^n$ with regular boundary Γ.*

(i) If $n > 2m$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, where $p \in \left[1, \frac{2n}{n-2m}\right]$.

(ii) If $n = 2m$, then $H^m(\Omega) \hookrightarrow L^p(\Omega)$, where $p \in [1, +\infty[$.

(iii) If $n = 1$ and $m \geq 1$, then $H^m(\Omega) \hookrightarrow L^\infty(\Omega)$.

Here the symbol $\hookrightarrow$ denotes continuous embedding.

Lemma 1.3.5 (Rellich-Kondrachov). *Let Ω be a bounded open of $\mathbb{R}^n$ with regular boundary Γ.*

(i) If $n \geq 2m$, then $H^m(\Omega)$ is compactly embedding in $L^p(\Omega)$, to $p \in \left[1, \frac{2n}{n-2m}\right[$.

(ii) If $n = 2m$, then $H^m(\Omega)$ is compactly embedding in $L^p(\Omega)$, to $p \in [1, +\infty[$.

(iii) If $2m > n$ and $m \geq 1$, then $H^m(\Omega)$ is compactly embedding in $C^k(\overline{\Omega})$, where k is a non-negative integer such that $k < m - \frac{n}{2} \leq k+1$.

We will denote by $L^p(0,T;X)$, $1 \leq p < \infty$, the space of Banach of (classes of) functions u, defined in $]0,T[$ with values in X, that are strongly measurable and $\|u(t)\|^p_X$ is Lebesgue integrable in $]0,T[$, with the norm

$$\|u(t)\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|^p_X dt\right)^{\frac{1}{p}}.$$

Additionally, $L^\infty(0,T;X)$ represents the Banach space of (classes of) functions u, defined in $]0,T[$ with values in X, that are strongly measurable and $\|u(t)\|_X$ has supreme essential finite in $]0,T[$, with the norm

$$\|u(t)\|_{L^\infty(0,T;X)} = \operatorname{supess}_{t\in]0,T[}\|u(t)\|_X.$$

Remark 1.3.6. *When $p = 2$ and X is a Hilbert space, the space $L^2(0,T;X)$ is a Hilbert space, whose inner product is given by*

$$(u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X dt.$$

Consider the space $L^p(0,T;X),\ 1<p<\infty$, with X being Hilbert separable space, then we can make the following identification

$$[L^p(0,T;X)]' \approx L^q(0,T;X'),$$

where $\frac{1}{p}+\frac{1}{q}=1$. When $p=1$, we will do the identification

$$[L^1(0,T;X)]' \approx L^\infty(0,T;X').$$

Given a Banach space X. The vector space of linear and continuous maps of $\mathcal{D}(0,T)$ on X is called the Space of Vector Distributions on $]0,T[$ with values in X and denoted by $\mathcal{D}'(0,T;X)$.

Exemple 1.3.4. *Given $u \in L^p(0,T;X), 1 \leq p < \infty$, and $\varphi \in \mathcal{D}(0,T)$ the application $Tu : \mathcal{D}(0,T) \to X$, defined by*

$$Tu(\varphi) = \int_0^T u(t)\varphi(t)dt,$$

Bochner's integral on X, is linear and continuous in the sense of convergence of $\mathcal{D}(0,T)$, so a vector distribution. The map $u \mapsto Tu$ is injective, so we can identify u with Tu and, in this sense, we have

$$L^p(0,T;X) \subset \mathcal{D}'(0,T;X).$$

Definition 1.3.4. *Given $S \in \mathcal{D}'(0,T;X)$, we define the derivative of order n as being the vector distribution over $]0,T[$ with values in X given for*

$$\left\langle \frac{d^n S}{dt^n}, \varphi \right\rangle = (-1)^n \left\langle S, \frac{d^n \varphi}{dt^n} \right\rangle, \ \forall \varphi \in \mathcal{D}(0,T).$$

Let us consider the space

$$W^{m,p}(0,T;X) = \{u \in L^p(0,T;X); u^{(j)} \in L^p(0,T,X), j=1,\dots,m\},$$

where $u^{(j)}$ represents the $j-$th derivative of u in the sense of vector distributions, equipped with norm

$$\|u\|_{W^{m,p}(0,T;X)} = \left(\sum_{j=0}^{m} \|u^{(j)}\|^p_{L^p(0,T;X)} \right)^{\frac{1}{p}}.$$

The space $\left(W^{m,p}(0,T;X), \|\cdot\|_{W^{m,p}(0,T;X)}\right)$ is a Banach space.

Remark 1.3.7. *When $p=2$ and X is a Hilbert space, the space $W^{m,p}(0,T;X)$ will be denoted by $H^m(0,T;X)$, which, equipped with the inner product*

$$(u,v)_{H^m(0,T;X)} = \sum_{j=0}^{m} (u^{(j)}, v^{(j)})_{L^2(0,T;X)},$$

is a Hilbert space. It is denoted by $H_0^m(0,T;X)$ the closure, in $H^m(0,T;X)$, of $\mathcal{D}(0,T;X)$ and by $H^{-m}(0,T;X)$ the topological dual of $H_0^m(0,T;X)$.

The following lemma can be found at [7].

Lemma 1.3.6 (Aubin-Lions lemma)**.** *Let X_0, X and X_1 be three Banach spaces with $X_0 \subseteq X \subseteq X_1$. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. For $1 \leq p, q \leq \infty$, let*

$$W = \{u \in L^p([0,T], X_0); u' \in L^q([0,T]; X_1)\}.$$

(i) If $p < \infty$ then the embedding of W into $L^p([0,T], X)$ is compact.

(ii) If $p = \infty$ and $q > 1$ then the embedding of W into $C([0,T]; X)$ is compact.

1.3.2 Interpolation of Sobolev spaces

The results that we will enunciate from now on, as well as their demonstrations, can be found in [69].

Let X and Y be two separable Hilbert spaces, with continuous and dense embedding, $X \hookrightarrow Y$. Let $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ be the inner products of X and Y, respectively. We will denote by $D(S)$, the set of all functions u's defined in X, such that the application $v \longrightarrow (u,v)_X, v \in X$, is continuous in the topology induced by Y. Then, $(u,v)_X = (Su, v)_Y$ defines S, as an unbounded operator on Y with domain $D(S)$, dense in Y. Since S is a self-adjoint and strictly positive operator. Using the spectral decomposition of self-adjoint operators, we can define S^θ, $\theta \in \mathbb{R}$. In particular we will use $A = S^{\frac{1}{2}}$. The operator A, is self-adjoint, positive defined on Y , with domain X and

$$(u,v)_X = (Au, Av)_Y, \forall u, v \in X.$$

Definition 1.3.5. *With the previous assumptions, we define the intermediate space*

$$[X,Y]_\theta = D \ (\textit{domain of } A^{1-\theta}), \ 0 \leq \theta \leq 1,$$

with norm

$$\|u\|_{[X,Y]_\theta} = (\|u\|_Y^2 + \|A^{1-\theta}u\|_Y^2)^{\frac{1}{2}}.$$

Note that

1. $X \hookrightarrow [X,Y]_\theta \hookrightarrow Y$.

2. $\|u\|_{[X,Y]_\theta} \leq \|u\|_X^{1-\theta} \|u\|_Y^\theta$.

3. If $0 < \theta_0 < \theta_1 < 1$, then $[X,Y]_{\theta_0} \hookrightarrow [X,Y]_{\theta_1}$.

4. $[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\theta = [X,Y]_{(1-\theta)\theta_0 + \theta\theta_1}$.

The following results are useful in this text.

Theorem 1.3.1. *Let Ω be a subset of $\mathbb{R}^n$, $s > 0$. Then,*

$$H_0^s(\Omega) = \{u \in H^s(\Omega); u|_\Gamma = 0\},$$

where $\Gamma = \partial\Omega$ and $u|_\Gamma$ is, by definition, the trace of u on Γ.

Theorem 1.3.2. *Let Ω be a subset of $\mathbb{R}^n$, $s_1 \geq s_2 \geq 0$, s_1 and s_2 not equal to $k+\dfrac{1}{2}$, $k \in \mathbb{Z}$. If $s = (1-\theta)s_1 + \theta s_2 \neq k + \dfrac{1}{2}$, then*

$$[H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)] = H_0^s(\Omega)$$

and

$$[H_0^m(\Omega), H^0(\Omega)]_\theta = H_0^s(\Omega), \quad s = (1-\theta)m \neq k + \frac{1}{2}$$

with equivalent norms.

1.3.3 Important inequalities

Now, let us present a series of inequalities that will be used throughout this thesis. The results are classical and the proofs will be omitted for more detais see, e.g., [1, 17] and in the references therein.

Lemma 1.3.7 (Young's Inequality)**.** *Let a and b be positive constants, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 1.3.8 (Generalized Young's Inequality)**.** *Let a and b be positive constants, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$, such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then for all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that*

$$ab \leq \epsilon a^p + C(\epsilon) b^q.$$

Lemma 1.3.9 (Gronwall Inequality)**.** *Let $u(t)$ be a non-negative differentiable function on $[0, T]$, satisfying*

$$u'(t) \leq f(t)u(t) + g(t)$$

where $f(t)$ and $g(t)$ are integrable functions over $[0, T]$. Then,

$$u(t) \leq e^{\int_0^t f(\tau)d\tau} \left[u(0) + \int_0^t g(s) e^{-\int_0^s f(\tau)d\tau} ds\right], \forall t \in [0, T].$$

If $f(t)$ and $g(t)$ are non-negative functions, then the expression becomes

$$u(t) \leq e^{\int_0^t f(\tau)d\tau} \left[u(0) + \int_0^t g(s) ds\right], \forall t \in [0, T].$$

Lemma 1.3.10 (Cauchy-Schwarz's Inequality)**.** *Let $(E, \langle\cdot,\cdot\rangle)$ be a vector space with an inner product and $\|\cdot\|$ the induced norm of the inner product, then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \ \forall x, y \in E.$$

Furthermore, equality holds if and only if x and y are linearly independent.

Lemma 1.3.11 (Hölder's Inequality)**.** *Let $f \in L^p(\Omega)$ e $g \in L^q(\Omega)$, com $1 \leq p \leq \infty$ e $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} = \int_\Omega |fg| \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

1.3.4 Semigroup theory

Some definitions and results of the semigroup theory that will be used in the chapters of this thesis, will be presented. The results contained here can be found in [79]. In what follows, we will denotes by $(X, \|\cdot\|_X)$ a Banach space.

Definition 1.3.6. *Let $\mathcal{L}(X)$ the algebra of linear operators bounded over X. We said that $S : \mathbb{R}^+ \to \mathcal{L}(X)$ is a C_0-semigroup of bounded operators on X if:*

(i) $S(0) = I$*, where I is the identity operator on X;*

(ii) $S(t+s) = S(t)S(s)$*, for all $t, s \in \mathbb{R}^+$;*

(iii) $\lim_{t\to 0^+} \|(S(t) - I)x\|_X = 0$*, for all $x \in X$.*

Proposition 1.3.1. *Let $S : \mathbb{R}^+ \to \mathcal{L}(X)$ be a C_0-semigroup, then*

$$\lim_{t\to\infty} \frac{ln\|S(t)\|_{\mathcal{L}(X)}}{t} = \inf_{t>0} \frac{ln\|S(t)\|_{\mathcal{L}(X)}}{t} = \omega_0.$$

Furthermore, for every $\omega > \omega_0$, there exists a constant $M \geq 1$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \ \textit{for all } t \geq 0. \tag{1.2}$$

Remark 1.3.8. *When $\omega_0 < 0$, we can take $\omega_0 < \omega < 0$ and by (1.2), there exists $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(X)} \leq M, \ \textit{for all } t \geq 0.$$

Besides, when $M \leq 1$, we said that $S : \mathbb{R}^+ \to \mathcal{L}(X)$ is a C_0-semigroup of contractions.

Definition 1.3.7. *Let $S : \mathbb{R}^+ \to \mathcal{L}(X)$ be a C_0-semigroup. The operator*

$$A : D(A) \subset X \to X,$$

where its domain $D(A)$ and value in x are defined, respectively, by

$$D(A) := \left\{ x \in X; \exists \lim_{h \to 0^+} \left(\frac{S(h) - I}{h} \right) x \right\}$$

and

$$Ax := \lim_{h \to 0} \left(\frac{S(h) - I}{h} \right) x,$$

is the infinitesimal generator of the C_0-semigroup $S(t)$.

Remark 1.3.9. *It is easy to see that if $D(A) \subset X$ is a nonempty subset, then $D(A)$ is a subspace of X and A is a linear operator.*

Proposition 1.3.2. *Let $S : \mathbb{R}^+ \to \mathcal{L}(X)$ be a C_0-semigroup and $A : D(A) \subset X \to X$ its infinitesimal generator, then:*

(i) If $x \in D(A)$, then $S(t)x \in D(A)$, for all $t \geq 0$ and also

$$\frac{d}{dt} S(t)x = AS(t)x = S(t)Ax, \forall t \geq 0.$$

(ii) If $x \in D(A)$, then

$$S(t)x - S(s)x = \int_s^t AS(\xi)x d\xi = \int_s^t S(\xi)Ax d\xi, \ 0 \leq s \leq t.$$

(iii) If $x \in X$, then $\int_0^t S(\xi)x d\xi \in D(A)$ and

$$A \int_0^t S(\xi)x d\xi = S(t)x - x.$$

Definition 1.3.8. *Let $A : D(A) \subset X \to X$ be a linear operator. The resolving set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, that is, $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operators is called the solvent of A.*

1.3.4.1 The Hille-Yosida and Lumer-Phillips theorems

This subsection presents two theorems that establish necessary and sufficient conditions for a linear operator $A : D(A) \subset X \to X$ to generate a C_0-semigroup.

Theorem 1.3.3. *(Hille-Yosida) A linear operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup that satisfy $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \ t \geq 0$ if and only if:*

(i) A is closed and $\overline{D(A)} = X$;

(ii) The resolvent set $\rho(A)$ of A is such that $\{\lambda; Im(\lambda) = 0, \ \lambda > \omega\} \subset \rho(A)$ and for such λ we have

$$\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq \frac{M}{\lambda - \omega}.$$

In order to A be the infinitesimal generator of a C_0-semigroup of contractions, we must replace (ii) by the following condition:

(iii) For all real number $\lambda > 0$, we have $\lambda \in \rho(A)$ and

$$\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

Before we presented the next result, we need another concept. Let X be a Banach space and let X^* its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{ x^*;\ x^* \in X^* \text{ and } \langle x^*, x \rangle = \|x\|_X^2 = \|x^*\|_{X^*}^2 \right\}.$$

Remark 1.3.10. *From the Hahn-Banach theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$.*

Definition 1.3.9. *A linear operator $A : D(A) \subset X \to X$ is dissipative if for every $x \in D(A)$ there is a $x^* \in F(X)$ such that $Re\langle Ax, x^* \rangle \leq 0$.*

Theorem 1.3.4. *(Lumer-Phillips) Let $A : D(A) \subset X \to X$ be a linear operator, with $\overline{D(A)} = X$.*

(a) If A is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(b) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$,

$$Re\langle Ax, x^* \rangle \leq 0.$$

Corolary 1.3.1. *Let $A : D(A) \subset X \to X$ be a linear closed operator, with $\overline{D(A)} = X$. If A and A^* (adjoint of A) are dissipative, then A is a generator of a C_0-semigroup of contractions on X.*

1.3.4.2 The abstract Cauchy problem: Linear case

Let $A : D(A) \subset X \to X$ be a linear operator. Given $u_0 \in X$, the abstract Cauchy problem for A with initial data u_0 consists of finding a solution $u(t)$ to the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t),\ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

Now, let us introduce a notion of a solution to the problem (1.3).

Definition 1.3.10. *(Classical solution) A function $u : \mathbb{R}^+ \to X$ is a classical solution of (1.3) for all $t \geq 0$ if u is continuous for all $t \geq 0$, continuously differentiable on $\mathbb{R}^+$, $u(t) \in D(A)$ for all $t \in \mathbb{R}^+$ and (1.3) is satisfied for all $t \geq 0$.*

Remark 1.3.11. *Consider $S : \mathbb{R}^+ \to X$ be a C_0-semigroup. Thanks to the Proposition 1.3.2, if $u_0 \in D(A)$ and A its infinitesimal generator, then $u(\cdot) = S(\cdot)u_0 : \mathbb{R}^+ \to D(A)$ is a classical solution of (1.3). Moreover, $S(\cdot)u_0$ is the only solution of (1.3).*

We consider the inhomogeneous Cauchy problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + f(t), \ t > 0 \\ u(0) = u_0. \end{cases} \tag{1.4}$$

where $f : \mathbb{R}^+ \to X$ is a function. Similarly to the Definition 1.3.10 we have the following definition for the classical solution of the problem (1.4), with $T > 0$ being a fixed constant.

Definition 1.3.11. *(Classical solution) A function $u : [0, T[\to X$ is a classical solution of (1.4) for all $t \in [0, T[$ if u is continuous on $[0, T[$, continuously differentiable on $]0, T[$, $u(t) \in D(A)$ for all $t \in]0, T[$ and (1.4) is satisfied for all $t \in [0, T[$.*

We will assume that A is an infinitesimal generator of a C_0- semigroup S and $u(t)$, is a classical solution of (1.4). Then $v(s) = S(t-s)u(s)$ is differentiable for $0 < s < t$ and

$$\frac{dv}{ds} = S(t-s)f(s). \tag{1.5}$$

Hence, If $f \in L^1(0, T; X)$ then $S(t-s)f(s)$ is integrable on $[0, t]$ and integrating (1.5) from 0 to t yields

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds. \tag{1.6}$$

As a consequence, the equation (1.6) has at most one solution u. Also, if u exists, then $u \in C([0, T]; X)$. So, it is natural to consider as a generalized solution of (1.4) as follows:

Definition 1.3.12. *Let $u_0 \in X$ and $f \in L^1([0, T]; X)$. The function $u \in C([0, T]; X)$ given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \ 0 \leq t \leq T.$$

is the mild solution of the inhomogeneous Cauchy problem (1.4) on $[0, T]$.

Remark 1.3.12. *In general, the homogeneous Cauchy problem (1.3) does not have a classical solution, once $u_0 \notin D(A)$. So making $f \equiv 0$ on Definition 1.3.12, $u(\cdot) = S(\cdot)u_0$ is the mild solution of (1.3), since that $u_0 \in X$. It is therefore clear that not every mild solution of (1.4) is indeed a classical solution even in the case $f \equiv 0$.*

Now, let us present another notion of solution to the Cauchy problem (1.4):

Definition 1.3.13. *(Strong solution): A function u which is differentiable almost everywhere on $[0,T]$ such that $\frac{du}{dt} \in L^1([0,T];X)$ is called a strong solution of the Cauchy problem (1.4) if $u(0) = u_0$ and*

$$\frac{du}{dt} = Au + f,$$

almost everywhere on $[0,T]$.

Remark 1.3.13. *Observe that if $A = 0$ and $f \in L^1([0,T];X)$, the Cauchy problem (1.4) has usually no solution unless $f \in C([0,T];X)$. However, (1.4) has always a strong solution given by*

$$u(t) = u_0 + \int_0^t f(s)ds.$$

Besides, it is easy to show that if u is a strong solution of (1.4) and $f \in L^1([0,T];X)$, then such u is a mild solution as well.

1.3.4.3 The abstract Cauchy problem: Nonlinear case

Let $(X, \|.\|_X)$ be a reflexive Banach space. Consider the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(u(t)), \ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.7}$$

where $F : X \to X$ is a continuous function and $A : D(A) \subset X \to X$ is an infinitesimal generator of a C_0-semigroup $S : \mathbb{R}^+ \to \mathcal{L}(X)$ such that $\|S(t)\|_{\mathcal{L}(X)} \leq M, \forall t \geq 0$. If u is a classical solution or a strong solution of (1.7), it is not difficult to see that u satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds,$$

which means that u is a mild solution. The next result gives some conditions over $F(u(t))$ such that (1.7) has a solution in the sense defined before.

Theorem 1.3.5. *Let $F : X \to X$ be a Lipschitz function, i.e., there exists $L > 0$ such that*

$$\|F(u) - F(v)\|_X \leq L\|u - v\|_X, \ \forall u, v \in X.$$

So, for every $u_0 \in X$, there exists an unique mild solution $u \in C(\mathbb{R}^+; X)$. Besides:

(i) If $u_0, v_0 \in X$ are initial data and u, v are its respective mild solutions of (1.7), then

$$\|u(t) - v(t)\|_X \leq Me^{LMt}\|u_0 - v_0\|_X.$$

(ii) If $u_0 \in D(A)$, then u is a strong solution of (1.7) on $[0,T]$, for $T > 0$.

1.4 Stabilization theory

The results of this subsection can be seen in [39, 67, 68, 85–87, 102].

We will give some tools, as well as techniques, used to achieve the stabilization problems for evolution equations that were used in this thesis.

1.4.1 Observability inequality

Let $(X, \| \ \|)$ be a Banach space, X' the dual space of X, where $\langle\langle \ , \ \rangle\rangle$ indicates the duality between X' and X, and $A : D(A) \subset X \to X$ be a linear operator. Defining

$$D(A^*) = \{u^* \in X'; \exists v^* \in X' \text{ who checks } \langle\langle u^*, Au\rangle\rangle = \langle\langle v^*, u\rangle\rangle, \forall u \in D(A)\},$$

when $D(A)$ is dense in X the vector v^* that corresponds to u^* is unique, which allows us to define the adjoint operator A^* as follows

$$\begin{array}{rcl} A^* : D(A^*) \subset X' & \longrightarrow & X' \\ u^* & \longmapsto & A^*u^* = v^*. \end{array}$$

Considering $(X, \langle \ , \ \rangle)$ as a Hilbert space, you can identify its dual with the space itself X. In this case, the duality can be charged as the inner product on X, as follows

$$\langle\langle u^*, v\rangle\rangle = \langle u^*, v\rangle.$$

So, the adjoint of the operator A is the operator A^* with domain

$$\mathcal{D}(A^*) = \{z \in X : \exists C \in \mathbb{R}^+; |\langle Ay, z\rangle_X| \leq C\|y\|_X, \forall y \in \mathcal{D}(A)\}$$

which is defined by

$$\langle Ay, z\rangle_X = \langle y, A^*z\rangle_X, \ \forall y \in \mathcal{D}(A), \ \forall z \in \mathcal{D}(A^*).$$

Furthermore, if A generates a continuous semigroup $(e^{tA})_{t\geq 0}$, then A^* also generates a continuous semigroup $(e^{tA^*})_{t\geq 0}$ fulfilling

$$e^{tA^*} = (S(t))^*, \forall t \geq 0.$$

If $A^* = A$ (resp. $A^* = -A$) the operator A is said self-adjoint (resp. skew-adjoint)[2].

Let us consider the abstract system described by

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & 0 < t < T, \\ y(0) = y_0, \end{cases}$$

where A generates a strongly continuous group on a Hilbert space X(state space) and $B \in L(X, X)$. We also will consider the observed system

$$\begin{cases} \varphi'(t) = -A^*\varphi(t), & 0 < t < T, \\ \varphi(T) = \varphi_T. \end{cases} \tag{1.8}$$

So, we have the following definition.

[2] A skew-adjoint operator generates a continuous group of isometries (e.g. [79]).

Definition 1.4.1. *The system described by* (1.8) *is said to be* observable *in time $T > 0$ if there exists $C > 0$ such that*

$$\int_0^T \|B^*\varphi\|dt \geq C\|\varphi_T\|, \ \forall \varphi_T \in X, \tag{1.9}$$

where φ is the solution of (1.8). *The inequality* (1.9) *is called the observability inequality.*

Remark 1.4.1. *Inequality* (1.9) *is equivalent to the following unique continuation principle:*

$$B^*\varphi(t) = 0, \forall t \in [0, T] \Rightarrow \varphi_T = 0.$$

1.4.2 Lyapunov theory

Given a control system, the first and most important question about its various properties is whether it is stable, because, in the words of Slotine and Li (see [87]), *an unstable control system is typically useless and potentially dangerous.* Here, we will present the concept of a stable system according to the Russian mathematician Alexandr Mikhailovich Lyapunov (1857-1917), presenting some definitions and the *direct method* introduced by him at the end of the 19th century in the work *The General Problem of Motion Stability*, which includes two methods for stability analysis (the so-called linearization method and the direct method) and which was first published in 1892.

The direct method determines the stability properties of a nonlinear system by constructing an "energy-like" scalar function for the system and examining the time variation of the function. This function is known as the Lyapunov function, as we will see later.

Definition 1.4.2. *A nonlinear dynamic system can usually be represented by a set of nonlinear differential equations in the form*

$$u'(t) = g(u, t) \tag{1.10}$$

where g is a nonlinear vector function and u is the state vector in $\mathbb{R}^n$.

A particular value of the state vector is also called a point because it corresponds to a point in the state space. The number n is called the order of the system. A solution $u(t)$ of the equations (1.10) generally corresponds to a state-space curve for t ranging from zero to infinity. This curve is often called a state trajectory or a system trajectory.

Note that although the equation (1.10) does not explicitly contain the control input as a variable, it is directly applicable to feedback control systems. The reason is that the equation (1.10) can represent the closed-loop dynamics of a feedback control system, with the control input being a function of state u and time t, disappearing in the loop dynamics closed. Specifically, if $u' = g(u, v, t)$ where $v = h(u, t)$ is a selected control law, then the closed-loop dynamics is $u' = g[u, h(u,t), t]$ which can be rewritten in the form (1.10). The equation (1.10) can also represent dynamic systems where no control signal is involved.

Definition 1.4.3. *The nonlinear system* (1.10) *is said to be autonomous if g does not explicitly depend on time, so*

$$u' = g(u). \tag{1.11}$$

Otherwise, the system is called non-autonomous.

The fundamental difference between autonomous and non-autonomous systems resides in the fact that the trajectory of the autonomous system is independent of the initial time, whereas that of a non-autonomous system generally is not.

Definition 1.4.4. *A state u^* is an equilibrium point of the system* (1.10) *if $u(t_0) = u^*$ implies*

$$g(u,t) = 0, \;\; \forall t \geq t_0.$$

In other words, $u(t) = u^*, \forall t \geq t_0$.

To simplify the notation and analysis of the stability of the system (1.10) at a specific equilibrium point, we can assume that such equilibrium point is the origin, since introducing a new variable $y = u - u^*$ and replacing $u = y + u^*$ in the equations of the system (1.10) a new set of equations in the variable y is obtained

$$y' = g(y + u^*, t) = h(y,t). \tag{1.12}$$

It can be easily verified that there is a one-to-one correspondence between the solutions of (1.10) and those of (1.12) and that, in addition, $y = 0$, the solution corresponding to $u = u^*$ is an equilibrium point of (1.12). Therefore, instead of studying the behavior equation of (1.10) in the neighborhood of u^*, one can equivalently study the behavior of the equations (1.12) in the neighborhood of the origin. In the remainder of this subsection we will assume that u is a solution to the system (1.10).

Definition 1.4.5. *The equilibrium point* 0 *is stable at t_0 if for any $R > 0$, there exists a positive scalar $r := r(R, t_0)$ such that*

$$\|u(t_0)\| < r \Rightarrow \|u(t)\| < R, \quad \forall t \geq t_0.$$

Otherwise, the equilibrium point 0 is unstable.

Definition 1.4.6. *The equilibrium point* 0 *is asymptotically stable at a time t_0 if*

- *it is stable;*

- $\exists r(t_0) > 0$ *such that* $\|u(t_0)\| < r(t_0) \Rightarrow \|u(t)\| \to 0$ *as* $t \to \infty$.

The concepts of stability and asymptotic stability presented here are for non-autonomous systems where the radius r of the initial ball, which is included in the definitions of this subsection, may depend on the initial time t_0. Here, asymptotic stability requires that there is an attractive region for each initial time t_0. The size of the attractive region and the trajectory convergence speed may depend on the initial time t_0. In the case of autonomous systems, the definitions differ in that r does not depend on the considered initial time $t_0 = 0$.

Definition 1.4.7. *The equilibrium point* 0 *is exponentially stable if there exist two positive numbers,* α *and* λ*, such that for sufficiently small* $u(t_0)$*,*

$$\|u(t)\| \leq \alpha \|u(t_0)\| e^{-\lambda(t-t_0)}, \ \forall t \geq t_0.$$

The positive number λ *is often called the* rate *of exponential convergence.*

Definition 1.4.8. *The equilibrium point* 0 *is globally asymptotically stable if*

$$u(t) \to 0 \quad \text{as } t \to \infty, \ \forall u(t_0).$$

Remark 1.4.2. *Throughout the next chapters, we will say that a system of differential equations is stable, asymptotically stable, exponentially stable, or globally asymptotically stable when the origin is an equilibrium point, respectively, stable, asymptotically stable, exponentially stable or globally asymptotically stable.*

Given a set of nonlinear differential equations, the basic procedure of Lyapunov's direct method is to generate an "energy-like" scalar function for the dynamical system and examine the time variation of this scalar function. In this way, conclusions can be drawn about the stability of the set of differential equations without using difficult stability definitions or requiring explicit knowledge of the solutions.

1.4.2.1 Lyapunov's direct method for autonomous systems

To introduce Lyapunov's direct method for autonomous systems, the first property to be formalized is the notion of positive definite functions, and the second is the concept of so-called Lyapunov functions.

Definition 1.4.9. *A continuous scalar function* $V(x)$ *is said to be locally positive definite if* $V(0) = 0$ *and, on a ball* $B_R = \{x \in X; \|x\| \leq R\}$*,*

$$x \in B_R, \ x \neq 0 \ \Rightarrow \ V(x) > 0.$$

If, on the other hand, the above properties hold for the entire state space, then $V(x)$ *is said to be globally positive definite. A function* $V(x)$ *is said to be negative definite if* $-V(x)$

is positive definite and a function $V(x)$ is globally negative definite if $-V(x)$ is globally positive definite.

Similarly to the previous definition, we have the following definition.

Definition 1.4.10. *A continuous scalar function $V(x)$ is said to be locally positive semi-definite if $V(0) = 0$ and, on a ball B_R,*

$$x \neq 0 \;\Rightarrow\; V(x) \geq 0.$$

If, on the other hand, the above properties hold for the entire state space, then $V(x)$ is said to be globally positive semi-definite. A function $V(x)$ is said to be negative semi-definite if $-V(x)$ is positive semi-definite and a function $V(x)$ is globally negative semi-definite if $-V(x)$ is globally positive semi-definite.

Definition 1.4.11. *If, on a ball B_R, the function $V(x)$ is positive definite and continuously differentiable (i.e. has continuous partial derivatives), and if its time derivative along any path of state of the system (1.11) is negative semi-definite, that is, for a trajectory $u(t)$ of the system taking $V(t) := V(u(t))$, we have*

$$V'(t) = \nabla V(u(t)) \cdot u'(t) < 0 \;(\text{ or } V'(x) = \nabla V(x) \cdot g(x) < 0, \forall x \in B_R)$$

then $V(x)$ is said to be a Lyapunov function for the system (1.11) .

The next two results ensure when the local and global stability holds.

Theorem 1.4.1 (Lyapunov theorem for local stability)**.** *If, on a ball B_R, the function $V(x)$ is a Lyapunov function for the system (1.11), then the equilibrium point 0 is stable. If, in addition, $V'(x) < 0, \forall x \in B_R$, then the equilibrium point 0 is asymptotically stable.*

Theorem 1.4.2 (Lyapunov theorem for global stability)**.** *Assume that there exists a scalar function V of the state x, with continuous first-order derivatives such that*

- *$V(x)$ is positive definite;*
- *$V'(x)$ is negative definite;*
- *$V(x) \to \infty$ as $\|x\| \to \infty$,*

then the equilibrium at the origin is globally asymptotically stable.

2 Problems and main results

In this chapter, we will present the problems of the well-posedness and stabilization for the Kawahara and Biharmonic Schrödinger equations. Precisely, we will provide a summary of the main results of this thesis.

2.1 Massera's theorems for a higher order dispersive system

The first work of this thesis, in collaboration with Roberto Capistrano Filho [25], presented some properties of the solutions for the Kawahara equation in a bounded domain. Let us introduce the problem.

Consider the following Kawahara system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0 & (x,t) \in I \times \mathbb{R} \\ u(0,t) = \varphi(t),\ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R} \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \end{cases} \tag{2.1}$$

with a boundary force $\varphi(t)$, a bounded domain $I = (0,1)$ and a damping term $\alpha u_{xx}(0,t)$, where $|\alpha| < 1$. Specially, we are interested to answer the following question:

Question $\mathcal{A}$: Are there periodic solutions for the system (2.1)?

The question before is presented in several contexts in Chapter 3. However, the main point in the analysis is to prove that if the boundary force is small enough the solutions of (2.1) are bounded. The result can be read as follows.

Theorem 2.1.1. *There exists a constant $\epsilon > 0$ such that for all $\varphi \in C_b^1(\mathbb{R})$ satisfying $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, the system (2.1) admits a unique solution u such that*

$$\|u\|_X \leq C\epsilon,$$

where $C > 0$ is a constant independent of ϵ.

Thanks to this strong theorem, the first result concerning the periodic solutions for (2.1) states that, if a boundary force is $T-$periodic (that is, $\varphi(t+T) = \varphi(t), \forall t \in \mathbb{R}$) and small enough, then the solution of (2.1) is $T-$periodic (that is, $u(t+T) = u(t),\ \forall t \in \mathbb{R}$).

Theorem 2.1.2. *Let $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, where ϵ is the constant determined by Theorem 2.1.1. If φ is a function T–periodic, thus u solution of (2.1), given by Theorem 2.1.1, is also a function T–periodic.*

As a next result, we can ensure that if the boundary force is sufficiently small and ϖ–quasi-periodic function, then the solution of (2.1) is ϖ–quasi-periodic function.

Theorem 2.1.3. *Let $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, where ϵ is the constant determined by Theorem 2.1.1. If φ is $\overline{\omega}$–quasi-periodic function in t, thus the solution u of (2.1), obtained in Theorem 2.1.1, is also $\overline{\omega}$–quasi-periodic function in t.*

Finally, the last result contained in Chapter 3 provides that the solutions of (2.1) are almost periodic.

Theorem 2.1.4. *Let $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, where $0 < \epsilon \ll 1$ is obtained via Theorem 2.1.1. If φ, φ' are almost periodic functions, the solution u of (2.1), given by Theorem 2.1.1, is also an almost periodic function.*

2.2 On the stability of the Kawahara equation with a distributed infinite memory

The second work of this thesis, obtained in collaboration with Roberto Capistrano Filho and Boumediène Chentouf (Kuwait University) [20], dealt with the well-posedness and stabilization problem for the Kawahara equation.

To present the results of [20], let us consider the following system

$$\begin{cases} \partial_t u(x,t) + \partial_x^3 u(x,t) - a_0 \partial_x^5 u(x,t) + u(x,t)\partial_x u(x,t) & \\ \quad +a_1 \partial_x u(x,t) + (-1)^k \displaystyle\int_0^\infty f(s) \partial_x^{2k} u(x,t-s) ds = 0, & (x,t) \in I \times (0,\infty), \\ u(0,t) = u(L,t) = \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t > 0, \\ u(x,-t) = u_0(x,t), & x \in I,\ t \geq 0. \end{cases} \tag{2.2}$$

Here u represents the amplitude of the dispersive wave, $k \in \{0,1,2\}$, $L > 0$, $I = (0,L)$, while $a_1 \in \mathbb{R}$ and $a_0 > 0$ are physical parameter of the dispersive equation. Moreover, u_0 is the initial condition and f is the memory kernel satisfying $f : \mathbb{R}_+ := [0,\infty) \to \mathbb{R}$ so as there exists a positive constant c_0 such that:

$$f \in C^2(\mathbb{R}_+), \quad f' < 0, \quad 0 \leq f'' \leq -c_0 f', \quad f(0) > 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{2.3}$$

Note that with this structure, before present, the energy associated with (2.2) is given by

$$E(t) = \frac{1}{2}\left(\|u(t)\|^2 + \int_0^\infty g(s)\|\partial_x^k \eta^t(\cdot,s)\|^2 ds\right),\ t \in \mathbb{R}_+.$$

Here η^t and its initial data η^0 are defined by

$$\eta^t(x,s) = \int_{t-s}^t u(x,\tau)d\tau \text{ and } \eta^0(x,s) = \int_0^s u_0(x,\tau)d\tau,\ x \in I, s,t \in \mathbb{R}_+.$$

and satisfies

$$\begin{cases} \partial_t \eta^t(x,s) + \partial_s \eta^t(x,s) = u(x,t), & x \in I,\ s,t \in \mathbb{R}_+, \\ \eta^t(0,s) = \eta^t(L,s) = 0, & s,t \in \mathbb{R}_+, \\ \eta^t(x,0) = 0, & x \in I,\ t \in \mathbb{R}_+. \end{cases}$$

So a natural question arises:

Does the energy $E(t)$ decay to 0 as t is sufficiently large? If so, can we provide a decay rate?

The answers to these questions, as well as their proofs, are presented in Chapter 4. With the additional hypothesis of the existence of a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\xi \in C^1(\mathbb{R}^+), \ \xi' \leq 0, \ \int_0^\infty \xi(s)ds = \infty \text{ and } \ g' \leq -\xi g. \tag{2.4}$$

where $g = -f'$, then we can prove the following theorem, which guarantees the stability of the system at a polynomial rate when ξ is constant and at an "exponential" rate when ξ is not constant.

Theorem 2.2.1. *Assume that $a_0 > 0$. Also, suppose that (2.3) and (2.4) are verified. If $U_0 \in \mathcal{H}$ satisfies*

$$a_1 M_P^2 + \frac{2}{3} M_P(M_P + 1)\sqrt{L} M_S \|U_0\| < 5a_0,$$

then there exist positive constants c and $\tilde{c}$ such that the solution U of the system associated with (2.2) satisfies the following stability estimates

(i) If ξ is a constant function, we have

$$E(t) \leq \tilde{c} e^{-ct}, \ t \in \mathbb{R}_+.$$

(ii) If ξ is not a constant function, yields that

$$E(t) \leq \tilde{c} e^{-c\int_0^t \xi(\tau)d\tau} \left(1 + \int_0^t e^{c\int_0^\sigma \xi(\tau)d\tau} \xi(\sigma) \int_\sigma^\infty g(s)h(\sigma, s)dsd\sigma\right), t \in \mathbb{R}_+,$$

where

$$h(t, s) = t^2 + t + \left\| \int_0^{t-s} \partial_x^k u_0(\cdot, \tau)d\tau \right\|,$$

for $0 \leq t \leq s$.

2.3 Asymptotic behavior of Kawahara equation with memory effect

The third work of this thesis, in collaboration with Roberto Capistrano Filho and Boumediène Chentouf (Kuwait University) [21], gave the stabilization results of the Kawahara system with a memory effect.

Consider the Kawahara system posed in a bounded domain $\Omega = (0, \ell)$, where $\ell > 0$ is the spatial length, under the action of a feedback law:

$$\begin{cases} \partial_t \omega(t,x) + \alpha \partial_x \omega(t,x) + \beta \partial_x^3 \omega(t,x) - \partial_x^5 \omega(t,x) & \\ \qquad + \omega^p(t,x) \partial_x \omega(t,x) = 0, & x \in \Omega,\ t > 0, \\ \omega(t,0) = \omega(t,\ell) = 0, & t > 0, \\ \partial_x \omega(t,0) = \partial_x \omega(t,\ell) = 0, & t > 0, \\ \partial_x^2 \omega(t,\ell) = \mathcal{F}(t), & t > 0, \\ \partial_x^2 \omega(t,0) = z_0(t), & t \in \mathcal{I}, \\ \omega(0,x) = \omega_0(x), & x \in \Omega, \end{cases} \tag{2.5}$$

with ω_0, z_0 are initial data and the feedback law is a linear combination of the damping and finite memory terms given by

$$\mathcal{F}(t) := \nu_1 \partial_x^2 \omega(t,0) + \nu_2 \int_{t-\tau_2}^{t-\tau_1} \sigma(t-s) \partial_x^2 \omega(s,0)\, ds. \tag{2.6}$$

Here, $\alpha > 0$ and $\beta > 0$ are physical parameters, $p \in \{1, 2\}$, whereas ν_1 and ν_2 are nonzero real numbers. In turn, $0 < \tau_1 < \tau_2$ correspond to the finite memory interval $(t - \tau_1, t - \tau_2)$. Moreover, $\mathcal{I} = (-\tau_2, 0)$, and the memory kernel is denoted by $\sigma(s)$.

Chapter 5 is devoted to studying the stabilization problem for the system (2.5) in the presence of the feedback law (2.6). Rigorously speaking, we would like to see if, in the presence of the boundary memory feedback law, the energy of the system given by

$$\mathcal{E}(t) = \int_\Omega \omega^2(t,x) dx + |\nu_2| \int_{\mathcal{M}} s\sigma(s) \left(\int_{\Omega_0} (\partial_x^2 \omega)^2 (t - s\phi, 0)\, d\phi \right) ds,\ \ t \geq 0 \tag{2.7}$$

tends towards the zero state with some specific decay rate, when t goes to ∞. To give a positive perspective in this direction, we will assume that the memory kernel σ obeys the following conditions:

Assumptions 1. *The function $\sigma \in \ell^\infty(\mathcal{M})$, where $\mathcal{M} := (\tau_1, \tau_2)$. In turn, we assume that*

$$\sigma(s) > 0, \quad a.e. \text{ in } \mathcal{M}.$$

Moreover, the feedback gains ν_1 and ν_2 together with the memory kernel satisfy

$$|\nu_1| + |\nu_2| \int_{\mathcal{M}} \sigma(s)\, ds < 1.$$

With this in hand, the next theorem is shown using a properly chosen Lyapunov functional, which assures us that the system is exponentially stable.

Theorem 2.3.1. *Under the assumptions 1 and assuming that the length ℓ fulfills the smallness condition*

$$0 < \ell < \pi \sqrt{\frac{3\beta}{\alpha}}, \tag{2.8}$$

there exists $r > 0$ sufficiently small, such that for every $(\omega_0, z_0) \in H$ with $\|(\omega_0, z_0)\|_H < r$, the energy of the system (2.5), given by (2.7), is exponentially stable. In other words, there exist two positive constants κ and μ such that

$$\mathcal{E}(t) \leq \kappa \mathcal{E}(0) e^{-2\mu t}, \ t > 0,$$

where $\mathcal{E}(t)$ is defined by (2.7).

In turn, the requirement (2.8) can be relaxed by using a compactness-uniqueness argument [68], which reduces the problem to study the following spectral system

$$\begin{cases} \zeta\omega(x) + \omega'(x) + \omega'''(x) - \omega'''''(x) = 0, & x \in \Omega, \\ \omega(x) = \omega'(x) = \omega''(x) = 0, & x \in \{0, \ell\}. \end{cases} \tag{2.9}$$

So, thanks to a detailed study of this spectral system, the following result is proved in Chapter 5.

Theorem 2.3.2. *Suppose that assumptions 1 hold. Moreover, we choose $\ell > 0$ so that the system (2.9) has only the trivial solution. Then, there exists $\varrho > 0$ such that for every $(\omega_0, z_0) \in H$ satisfying $\|(\omega_0, z_0)\|_H \leq \varrho$, the energy (2.7) of the problem (2.5) decays exponentially.*

2.4 Infinite memory effects on the stabilization of a Biharmonic Schrödinger equation

The fourth, and last work, of this thesis, in collaboration with Roberto Capistrano Filho and Victor Hugo Gonzalez Martinez (Federal University of Pernambuco) [22], presented the well-posedness and stability of a Schrödinger type system. Let us summarize the results.

Consider the following system

$$\begin{cases} i\partial_t y(x,t) + \Delta y(x,t) - \Delta^2 y(x,t) \\ \qquad +(-1)^j i \displaystyle\int_0^\infty f(s)\Delta^j y(x, t-s)ds = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ y(x,t) = \nabla y(x,t) = 0, & (x,t) \in \Gamma \times \mathbb{R}_+^*, \\ y(x,-t) = y_0(x,t), & (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{2.10}$$

well known as Biharmonic Schrödinger equation. Here $j \in \{0, 1, 2\}$, $\Omega \subset \mathbb{R}^n$ is a n-dimensional open bounded domain with a smooth boundary Γ, and $f : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$ is the kernel (or relaxation) function.

It is important to point out that in the theory developed in [22] for each j the memory term present in (2.10) is modified. Moreover, the memory kernel f satisfies the following assumptions:

Assumption 1. *Consider $f \in C^2(\mathbb{R}_+)$. For some positive constant c_0, we have the following conditions*

$$f' < 0, \quad 0 \leq f'' \leq -c_0 f', \quad f(0) > 0 \quad \textit{and} \quad \lim_{s\to\infty} f(s) = 0.$$

Under the Assumption 1, let us introduce the following energy functionals associated with the solutions of (2.10) :

$$E_j(t) = \frac{1}{2}\left(\|y\|^2 + \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds\right), \tag{2.11}$$

with $j \in \{0, 1, 2\}$ and $g = -f'$. So $g \in C^1(\mathbb{R}_+)$, satisfies

$$g > 0, \quad 0 \leq -g' \leq c_0 g, \quad g_0 = \int_0^\infty g(s)ds = f(0) > 0 \tag{2.12}$$

and

$$\lim_{s\to\infty} g(s) = 0.$$

It is worth mentioning that the abuse of notation $\Delta^{\frac{j}{2}}$ here means the identity operator for $j = 0$, the operator ∇ for $j = 1$ and the Laplacian operator for $j = 2$.

Therefore, taking into account the action of the infinite memory term on the behavior of the solutions of the fourth order Schrödinger system (2.10), it is crucial to understand the answer to the following problem:

Are the solutions to our system stable despite the action of the memory term? If yes, then how robust is the stabilization property of the solutions?

To answer the previous question, let us introduce another assumption.

Assumption 2. *Assume there is a positive constant α_0 and a strictly convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying*

$$G(0) = G'(0) = 0 \quad \textit{and} \quad \lim_{t\to\infty} G'(t) = \infty$$

such that

$$g' \leq -\alpha_0 g \tag{2.13}$$

or

$$\int_0^\infty \frac{s^2 g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{2.14}$$

Additionally, when (2.13) is not verified, we will assume that y_0 satisfies,

$$\sup_{t\in\mathbb{R}_+} \max_{k\in\{0,\ldots,n+1\}} \int_t^\infty \frac{g(s)}{G^{-1}(-g'(s))} \left\|\int_0^{s-t} \Delta^{\frac{j}{2}}\partial_s^k y_0(\cdot,\tau)d\tau\right\|^2 ds < \infty.$$

for $j \in \{0, 1, 2\}$.

With this framework in hand, Chapter 6 presented an answer to the stabilization problem to the system (2.10) in the n-dimensional case. The result can be read as follows:

Theorem 2.4.1. *Assume* (2.12) *and that the Assumption 2 holds. Let* $n \in \mathbb{N}^*$, $U_0 \in D(\mathcal{A}_j^{2n})$ *when* $j = 0$, *and* $U_0 \in D(\mathcal{A}_j^{2n+2})$ *when* $j \in \{1, 2\}$. *Thus, there exists positive constants* $\alpha_{j,n}$ *such that the energy* (2.11) *associated with* (2.10) *satisfies*

$$E_j(t) \leq \alpha_{j,n} G_n \left(\frac{\alpha_{j,n}}{t} \right), \quad t \in \mathbb{R}_+^*, \ j \in \{0, 1, 2\}.$$

Here, G_n *is defined, recursively, as follows:*

$$G_m(s) = G_1(sG_{m-1}(s)), m = 2, 3, \ldots, n, \ G_1 = G_0^{-1},$$

where $G_0(s) = s$ *if* (2.13) *is verified, and* $G_0(s) = sG'(s)$ *if* (2.14) *holds.*

3 Massera's theorems for a higher order dispersive system

3.1 Introduction

3.1.1 Problem under consideration

Results of the existence of periodic solutions for differential equations date back to the 50s, when in 1950, J. L. Massera published a remarkable paper [71] on the existence of periodic solutions to ordinary differential equations (ODE) with periodic right-hand sides. Precisely, the corresponding linear setup Massera's theorem is as follows: Consider the ODE of the form

$$\dot{x} = A(t)x + b(t), \quad x \in \mathbb{R}^m, \tag{3.1}$$

with the matrix $A(t)$ and the vector $b(t)$ continuous on $\mathbb{R}_+$ and periodic with the same period τ. Then, the system (3.1) has a periodic solution with period τ if and only if it has a bounded solution on $\mathbb{R}_+$.

So, in this context, we are interested to prove some periodic properties of the following Kawahara equation in a bounded domain

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0 & (x,t) \in I \times \mathbb{R} \\ u(0,t) = \varphi(t), \ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R} \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \end{cases} \tag{3.2}$$

with a boundary force $\varphi(t)$ in a bounded domain $I = (0,1)$ and a damping term $\alpha u_{xx}(0,t)$, where $|\alpha| < 1$. Precisely, we are interested to understand if the system (3.2) has good properties when we investigate its solutions, considering the context introduced to Massera. Roughly speaking, we are interested in the study of the existence and qualitative property of recurrent solutions. This kind of property may be reformulated in the following question.

Question $\mathcal{A}$: Are there periodic solutions for the system (3.2)?

3.1.2 Physical motivation

Under suitable assumption on amplitude, wavelength, wave steepness, and so on, the properties of the asymptotic models for water waves has been extensively studied to understand the full water wave system[1]. In this spirit, formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where

[1] See for instance [4, 14, 65] and references therein, for a rigorous justification of various asymptotic models for surface and internal waves.

the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces.

The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long-wave approximations according to relations between ε and δ. So, if we consider

$$\varepsilon = \delta^4 \ll 1 \quad \text{and} \quad \mu = \frac{1}{3} + \nu \varepsilon^{\frac{1}{2}},$$

and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation which is an equation derived by Hasimoto and Kawahara in [51, 62] that take the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0.$$

Rescaling this equation, we will study in this chapter the following system

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0.$$

3.1.3 Historical background

Before answering the Question $\mathcal{A}$, let us introduce a state of the art related to the Kawahara equation. As mentioned before, problems related to higher-order dispersive systems are extensively studied. Precisely, stabilization and control problems have been studied in recent years.

A pioneer work is due to Silva and Vasconcellos [92]. The authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second work in this way is due Capistrano-Filho *et al.* [6]. In this work, the authors considered the Kawahara equation in a bounded domain $Q_T = (0, T) \times (0, L)$,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t, x) & \text{in } Q_T, \\ u(t, 0) = u(t, L) = u_x(t, 0) = u_x(t, L) = u_{xx}(t, L) = 0 & \text{on } [0, T], \\ u(0, x) = u_0(x) & \text{in } [0, L]. \end{cases} \tag{3.3}$$

In this article, the authors were able to introduce an internal feedback law in (3.3), considering general nonlinearity $u^p u_x$, $p \in [1, 4)$, instead of uu_x. They proved that under the effect of the damping mechanism the energy associated with the solutions of the system decays exponentially.

Related to internal control issues in a bounded domain, Chen [31] presented results considering the Kawahara equation (3.3) posed on a bounded interval with a distributed control $f(t, x)$ and homogeneous boundary conditions. She showed the result by taking advantage of a Carleman estimate associated with the linear operator of the Kawahara equation with an internal observation. With this in hand, she was able to get a null controllable result when f is effective in a $\omega \subset (0, L)$. In [23], considering the system (3.3) with an internal control $f(t, x)$ and homogeneous boundary conditions, the authors can show that the equation in consideration is exactly controllable in L^2-weighted Sobolev spaces and, additionally, the Kawahara equation is controllable by regions on L^2-Sobolev space.

Recently, a new tool to find control properties for the Kawahara operator was proposed in [27, 28]. First, in [27], the authors showed a new type of controllability for the Kawahara equation, what they called *overdetermination control problem*. The method consist of finding a control acting at the boundary such that the solution of the problem under consideration satisfies an integral condition. In addition, when the control acts internally in the system, instead of in the boundary, the authors proved that this condition is also satisfied. After that, in [28], the authors extend this idea for the internal control problem for the Kawahara equation on unbounded domains. Precisely, under certain hypotheses over the initial and boundary data, they can prove that there exists an internal control input such that solutions of the Kawahara equation satisfies an integral overdetermination condition considering the Kawahara equation posed in the real line, left half-line, and right half-line.

We finish presenting the last works in control theory related to the Kawahara equation. In [24,36], under suitable assumptions on the time delay coefficients, the authors can prove that solutions of the Kawahara system are exponentially stable. The results are obtained using the Lyapunov approach and compactness-uniqueness argument. We caution that this is only a small sample of the extant work on control theory for the Kawahara equation.

3.1.4 Notation and auxiliary results

Before presenting the main results of the chapter, let us introduce some notation and two auxiliary results. Denote by $C_b(\mathbb{R})$ a space whose elements are continuous functions of complex value on $\mathbb{R}$ whose associated norm is given by $\|\phi\|_{C_b(\mathbb{R})} = \sup_{t\in\mathbb{R}} |\phi(t)|$. and by $C^1(\mathbb{R})$ a function space whose elements are continuously differentiable complex-valued functions on $\mathbb{R}$. Set

$$C_b^1(\mathbb{R}) = \{u \in C^1(\mathbb{R}); u, u_t \in C_b(\mathbb{R})\}.$$

and define

$$\|u\|_{C_b^1(\mathbb{R})} := \left(\|u\|_{C_b(\mathbb{R})} + \|u_t\|_{C_b(\mathbb{R})}\right)^{\frac{1}{2}}, \quad \forall u \in C^1(\mathbb{R}).$$

Consider by $L^2(I)$ the space of all Lebesgue square integrable complex-valued functions on I with the following inner product

$$(u,v) := \int_0^1 uvdx, \quad \forall u,v \in L^2(I),$$

where $\overline{v}$ denotes the conjugate of v. With the previous inner product, we define in $L^2(I)$ the following norm

$$\|u\| = (u,u)^{\frac{1}{2}}.$$

Additionally, let $H^s(I), s \geq 0$, be the classical Sobolev spaces of complex-valued functions on I with its classical inner product and norm, denoted by $\|\cdot\|_{H^s(I)}$. Finally, consider $H^s_\alpha(I)$ given by

$$\{u \in H^s(I)|u^{(5i)}(0) = u^{(5i)}(0) = u^{(5i+1)}(0) = u^{(5i+1)}(1) = 0, u^{(5i+2)}(1) = \alpha u^{(5i+2)}(0)\}$$

where $i \in \mathbb{N}$ are such that the derivatives $5i,\ 5i+1,\ 5i+2$ are of order less than or equal to $s-1$. The norm and inner product of $H^s_\alpha(I)$ are inherited from $H^s(I)$. For instance,

$$H^0_\alpha(I) = L^2(I),\ H^1_\alpha(I) = H^1_0(I),\ H^2_\alpha(I) = H^2_0(I)$$

and

$$H^k_\alpha(I) = \{u \in H^k(I); u \in H^2_0(I), u''(1) = \alpha u''(0)\},\ k \in \{3,4,5\}.$$

The first result of this chapter is devoted to proving the well-posedness *via* semigroup theory, which is the key to proving the other main results of the chapter. Precisely, we first prove that the linear Kawahara operator generates a C_0–semigroup of contraction $\{S(t)\}_{t\geq 0}$ in $L^2(I)$.

Theorem 3.1.1. *There exists $\omega > 0$ such that for any $k = 0,1,2,3,4$ and 5, we can find a positive constant $C_k > 0$ which the semigroup associated to the linear Kawahara operator satisfies*

$$\|S(t)u_0\|_{H^k_\alpha(I)} \leq C_k e^{-\omega t}\|u_0\|_{H^k_\alpha(I)},$$

for all $t > 0$.

The previous theorem is the key to proving the existence of the bounded solution for the Kawahara equation (3.2). For that, pick

$$X := C_b(\mathbb{R}, H^2(I))$$

with a norm

$$\|u\|_X := \sup_{t\in\mathbb{R}} \|u(t)\|_{H^2(I)}$$

and define the following set

$$X_\rho := \{u \in X | \|u\|_X \leq \rho\}.$$

The next theorem, thanks to the previous one, ensures that the solutions of (3.2) are bounded.

Theorem 3.1.2. *There exists a constant $\epsilon > 0$ such that for all $\varphi \in C_b^1(\mathbb{R})$ satisfying $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, the system (3.2) admits a unique mild solution u such that $\|u\|_X \leq C\epsilon$, where $C > 0$ is a constant independent of ϵ.*

3.1.5 Massera-type theorems and structure of the chapter

With the previous background in hand it is clear that no results concerning the recurrent solutions for the Kawahara system are presented in the literature. This chapter is interesting to fill this gap by giving answers for the Question $\mathcal{A}$ before presented. Precisely, the next three theorems give us Massera-type theorems for a higher-order dispersive system that is, the result below ensures that the solution of (3.2) is T–periodic.

Theorem 3.1.3. *Let $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, where ϵ is the constant determined by Theorem 3.1.2. If φ is a function T–periodic, thus u solution of (3.2), given by Theorem 3.1.2, is also a function T–periodic.*

Additionally, the next Massera-type theorem gives some property of the periodicity of the solution to (3.2). The result can be read as follows.

Theorem 3.1.4. *Let $\Phi \in C_b(\mathbb{R}^k, \mathbb{R})$ such that*

1. $\Phi(u + 2\pi e_i) = \Phi(u), \ \forall i \in \{1, \cdots, k\}$,
2. $\|\Phi\|_{C_b(\mathbb{R}^k,\mathbb{R})} \leq \epsilon$,

where ϵ is the constant determined by Theorem 3.1.2 then the solution u of (3.2), obtained in Theorem 3.1.2, for $\varphi(t) = \Phi(t\overline{\omega})$, is a function $\overline{\omega}$–quasi-periodic on t.

Finally, let us present the last result of this chapter. Precisely, we can prove that the solutions of (3.2) are almost periodic.

Theorem 3.1.5. *Let $\|\varphi\|_{C_b^1(\mathbb{R})} \leq \epsilon$, where $0 < \epsilon \ll 1$ is obtained via Theorem 3.1.2. If φ, φ' are functions almost periodic, the solution u of (3.2), given by Theorem 3.1.2, is also an almost periodic function.*

The remainder of the chapter is organized as follows. In Section 3.2, we present the auxiliary results that are essential for the proof of the Massera-type theorems, precisely, we present the proof of Theorems 3.1.1 and 3.1.2. After that, in Section 3.3, we present the answer for the question $\mathcal{A}$ which is divided into three results, that is, we present the proof of Theorems 3.1.3, 3.1.4 and 3.1.5. Further comments are presented in Section 3.4. Finally, in Appendix 3.5, we give some properties of the energy associated with (3.2).

3.2 Preliminaries

In this section, we are interested to prove some properties of the following linear Kawahara system

$$
\begin{cases}
u_t + u_{xxx} - u_{xxxxx} = 0 & (x,t) \in I \times \mathbb{R} \\
u(0,t) = u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R} \\
u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R} \\
u(x,0) = u_0(x), & x \in I.
\end{cases} \tag{3.4}
$$

which are essential for the rest of the chapter.

3.2.1 Well-posedness: Linear system

From now on C with or without subscripts denotes positive constants whose value may change on different occasions. We will write the dependence of constant on parameters explicitly if it is essential. Additionally, we denote $\lambda_* > 0$ is the smallest constant such that the following inequality holds

$$\|u_x\|^2 \geq \lambda_* \|u\|^2, \; u \in H_0^1(I).$$

Consider the following operator $A : D(A) \subset L^2(I) \longrightarrow L^2(I)$ defined by

$$Au := -u_{xxx} + u_{xxxxx}$$

where

$$D(A) = \{u \in H^5(I) : u(0) = u(1) = u_x(0) = u_x(1) = 0, u_{xx}(1) = \alpha u_{xx}(0)\},$$

with $|\alpha| < 1$, and its adjoint $A^*v = v_{xxx} - v_{xxxxx}$ with

$$D(A^*) = \{v \in H^5(I) : v(0) = v(1) = v_x(0) = v_x(1) = 0, v_{xx}(0) = \alpha v_{xx}(1)\}.$$

Thus, the following property holds.

Proposition 3.2.1. *A generates a C_0 semigroup of contractions on $L^2(I)$.*

Proof. Since A is a continuous linear operator, using the closed graph theorem, A has the closed graph. Moreover, as $D(A)$ is dense in $L^2(I)$, if we prove that A and A^* are dissipative, thanks to [79, Corollary 4.4] we have that A generates a C_0 semigroup of contractions on $L^2(I)$. To do this, by using the definitions of A and A^*, we get, integrating by parts that

$$(Au, u) = \frac{1}{2}(\alpha^2 - 1)(u_{xx}(0))^2 \leq 0$$

and

$$(A^*v, v) = \frac{1}{2}(\alpha^2 - 1)(v_{xx}(1))^2 \leq 0,$$

that is, A and A^* are dissipative, and so the proof is finished. □

From now on, denote by $\{S(t)\}_{t\geq 0}$ the C_0-semigroup associated with A, so $u(t) = S(t)u_0$ is the mild solution of the linearized system (3.4). The next result ensures some properties of the solution of the linear Kawahara system.

Proposition 3.2.2. *Let u solution of (3.4) with $u_0 \in L^2(I)$. Then, we have for all $T > 0$ that*

(i) $\|u(\cdot, T)\| \leq \|u_0\|$;

(ii) $(1-\alpha^2)\displaystyle\int_0^T u_{xx}^2(0,t)dt \leq \|u_0\|^2$;

(iii) $\|u\|_{L^2(0,T;H^2(I))} \leq \sqrt{\dfrac{1}{3}\left(\dfrac{1}{1-\alpha^2} + 4T\right)}\|u_0\|$.

Proof. Assume that $u_0 \in D(A)$. The validity of the estimates for $u_0 \in L^2(I)$ follows from a standard density argument. Since $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of contractions, item (i) follows. Now, observe that

$$\frac{d}{dt}\left(\|u(t)\|^2_{L^2(I)}\right) = 2(u_t, u) = 2(Au, u) = (\alpha^2 - 1)(u_{xx}(0,t))^2,$$

and integrating in $(0,T)$ we get (ii).

Now, multiplying (3.4) by xu, integrating by parts, and using the boundary condition we have that

$$\begin{aligned}\frac{d}{dt}\left(\int_I xu^2dx\right) =& 2\int_I xuu_tdx = -2\int_I xuu_{xxx}dx + 2\int_I xuu_{xxxxx}dx \\ =& \int_I u^2dx - 3\int_I (u_x)^2dx + \alpha^2(u_{xx}(0,t))^2 - 5\int_I (u_{xx})^2dx.\end{aligned}$$

Integrating both side in $(0,T)$, holds that

$$\begin{aligned}3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt \leq& \int_I x(u_0)^2dx + \int_0^T\int_I u^2dxdt + \int_0^T \alpha^2(u_{xx}(0,t))^2dt \\ \leq& \int_I (u_0)^2dx + \int_0^T \|u_0\|^2dt + \int_0^T \alpha^2(u_{xx}(0,t))^2dt\end{aligned}$$

since $\|u(t)\| \leq \|u_0\|$ for all $t \geq 0$. So, using item (ii), we obtain

$$3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt \leq \left(1 + T + \frac{\alpha^2}{1-\alpha^2}\right)\|u_0\|^2.$$

Therefore,

$$\begin{aligned}3\|u\|^2_{L^2(0,T;H^2(I))} =& 3\int_0^T\int_I u^2dxdt + 3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt \\ \leq& 3T\|u_0\|^2 + \left(1 + T + \frac{\alpha^2}{1-\alpha^2}\right)\|u_0\|^2 = \left(\frac{1}{1-\alpha^2} + 4T\right)\|u_0\|^2,\end{aligned}$$

showing the result. □

The next result ensures the decay of the semigroup associated with the Kawahara operator. This can be proved by using the results shown in Appendix 3.5.

Proposition 3.2.3. *There exists $\omega > 0$ and $C > 0$ such that*

$$\|S(t)u_0\| \leq Ce^{-\omega t}\|u_0\|, \ t \geq 0.$$

Proof. Consider $E(t) = \frac{1}{2}\|u(t)\|^2$ the energy associated with (3.4). So, thanks to the Theorem 3.5.1, we have that

$$\|S(t)u_0\|^2 = 2E(t) \leq C\|u_0\|^2 e^{-\mu t},$$

and taking the square root of both sides in the previous inequality with $\omega = \frac{\mu}{2} > 0$ the results holds. □

The next result shows that the solutions of (3.4) are bounded.

Proposition 3.2.4. *There exists $C > 0$ such that for any $t > 0$,*

$$\|S(t)u_0\|_{H^2(I)} \leq C\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t} + 1}\|u_0\|,$$

holds for any $u_0 \in H^2(I)$.

Proof. Define the following function

$$g(t) := \int_0^t \|S(s)u_0\|^2_{H^2(I)} ds.$$

Applying the mean value theorem we have the existence of $\tau \in \left(0, \frac{t}{2}\right)$ such that

$$\|S(\tau)u_0\|^2_{H^2(I)} \cdot \left(\frac{t}{2}\right) = \int_0^{\frac{t}{2}} \|S(s)u_0\|^2_{H^2(I)} ds.$$

Thanks to item (iii) of Proposition 3.2.2, we get

$$\|S(\tau)u_0\|^2_{H^2(I)} \cdot \left(\frac{t}{2}\right) = \int_0^{\frac{t}{2}} \|S(s)u_0\|^2_{H^2(I)} ds \leq \frac{1}{3}\left(\frac{1}{1-\alpha^2} + 2t\right)\|u_0\|^2.$$

Thus,

$$\|S(\tau)u_0\|^2_{H^2(I)} \leq \frac{1}{3}\left(\frac{1}{1-\alpha^2} + 2t\right)\left(\frac{2}{t}\right)\|u_0\|^2 = \frac{4}{3}\left(\frac{1}{1-\alpha^2}\frac{1}{2t} + 1\right)\|u_0\|^2$$

and so,

$$\|S(\tau)u_0\|_{H^2(I)} \leq \frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{2t} + 1}\|u_0\| \leq \frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t} + 1}\|u_0\|.$$

Finally, semigroup properties ensure that

$$\|S(t)u_0\|_{H^2(I)} = \|S(t-\tau)S(\tau)u_0\|_{H^2(I)} \leq C_1\|S(\tau)u_0\|_{H^2(I)} \leq C_1\frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t} + 1}\|u_0\|,$$

and the proof is achieved. □

3.2.2 Proof of the well-posedness result

Proof of Theorem 3.1.1. Considering $k = 0$, thus the result is a consequence of Proposition 3.2.3. Now, taking $u_0 \in D(A) = H^5_\alpha(I)$, semigroup theory ensures that $u = S(t)u_0 \in D(A)$ and $u_t = Au = S(t)(Au_0)$, with $Au_0 \in L^2(I)$. Pick $v = u_t$, we have that v satisfies the following initial value problem

$$\begin{cases} v_t = Av, & (x,t) \in I \times (0,T), \\ v(x,0) = v_0(x) = (Au_0)(x) \in L^2(I), & x \in I. \end{cases}$$

Proposition 3.2.3 yields that

$$\|v(\cdot,t)\| \leq Ce^{-\omega t}\|v_0\| = Ce^{-\omega t}\|Au_0\|.$$

Since the following norms $\|u\| + \|Au\|$ and $\|u\|_{D(A)}$ are equivalents in $D(A)$, we ensure the existence of two constants $M_1, M_2 > 0$ such that

$$M_1\|u\|_{D(A)} \leq \|u\| + \|Au\| \leq M_2\|u\|_{D(A)}.$$

Thus,

$$\begin{aligned} \|S(t)u_0\|_{H^5_\alpha(I)} &\leq M_1^{-1}(\|S(t)u_0\| + \|A(S(t)u_0)\|) \\ &\leq M_1^{-1}(Ce^{-\omega t}\|u_0\| + Ce^{-\omega t}\|Au_0\|) \\ &\leq M_1^{-1}CM_2e^{-\omega t}\|u_0\|_{H^5_\alpha(I)}. \end{aligned}$$

The results for $k = 1, 2, 3$ and 4, are consequences of an interpolation argument. So, Theorem 3.1.1 is shown. $\square$

3.2.3 Well-posedness: Bounded solutions for the nonlinear system

Consider the following boundary value problem

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = \varphi(t), \; u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}. \end{cases} \tag{3.5}$$

Let us study the bounded solution to the system (3.5). To do this, we introduce the definition of a solution of (3.5). Let us consider

$$y(x,t) := u(x,t) + A(x)\varphi(t),$$

with the function A defined by

$$A(x) = -\frac{3(\alpha+1)}{\alpha-1}x^4 + \frac{4(\alpha+2)}{\alpha-1}x^3 - \frac{6}{\alpha-1}x^2 - 1$$

and $\varphi \in C_b^1(\mathbb{R})$. If we suppose that u satisfies (3.5), so we have that y satisfies

$$\begin{cases} y_t + y_{xxx} - y_{xxxxx} + yy_x + ay_x + by = f, & (x,t) \in I \times \mathbb{R}, \\ y(0,t) = y(1,t) = y_x(1,t) = y_x(0,t) = 0, & t \in \mathbb{R}, \\ y_{xx}(1,t) = \alpha y_{xx}(0,t) & t \in \mathbb{R}, \end{cases} \tag{3.6}$$

with

$$a(x,t) = -A(x)\varphi(t), \quad b(x,t) = -A'(x)\varphi(t) \tag{3.7}$$

and

$$f = A(x)\varphi'(t) - A(x)A'(x)\varphi^2(t) + A'''(x)\varphi(t). \tag{3.8}$$

Moreover, we consider that y is a mild solution of (3.6) if satisfies the integral equation

$$y(t) = S(t-r)y(r) + \int_r^t S(t-s)(-yy_x - ay_x - by + f)(s)ds,$$

for all $t \geq r$ and each $r \in \mathbb{R}$. Thus, as y is a mild solution of (3.6), we have that

$$u(x,t) = y(x,t) - A(x)\varphi(t)$$

is a mild solution of (3.5). With this in hand, we are in a position to prove our second auxiliary result of this chapter.

Proof of Theorem 3.1.2. A straightforward calculation shows that, for $\omega > 0$ given in Proposition 3.2.3, by using integration by parts, we get

$$\int_0^\infty e^{-\frac{\omega}{2}\tau}\sqrt{\frac{1}{\tau}}d\tau = \int_0^\infty \omega e^{-\frac{\omega}{2}\tau}\sqrt{\tau}d\tau = \int_0^\infty \omega e^{-\frac{\omega}{4}\tau}e^{-\frac{\omega}{4}\tau}\sqrt{\tau}d\tau.$$

Pick the function $h(\tau) = e^{-\frac{\omega}{2}\tau}\tau$, with $\tau \in \mathbb{R}$. Since

$$h'(\tau) = \left(1 - \frac{\omega}{2}\tau\right)e^{-\frac{\omega}{2}\tau} = 0 \Leftrightarrow \tau = \frac{2}{\omega},$$

$h'(t) > 0$ for $\tau < \dfrac{2}{\omega}$ and $h'(t) < 0$ for $\tau > \dfrac{2}{\omega}$, yields that

$$h(\tau) \leq h\left(\frac{2}{\omega}\right) = \frac{2e^{-1}}{\omega}, \quad \forall \tau \in \mathbb{R}.$$

So we have

$$e^{-\frac{\omega}{4}\tau}\sqrt{\tau} \leq \sqrt{\frac{2e^{-1}}{\omega}}.$$

Therefore,

$$\begin{aligned} \int_0^\infty e^{-\frac{\omega}{2}\tau}\sqrt{\frac{1}{\tau}}d\tau &= \int_0^\infty \omega e^{-\frac{\omega}{4}\tau}e^{-\frac{\omega}{4}\tau}\sqrt{\tau}d\tau \\ &\leq \omega\sqrt{\frac{2e^{-1}}{\omega}}\int_0^\infty e^{-\frac{\omega}{4}\tau}d\tau \\ &= 4\sqrt{\frac{2e^{-1}}{\omega}}. \end{aligned}$$

On the other hand,

$$\int_0^\infty e^{-\frac{\omega}{2}\tau}d\tau = \frac{2}{\omega}.$$

Thus, Theorem 3.1.1, Proposition 3.2.4 and Agmon inequality[2], ensure that

$$\begin{aligned}
&\left\|\int_{-\infty}^t S(t-s)y(\cdot,s)y_x(\cdot,s)ds\right\|_{H^2(I)} \leq \\
&\quad\int_{-\infty}^t \left\|S\left(\frac{t-s}{2}\right)S\left(\frac{t-s}{2}\right)y(\cdot,s)y_x(\cdot,s)\right\|_{H^2(I)} ds \\
&\quad\leq C_2\int_{-\infty}^t e^{-\frac{\omega}{2}(t-s)}\left\|S\left(\frac{t-s}{2}\right)y(\cdot,s)y_x(\cdot,s)\right\|_{H^2(I)} ds \\
&\quad\leq CC_2\int_{-\infty}^t e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{2}{t-s}+1}\,\|y(\cdot,s)y_x(\cdot,s)\|\,ds \\
&\quad\leq C\int_{-\infty}^t e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{2}{t-s}+1}\,\|y(\cdot,s)\|_{H^2(I)}\,\|y_x(\cdot,s)\|\,ds \\
&\quad\leq C\,\|y\|_X^2\int_{-\infty}^t e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}ds.
\end{aligned}$$

Thanks to the previous inequality, taking $\tau = t-s$, we get

$$\begin{aligned}
\left\|\int_{-\infty}^t S(t-s)y(\cdot,s)y_x(\cdot,s)ds\right\|_{H^2(I)} \leq& C\,\|y\|_X^2\int_0^\infty e^{-\frac{\omega}{2}\tau}\left(\sqrt{\frac{1}{1-\alpha^2}}\sqrt{\frac{1}{\tau}}+\sqrt{1}\right)d\tau \\
\leq& C\,\|y\|_X^2\int_0^\infty e^{-\frac{\omega}{2}\tau}\left(\sqrt{\frac{1}{\tau}}+1\right)d\tau \\
\leq& C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y\|_X^2\,.
\end{aligned}\tag{3.9}$$

Now, considering $\varphi \in C_b^1(\mathbb{R})$, we have $a, b \in X$. Therefore, we get, using the same computations as before, that

$$\begin{aligned}
\left\|\int_{-\infty}^t S(t-s)a(\cdot,s)y_x(\cdot,s)ds\right\|_{H^2(I)} \leq& C\,\|a\|_X\,\|y\|_X\int_0^\infty e^{-\frac{\omega}{2}\tau}\left(\sqrt{\frac{1}{\tau}}+1\right)d\tau \\
\leq& C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\,\|y\|_X
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
\left\|\int_{-\infty}^t S(t-s)b(\cdot,s)y(\cdot,s)ds\right\|_{H^2(I)} \leq& C\,\|b\|_X\,\|y\|_X\int_0^\infty e^{-\frac{\omega}{2}\tau}\left(\sqrt{\frac{1}{\tau}}+1\right)d\tau \\
\leq& C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\,\|y\|_X\,.
\end{aligned}\tag{3.11}$$

Using the change of variable y by $y-z$ in (3.10) and (3.11), respectively, yields that

$$\left\|\int_{-\infty}^t S(t-s)a(\cdot,s)\left[y(\cdot,s)-z(\cdot,s)\right]_x ds\right\|_{H^2(I)} \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\,\|y-z\|_X \tag{3.12}$$

[2] Agmon inequality in the one dimensional case: $\|u\|_{L^\infty(I)} \leq C\|u\|_{L^2(I)}^{\frac{3}{4}}\|u\|_{H^2(I)}^{\frac{1}{4}} \leq C\|u\|_{H^2(I)}$, $I = (0,1)$.

and

$$\left\| \int_{-\infty}^{t} S(t-s)b(\cdot,s)\left[y(\cdot,s)-z(\cdot,s)\right]ds \right\|_{H^2(I)} \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X \|y-z\|_X \,. \tag{3.13}$$

Additionally, thanks to Theorem 3.1.1, we have

$$\begin{aligned} \left\| \int_{-\infty}^{t} S(t-s)f(\cdot,s)ds \right\|_{H^2(I)} \leq & \int_{-\infty}^{t} \|S(t-s)f(\cdot,s)\|_{H^2(I)}\,ds \\ \leq & C_2 \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(\cdot,s)\|_{H^2(I)}\,ds \\ \leq & C\|f\|_X \int_{-\infty}^{t} e^{-\omega(t-s)}ds \\ = & C\|f\|_X \int_{0}^{\infty} e^{-\omega\tau}ds = \frac{C}{\omega}\|f\|_X \,, \end{aligned} \tag{3.14}$$

where in the last line we have used the following change of variable $\tau = t-s$. Also, note that by analogous process yields that

$$\begin{aligned} & \left\| \int_{-\infty}^{t} S(t-s)\left[y(\cdot,s)y_x(\cdot,s)-z(\cdot,s)z_x(\cdot,s)\right]ds \right\|_{H^2(I)} \\ \leq & \int_{-\infty}^{t} \left\| S\left(\frac{t-s}{2}\right) S\left(\frac{t-s}{2}\right)\left[y(\cdot,s)y_x(\cdot,s)-z(\cdot,s)z_x(\cdot,s)\right] \right\|_{H^2(I)} ds \\ \leq & C\int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}\left\| \frac{1}{2}\frac{d}{dx}(y^2-z^2)(\cdot,s) \right\| ds \\ \leq & C\int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}\,\|(y-z)(\cdot,s)\|_{H^1(I)}\|(y+z)(\cdot,s)\|_{H^1(I)}\,ds \\ \leq & C\|y-z\|_X\|y+z\|_X \int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}ds \\ \leq & C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y-z\|_X\|y+z\|_X \,. \end{aligned} \tag{3.15}$$

Set

$$(\Psi y)(t) := \int_{-\infty}^{t} S(t-s)(-yy_x-ay_x-by+f)(\cdot,s)ds.$$

Now, using (3.9), (3.10), (3.11) and (3.14), we get that

$$\begin{aligned} \|(\Psi y)(t)\|_{H^2(I)} \leq & \left\| \int_{-\infty}^{t} S(t-s)(yy_x)(\cdot,s)ds \right\|_{H^2(I)} + \left\| \int_{-\infty}^{t} S(t-s)(ay_x)(\cdot,s)ds \right\|_{H^2(I)} \\ & + \left\| \int_{-\infty}^{t} S(t-s)(by)(\cdot,s)ds \right\|_{H^2(I)} + \left\| \int_{-\infty}^{t} S(t-s)f(\cdot,s)ds \right\|_{H^2(I)} \\ \leq & C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y\|_X^2 + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\|y\|_X \\ & + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y\|_X + \frac{C}{\omega}\|f\|_X \end{aligned}$$

soon

$$\begin{aligned} \|\Psi y\|_X = \sup_{t\in\mathbb{R}} \|(\Psi y)(t)\|_{H^2(I)} \leq & C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y\|_X^2 + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\|y\|_X \\ & + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y\|_X + \frac{C}{\omega}\|f\|_X \,. \end{aligned}$$

Thanks to (3.12), (3.13) and (3.15), we get that

$$\begin{aligned}\|(\Psi y-\Psi z)(t)\|_{H^2(I)}\leq&\left\|\int_{-\infty}^{t}S(t-s)(yy_x-zz_x)(\cdot,s)ds\right\|_{H^2(I)}\\&+\left\|\int_{-\infty}^{t}S(t-s)\left(a(y-z)_x\right)(\cdot,s)ds\right\|_{H^2(I)}\\&+\left\|\int_{-\infty}^{t}S(t-s)\left(b(y-z)\right)(\cdot,s)ds\right\|_{H^2(I)}\\\leq&C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y-z\|_X\|y+z\|_X+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\|y-z\|_X\\&+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y-z\|_X.\end{aligned}$$

Soon

$$\begin{aligned}\|\Psi y-\Psi z\|_X=&\sup_{t\in\mathbb{R}}\|(\Psi y-\Psi z)(t)\|_{H^2(I)}\\\leq&C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y-z\|_X\|y+z\|_X+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\|y-z\|_X\\&+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y-z\|_X.\end{aligned}$$

Moreover, we have that

$$\|\Psi y\|_X\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\rho^2+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\rho+\frac{C}{\omega}\|f\|_X \tag{3.16}$$

and

$$\|\Psi y-\Psi z\|_X\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\rho\|y-z\|_X+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\|y-z\|_X, \tag{3.17}$$

for $y,z\in X_\rho$. By hypothesis

$$\|\varphi\|_{C_b^1(\mathbb{R})}\leq\epsilon,$$

since $a(x,t)=-A(x)\varphi(t)$ we get

$$\|a(\cdot,t)\|_{H^2(I)}^2=(\varphi(t))^2\int_0^1(A^2+A_x^2+A_{xx}^2)dx=(C\varphi(t))^2,$$

with $C^2:=\int_0^1(A^2+A_x^2+A_{xx}^2)dx$. So

$$\|a\|_X=\sup_{t\in\mathbb{R}}\{|\varphi(t)|C\}\leq C\|\varphi\|_{C_b^1(\mathbb{R})}\leq C\epsilon.$$

Analogously, taking $b(x,t)=-A_x(x)\varphi(t)$, ensures that

$$\|b(\cdot,t)\|_{H^2(I)}^2=(\varphi(t))^2\int_0^1(A_x^2+A_{xx}^2+A_{xxx}^2)dx=(C\varphi(t))^2.$$

Thus,

$$\|b\|_X=\sup_{t\in\mathbb{R}}\{|\varphi(t)|C\}\leq C\|\varphi\|_{C_b^1(\mathbb{R})}\leq C\epsilon.$$

As $f(x,t) = A(x)\varphi'(t) + A'(x)\varphi(t) - A(x)A'(x)\varphi^2(t)$ and $0 < \epsilon \leq 1$, follows that

$$\|f\|_X \leq C\epsilon,$$

where $C > 0$ independent of ϵ.

Finally, thanks to the previous inequality, let us consider

$$\rho = \frac{3C}{\omega}\|f\|_X \leq \frac{3C^2}{\omega}\epsilon.$$

For $\epsilon \ll 1$ small enough we have

$$\rho \ll 1,$$
$$C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)\rho \leq \frac{1}{3},$$

and

$$C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)(\|a\|_X + \|b\|_X) < \frac{1}{3}.$$

Therefore, thanks to (3.16) and (3.17), the following holds true

$$\|\Psi y\|_X \leq \rho$$

and

$$\|\Psi y - \Psi z\|_X \leq \frac{2}{3}\|y - z\|_X,$$

respectively. Therefore, using the Banach fixed-point theorem, there exists a unique $y \in X_\rho$ such that $\Psi y = y$. Note that for $t \geq r$ we have

$$\begin{aligned} y(t) &= (\Psi y)(t) = \int_{-\infty}^{t} S(t-s)(-yy_x - ay_x - by + f)(\cdot, s)ds \\ &= \int_{-\infty}^{r} S(t-r)S(r-s)(-yy_x - ay_x - by + f)(\cdot, s)ds \\ &\quad + \int_{r}^{t} S(t-s)(-yy_x - ay_x - by + f)(\cdot, s)ds \\ &= S(t-r)y(r) + \int_{r}^{t} S(t-s)(-yy_x - ay_x - by + f)(\cdot, s)ds. \end{aligned}$$

Thus, y is a mild solution for (3.6) such that

$$\|y\|_X = \|\Psi y\|_X \leq \rho.$$

As $y(x,t) = u(x,t) + A(x)\varphi(t)$, we get

$$\|u\|_X \leq C\epsilon,$$

showing the result. $\square$

3.3 Massera's theorems for the Kawahara operator

In this section, our goal is to present several Massera's type theorems associated with the system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = \varphi(t), \ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}. \end{cases} \tag{3.18}$$

This theorem ensures that this higher-order dispersive equation has recurrent solutions. Let us start proving the first result in this way.

3.3.1 T-periodic solution

Before proving the theorem we will give a definition of the $T-$periodic function.

Definition 3.3.1. *Let T be a positive real number and $I = (0,1)$. We say that $\varphi : \mathbb{R} \to \mathbb{R}$ is T-periodic when $\varphi(t+T) = \varphi(t)$, $\forall t \in \mathbb{R}$. We say that $u : I \times \mathbb{R} \to \mathbb{R}$ is T-periodic when $u(x,t+T) = u(x,t)$, $\forall (x,t) \in I \times \mathbb{R}$.*

Examples of $T-$periodic functions are $\varphi(t) = \cos\left(\frac{2t\pi}{T}\right)$ and $u(x,t) = x \cdot \cos\left(\dfrac{2t\pi}{T}\right)$.

Proof of Theorem 3.1.3. We have that $v(x,t) = u(x,t+T)$ is the unique solution of

$$\begin{cases} v_t + v_{xxx} - v_{xxxxx} + vv_x = 0, & (x,t) \in I \times \mathbb{R}, \\ v(0,t) = \varphi(t+T) = \varphi(t), & t \in \mathbb{R}, \\ v(1,t) = v_x(1,t) = v_x(0,t) = 0, & t \in \mathbb{R}, \\ v_{xx}(1,t) = \alpha v_{xx}(0,t), & t \in \mathbb{R}. \end{cases}$$

The system above is exactly (3.18), so the uniqueness of solutions gives us that

$$u(x,t+T) = v(x,t) = u(x,t),$$

for all $(x,t) \in I \times \mathbb{R}$, showing the result. □

3.3.2 Quasi-periodic solution

In this section, we are interested in analyzing the quasi-periodic solutions of (3.18). Before it, we present some definitions necessary for this study.

Definition 3.3.2. *We say that the real numbers $\omega_1, \omega_2, \ldots, \omega_k$ are rationally independent when*

$$m_1\omega_1 + \cdots + m_k\omega_k = 0,$$

only happens when $m_1 = \cdots = m_k = 0$, with $m_1, \ldots, m_k \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers.

Let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and e_i a unitary vector of $\mathbb{R}^k$ such that the i-*th* component is 1 and the others are zero. We have the following definition.

Definition 3.3.3. *A function $f(x,t)$ is denoted by $\overline{\omega}$–quasi-periodic in t uniformly with respect to $x \in I$, if there are $\omega_1, \dots, \omega_k \in \mathbb{R}$ rationally independent and a function $F(x,u) \in C(I \times \mathbb{R}^k, \mathbb{R})$ such that*

$$f(x,t) = F(x, t\omega_1, t\omega_2, \dots, t\omega_k) = F(x, t\overline{\omega}), \quad \forall t \in \mathbb{R} \text{ and } x \in I,$$

where $\overline{\omega} = (\omega_1, \dots, \omega_k)$ and $F(x, u + 2\pi e_i) = F(x,u)$, for all $u \in \mathbb{R}^k$ and $x \in I$, $i = 1, 2, \dots, k$. The numbers $\omega_1, \dots, \omega_k$ are called basic frequencies of f.

Definition 3.3.4. *We say that $\varphi(t)$ is $\overline{\omega}$–quasi-periodic in t if there is $\Phi \in C_b(\mathbb{R}^k, \mathbb{R})$ such that*

$$\Phi(u + 2\pi e_i) = \Phi(u), \quad \forall u \in \mathbb{R}^k, i = 1, \dots, k$$

and

$$\varphi(t) = \Phi(t\overline{\omega}) = \Phi(t\omega_1, \dots, t\omega_k) \quad \forall t \in \mathbb{R}.$$

Remarks 3.3.1. *Note that every $\omega \in \mathbb{R}^*$ is rationally independent. Thus, if a function φ is T–periodic, $T > 0$, then $\Phi(x) = \varphi\left(\frac{T}{2\pi}x\right)$ satisfies*

$$\Phi(x + 2\pi) = \Phi(x), \forall x \in \mathbb{R}.$$

As $\varphi(t) = \Phi\left(\frac{2\pi}{T}t\right)$ we have φ is $\frac{2\pi}{T}$–quasi-periodic. On the other hand, not every $\overline{w}$–quasi-periodic function is periodic. For example, $\varphi(t) = \sin(t) + \sin(\pi t)$ is not periodic, but for $\overline{w} = (1, \pi)$, the function φ is $\overline{w}$–quasi-periodic with $\Phi(x,y) = \sin x + \sin y$.

With these definitions in hand, we prove now the second main result of the work.

Proof of Theorem 3.1.4. Since $\varphi(t)$ is $\overline{\omega}$–quasi-periodic function in t, by definition, there exists $\Phi(u) \in C_b(\mathbb{R}^k, \mathbb{R})$ such that

$$\Phi(u + 2\pi e_i) = \Phi(u), \forall u \in \mathbb{R}^k, \ i = 1, \dots, k$$

and

$$\varphi(t) = \Phi(t\overline{\omega}) = \Phi(t\omega_1, \dots, t\omega_k), \quad \forall t \in \mathbb{R}.$$

Set $\overline{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ and $\varphi_{\overline{\alpha}}(t) = \Phi(\overline{\alpha} + t\overline{\omega})$. Thanks to the Theorem 3.1.2, for each boundary force $\varphi_{\overline{\alpha}}(t)$ the equation (3.18) has unique solution $u_{\overline{\alpha}} \in X_{C_\epsilon}$.

Pick now

$$U(x, \overline{\alpha}) := u_{\overline{\alpha}}(x, 0). \tag{3.19}$$

Thus, U is well-defined due to the uniqueness of the solutions. We prove the result by several claims.

Claim 1. $u_{\overline{\alpha}}(x, t+h) = u_{\overline{\alpha}+h\overline{\omega}}(x, t)$.

Indeed, noting that

$$\varphi_{h\overline{\omega}+\overline{\alpha}}(t) = \Phi(t\overline{\omega} + h\overline{\omega} + \overline{\alpha}) = \Phi((t+h)\overline{\omega} + \overline{\alpha}) = \varphi_{\overline{\alpha}}(t+h),$$

we have that $u_{\overline{\alpha}}(x, t+h)$ and $u_{\overline{\alpha}+h\overline{\omega}}(x, t)$ are solutions of (3.18) with boundary force $\varphi_{h\overline{\omega}+\overline{\alpha}}(t)$. The uniqueness of solutions ensures that

$$u_{\overline{\alpha}}(x, t+h) = u_{\overline{\alpha}+h\overline{\omega}}(x, t),$$

and Claim 1 is proved.

Claim 2. $U(x, \overline{\alpha}) = U(x, \alpha_1, \ldots, \alpha_k)$ has period 2π with respect to each argument α_i.

In fact, taking $t = 0$, in Claim 1, we get

$$u_{\overline{\alpha}}(x, h) = u_{\overline{\alpha}+h\overline{\omega}}(x, 0) = U(x, \overline{\alpha} + h\overline{\omega}), \quad \forall h \in \mathbb{R}.$$

As h is arbitrary, we have that

$$u_{\overline{\alpha}}(x, t) = U(x, \overline{\alpha} + t\overline{\omega}), \quad \forall t \in \mathbb{R}.$$

So, (3.19) help us to ensure that

$$U(x, \overline{\alpha} + 2\pi e_i) = u_{\overline{\alpha}+2\pi e_i}(x, 0),$$

where $\{e_1, \ldots, e_k\}$ is the standard basis in $\mathbb{R}^k$. Since

$$\varphi_{\overline{\alpha}+2\pi e_i}(t) = \Phi(t\overline{\omega} + \overline{\alpha} + 2\pi e_i) = \Phi(t\overline{\omega} + \overline{\alpha}) = \varphi_{\overline{\alpha}}(t),$$

holds true, the uniqueness of solution guaranteed by Theorem 3.1.2, gives

$$u_{\overline{\alpha}+2\pi e_i} = u_{\overline{\alpha}},$$

therefore

$$U(x, \overline{\alpha} + 2\pi e_i) = u_{\overline{\alpha}+2\pi e_i}(x, 0) = u_{\overline{\alpha}}(x, 0) = U(x, \overline{\alpha}),$$

and Claim 2 is shown.

Finally, taking $\overline{\alpha} = (0, \ldots, 0) \in \mathbb{R}^k$, we have that the external force $\varphi_{\overline{\alpha}}(t) = \varphi(t)$ and $u(x, t) = U(x, t\overline{\omega})$. Thus, we get u is a $\overline{\omega}$–quasi-periodic solution in t. $\square$

3.3.3 Almost periodic solution

In this section the goal is to prove that (3.18) have almost periodic solutions. To do that, let us begin this subsection with the following definition.

Definition 3.3.5. *Let (Y, d) be a separable and complete metric space and $f : \mathbb{R} \longrightarrow Y$ be a continuous mapping. The function f is said to be almost periodic if, for every $\delta > 0$, there exists a constant $l(\delta) > 0$ such that any interval of length $l(\delta)$ contains at least a number τ for which*

$$\sup_{t\in\mathbb{R}} d(f(t+\tau), f(t)) < \delta.$$

Note that every continuous function $f : \mathbb{R} \to \mathbb{R}$ that is T–periodic is also a almost periodic function, where $l(\delta) = 2T$ and $\tau = mT$ for some $m \in \mathbb{N}$. Indeed, given $\delta > 0$ and $a \in \mathbb{R}$, take the largest value $m \in \mathbb{Z}$ such that $mT < a$, then $(m+1)T \geq a$ and

$$(m+1)T < mT + 2T < a + 2T.$$

Thus, for $\tau_a = (m+1)T$, we have $d(f(t+\tau_a), f(t)) = 0, \forall t \in \mathbb{R}$.

Now, we are in a position to prove the last result of the chapter.

Proof of Theorem 3.1.5. Consider y, a, b and f satisfying (3.6), (3.7) and (3.8). The straightforward calculation, thanks to the following change of variable $\tau = s - \sigma$, shows that

$$\begin{aligned}\|y(t+\sigma) - y(t)\|_{H^2(I)} =& \left\| \int_{-\infty}^{t+\sigma} S(t+\sigma-s)(-yy_x - ay_x - by + f)(s)ds \right.\\ &\left. - \int_{-\infty}^{t} S(t-s)(-yy_x - ay_x - by + f)(s)ds \right\|_{H^2(I)}\\ \leq& \left\| \int_{-\infty}^{t} S(t-s)\left(y(s+\sigma)y_x(s+\sigma) - y(s)y_x(s)\right) ds \right\|_{H^2(I)}\\ &+ \left\| \int_{-\infty}^{t} S(t-s)\left(a(s+\sigma)y_x(s+\sigma) - a(s)y_x(s)\right) ds \right\|_{H^2(I)}\\ &+ \left\| \int_{-\infty}^{t} S(t-s)\left(b(s+\sigma)y(s+\sigma) - b(s)y(s)\right) ds \right\|_{H^2(I)}\\ &+ \left\| \int_{-\infty}^{t} S(t-s)\left(f(s+\sigma) - f(s)\right) ds \right\|_{H^2(I)}.\end{aligned} \tag{3.20}$$

Set $z(\cdot, s) = y(\cdot, s+\sigma)$ in the expression (3.15), we get

$$\begin{aligned}&\left\| \int_{-\infty}^{t} S(t-s)\left(y(s+\sigma)y_x(s+\sigma) - y(s)y_x(s)\right) ds \right\|_{H^2(I)}\\ &\qquad \leq C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|y(\cdot+\sigma) - y(\cdot)\|_X^2.\end{aligned} \tag{3.21}$$

Therefore, follows by (3.12) and (3.10) with $a(\cdot+\sigma) - a(\cdot)$ instead of a that

$$\begin{aligned}&\left\| \int_{-\infty}^{t} S(t-s)\left(a(s+\sigma)y_x(s+\sigma) - a(s)y_x(s)\right) ds \right\|_{H^2(I)}\\ &\qquad \leq C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|a\|_X \|y(\cdot+\sigma) - y(\cdot)\|_X\\ &\qquad +C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|a(\cdot+\sigma) - a(\cdot)\|_X \|y\|_X,\end{aligned} \tag{3.22}$$

Analogously, we get

$$\begin{aligned}\left\|\int_{-\infty}^{t} S(t-s)\left(b(s+\sigma)y(s+\sigma)-b(s)y(s)\right)ds\right\|_{H^2(I)}&\\ \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y(\cdot+\sigma)-y(\cdot)\|_X&\\ +C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b(\cdot+\sigma)-b(\cdot)\|_X\|y\|_X,&\end{aligned} \tag{3.23}$$

where we have used (3.13) and (3.11) with $b(\cdot+\sigma)-b(\cdot)$ instead of b. Due to the Theorem 3.1.1 and using the change of variables $\tau = t-s$, yields that

$$\left\|\int_{-\infty}^{t} S(t-s)\left(f(\cdot,s+\sigma)-f(\cdot,s)\right)ds\right\|_{H^2(I)} \leq \frac{C}{\omega}\left\|f(\cdot+\sigma)-f(\cdot)\right\|_X. \tag{3.24}$$

Now, replacing (3.21), (3.22), (3.23) and (3.24) into (3.20), we ensures that

$$\begin{aligned}\|y(\cdot+\sigma)-y(\cdot)\|_X \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X^2&\\ +C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\|y(\cdot+\sigma)-y(\cdot)\|_X&\\ +C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\left(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X\right)\|y\|_X&\\ +\frac{C}{\omega}\left\|f(\cdot+\sigma)-f(\cdot)\right\|_X.&\end{aligned} \tag{3.25}$$

Thus, taking $y \in X_\rho$ and $0<\epsilon \ll 1$ in the proof of Theorem 3.1.2 such that

$$2C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\rho < \frac{1}{3}$$

and

$$C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X) < \frac{1}{3},$$

we have that Theorem 3.1.2 is still valid and also is verified that

$$C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X^2 \leq 2\rho C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X < \frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X.$$

Finally, applying it in (3.25) we have

$$\begin{aligned}\|y(\cdot+\sigma)-y(\cdot)\|_X \leq& 2\rho C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X\\ &+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\|y(\cdot+\sigma)-y(\cdot)\|_X\\ &+C\rho\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\left(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X\right)\\ &+\frac{C}{\omega}\left\|f(\cdot+\sigma)-f(\cdot)\right\|_X\\ \leq& \frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X+\frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X\\ &+\frac{1}{3}\left(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X\right)\\ &+\frac{C}{\omega}\left\|f(\cdot+\sigma)-f(\cdot)\right\|_X\end{aligned}$$

and follows that

$$\|y(\cdot+\sigma)-y(\cdot)\|_X \le \|a(\cdot+\sigma)-a(\cdot)\|_X + \|b(\cdot+\sigma)-b(\cdot)\|_X + \frac{3C}{\omega}\|f(\cdot+\sigma)-f(\cdot)\|_X .$$

Since a, b, and f are almost periodic functions, y is also an almost periodic function. According to the fact that

$$y(x,t) := u(x,t) - A(x)\varphi(t),$$

we obtain that u is also an almost periodic function. Thus, u is almost a periodic solution of (3.18), and the Theorem is achieved. □

3.4 Further comments

In this chapter, we present properties for a higher-order dispersive system, namely, the Kawahara equation, posed on a bounded domain. Many results in the literature, as we saw in the introduction, treated this equation from a control point of view. Here, we provide periodic properties for the following initial boundary value problem

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R} \\ u(0,t) = \varphi(t),\ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R} \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{3.26}$$

with a forcing boundary term $\varphi(t)$ and a term $\alpha u_{xx}(0,t)$ acting as a damping mechanism. Thus, we have succeeded to prove Massera-type theorems for the solution of (3.26). About the generality of the work done in this chapter, let us make some additional comments.

- Theorems 3.1.3, 3.1.4 and 3.1.5 can be obtained for more general nonlinearities. Indeed, if we consider $u \in \mathcal{B} := C(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ and the nonlinearity $u^p u_x$, $p \in (2,4]$, we have that

$$\int_0^T \int_0^1 |u^{p+2}| dx dt \leqslant C \|u\|^p_{C([0,T];L^2(0,1))} \int_0^T \|u_x\|^2 dt \leqslant C \|u\|^{p+2}_{\mathcal{B}},$$

  by the Gagliardo–Nirenberg inequality. Moreover, recently, Zhou [101] showed the well-posedness of the following initial boundary value problem

$$\begin{cases} u_t - u_{xxxxx} = c_1 uu_x + c_2 u^2 u_x + b_1 u_x u_{xx} + b_2 uu_{xxx}, & x \in (0,L), t \in \mathbb{R}^+, \\ u(t,0) = h_1(t), \quad u(t,L) = h_2(t), \quad u_x(t,0) = h_3(t), & t \in \mathbb{R}^+, \\ u_x(t,L) = h_4(t), \quad u_{xx}(t,L) = h(t), & t \in \mathbb{R}^+, \\ u(0,x) = u_0(x), & x \in (0,L), \end{cases}$$

  Thus, due to the previous inequality and the results proved in [101], when we consider $b_1 = b_2 = 0$ and the combination $c_1 uu_x + c_2 u^2 u_x$ instead of uu_x on (3.26), the main results of this work remains valid.

- As in the classical framework of Massera's theorem, a principal point is to prove that the initial boundary value problem (3.26) admits a bounded solution, to do that, an important step is the study of the energy associated with the linear system under consideration, this analysis was made in the Appendix 3.5.

- An important point of the previous remark is to deal with the energy of (3.26) we analyze the Kawahara operator removing the drift term u_x. This term presents an extra problem because a critical set appears, see [6] for details. In this way, to overcome this difficulty it was necessary to remove the drift term. Thus, an interesting open problem is to extend the result presented in this chapter for the Kawahara equation (3.26) with the drift term taking into account that this equation, with $\varphi(t) = 0$, has the critical set phenomena, as conjectured in [6].

- It is important to point out that the Massera-type theorem has been extended to many differential equations as we can see in [32, 47, 52, 70, 103] and the references therein. The method employed in these works is to prove the existence of periodic solutions if the solution of the equation under consideration is bounded.

- Finally, there are two important points related to the Massera-type theorems for the Kawahara equation. The first one is that we can work with more general non-linearities, as mentioned before. The second one is the strong relationship between the damping mechanism (stabilization problem) and the Massera-type theorems in our case.

Appendix

3.5 Additional properties

In this appendix, we present some additional properties of the linear Kawahara system. For the sake of simplicity, we present the results for the linear system, however, the results obtained here can be also extended for the nonlinear system. Precisely, let us study the energy properties for the following linear system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{3.27}$$

where $|\alpha| < 1$. Note that multiplying (3.27) by u and integrating over $(0,1)$ yields

$$\frac{d}{dt}\int_0^1 |u(x,t)|^2\,dx = \frac{1}{2}(\alpha^2 - 1)(u_{xx}(0))^2 \leq 0, \ \forall t \geq 0.$$

This indicates that the energy $E(t) = \frac{1}{2}\|u\|^2(t)$ associated with (3.27) is not increasing, and the term $\alpha u_{xx}(0,t)$ designs a damping mechanism. To ensure that this energy decays exponentially is natural to show an *observability inequality* associated with the solutions of (3.27). Before presenting it, let us first prove *a weak observability inequality.*

Proposition 3.5.1. *Consider u solution of (3.27) belonging in*

$$C(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)).$$

Thus, we have

$$\frac{1}{2}\|u_0\|^2 \leq \frac{1}{2T}\int_0^T\int_0^1 |u(x,t)|^2 dxdt + \frac{1-\alpha^2}{2}\int_0^T |u_{xx}(0,t)|^2 dt, \tag{3.28}$$

for all $T > 0$.

Proof. We prove the result for the initial data $u_0 \in D(A)$, the result in $L^2(0,1)$ follows by density. First, multiplying the system (3.27) by $(T-t)u$, integrating by parts in $(0,T) \times (0,1)$ and using the boundary conditions we have

$$-\frac{T}{2}\|u_0\|^2 + \frac{1}{2}\int_0^T\int_0^1 u^2 dxdt + \frac{1}{2}(\alpha^2-1)\int_0^T (T-t)(u_{xx}(0,t))^2 dt = 0,$$

or equivalently,

$$\frac{1}{2}\int_0^T\int_0^1 u^2 dxdt + \frac{1}{2}(\alpha^2-1)\int_0^T (T-t)(u_{xx}(0,t))^2 dt = \frac{T}{2}\|u_0\|^2.$$

Thus, we get

$$\frac{1}{2}\|u_0\|^2 \leq \frac{1}{2T}\int_0^T\int_0^1 u^2 dxdt + \frac{(\alpha^2-1)}{2}\int_0^T |u_{xx}(0,t)|^2 dt,$$

showing the proposition. □

Now, we are in a position to prove that the energy associated with (3.27) decays exponentially.

Theorem 3.5.1. *There exists $C > 0$ and $\mu > 0$ such that*

$$E(t) \leq C\|u_0\|^2 e^{-\mu t},$$

for all $t \geq 0$ and u solution of (3.27) with $u_0 \in L^2(0,1)$.

Proof. This result is a consequence of the following observability inequality

$$\frac{1}{2}\int_0^T\int_0^1 u^2 dxdt \leq c_1 \int_0^T |u_{xx}(0,t)|^2 dt, \tag{3.29}$$

for some constant $c_1 > 0$ independent of the solution u.

In fact, replacing (3.29) in (3.28), we get

$$\begin{aligned}
\frac{1}{2}\|u_0\|^2 \leq & \frac{1}{2T}\int_0^T\int_0^1 |u(x,t)|^2 dxdt + \left(\frac{1-\alpha^2}{2}\right)\int_0^T |u_{xx}(0,t)|^2 dt \\
\leq & \frac{1}{2T}\int_0^T |u_{xx}(0,t)|^2 dt + \left(\frac{1-\alpha^2}{2}\right)\int_0^T |u_{xx}(0,t)|^2 dt \\
= & C\int_0^T |u_{xx}(0,t)|^2 dt,
\end{aligned}$$

where $C = C(T,\alpha) > 0$. As we have that

$$E'(t) = \frac{1}{2}(\alpha^2-1)(u_{xx}(0,t))^2 \leq 0,$$

integrating in $(0,t)$ the previous equation and multiplying by $(1+C)$, where C is the same constant obtained previously, we have that

$$(1+C)E(T) \leq CE(0).$$

Thus,

$$E(T) \leq \gamma E(0), \text{ where } 0 < \gamma = \frac{C}{1+C} < 1,$$

with $\gamma := \gamma(T,\alpha)$. Now, the same argument used on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots$, yields that

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0)$$

Thus, we have

$$E(mT) \leq e^{-\nu mT} E(0)$$

with

$$\nu = \frac{1}{T} \ln\left(1 + \frac{1}{C}\right) > 0.$$

For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0),$$

showing the result. □

Let us now prove the observability inequality.

Proof of (3.29). We argue by contradiction. Suppose that (3.29) does not hold. Thus, there exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ of (3.27) such that

$$\lim_{n\longrightarrow+\infty} \frac{\displaystyle\int_0^T \int_0^1 u^2 dxdt}{\displaystyle\int_0^T |u_{xx}(0,t)|^2 dt} = +\infty. \tag{3.30}$$

Now on, taking $\lambda_n = \sqrt{\int_0^T \int_0^1 |u_n(x,t)|^2 dxdt}$ and $v_n(x,t) = \dfrac{u_n(x,t)}{\lambda_n}$, we have that $\{v_n\}_{n\in\mathbb{N}}$ is a sequence satisfying (3.27) with initial data $v_n(x,0) = \dfrac{u_n(x,0)}{\lambda_n}$ and

$$\int_0^T \int_0^1 |v_n(x,t)|^2 dxdt = \frac{1}{\lambda_n^2} \int_0^T \int_0^1 |u_n(x,t)|^2 dxdt = 1. \tag{3.31}$$

Thanks to the equation (3.30), we have

$$\begin{aligned} \lim_{n\to+\infty} \int_0^T |(v_n)_{xx}(0,t)|^2 dt &= \lim_{n\to+\infty} \frac{\displaystyle\int_0^T |(u_n)_{xx}(0,t)|^2 dt}{\lambda_n^2} \\ &= \lim_{n\to+\infty} \frac{\displaystyle\int_0^T |(u_n)_{xx}(0,t)|^2 dt}{\displaystyle\int_0^T \int_0^1 |u_n(x,t)|^2 dxdt} = 0. \end{aligned} \tag{3.32}$$

Due the relation (3.28), since (3.31) and (3.32) are verified, we have that $\{v_n(x,0)\}_{n\in\mathbb{N}}$ is a sequence bounded in $L^2(0,1)$. Therefore, Propositions 3.2.1 and 3.2.2 gives the existence of a constant $M > 0$ such that

$$\|v_n\|^2_{L^2(0,T;H_0^2(0,1))} \leq M,$$

for all $n \in \mathbb{N}$. Since $H_0^2(0,1) \hookrightarrow L^2(0,1)$ compactly, we have $\{v_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2(0,1))$. Thus, there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(0,T;H_0^2(0,1))$$

Moreover, since $v_{n,t}$ is bounded in $L^2(0,T;H^{-3}(0,1))$, so thanks to the Aubin-Lions's theorem we have

$$v_n \to v \text{ strongly in } L^2(0,T;L^2(0,1)). \tag{3.33}$$

By (3.31), we get

$$\|v\|_{L^2(0,T;L^2(0,1))} = 1, \tag{3.34}$$

and so using (3.32) and (3.33), verifies that

$$\int_0^T |v_{xx}(0,t)|^2 dt \leq \liminf_{n\to+\infty} \int_0^T |(v_n)_{xx}(0,t)|^2 dt = 0,$$

which ensures $v_{xx}(0,t) = 0$, for all $t \in (0,T)$. Therefore, the function v satisfies

$$\begin{cases} v_t + v_{xxx} - v_{xxxxx} = 0, & (x,t) \in I \times \mathbb{R}, \\ v(0,t) = v(1,t) = v_x(1,t) = v_x(0,t) = v_{xx}(1,t) = v_{xx}(0,t) = 0, & t \in \mathbb{R}, \\ v(x,0) = v_0, & x \in I. \end{cases}$$

It follows from a result by (Vasconcellos; Silva, 2010, Asymptotic Analysis) that $v = 0$. Contradicting the hypothesis (3.34).

The result follows from a result by [93] that $v = 0$, contradicting the hypotheses (3.34). Thus, the observability inequality holds. □

4 On the stability of the Kawahara equation with a distributed infinite memory

4.1 Introduction

4.1.1 Model under consideration and objective

The fifth-order nonlinear dispersive equation

$$\pm 2\partial_t u + 3u\partial_x u - \nu\partial_x^3 u + \frac{1}{45}\partial_x^5 u = 0, \tag{4.1}$$

models numerous physical phenomena. Considering suitable assumptions on the amplitude, wavelength, wave steepness, and so on, the properties of the asymptotic models for water waves have been extensively studied in the last years, through (4.1), to understand the full water wave system. For a rigorous justification of various asymptotic surface and internal waves models, we suggest the reader consult the following references [4, 14, 65].

On the other hand, we can formulate the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form with at least two parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, non-dimensional, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. In turn, if we introduce another non-dimensional parameter μ, so-called the Bond number, which measures the importance of gravitational forces compared to surface tension forces, then the physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves. On the other hand, there are several long-wave approximations depending on the relations between ε and δ. For instance, if we consider $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation, represented by (4.1), and derived by Hasimoto and Kawahara in [51, 62].

The main concern of this chapter is to deal with the well-posedness and stability of an initial-boundary-value problem related to (4.1). Specifically, we are concerned with a fifth-order dispersive partial differential equation with a memory term

$$\begin{cases} \partial_t u(x,t) + \partial_x^3 u(x,t) - a_0\partial_x^5 u(x,t) + u(x,t)\partial_x u(x,t) & \\ \qquad +a_1\partial_x u(x,t) + (-1)^k \displaystyle\int_0^\infty f(s)\partial_x^{2k} u(x,t-s)ds = 0, & (x,t) \in I \times (0,\infty), \\ u(0,t) = u(L,t) = \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t > 0, \\ u(x,-t) = u_0(x,t), & x \in I,\ t \geq 0. \end{cases} \tag{4.2}$$

Here u represents the amplitude of the dispersive wave, $k \in \{0, 1, 2\}$, $L > 0$, $I = (0, L)$, while $a_1 \in \mathbb{R}$ and $a_0 > 0$ are physical parameter of the dispersive equation. Moreover, u_0

is the initial condition and f is the memory kernel satisfying $f : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$ so as there exists a positive constant c_0 such that:

$$f \in C^2(\mathbb{R}_+), \quad f' < 0, \quad 0 \le f'' \le -c_0 f', \quad f(0) > 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{4.3}$$

Thereafter, the energy associated with the system (4.2) is

$$E(t) = \frac{1}{2}\left(\|u(t)\|^2 + \int_0^\infty g(s)\|\partial_x^k \eta^t(\cdot, s)\|^2 ds \right),\ t \in \mathbb{R}_+. \tag{4.4}$$

Observe that $E' \le 0$ and hence the energy of our system is non-increasing (see Lemma 4.2.1). This means that the localized damping mechanism and the memory term constitute a damping mechanism and consequently one has to study the decay of the solutions of (4.2). Notwithstanding, it has been noticed that the stability property of solutions of numerous physical systems may be lost when a memory effect occurs [75]. Thus, our concern is to provide an answer to the following questions:

Does the energy $E(t)$ decay to 0 as t is sufficiently large? If so, can we provide a decay rate?

4.1.2 Historical background

Let us first present a review of the main results available in the literature for the analysis of the Kawahara equation in a bounded interval. A pioneer work is due to Silva and Vasconcellos [92,93], where the authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second endeavor, in this line, was completed by Capistrano-Filho *et al.* [6], where the generalized Kawahara equation in a bounded domain $Q_T = (0, T) \times (0, L)$ considered the following system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + a(x)u = 0, & \text{in } Q_T, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, L) = 0, & \text{on } [0, T], \\ u(0, x) = u_0(x), & \text{in } [0, L], \end{cases} \tag{4.5}$$

with $p \in [1, 4)$ and $a(x)$ is a nonnegative function and positive only on an open subset of $(0, L)$. It is proven that the solutions of the above system decay exponentially.

The internal controllability problem has been tackled by Chen [31] for the Kawahara equation with homogeneous boundary conditions. Using Carleman estimates associated with the linear operator of the Kawahara equation with an internal observation, a null controllable result was shown when the internal control is effective in a subdomain $\omega \subset (0, L)$. In [23], considering the system (4.5) with an internal control $f(t, x)$ and homogeneous boundary conditions, the equation is exactly shown to be controllable in L^2-weighted Sobolev spaces and, additionally, controllable by regions in L^2-Sobolev space.

Recently, a new tool for the control properties of the Kawahara operator was proposed. In [27], the authors showed a new type of controllability for the Kawahara equation, what they called *overdetermination control problem*. A boundary control was designed so that the solution to the problem under consideration satisfies an integral condition.

The last works on the stabilization of the Kawahara equation with a localized time-delayed interior control. In [24, 36], under suitable assumptions on the time delay coefficients, the authors were able to prove that solutions of the Kawahara system are exponentially stable. The results were obtained using the Lyapunov approach and a compactness-uniqueness argument. More recently, the authors in [19] gave an analysis to better understand the stabilization issue for the Kawahara equation. Indeed, it is shown that the Kawahara equation under the action of a time-delayed boundary control system remains exponentially stable under a condition on the length of the spatial domain. Such a desirable property is proved using two different approaches. It is also worth mentioning that the stability of the solutions to the Kawahara equation has been extensively studied in the context of periodic or non-periodic bounded domain [44, 45, 53, 61] and also in the case when the spacial variable lies in $(-\infty, \infty)$ or $[0, \infty)$ [37, 38, 41, 46, 56, 66].

We end the literature review by mentioning that the occurrence of a memory phenomenon in the Kawahara problem (4.2) could be explained in practice by the fact that numerous compressible and incompressible fluids are intrinsically viscoelastic and therefore the influence of the past values of the amplitude of the dispersive wave of the fluid is unavoidable [5, 33, 42, 76].

Regarding the main contribution of this chapter, we can claim that we can go one step further in the study of the stabilization problem for the fifth-order Korteweg-de-Vries type system. Compared to the recent works [6, 19, 24, 36], where damping mechanisms and delay controls are used, the results of this chapter closes the gap since it is the first work that treats exponential stability using only infinite memory. It is also noteworthy that the current work shows that a memory term plays a role of a damping control in the sense that it leads to the stability of the system without any additional damping such as $a(x)u$ used in [6, 36, 91] to get the stability property of the system. Finally, note that our results remain valid if $a_1 = 0$ and hence the drift term $\partial_x u(x, t)$ can be omitted.

4.1.3 Notations and main result

Throughout this chapter, C denotes a constant that can be different from one step to another in the demonstrations presented here. Let us use $\langle \, , \, \rangle$ and $\|\cdot\|$ to denote the standard real inner product in $L^2(I)$ and its corresponding norm given by

$$\langle u, v\rangle = \int_0^L u(x)v(x)dx \quad \text{and} \quad \|u\| = \left(\int_0^L |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

As in [42], we use the history approach by introducing the following variable η^t and its initial data η^0 defined by

$$\eta^t(x,s)=\int_{t-s}^{t}u(x,\tau)d\tau \text{ and } \eta^0(x,s)=\int_0^s u_0(x,\tau)d\tau,\ x\in I, s,t\in\mathbb{R}_+.$$

Direct and formal computations show that the functional η^t satisfies

$$\begin{cases}\partial_t\eta^t(x,s)+\partial_s\eta^t(x,s)=u(x,t), & x\in I,\ s,t\in\mathbb{R}_+,\\ \eta^t(0,s)=\eta^t(L,s)=0, & s,t\in\mathbb{R}_+,\\ \eta^t(x,0)=0, & x\in I,\ t\in\mathbb{R}_+.\end{cases} \tag{4.6}$$

In order to express the memory term in (4.2) in terms of η^t, pick $g:=-f'$. Thus, according to (4.3), we get $g\in C^1(\mathbb{R}_+)$,

$$g>0,\ 0\le -g'\le c_0 g,\ g_0=\int_0^\infty g(s)ds=f(0)>0 \text{ and } \lim_{s\to\infty}g(s)=0. \tag{4.7}$$

On the other hand, integrating by parts with respect to s and using that $\eta^t(x,0)=0$ and the limit (4.3), we have that

$$\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds=\int_0^\infty f(s)\partial_x^{2k}u(x,t-s)ds. \tag{4.8}$$

Note that, with (4.8) in hands, we rewrite (4.2) as follows

$$\partial_t u+\partial_x^3 u-a_0\partial_x^5 u+u\partial_x u+a_1\partial_x u+a(x)u+(-1)^k\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)ds=0. \tag{4.9}$$

Thereafter, we introduce a variable U and its initial data U_0 defined by

$$U=(u,\eta^t)^T \quad \text{and} \quad U_0(x,s)=(u_0(x),\eta^0(x,s))^T$$

with

$$u\in L^2(I) \text{ and } \eta^t\in L_k=L_g^2(\mathbb{R}_+;H_k):=\left\{v:\mathbb{R}_+\longrightarrow H_k;\int_0^\infty g(s)\|\partial_x^k v(s)\|^2ds<+\infty\right\},$$

where the space H_k is defined as

$$H_k=\begin{cases}L^2(I), & \text{if } k=0,\\ H_0^1(I), & \text{if } k=1,\\ H_0^2(I), & \text{if } k=2.\end{cases}$$

Furthermore, we will consider in the set L_k, defined above, the inner product and norm are given by

$$\langle v,w\rangle_{L_k}=\int_0^\infty g(s)\langle\partial_x^k v(s),\partial_x^k w(s)\rangle ds \text{ and } \|v(s)\|_{L_k}=\left(\int_0^\infty g(s)\|\partial_x^k v(s)\|^2ds\right)^{\frac12},$$

respectively and we define the energy space as $\mathcal{H} = L^2(I) \times L_k$, which will be equipped with the following inner product and its corresponding norm

$$\langle (v_1, v_2), (w_1, w_2) \rangle_{\mathcal{H}} = \langle v_1, w_1 \rangle + \langle v_2, w_2 \rangle_{L_k}$$

and

$$\|(v(s), w(s))\|_{\mathcal{H}} = \left(\|v(s)\|^2 + \|w(s)\|^2_{L_k} \right)^{\frac{1}{2}}.$$

Additionally, to get our stability results we assume the following additional hypothesis to g. There exists a function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\xi \in C^1(\mathbb{R}^+), \ \xi' \leq 0, \ \int_0^\infty \xi(s) ds = \infty \text{ and } \ g' \leq -\xi g. \tag{4.10}$$

In the sequel, M_P is the smallest positive constant satisfying the Poincaré's Inequality

$$\|v\|^2 \leq M_P \|\partial_x v\|^2,$$

for all $v \in H^1_0(I)$. Furthermore, let us denote by M_S the positive constant of the Sobolev embedding $H^1(I) \hookrightarrow L^\infty(I)$

$$\|v\|^2_{L^\infty(I)} \leq M_S \|v\|^2_{H^1(I)}, \quad v \in H^1(I).$$

With this in hand, we will announce the main result of this chapter, precisely, the stability results for the solutions of (4.2). For that, let us reformulate our problem (4.2) and (4.7) as an abstract initial value problem, namely,

$$\begin{cases} \partial_t U(t) = \mathcal{A} U, \\ U(0) = U_0. \end{cases} \tag{4.11}$$

The main result of the chapter can be read as follows.

Theorem 4.1.1. *Assume that $a_0 > 0$. Also, suppose that (4.3) and (4.10) are verified. If $U_0 \in \mathcal{H}$ satisfies*

$$a_1 M_P^2 + \frac{2}{3} M_P (M_P + 1) \sqrt{L} M_S \|U_0\| < 5 a_0, \tag{4.12}$$

then there exist positive constants c and $\tilde{c}$ such that the solution U of (4.11) satisfies the following stability estimates

(i) If ξ is a constant function, we have

$$E(t) \leq \tilde{c} e^{-ct}, \ t \in \mathbb{R}_+.$$

(ii) If ξ is not a constant function, yields that

$$E(t) \leq \tilde{c} e^{-c \int_0^t \xi(\tau) d\tau} \left(1 + \int_0^t e^{c \int_0^\sigma \xi(\tau) d\tau} \xi(\sigma) \int_\sigma^\infty g(s) h(\sigma, s) ds d\sigma \right), t \in \mathbb{R}_+,$$

where

$$h(t, s) = t^2 + t + \left\| \int_0^{t-s} \partial_x^k u_0(\cdot, \tau) d\tau \right\|,$$

for $0 \leq t \leq s$.

Remark 4.1.1. *Functions g satisfying (4.7) and (4.10) are very wide and contain, for example, the ones which converge to zero exponentially like*

$$g(s) = d_1 e^{-q_1 s},$$

where $\xi(s) = q_1 = \xi_0$ with $d_1 > 0$ and $q_1 > 0$, or polynomial like

$$g(s) = d_1(1+s)^{-q_1},$$

where $\xi(s) = \dfrac{q_1}{s+1}$, $\xi_0 = q_1$ with $d_1 > 0$ and $q_1 > 1$, or between them like the following one

$$g(s) = d_1 e^{-q_1(s+1)^{p_1}},$$

where $\xi(s) = q_1 p_1 (s+1)^{p_1-1}$, $\xi_0 = q_1 p_1$, with $d_1 > 0$, $q_1 > 0$ and $p_1 \in (0,1)$.

This chapter is outlined as follows: Section 4.2 is devoted to presenting preliminary results which are essential for the rest of the chapter. In Section 4.3 we prove the well-posedness of the damping-memory problem (4.2). After that, the main result of the chapter, namely, Theorem 4.1.1 is shown in Section 4.4.

4.2 Preliminaries

As mentioned before, (4.2) and (4.7) can be seen as the following Cauchy problem:

$$\begin{cases} \partial_t U(t) = \mathcal{A}U \\ U(0) = U_0, \end{cases} \tag{4.13}$$

with the operator $\mathcal{A}$ given by

$$\mathcal{A}(U) = \begin{pmatrix} -\partial_x^3 u + a_0 \partial_x^5 u - u\partial_x u - a_1 \partial_x u - (-1)^k \displaystyle\int_0^\infty g(s)\partial_x^{2k}\eta^t(\cdot,s)ds \\ u - \partial_s \eta^t \end{pmatrix}.$$

Here, let us consider the domain of $\mathcal{A}$ as follows

$$D(\mathcal{A}) = \{U \in \mathcal{H}; \mathcal{A}(U) \in \mathcal{H}, u \in H_0^2(I), \partial_x^2 u(L) = 0, \eta^t(x,0) = 0\}.$$

Additionally, for $T > 0$, we introduce the space

$$\mathcal{B} = C([0,T]; L^2(I)) \cap L^2(0,T; H^2(I))$$

whose considered norm is

$$\|\cdot\|_{\mathcal{B}} = \|\cdot\|_{C([0,T];L^2(I))} + \|\cdot\|_{L^2(0,T;H^2(I))}.$$

The next lemma gives us a formal calculation of the derivative of $E(t) = \dfrac{1}{2}\|U(t)\|_{\mathcal{H}}^2$, defined by (4.4), that will be important in the work presented in this chapter (the calculations will be rigorously justified later).

Lemma 4.2.1. *Let us consider* $I = (0, L)$ *and* $a_0 > 0$. *Assume that* (4.3) *hold, then the derivative in time of the energy functional* E *satisfies*

$$E'(t) = -\frac{1}{2}a_0 \left[(\partial_x^2 u)(0)\right]^2 + \frac{1}{2}\int_0^\infty g'(s)\|\partial_x^k \eta^t\|^2 ds. \tag{4.14}$$

Proof. Observe that it follows from (4.3) that

$$E'(t) = \langle \partial_t u, u\rangle + \frac{1}{2}\partial_t \left(\int_0^\infty g(s)\left\|\partial_x^k \eta^t(\cdot, s)\right\|^2 ds\right). \tag{4.15}$$

We will analyze each part of the $E'(t)$ separately. First, note that by multiplying (4.9) by u, integrating by parts in and using the boundary condition of (4.2), we have

$$\int_0^L u\partial_t u dx = -a_0 \frac{1}{2}\left(\partial_x^2 u(0)\right)^2 - (-1)^k \int_0^L u \int_0^\infty g(s)\partial_x^{2k}\eta^t(x, s)dsdx. \tag{4.16}$$

Now, multiplying (4.6) by $(-1)^k g(s)\partial_x^{2k}\eta^t$ and again, integrating by parts in $I \times \mathbb{R}_+$, we get

$$\begin{aligned}\int_0^L \int_0^\infty (-1)^k g(s)\partial_x^{2k}\eta^t \partial_t \eta^t dsdx + \int_0^L \int_0^\infty (-1)^k g(s)\partial_x^{2k}\eta^t \partial_s \eta^t dsdx \\ = \int_0^L \int_0^\infty (-1)^k u g(s)\partial_x^{2k}\eta^t dsdx\end{aligned} \tag{4.17}$$

thanks to the boundary conditions of (4.6). When $k = 0$, (4.14) holds directly from (4.15), (4.16) and (4.17). For the case $k = 1$ or $k = 2$, note that once again integrating by parts $k-$times with respect to the variable x, in the two left-hand parcels of (4.17) and, only one time with respect to the variable s in the second left-hand parcel of (4.17), we find that

$$\begin{aligned}\frac{1}{2}\partial_t \left(\int_0^\infty g(s)\left\|\partial_x^k \eta^t\right\|^2 ds\right) = \frac{1}{2}\int_0^\infty g'(s)\left\|\partial_x^k \eta^t\right\|^2 ds \\ + (-1)^k \int_0^L u \int_0^\infty g(s)\partial_x^{2k}\eta^t dsdx,\end{aligned} \tag{4.18}$$

since we have that $\eta^t(x, 0) = 0$ and that the limit (4.7) holds. Hence, in this case, to get (4.14) just add (4.16) and (4.18). □

Remark 4.2.1. *Let us give some comments.*

(i) Since $a_0 > 0$ *and due to the assumptions on* g' *(see* (4.7)*), it follows from* (4.14) *that* $E'(t) \leq 0$. *Hence the memory is acting as a mechanism of damping feedback.*

(ii) Note that the integral term of (4.14) *is well-defined. In fact, observe that since* $0 \leq -g' \leq c_0 g$, *we have*

$$\begin{aligned}\left|\int_0^\infty g'(s)\|\partial_x^k \eta^t\|^2 ds\right| &= -\int_0^\infty g'(s)\|\partial_x^k \eta^t\|^2 ds \\ &\leq c_0 \int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds = c_0\|\partial_x^k \eta^t\|_{L_k}^2 < \infty,\end{aligned}$$

for any $\eta^t \in L_k$, *showing our claim.*

4.3 Well-posedness of the memory problem

In this section, we will study the well-posedness of the system (4.2). Precisely, we will initially study the well-posedness of the linearized system associated with (4.2), and then, we will show that the system with source term is well-posed and, finally, we prove that the original nonlinear system (4.2) is well-posed.

4.3.1 Well-posedness: The linearized problem

In this subsection, we give the details about the well-posedness of the linearized system associated with (4.2), namely

$$\begin{cases} \partial_t u + \partial_x^3 u - a_0 \partial_x^5 u + a_1 \partial_x u & \\ \quad +(-1)^k \int_0^\infty g(s) \partial_x^{2k} \eta^t(x,s) ds = 0, & (x,t) \in I \times (0,\infty), \\ \partial_t \eta^t(x,s) + \partial_s \eta^t(x,s) - u(x,t) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ \eta^t(0,s) = \eta^t(L,s) = \eta^t(x,0) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ u(0,t) = u(L,t) = \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{4.19}$$

with some initial data (u_0, η^0). Note that the system (4.19) can be written in an abstract form in $\mathcal{H}$ as follows

$$\begin{cases} \partial_t \Phi(t) = A\Phi(t), \ t > 0 \\ \Phi(0) = \Phi_0, \end{cases} \tag{4.20}$$

with $\Phi = (u, \eta^t), \ \Phi_0 = (u_0, \eta^0)$ and A is a linear operator giving by

$$A(\Phi) = \begin{pmatrix} -\partial_x^3 u + a_0 \partial_x^5 u - a_1 \partial_x u - (-1)^k \int_0^\infty g(s) \partial_x^{2k} \eta^t(x,s) ds \\ u - \partial_s \eta^t \end{pmatrix} \tag{4.21}$$

with domain

$$D(A) = \{\Phi \in \mathcal{H}; A(\Phi) \in \mathcal{H}, u \in H_0^2(I), \partial_x^2 u(L) = 0, \eta^t(x,0) = 0\}.$$

In turn, recall that, in this section, the generic positive constant C is independent of the initial data Φ_0 but may depend on T, g_0 and the system's parameters $a_i (i = 0, 1)$. The following result ensures the well-posedness of the linearized system.

Theorem 4.3.1. *Let us consider* $I = (0, L)$, $a_1 \in \mathbb{R}$ *and* $a_0 > 0$. *Suppose that* (4.3) *is verified, then the following assertions can be verified:*

(i) The linear operator A *defined by* (4.21) *generates a* C_0*-semigroup of contractions* $S(t)$. *Moreover, given an initial data* $\Phi_0 \in D(A)$, *the problem* (4.20) *admits a unique classical solution*

$$\Phi \in C(\mathbb{R}_+; D(A)) \cup C^1(\mathbb{R}_+, \mathcal{H}). \tag{4.22}$$

In turn, if $\Phi_0 \in \mathcal{H}$, *then* (4.20) *have a unique mild solution*

$$\Phi \in C(\mathbb{R}_+; \mathcal{H}). \tag{4.23}$$

(ii) *For any* $\Phi_0 \in \mathcal{H}$ *and* $T > 0$, *the following estimates holds*

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leq C\|(u_0, \eta^0)\|^2_{\mathcal{H}}, \tag{4.24}$$

for some positive constant C. *Additionally, the mapping*

$$\Delta : \Phi_0 = (u_0, \eta^0)^T \in \mathcal{H} \to \Phi(\cdot) := S(\cdot)\Phi_0 \in \mathcal{B} \times C([0,T]; L_k)$$

is continuous.

Proof. In order to show (i), consider $\Phi = (u, \eta^t) \in D(A)$. Thanks to (4.14) and (4.20), we find

$$\langle A(\Phi), \Phi\rangle_{\mathcal{H}} = \langle \partial_t\Phi, \Phi\rangle_{\mathcal{H}} = \left(\frac{1}{2}\|\Phi\|^2_{\mathcal{H}}\right)' = E'(t) < 0.$$

Thus, A is dissipative thanks to Remark 4.2.1. On the other hand, we can check that the adjoint operator of A is defined by

$$A^*\Psi = \begin{pmatrix} \partial_x^3 v - a_0\partial_x^5 v + a_1\partial_x v + (-1)^k \displaystyle\int_0^\infty g(s)\partial_x^{2k}\zeta^t(x,s)ds \\ -v + \dfrac{g'(s)}{g(s)}\zeta^t + \partial_s\zeta^t \end{pmatrix}$$

with domain

$$D(A^*) = \{\Psi \in \mathcal{H}; A^*(\Psi) \in \mathcal{H}, v \in H_0^2(I), \partial_x^2 v(0) = 0, \zeta^t(x,0) = 0\}.$$

The same line of thought may be applied to obtain

$$\langle A^*(\Psi), \Psi\rangle_{\mathcal{H}} = -a_0\frac{(\partial_x^2 v(L))^2}{2} + \frac{1}{2}\int_0^\infty g'(s)\|\partial_x^k\zeta^t\|^2 ds \leq 0, \forall\Psi \in D(A^*)$$

and hence A^* is also dissipative. Now, since A is densely defined and closed, the assertion in (i) is a direct consequence of the semigroup theory of linear operator, for details see [79].

From now on we will show (ii). Let $\Phi_0 = (u_0, \eta^0) \in \mathcal{H}$. As we know that $S(t)$ is a C_0- semigroup of contractions we have

$$\|S(t)\Phi_0\|_{\mathcal{H}} = \|(u, \eta^t)\|_{\mathcal{H}} \leq \|\Phi_0\|_{\mathcal{H}} = \|(u_0, \eta^0)\|_{\mathcal{H}}, \quad \forall t \in [0,T]. \tag{4.25}$$

Next, consider the function $p(x,t)$, to be chosen later, and consider a regular solution $\Phi = (u, \eta^t)$ of (4.20) with initial data $\Phi_0 \in D(A)$. In this case, Φ has the regularity (4.22).

Then, multiplying the equation (4.19) by $2xu$, integrating by parts over $[0,T]\times I$ and using the boundary condition we have

$$\begin{aligned}
&4\int_0^T \|\partial_x u\|^2 dt + 5a_0\int_0^T \|\partial_x^2 u\|^2 dt\\
&=\int_0^L x u_0^2 dx - \int_0^L xu^2(x,T)dx\\
&-(-1)^k\int_0^T\int_0^L 2xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdxdt + a_1\int_0^T\|u\|^2dt\\
&\leq L\|u_0\|^2 + a_1\int_0^T \|u\|^2 dt - (-1)^k\int_0^T\int_0^L 2xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdxdt.
\end{aligned} \tag{4.26}$$

Let us treat the case $k=0$ and $k\in\{1,2\}$ separately.

Case 1: $k=0$.

First, note that

$$\begin{aligned}
-\int_0^T\int_0^L\int_0^\infty 2g(s)xu\eta^t(x,s)dsdxdt \leq & L^2\int_0^T\int_0^\infty g(s)\|u\|^2dsdt + \int_0^T \|\eta^t(\cdot,s)\|^2_{L_k}dt\\
=&L^2 g_0\int_0^T\|u\|^2dt + \int_0^T\|\eta^t(\cdot,s)\|^2_{L_k}dt.
\end{aligned} \tag{4.27}$$

Thus, amalgamating (4.26) and (4.27), we deduce that

$$\int_0^T\left(\|u\|^2 + \|\partial_x u\|^2 + \|\partial_x^2 u\|^2\right)dt \leq C\|(u_0,\eta^0)\|^2_{\mathcal{H}},$$

where $C=C(T,L,a_0,a_1,g_0)>0$, showing (4.24) for $\Phi\in D(A)$. Finally, the result for $\Phi\in\mathcal{H}$ follows by a density argument. This, together with (4.25) implies the continuity of Δ.

Case 2: $k=1$.

Now, considering $k=1$, integrating the last term of (4.26) by parts and using Hölder's inequality and Young's inequality we get

$$\begin{aligned}
(4-\epsilon g_0L^2)\int_0^T\|\partial_x u\|^2dt + 5a_0\int_0^T\|\partial_x^2u\|^2dt + 2\int_0^T\int_0^L a(x)xu^2dxdt\\
\leq L\|u_0\|^2 + (a_1+g_0)\int_0^T\|u\|^2dt + \left(1+\frac{1}{\epsilon}\right)\int_0^T\|\eta^t\|^2_{L_k}dt.
\end{aligned}$$

Taking $\epsilon = \dfrac{3}{g_0L^2}>0$, we have

$$\int_0^T\left(\|u\|^2 + \|\partial_x u\|^2 + \|\partial_x^2 u\|^2\right)dt \leq C\|(u_0,\eta^0)\|^2_{\mathcal{H}},$$

where $C=C(T,L,a_0,a_1,g_0)>0$ showing (4.24) for $\Phi\in D(A)$. By density argument, (4.24) holds for $\Phi\in\mathcal{H}$, and also the continuity of the mapping Δ is verified.

Case 3: $k=2$.

One has merely to argue as in the previous case. The only difference is that we need to handle the term involving $\|\partial_x^2 u\|^2$ in addition to $\|\partial_x u\|^2$. □

4.3.2 Well-posedness: The equation with source term

The goal of this part is to deal with the well-posedness of the system (4.19) with a source term $\varphi(x,t)$

$$\begin{cases} \partial_t u + \partial_x^3 u - a_0 \partial_x^5 u + a_1 \partial_x u & \\ \quad +(-1)^k \int_0^\infty g(s) \partial_x^{2k} \eta^t(x,s) ds = \varphi(x,t), & (x,t) \in I \times (0,\infty), \\ \partial_t \eta^t(x,s) + \partial_s \eta^t(x,s) - u(x,t) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ \eta^t(0,s) = \eta^t(L,s) = \eta^t(x,0) = 0, & x \in I, \ s,t \in \mathbb{R}_+, \\ u(0,t) = u(L,t) = \partial_x u(0,t) = \partial_x u(L,t) = \partial_x^2 u(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{4.28}$$

We have the following result:

Theorem 4.3.2. *Let us consider $T > 0$ and $a_0 > 0$. If (4.3) is verified, then we have:*

(*i*) *If $\Phi_0 = (u_0, \eta^0)^T \in \mathcal{H}$ and $\varphi \in L^1(0,T;L^2(I))$, then there exists a unique mild solution $\Phi = (u, \eta^t)^T$ of (4.28) such that $\Phi \in \mathcal{B} \times C([0,T];L_k)$,*

$$\|(u,\eta^t)\|^2_{C([0,T];\mathcal{H})} \leq C_0 \left(\|(u_0,\eta^0)\|^2_{\mathcal{H}} + \|\varphi\|^2_{L^1(0,T;L^2(I))} \right) \tag{4.29}$$

and

$$\|u\|^2_{\mathcal{B}} \leq C_1 \left(\|(u_0,\eta^0)\|^2_{\mathcal{H}} + \|\varphi\|^2_{L^1(0,T;L^2(I))} \right), \tag{4.30}$$

for some positive constants C_0, C_1 independent of Φ_0 and φ.

(*ii*) *Given $u \in L^2(0,T;H^2(I))$, we have $u\partial_x u \in L^1(0,T;L^2(I))$ and the map*

$$\Theta : u \in L^2(0,T;H^2(I)) \to u\partial_x u \in L^1(0,T;L^2(I))$$

is continuous.

Proof. (i) Since A generates a C_0-semigroup of contractions $S(t)$ and $\varphi \in L^1(0,T;L^2(I))$ and to ensure the validity of the computations, we shall work with a regular solution Φ of (4.28) stemmed from an initial data $\Phi_0 = (u_0,\eta^0)^T \in D(A)$. It is well-known from the semigroups theory [79] that the solution of (4.28) satisfies

$$\|(u,\eta^t)\|_{\mathcal{H}} \leq C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \int_0^t \|\varphi\| dt \right) \leq C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \|\varphi\|_{L^1(0,T;L^2(I))} \right), \tag{4.31}$$

and consequently (4.29) holds. We also have, thanks to (4.31), that

$$\|u\|_{C([0,T];L^2(I))} \leq C \left(\|(u_0,\eta^0)\|_{\mathcal{H}} + \|\varphi\|_{L^1(0,T;L^2(I))} \right).$$

Therefore, to obtain the H^2-norm of the solution, that is, (4.30), we use an analogous process as in the proof of (4.24), and hence we will omit it. On the other hand, a density argument allows us to extend the results to any initial condition $\Phi_0 \in \mathcal{H}$.

(ii) First, consider $y, z \in L^2(0,T;H^2(I))$. We have

$$\begin{aligned}\|y\partial_x y\|_{L^1(0,T;L^2(I))} &\leq K\int_0^T \|y\|_{H^2(I)}\|\partial_x y\|dt \\ &\leq K\int_0^T \|y\|^2_{H^2(I)}dt = K\|y\|^2_{L^2(0,T;H^2(I))},\end{aligned} \tag{4.32}$$

where K is the positive constant of the Sobolev embedding $H^2 \hookrightarrow L^\infty(I)$. Thus,

$$y\partial_x y \in L^1(0,T;L^2(I)),$$

for each $y \in L^2(0,T;H^2(I))$.

In turn, using triangle inequality together with Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|\Theta(y)-\Theta(z)\|_{L^1(0,T;L^2(I))} \leq & K\int_0^T \|y-z\|_{H^2(I)}\|y\|_{H^2(I)}dt \\ & + K\int_0^T \|z\|_{H^2(I)}\|y-z\|_{H^2(I)}dt \\ \leq & K\|y-z\|_{L^2(0,T;H^2(I))}\|y\|_{L^2(0,T;H^2(I))} \\ & + K\|z\|_{L^2(0,T;H^2(I))}\|y-z\|_{L^2(0,T;H^2(I))} \\ = & K\|y-z\|_{L^2(0,T;H^2(I))}\left(\|y\|_{L^2(0,T;H^2(I))} + \|z\|_{L^2(0,T;H^2(I))}\right)\end{aligned}$$

Thus, the mapping Θ is continuous concerning the corresponding topologies. □

4.3.3 Well-posedness: The nonlinear problem

The next result ensures the well-posedness of the system (4.2), which is represented by the problem (4.13).

Theorem 4.3.3. *Let us consider $T > 0$ and $a_0 > 0$. If (4.3) is verified, then there exists a positive constant C such that, for every $U_0 \in \mathcal{H}$ with*

$$\|U_0\|^2 < \frac{1}{16C_1^2K^2} \tag{4.33}$$

where C_1 is as in (4.30) and K is the positive constant of the Sobolev embedding $H^2 \hookrightarrow L^\infty(I)$, the problem (4.13) has a unique global solution U satisfying the regularity (4.23) and consequently, the problem (4.2) admits a unique global solution $u \in \mathcal{B}$.

Proof. First, consider $U_0 = (u_0, \eta^0) \in \mathcal{H}$ satisfying (4.33). Next, define the map $\Gamma : \mathcal{B} \to \mathcal{B}$ by $\Gamma(z) = u$, where u is a solution of (4.28) with source term $\varphi(x,t) = -z(x,t)\partial_x z(x,t)$ and initial data U_0.

Claim 1: Γ is well-defined.

In fact, take $\alpha > 0$ such that

$$\|U_0\|^2_{\mathcal{B}} \leq \alpha < \frac{1}{16C_1^2K^2}.$$

Theorem 4.3.2 ensures that for the initial data U_0, there exists a unique solution $U = (u, \eta^t)$ of (4.28) satisfying, thanks to (4.33), the estimate

$$\|\Gamma(z)\|_{\mathcal{B}} \leq C_1(\alpha + \|z\partial_x z\|^2_{L^1(0,T;L^2(I))}). \tag{4.34}$$

Moreover, by using (4.32), we get

$$\begin{aligned}\|\Gamma(z)\|^2_{\mathcal{B}} \leq & C_1\left(\|(u_0,\eta^0)\|^2_{\mathcal{H}} + \|z\partial_x z\|^2_{L^1(0,T;L^2(I))}\right) \leq C_1(\alpha + K^2\|z\|^4_{L^2(0,T;H^2(I))}) \\ \leq & C_1(\alpha + K^2\|z\|^4_{\mathcal{B}}),\end{aligned} \tag{4.35}$$

for all $z \in \mathcal{B}$, showing the claim 1.

Claim 2: Γ is a contraction.

Indeed, we have

$$\begin{aligned}\|\Gamma(y) - \Gamma(z)\|^2_{\mathcal{B}} \leq & 2K^2\|y - z\|^2_{L^2(0,T;H^2(I))}\left(\|y\|^2_{L^2(0,T;H^2(I))} + \|z\|^2_{L^2(0,T;H^2(I))}\right) \\ \leq & 2K^2\|y - z\|^2_{\mathcal{B}}\left(\|y\|^2_{\mathcal{B}} + \|z\|^2_{\mathcal{B}}\right).\end{aligned}$$

Then, consider the restriction of Γ to the closed ball $B = \{z \in \mathcal{B}; \|z\|^2_{\mathcal{B}} \leq r\}$, with $r = \dfrac{\sqrt{\alpha}}{2K}$. Thus, (4.34) and (4.35) yields that

$$\|\Gamma(z)\|^2_{\mathcal{B}} \leq C_1(\alpha + K^2\|z\|^4_{\mathcal{B}}) \leq C_1(\alpha + K^2r^2) < 2C_1\alpha < r$$

and

$$\|\Gamma(y) - \Gamma(z)\|^2_{\mathcal{B}} \leq 4rK^2\|y - z\|^2_{\mathcal{B}} \leq \frac{1}{2}\|y - z\|^2_{\mathcal{B}}.$$

The mapping Γ is well-defined and contractive on the ball B according to the choice (4.33), showing claim 2.

Therefore, using the Banach Fixed Point Theorem, we deduce that Γ has a unique fixed element $u \in B$, which turns out to be the unique solution to our problem (4.2). Lastly, the system (4.2) being dissipative as its energy is decreasing, the solution is global. □

4.4 Proof of the main result

This section is devoted to the proof of the main result in this chapter, namely, Theorem 4.1.1. The main ingredient of the proof is the utilization of the energy method.

Proof of Theorem 4.1.1. First, multiplying (4.9) by xu, integrating by parts several times, observing that $\partial_x(xu) = u + x\partial_x u$, and thanks to the boundary conditions of (4.2), we have that

$$\begin{aligned}\frac{5a_0}{2}\|\partial_x^2 u\|^2 = & -\partial_t\left(\frac{1}{2}\int_0^L xu^2dx\right) - \frac{3}{2}\|\partial_x u\|^2 + \frac{1}{3}\int_0^L u^3dx + \frac{a_1}{2}\|u\|^2 \\ & - (-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx.\end{aligned} \tag{4.36}$$

We are now in a position to estimate the terms of the right-hand side of (4.36).

Estimate 1: First, the Sobolev embedding yields that

$$\left|\int_0^L u^3dx\right| \leq \|u\|^2_{L^\infty(I)}\int_0^L |u|dx \leq M_S\|u\|^2_{H_1(I)}\int_0^L |u|dx.$$

Thus, the previous inequality together with Hölder's and Poincaré's inequalities give us

$$\begin{aligned}
\left|\int_0^L u^3dx\right| \leq & M_S\|u\|^2_{H_1(I)}\left(\int_0^L 1^2dx\right)^{\frac{1}{2}}\left(\int_0^L |u|^2dx\right)^{\frac{1}{2}}\\
\leq & M_S L^{\frac{1}{2}}\left(\|u\|^2+\|\partial_x u\|^2\right)\|u\|\\
\leq & M_S L^{\frac{1}{2}}\left(M_P+1\right)\|\partial_x u\|^2(2E(t))^{\frac{1}{2}}\\
\leq & M_S L^{\frac{1}{2}}\left(M_P+1\right)(2E(0))^{\frac{1}{2}}\|\partial_x u\|^2.
\end{aligned}$$

Estimate 2: We claim that for each $\epsilon>0$, there exists a constant $C_\epsilon>0$ such that

$$\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s)dsdx\right| \leq \epsilon\|\partial_x^2 u\|^2 + C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2ds.$$

We split these estimates into three parts, namely, $k=0, k=1$, and $k=2$.

Indeed, for the case $k=0$, using the Young's and Poincaré's inequalities we have

$$\begin{aligned}
\left|\int_0^L xu\int_0^\infty g(s)\eta^t dsdx\right| \leq & \int_0^\infty g(s)\int_0^L |xu|\left|\eta^t\right|dxds\\
\leq & L\int_0^\infty g(s)\int_0^L\left(\delta|u|^2+\frac{1}{4\delta}\left|\eta^t\right|^2\right)dxds\\
\leq & L\delta\underbrace{\int_0^\infty g(s)ds}_{=g_0}\|u\|^2 + L\frac{1}{4\delta}\int_0^\infty g(s)\|\eta^t\|^2ds\\
\leq & \underbrace{LM_P^2g_0\delta}_{\epsilon}\|\partial_x^2u\|^2 + \underbrace{L\frac{1}{4\delta}}_{C_\epsilon}\int_0^\infty g(s)\|\eta^t\|^2ds\\
\leq & \epsilon\|\partial_x^2 u\|^2 + C_\epsilon\int_0^\infty g(s)\|\eta^t\|^2ds,
\end{aligned}$$

where $\delta=\dfrac{\epsilon}{Lc_p^2g_0}>0$, showing the estimate 2.

Now, we turn to the case $k=1$. Applying once again Young's and Poincaré's inequalities gives the following

$$\begin{aligned}
\left|-\int_0^\infty g(s)\int_0^L u\partial_x\eta^t(x,s)dxds\right| \leq & \int_0^\infty\int_0^L\left|g(s)u\partial_x\eta^t(x,s)\right|dxds\\
\leq & \delta\int_0^\infty g(s)\|u\|^2ds+\frac{1}{4\delta}\int_0^\infty g(s)\|\partial_x\eta^t\|^2ds\\
\leq & \delta\int_0^\infty g(s)\|u\|^2ds+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}\\
\leq & \delta g_0M_P^2\|\partial_x^2u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}.
\end{aligned} \tag{4.37}$$

Similarly, we also have the following estimate

$$\begin{aligned}\left|-\int_0^\infty g(s)\int_0^L x\partial_x u\partial_x\eta^t(x,s)dxds\right| \leq &\delta\int_0^\infty g(s)\|x\partial_x u\|^2ds\\ &+\frac{1}{4\delta}\int_0^\infty g(s)\|\partial_x\eta^t\|^2ds\\ \leq&\delta L^2\int_0^\infty g(s)\|\partial_x u\|^2ds+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}\\ =&\delta g_0L^2\|\partial_x u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}\\ \leq&\delta g_0M_PL^2\|\partial_x^2 u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}.\end{aligned} \tag{4.38}$$

Now, observe that

$$\begin{aligned}\left|(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^2\eta^tdsdx\right| \leq &\left|-\int_0^\infty g(s)\int_0^L u\partial_x\eta^t(x,s)dxds\right|\\ &+\left|-\int_0^\infty g(s)\int_0^L x\partial_x u\partial_x\eta^t(x,s)dxds\right|.\end{aligned}$$

Thus, thanks to the inequalities (4.37) and (4.38) applied on the right-hand side of the previous inequality we have

$$\begin{aligned}\left|(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^2\eta^tdsdx\right| \leq&\delta g_0M_P^2\|\partial_x^2 u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}\\ &+\delta g_0M_PL^2\|\partial_x^2 u\|^2+\frac{1}{4\delta}\|\eta^t\|^2_{L_k}\\ =&\underbrace{\delta g_0(M_P+M_P^2)}_{\epsilon}\|\partial_x^2 u\|^2+\underbrace{\frac{1}{2\delta}}_{C_\epsilon}\|\eta^t\|^2_{L_k},\end{aligned}$$

where $\delta=\dfrac{\epsilon}{g_0(M_P+M_P^2)}>0$, showing the estimate 2.

Now, we turn to the case $k=2$. Thanks to Young's and Poincaré's inequalities, we have

$$\left|2\int_0^\infty g(s)\int_0^L \partial_x u\partial_x^2\eta^t(x,s)dxds\right| \leq\delta g_0M_P\|\partial_x^2 u\|^2+\frac{1}{\delta}\|\eta^t\|^2_{L_k}. \tag{4.39}$$

Using the same arguments as for the case $k=1$, we have

$$\left|-\int_0^\infty g(s)\int_0^L x\partial_x^2 u\partial_x^2\eta^t(x,s)dxds\right| \leq\delta g_0L^2\|\partial_x^2 u\|^2+\frac{1}{\delta}\|\eta^t\|^2_{L_k}. \tag{4.40}$$

Moreover, since

$$\begin{aligned}\left|-(-1)^k\int_0^L xu\int_0^\infty g(s)\partial_x^4\eta^tdsdx\right| \leq &\left|2\int_0^\infty g(s)\int_0^L \partial_x u\partial_x^2\eta^t(x,s)dxds\right|\\ &+\left|\int_0^\infty g(s)\int_0^L x\partial_x^2 u\partial_x^2\eta^t(x,s)dxds\right|,\end{aligned}$$

and thanks to (4.39) and (4.40), we reach

$$\begin{aligned}\left|-(-1)^k \int_0^L xu \int_0^\infty g(s)\partial_x^4\eta^t dsdx\right| \leq & \delta g_0 M_P \|\partial_x^2 u\|^2 + \delta g_0 L^2 \|\partial_x^2 u\|^2 + \frac{2}{\delta}\|\eta^t\|_{L_k}^2 \\ = & \underbrace{\delta g_0(M_P + L^2)}_{\epsilon} \|\partial_x^2 u\|^2 + \underbrace{\frac{2}{\delta}}_{C_\epsilon} \|\eta^t\|_{L_k}^2,\end{aligned}$$

where $\delta = \dfrac{\epsilon}{g_0(M_P + L^2)} > 0$, showing the estimate 2.

Estimate 3: There are two constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|\partial_x^2 u\|^2 \leq -C_1\partial_t\left(\int_0^L xu^2 dx\right) + C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds. \tag{4.41}$$

Indeed, estimates 1 and 2 together with the Poincaré inequality yield

$$\begin{aligned}\frac{5a_0}{2}\|\partial_x^2 u\|^2 \leq & -\partial_t\left(\frac{1}{2}\int_0^L xu^2 dx\right) + \frac{1}{3}\left|\int_0^L u^3 dx\right| \\ & + \frac{a_1}{2}\|u\|^2 + \left|-(-1)^k \int_0^L xu \int_0^\infty g(s)\partial_x^{2k}\eta^t(x,s) dsdx\right| \\ \leq & -\partial_t\left(\frac{1}{2}\int_0^L xu^2 dx\right) + \frac{1}{3}M_S L^{\frac{1}{2}}(M_P+1)(E(0))^{\frac{1}{2}}\|\partial_x u\|^2 \\ & + \frac{a_1}{2}\|u\|^2 + \epsilon\|\partial_x^2 u\|^2 + C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds \\ \leq & -\partial_t\left(\frac{1}{2}\int_0^L xu^2 dx\right) + \frac{1}{3}M_S L^{\frac{1}{2}} M_P(M_P+1)(E(0))^{\frac{1}{2}}\|\partial_x^2 u\|^2 \\ & + \frac{a_1 M_P^2}{2}\|\partial_x^2 u\|^2 + \epsilon\|\partial_x^2 u\|^2 + C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds\end{aligned}$$

which ensures

$$\begin{aligned}\underbrace{\left(5a_0 - \frac{2}{3}M_S L^{\frac{1}{2}} M_P(M_P+1)(E(0))^{\frac{1}{2}} - a_1 M_P^2 - 2\epsilon\right)}_{:=D}\|\partial_x^2 u\|^2 \leq \\ -\partial_t\left(\int_0^L xu^2 dx\right) + 2C_\epsilon\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds.\end{aligned}$$

Now on, taking $\epsilon > 0$ small enough, it follows from (4.12) that $D > 0$ and hence (4.41) holds true for $C_1 = \dfrac{1}{D} > 0$ and $C_2 = \dfrac{2C_\epsilon}{D} > 0$. Thus, the estimate 3 is achieved.

To conclude the proof, consider the following function

$$F(t) = \mu E(t) + C_1\xi(t)\left(\int_0^L xu^2 dx\right),$$

where $\mu = 2\left(C_2 + \frac{1}{M_P^2}\right)$. As $\xi' \leq 0$, we have

$$0 \leq \xi(t)\left(\int_0^L xu^2 dx\right) \leq \xi(0)\left(\int_0^L xu^2 dx\right) \leq \xi(0)L\|u\|^2 \leq 2L\xi(0)E(t).$$

Consequently, owing to the previous inequality, we get

$$\begin{aligned}\mu E(t) \leq & F(t) \leq \mu E(t) + C_1\xi(t)\left(\int_0^L xu^2 dx\right) \\ \leq & \mu E(t) + 2LC_1\xi(0)E(t) = \left(\mu + 2LC_1\xi(0)\right)E(t).\end{aligned} \tag{4.42}$$

Observe that $\xi' \leq 0$ ensures that

$$\begin{aligned}F'(t) = & \mu E'(t) + C_1\xi'(t)\left(\int_0^L xu^2 dx\right) + C_1\xi(t)\partial_t\left(\int_0^L xu^2 dx\right) \\ \leq & \mu E'(t) + C_1\xi(t)\partial_t\left(\int_0^L xu^2 dx\right).\end{aligned} \tag{4.43}$$

Now, putting (4.41) into (4.43) and using the Poincaré's inequality, we get

$$\begin{aligned}F'(t) \leq & \mu E'(t) + \xi(t)\left(C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \|\partial_x^2 u\|^2\right) \\ \leq & \mu E'(t) + \xi(t)\left(C_2\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \frac{1}{M_P^2}\|u\|^2\right) \\ = & \mu E'(t) + \xi(t)\left(C_2 + \frac{1}{M_P^2}\right)\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \frac{2}{M_P^2}\xi(t)E(t) \\ \leq & \mu E'(t) + \xi(t)\left(C_2 + \frac{1}{M_P^2}\right)\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds - \lambda_0\xi(t)F(t),\end{aligned} \tag{4.44}$$

where in the last inequality we have used (4.42). Here,

$$\lambda_0 = \frac{2}{M_P^2\left[\mu + 2LC_1\xi(0)\right]}.$$

Now on, we shall distinguish two cases.

Case 1: ξ is a constant function.

In this case, taking into account (4.10) and (4.14), we have

$$\xi\int_0^\infty g(s)\|\partial_x^k\eta^t\|^2 ds \leq -\int_0^\infty g'(s)\|\partial_x^k\eta^t\|^2 ds \leq -2E'(t)$$

that substituting in (4.44) gives us

$$F'(t) \leq \mu E'(t) - 2\left(C_2 + \frac{1}{M_P^2}\right)E'(t) - \lambda_0\xi F(t) = -\lambda_0\xi F(t)$$

implying $F(t) \leq e^{-ct}F(0)$, with $c = \lambda_0\xi$. Finally, (4.42) yields

$$E(t) \leq \frac{1}{\mu}F(t) = \frac{F(0)}{\mu}e^{-ct} \leq \frac{2\xi(0)E(0)}{\mu}e^{-ct},$$

which ensures item (i) of the Theorem 4.1.1.

Case 2: ξ is not a constant function.

First, observe that integrating (4.41) over $[0, t]$ and using the definition of $E(t)$, since E is decreasing, we get

$$\begin{aligned}\int_0^t \|\partial_x^2 u\|^2 ds \leq & - C_1 \int_0^t \partial_\tau \left(\int_0^L xu^2 dx\right) d\tau + C_2 \int_0^t \int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds d\tau \\ \leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t \int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds d\tau \\ \leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t 2E(\tau) d\tau \\ \leq & C_1 \left(\int_0^L xu_0^2 dx\right) + C_2 \int_0^t 2E(0) d\tau := C_3(1+t),\end{aligned} \tag{4.45}$$

where $C_3 = \max\left\{C_1 \left(\int_0^L xu_0^2 dx\right), 2C_2 E(0)\right\}$. Now, Young's and Hölder's inequalities together with (4.45) , ensures that

$$\begin{aligned}\left\|\int_{t-s}^t \partial_x^k u(\cdot,\tau) d\tau\right\|^2 \leq & 2\left\|\int_{t-s}^0 \partial_x^k u(\cdot,\tau) d\tau\right\|^2 + 2\left\|\int_0^t \partial_x^k u(\cdot,\tau) d\tau\right\|^2 \\ \leq & 2\left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau) d\tau\right\|^2 + 2t\int_0^L \int_0^t (\partial_x^k u)^2(\cdot,\tau) d\tau dx \\ \leq & 2\left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau) d\tau\right\|^2 + 2t\int_0^t \|\partial_x^k u\|^2 d\tau := c_1 h(t,s),\end{aligned}$$

for $0 \leq t \leq s$. Here $c_1 = 2\max\{1, M_P^{2-k} C_3\}$ and $h(t,s) = t^2 + t + \left\|\int_0^{t-s} \partial_x^k u_0(\cdot,\tau) d\tau\right\|^2$. On the other hand, thanks to (4.14)

$$\begin{aligned}\xi(t)\int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds = & \xi(t)\int_0^t g(s)\|\partial_x^k \eta^t\|^2 ds + \xi(t)\int_t^\infty g(s)\|\partial_x^k \eta^t\|^2 ds \\ \leq & -\int_0^t g'(s)\|\partial_x^k \eta^t\|^2 ds + \xi(t)\int_t^\infty g(s)\|\partial_x^k \eta^t\|^2 ds \\ \leq & -\int_0^t g'(s)\|\partial_x^k \eta^t\|^2 ds + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds \\ \leq & -\int_0^\infty g'(s)\|\partial_x^k \eta^t\|^2 ds + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds \\ \leq & -2E'(t) + c_1\xi(t)\int_t^\infty g(s)h(t,s) ds.\end{aligned}$$

Recall that

$$F(t) = \mu E(t) + C_1 \xi(t)\left(\int_0^L xu^2 dx\right).$$

Analogously to the previous case, we have

$$\begin{aligned}F'(t) \leq & -\lambda_0 \xi(t) F(t) + \mu E'(t) + \xi(t)\frac{\mu}{2}\int_0^\infty g(s)\|\partial_x^k \eta^t\|^2 ds \\ \leq & -\lambda_0 \xi(t) F(t) + \mu E'(t) - \mu E'(t) + c_1\frac{\mu}{2}\xi(t)\int_t^\infty g(s)h(t,s) ds.\end{aligned} \tag{4.46}$$

Setting $c := \lambda_0$, (4.46) ensures that

$$F'(t)+c\xi(t)F(t) \leq \mu E'(t)-\mu E'(t)+\frac{c_1\mu}{2}\xi(t)\int_t^\infty g(s)h(t,s)ds = \frac{c_1\mu}{2}\xi(t)\int_t^\infty g(s)h(t,s)ds,$$

or equivalently,

$$\left(e^{c\int_0^t \xi(\tau)d\tau}F(t)\right)' \leq \frac{c_1\mu}{2}e^{c\int_0^t \xi(\tau)d\tau}\xi(t)\int_t^\infty g(s)h(t,s)ds.$$

Finally, the previous inequality and $E(t) \leq \dfrac{1}{\mu}F(t)$, gives us

$$E(t) \leq \tilde{c}e^{-c\int_0^t \xi(\tau)d\tau}\left(1+\int_0^t e^{c\int_0^\sigma \xi(\tau)d\tau}\xi(\sigma)\int_\sigma^\infty g(s)h(\sigma,s)dsd\sigma\right),$$

where $\tilde{c} = \dfrac{\max\left\{F(0), \dfrac{c_1\mu}{2}\right\}}{\mu}$. Thereby, the proof of the second part (ii) of Theorem 4.1.1 is complete. $\square$

4.5 Conclusion

In this chapter, we considered the well-known Kawahara equation under the presence of only an internal infinite memory term. Then, it is shown that the energy of the system decays under some assumptions of the memory kernel. Moreover, an estimate of the energy decay is provided depending on the property of the kernel. The main ingredient of the proof is the utilization of the Fixed Point Theorem and the energy method. Based on this outcome, one can conclude that the distributed memory term creates enough dissipation for the energy of the system so that the exponential stability holds. On the other hand, we believe that our results remain valid if the memory term occurs in a boundary condition. Of course, this could be the subject of future work to ascertain the claim.

5 Asymptotic behavior of Kawahara equation with memory effect

5.1 Introduction

5.1.1 Background and literature review

Water wave models have been studied by many scientists from numerous disciplines such as hydraulic engineering, fluid mechanics engineering, physics as well as mathematics. These models are in general hard to derive, and complex to obtain qualitative information on the dynamics of the waves. This makes their studies interesting and challenging. Recently, appropriate assumptions on the amplitude, wavelength, wave steepness, and so on, are invoked to investigate the asymptotic models for water waves and understand the full water wave system (see, for instance, [4, 14, 65] and references therein for a rigorous justification of various asymptotic models for surface and internal waves).

It has been noticed that the water waves can be considered as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form. This means that there are two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength, and the amplitude of the free surface are respectively denoted by h, λ and a. On the other hand, the parameter μ, known as the Bond number, measures the importance of gravitational forces compared to surface tension forces. We also note that the long waves (also called shallow water waves) are mathematically characterized by the condition $\delta \ll 1$. There are several long-wave approximations depending on the relation between ε and δ.

The above discussion led to, instead of studying models that do not give good asymptotic properties, we can rescale the parameters mentioned above to find systems that reveal asymptotic models for surface and internal waves, like the Kawahara model. Precisely, letting $\varepsilon = \delta^4 \ll 1$, $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and the critical Bond number $\mu = \frac{1}{3}$, the so-called equation Kawahara equation is put forward. Such an equation was derived by Hasimoto and Kawahara [51, 62] and takes the form

$$\pm 2W_t + 3WW_x - \nu W_{xxx} + \frac{1}{45}W_{xxxxx} = 0,$$

or, after re-scaling,

$$W_t + \alpha W_x + \beta W_{xxx} - W_{xxxxx} + WW_x = 0. \tag{5.1}$$

The latter is also seen as the fifth-order KdV equation [16], or singularly perturbed KdV equation [81]. It describes a dispersive partial differential equation with numerous wave physical phenomena such as magneto-acoustic waves in a cold plasma [58], the propagation of long waves in a shallow liquid beneath an ice sheet [55], gravity waves on the surface of a heavy liquid [40], etc.

Note that valuable efforts in the last decays were made to understand this model in various research frameworks. For example, numerous works focused on the analytical and numerical methods for solving (5.1). These methods include the tanh-function method [9], extended tanh-function method [10], sine-cosine method [98], Jacobi elliptic functions method [54], direct algebraic method and numerical simulations [80], decompositions methods [63], as well as the variational iterations and homotopy perturbations methods [57]. Another direction is the study of the Kawahara equation from the point of view of control theory and specifically, the controllability and stabilization problem [6], which is our motivation.

Whereupon, we are interested in a detailed qualitative analysis for the system (5.1) in a bounded region. More precisely, our primary concern is to analyze the qualitative properties of solutions to the initial-boundary value problem associated with the model (5.1) posed on a bounded interval under the presence of damping and memory-type controls.

Now, we will present some previous results that dealt with the asymptotic behavior of solutions for the Kawahara model (5.1) in a bounded domain. One of the first outcomes is due to Silva and Vasconcellos [92, 93], where the authors studied the stabilization of global solutions of the linear Kawahara equation, under the effect of a localized damping mechanism. The second endeavor is completed by Capistrano-Filho *et al.* [6], where the generalized Kawahara equation in a bounded domain is considered, that is, a more general nonlinearity $W^p\partial_x W$ with $p \in [1, 4)$ is taken into account. It is proved that the solutions of the Kawahara system decay exponentially when an internal damping mechanism is applied.

Recently, a new tool for the control properties of the Kawahara operator was proposed in [27]. In this work, the authors treated the so-called overdetermination control problem for the Kawahara equation. Precisely, a boundary control is designed so that the solution to the problem under consideration satisfies an integral condition. Furthermore, when the control acts internally in the system, instead of the boundary, the authors proved that this integral condition is also satisfied.

We conclude the literature review with three recent works. In [24,36] the stabilization of the Kawahara equation with a localized time-delayed interior control is considered. Under suitable assumptions on the time delay coefficients, the authors proved that the solutions of the Kawahara system are exponentially stable. This result is established using either the Lyapunov method or a compactness-uniqueness argument. More recently, the Kawahara equation in a bounded interval and with a delay term in one of the boundary conditions was studied in [19]. The authors used two different approaches to prove that the solutions of (5.1) are exponentially stable under a condition on the length of the spatial domain. We point out that this is a small sample of papers that were concerned with the stabilization problem of the Kawahara equation in a bounded interval. Of course, we suggest that the reader, who is interested in more details on the topic, consult the papers cited above and the references therein.

Let us now present the framework of this chapter.

5.1.2 Problem setting and main results

Consider the system (5.1) in a bounded domain $\Omega = (0,\ell)$, where $\ell > 0$ is the spatial length, under the action of the following feedback:

$$\begin{cases} \partial_t\omega(t,x)+\alpha\partial_x\omega(t,x)+\beta\partial_x^3\omega(t,x)-\partial_x^5\omega(t,x) & \\ \qquad\qquad +\,\omega^p(t,x)\partial_x\omega(t,x)=0, & x\in\Omega,\ t>0,\\ \omega(t,0)=\omega(t,\ell)=0, & t>0,\\ \partial_x\omega(t,0)=\partial_x\omega(t,\ell)=0, & t>0,\\ \partial_x^2\omega(t,\ell)=\mathcal{F}(t), & t>0,\\ \partial_x^2\omega(t,0)=z_0(t), & t\in\mathcal{I},\\ \omega(0,x)=\omega_0(x), & x\in\Omega, \end{cases} \tag{5.2}$$

with ω_0, z_0 are initial data and the feedback law is a linear combination of the damping and finite memory terms given by

$$\mathcal{F}(t) := \nu_1\partial_x^2\omega(t,0)+\nu_2\int_{t-\tau_2}^{t-\tau_1}\sigma(t-s)\partial_x^2\omega(s,0)\,ds. \tag{5.3}$$

Here, $\alpha > 0$ and $\beta > 0$ are physical parameters, $p \in \{1,2\}$, whereas ν_1 and ν_2 are nonzero real numbers. In turn, $0 < \tau_1 < \tau_2$ correspond to the finite memory interval $(t-\tau_1, t-\tau_2)$. Moreover, $\mathcal{I} = (-\tau_2, 0)$, and the memory kernel is denoted by $\sigma(s)$. Of course, the presence of a memory term is usually ubiquitous in practice. Particularly, memory is of great importance in systems control as they are governed by equations, where the past values of one or more variables involved in the system play a crucial role. On the other hand, the impact of a memory term in some systems can be deleterious as it can affect their performance [34, 35, 75]. Last but not least, we indicate that the memory term, that arises in the boundary control (5.3), could reflect the case of a compressible (or incompressible) viscoelastic fluid. The latter is regarded as the simplest material with memory [5, 42] .

On the other hand, let us note that the energy associated with the full system (5.2) is given by

$$\mathcal{E}(t)=\int_\Omega \omega^2(t,x)dx+|\nu_2|\int_{\mathcal{M}} s\sigma(s)\left(\int_{\Omega_0}(\partial_x^2\omega)^2(t-s\phi,0)\,d\phi\right)ds,\ t\geq 0. \tag{5.4}$$

Naturally, as we are interested in the behavior of the system (5.2), we need to check whether the feedback law, given by (5.3), represents a damping mechanism. In other words, we would like to see if, in the presence of the boundary memory-type feedback law, the energy of the system (5.4) tends to zero state with some specific decay rate, when t goes to 0. This situation can be presented in the following question:

Question: *Does $\mathcal{E}(t) \longrightarrow 0$, as $t \to \infty$? If it is the case, is it possible to come up with a decay rate?*

To answer the previous question for the system (5.2), we will assume, from now on, that the memory kernel σ obeys the following conditions:

Assumptions 2. *The function $\sigma \in \ell^\infty(\mathcal{M})$, where $\mathcal{M} := (\tau_1, \tau_2)$. In turn, we assume that*

$$\sigma(s) > 0, \quad a.e. \ in \ \mathcal{M}.$$

Moreover, the feedback gains ν_1 and ν_2 together with the memory kernel satisfy

$$|\nu_1| + |\nu_2| \int_{\mathcal{M}} \sigma(s)\, ds < 1. \tag{5.5}$$

Some notations, that we will use throughout this chapter, are presented below:

(i) We denote by $(\cdot,\cdot)_{\mathbb{R}^2}$ the canonical inner product of $\mathbb{R}^2$, whereas $\langle \cdot, \cdot \rangle$ denotes the canonical inner product of $\ell^2(\Omega)$ whose induced norm is $\|\cdot\|$.

(ii) For $T > 0$, consider the space of solutions

$$Y_T = C(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^2(\Omega))$$

equipped with the norm

$$\|v\|^2_{Y_T} = \left(\max_{t\in(0,T)} \|v(t,\cdot)\| \right)^2 + \int_0^T \|v(t,\cdot)\|^2_{H_0^2(\Omega)} dt.$$

(iii) Let $\Omega_0 = (0,1)$ and $\mathcal{Q} := \Omega_0 \times \mathcal{M}$. Then, we consider the spaces

$$H := L^2(\Omega) \times L^2(\mathcal{Q}), \quad \mathcal{H} := L^2(\Omega) \times L^2(\mathcal{I} \times \mathcal{M}),$$

respectively equipped with the following inner product:

$$\begin{cases} \langle (\omega, z), (v, y) \rangle_H = \langle \omega, v \rangle + |\nu_2| \displaystyle\int_{\mathcal{M}} \int_{\Omega_0} s\sigma(s) z(\phi, s) y(\phi, s)\ d\phi ds, \\ \langle (\omega, z), (v, y) \rangle_{\mathcal{H}} = \langle \omega, v \rangle + |\nu_2| \displaystyle\int_{\mathcal{I}} \int_{\mathcal{M}} \sigma(s) z(r, s) y(r, s)\, ds dr. \end{cases}$$

Subsequently, we can state our first main result:

Theorem 5.1.1. *Under the assumptions 2 and assuming that the length ℓ fulfills the smallness condition*

$$0 < \ell < \pi\sqrt{\frac{3\beta}{\alpha}}, \tag{5.6}$$

there exists $r > 0$ sufficiently small, such that for every $(\omega_0, z_0) \in H$ with $\|(\omega_0, z_0)\|_H < r$, the energy of the system (5.2), given by (5.4), is exponentially stable. In other words, there exist two positive constants κ and μ such that

$$\mathcal{E}(t) \leq \kappa\mathcal{E}(0)e^{-2\mu t}, \ t > 0, \tag{5.7}$$

where $\mathcal{E}(t)$ is defined by (5.4).

The proof of this result uses an appropriate Lyapunov function, which requires the condition (5.6). In turn, such a requirement can be relaxed by using a compactness-uniqueness argument [83] (see [6, 19, 25, 92, 93]). The proof is based on the following outcome [19]:

Lemma 5.1.1. *Let $\ell > 0$ and consider the assertion: There exist $\zeta \in \mathbb{C}$ and $\omega \in H_0^2(\Omega) \cap H^5(\Omega)$ such that*

$$\begin{cases} \zeta\omega(x) + \omega'(x) + \omega'''(x) - \omega'''''(x) = 0, & x \in \Omega, \\ \omega(x) = \omega'(x) = \omega''(x) = 0, & x \in \{0, \ell\}. \end{cases}$$

If $(\zeta, \omega) \in \mathbb{C} \times H_0^2(\Omega) \cap H^5(\Omega)$ is solution of (5.1.1), then $\omega = 0$.

We have:

Theorem 5.1.2. *Suppose that assumptions 2 hold. Moreover, we choose $\ell > 0$ so that the problem in Lemma 5.1.1 has only the trivial solution. Then, there exists $\varrho > 0$ such that for every $(\omega_0, z_0) \in H$ satisfying $\|(\omega_0, z_0)\|_H \leq \varrho$, the energy (5.4) of the problem (5.2) decays exponentially.*

5.1.3 Further comments and chapter's outline

As mentioned above, the exponential stability result of the system (5.2) will be established using two different methods. The first one evokes a Lyapunov function and requires an explicit smallness condition on the length of the spatial domain ℓ. The second one is obtained via a classical compactness-uniqueness argument, where critical lengths phenomena appear with a relation with the Möbius transforms (see for instance [19]). This permits us to answer the question raised in the introduction.

Remarks 5.1.1. *Let us point out some important comments:*

- *Considering $\nu_2 = 0$ and $\alpha = 0$, the authors in [25] showed the stabilization property for (5.2) using the compactness-uniqueness argument. Since they removed the drift term $\alpha\partial_x\omega$, the critical lengths phenomena did not appear.*

- *The main concern of this work is to deal with the feedback law of memory type as in (5.3). One needs to control this term to ensure well-posedness and stabilization results.*

- *Our results are valid for the general nonlinearities $u^p\partial_x u$, $p \in \{1, 2\}$, and also can be extended for linearity like $c_1 u\partial_x u + c_2 u^2\partial_x u$. To draw more attention to the first general nonlinearity, the decay rate in (5.7) depends on the values of p since we have*

$$\mu < \min\left\{\frac{\mu_2|\nu_2|e^{-\delta\tau_2}\delta}{2(1+\mu_1|\nu_2|)}, \frac{\mu_1}{2\ell^2(1+\ell\mu_1)(p+2)}\left[(p+2)(3\pi^2\beta - \alpha\ell^2) - 2\pi^2\ell^{2-\frac{p}{2}}r^p\right]\right\}.$$

We finish our introduction with the chapter's outline: The work consists of three parts including the Introduction. Section 5.2 discusses the existence of solutions for the full system (5.2). Section 5.3 is devoted to proving the stabilization results, that is, Theorem 5.1.1 and Theorem 5.1.2.

5.2 Well-posedness theory

In this section, we are interested in analyzing the well-posedness property of the system (5.2). The first and the second subsections are devoted to proving the existence of solutions for the linearized (homogenous and non-homogeneous) system associated with (5.2), respectively. The third subsection concerns the well-posedness of the full system (5.2).

5.2.1 Linear problem

As in the literature (see for instance the references [97] and [74]), the homogenous linear system associated with (5.2) can be viewed as follows:

$$\begin{cases} \partial_t\omega(t,x) + \alpha\partial_x\omega(t,x) + \beta\partial_x^3\omega(t,x) - \partial_x^5\omega(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ s\partial_t z(t,\phi,s) + \partial_\phi z(t,\phi,s) = 0, & (t,\phi,s) \in \mathbb{R}^+ \times \Omega_0 \times \mathcal{M}, \\ \omega(t,0) = \omega(t,\ell) = \partial_x\omega(t,0) = \partial_x\omega(t,\ell) = 0, & t > 0, \\ \partial_x^2\omega(t,\ell) = \nu_1\partial_x^2\omega(t,0) + \nu_2\displaystyle\int_{\mathcal{M}} \sigma(s)z(t,1,s)\,ds, & t > 0, \\ \omega(0,x) = \omega_0(x), & x \in \Omega, \\ z(0,\phi,r) = z_0(-\phi r), & (\phi,r) \in \Omega_0 \times (0,\tau_2), \end{cases} \tag{5.8}$$

where $z(t,\phi,s)=\partial_x^2\omega(t-\phi s,0)$ satisfies a transport equation (see $(5.8)_2$). Defining $\Lambda(t)=\begin{bmatrix}\omega(t,\cdot)\\ z(t,\cdot,\cdot)\end{bmatrix}$, $\Lambda_0=\begin{bmatrix}\omega_0\\ z_0(-\phi\cdot)\end{bmatrix}$, one can rewrite this system abstractly:

$$\begin{cases}\Lambda_t(t)=A\Lambda(t), & t>0,\\ \Lambda(0)=\Lambda_0\in H,\end{cases}$$

where

$$A=\begin{bmatrix}-\alpha\partial_x-\beta\partial_x^3+\partial_x^5 & 0\\ 0 & -\dfrac{1}{s}\partial_\phi\end{bmatrix},$$

whose domain is given by

$$D(A):=\left\{(\omega,z)\in H \;\middle|\; \begin{array}{l}\omega\in H^5(\Omega)\cap H_0^2(\Omega),\ z\in L^2\Big(\mathcal{M};H^1(\Omega_0)\Big),\\ \partial_x^2\omega(0)=z(0,\cdot),\ \partial_x^2\omega(\ell)=\nu_1\partial_x^2\omega(0)+\nu_2\displaystyle\int_{\mathcal{M}}\sigma(s)z(1,s)\,ds\end{array}\right\}.$$

The following result ensures the well-posedness of the linear homogeneous system.

Proposition 5.2.1. *Under the assumption (2), we have:*

i. The operator A is densely defined in H and generates a C_0-semigroup of contractions e^{tA}. Thereby, for each $\Lambda_0\in H$, there exists a unique mild solution $\Lambda\in C([0,+\infty),H)$ for the linear system associated with (5.2). Moreover, if $\Lambda_0\in D(A)$, then we have a unique classical solution with the regularity

$$\Lambda\in C([0,+\infty),D(A))\cap C^1([0,+\infty),H).$$

ii. Given $\Lambda_0=(\omega_0,z_0(\cdot))\in H$, the following estimates hold:

$$\|\partial_x^2\omega(0,\cdot)\|^2_{L^2(0,T)}+\int_0^T\int_{\mathcal{M}}s\sigma(s)z^2(t,1,s)\,dsdt\le C\|(\omega_0,z_0(\cdot))\|^2_H,\tag{5.9}$$

$$\|\partial_x^2\omega(\cdot)\|^2_{L^2(0,T;L^2(\Omega))}\le C\|(\omega_0,z_0(\cdot))\|^2_H,\tag{5.10}$$

$$\|z_0(\cdot)\|^2_{L^2(\mathcal{Q})}\le\|z(T,\cdot,\cdot)\|^2_{L^2(\mathcal{Q})}+\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(t,1,s)\,dsdt,\tag{5.11}$$

and

$$T\|\omega_0(\cdot)\|^2\le\|\omega\|^2_{L^2(0,T;L^2(\Omega))}+T\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}.\tag{5.12}$$

iii. The map

$$\mathcal{G}:\Lambda_0=(\omega_0,z_0(\cdot))\in H\mapsto\Lambda(\cdot)=e^{\cdot A}\Lambda_0\in Y_T\times C\left([0,T];\,L^2(\mathcal{Q})\right)$$

is continuous.

Proof. **Proof of item i.** This part can be proved by using the semigroup theory. In fact, note first that for given $\Lambda = (\omega, z) \in D(A)$, it follows from the Cauchy-Schwarz inequality that

$$\int_{\mathcal{M}} \sigma(s) z(1,s) ds \leq \left(\int_{\mathcal{M}} \sigma(s) ds \right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \sigma(s) (z(1,s))^2 ds \right)^{\frac{1}{2}}. \tag{5.13}$$

Thus, using integration by parts and (5.13) yields that

$$\begin{aligned}
\langle A\Lambda, \Lambda \rangle =& \frac{1}{2} \left[\left(\nu_1 \partial_x^2 \omega(0) + \nu_2 \int_{\mathcal{M}} \sigma(s) z(1,s)\ ds \right)^2 - \left(\partial_x^2 \omega(0) \right)^2 \right. \\
& \left. - |\nu_2| \int_{\mathcal{M}} \sigma(s) \left(z(1,s) \right)^2\ ds + |\nu_2| \left(\partial_x^2 \omega(0) \right)^2 \int_{\mathcal{M}} \sigma(s)\ ds \right] \\
\leq & \frac{1}{2} \left[\left(\partial_x^2 \omega(0) \right)^2 \left(\nu_1^2 - 1 + |\nu_2| \int_{\mathcal{M}} \sigma(s)\ ds \right) \right. \\
& + 2\nu_1 \nu_2 \left(\partial_x^2 \omega(0) \right) \left(\int_{\mathcal{M}} \sigma(s) z(1,s)\ ds \right) \\
& \left. + \left(\nu_2^2 - \frac{|\nu_2|}{\| \sqrt{\sigma(s)} \|^2} \right) \left(\int_{\mathcal{M}} \sigma(s) z(1,s)\ ds \right)^2 \right] = \frac{1}{2} \langle GX, X \rangle_{\mathbb{R}^2},
\end{aligned} \tag{5.14}$$

where

$$X = \begin{pmatrix} \partial_x^2 \omega(0) \\ \displaystyle\int_{\mathcal{M}} \sigma(s) z(1,s)\ ds \end{pmatrix}$$

and

$$G = \begin{pmatrix} \nu_1^2 - 1 + |\nu_2| \displaystyle\int_{\mathcal{M}} \sigma(s)\ ds & \nu_1 \nu_2 \\ \nu_1 \nu_2 & \nu_2^2 - \dfrac{|\nu_2|}{\| \sqrt{\sigma(s)} \|^2} \end{pmatrix}.$$

Due to (5.5), we have

$$\det G = |\nu_2| \left(\int_{\mathcal{M}} \sigma(s)\ ds \right)^{-1} \left\{ \left[1 - |\nu_2| \left(\int_{\mathcal{M}} \sigma(s)\ ds \right) \right]^2 - \nu_1^2 \right\} > 0$$

and

$$\mathrm{tr} G \leq |\nu_1| (|\nu_1| - 1) - |\nu_1| |\nu_2| \left(\int_{\mathcal{M}} \sigma(s)\ ds \right)^{-1} < 0,$$

since $|\nu_1| < 1$. Moreover, is not difficult to see that G is a negative definite matrix. Putting these previous information together in (5.14) we have that A is dissipative. Analogously, considering the adjoint operator of A as follows

$$A^*(v, y) = \left(\alpha \partial_x v + \beta \partial_x^3 v - \partial_x^5 v, \frac{1}{s} \partial_\phi y \right)$$

with domain

$$D(A^*) := \left\{ (v, y) \in H \ \middle| \begin{array}{l} \omega \in H^5(\Omega) \cap H_0^2(\Omega),\ y \in L^2\Big(\mathcal{M}; H^1(\Omega_0)\Big), \\ \partial_x^2 v(\ell) = \dfrac{|\nu_2|}{\nu_2} y(1,s),\ \partial_x^2 v(0) = \nu_1 \partial_x^2 v(\ell) + |\nu_2| \displaystyle\int_{\mathcal{M}} \sigma(s) y(0,s)\ ds \end{array} \right\},$$

we have that for $(v, y) \in D(A^*)$,

$$\begin{aligned} &\langle A^*(v,y),(v,y)\rangle + \left[|\nu_2|^2 - |\nu_2|\|\sqrt{\sigma}\|^2_{L^2(\mathcal{M})}\right]\left(\int_{\mathcal{M}}\sigma(s)y(0,s)ds\right)^2 \\ &=\frac{1}{2}\langle G_* Z, Z\rangle, \end{aligned} \tag{5.15}$$

where

$$Z = \begin{pmatrix} \partial_x^2 v(\ell) \\ \int_{\mathcal{M}}\sigma(s)y(0,s)ds \end{pmatrix}$$

and

$$G_* = \begin{pmatrix} \nu_1^2 - 1 + |\nu_2|\int_{\mathcal{M}}\sigma(s)\ ds & \nu_1|\nu_2| \\ \nu_1|\nu_2| & \nu_2^2 - \dfrac{|\nu_2|}{\|\sqrt{\sigma(s)}\|^2} \end{pmatrix}.$$

Again, thanks to the relation (5.5), we have $\det G_* = \det G > 0$ and $\mathrm{tr} G_* = \mathrm{tr} G < 0$, since $|\nu_1| < 1$. Thus, using the fact that G_* is negative definite in (5.15), we have that A^* is also dissipative, showing the item i.

Proof of item ii. First, remember that e^{tA} is a semigroup of contractions and therefore, for each $\Lambda_0 = (\omega_0, z_0) \in H$, the following estimate is valid

$$\|(\omega(t), z(t,\cdot,\cdot))\|_H^2 = \|\omega(t)\|^2 + \|z(t,\cdot,\cdot)\|^2_{L^2(\mathcal{Q})} \leq \|\omega_0\|^2 + \|z_0(-\cdot)\|^2_{L^2(\mathcal{Q})}, \forall t \in [0,T]. \tag{5.16}$$

Moreover, the following inequality holds

$$\begin{aligned} \int_0^T\int_{\mathcal{M}} s\sigma(s)\left[z(t,1,s)\right]^2\ dsdt \ \leq\ & \frac{\tau_2}{|\nu_2|}\int_{\Omega_0}\int_{\mathcal{M}}|\nu_2|s\sigma(s)\left[z_0^2(-\phi s)\right]dsd\phi \\ & + \frac{\tau_2}{\tau_1|\nu_2|}\int_0^T\int_{\Omega_0}\int_{\mathcal{M}}|\nu_2|s\sigma(s)z^2\ dsd\phi dt. \end{aligned} \tag{5.17}$$

Indeed, multiplying the second equation of (5.8) by $\phi\sigma(s)z$, rearranging the terms, integrating by parts and taking into account that $s \in \mathcal{M} = (\tau_1, \tau_2)$, we have

$$\begin{aligned} \int_0^T\int_{\mathcal{M}} s\sigma(s)\left(z(t,1,s)\right)^2\ dsdt \leq & \frac{\tau_2}{|\nu_2|}\int_0^T\int_{\Omega_0}\int_{\mathcal{M}}|\nu_2|\sigma(s)\left(z(t,\phi,s)\right)^2\ dsd\phi dt \\ & + \frac{\tau_2}{|\nu_2|}\int_{\Omega_0}\int_{\mathcal{M}}\phi|\nu_2|\sigma(s)s\left(z(0,\phi,s)\right)^2\ dsd\phi \\ & - \frac{\tau_2}{|\nu_2|}\int_{\Omega_0}\int_{\mathcal{M}}|\nu_2|\phi\sigma(s)s\left(z(T,\phi,s)\right)^2\ dsd\phi \\ \leq & \frac{\tau_2}{\tau_1|\nu_2|}\int_0^T\int_{\Omega_0}\int_{\mathcal{M}}s|\nu_2|\sigma(s)\left(z(t,\phi,s)\right)^2\ dsd\phi dt \\ & + \frac{\tau_2}{|\nu_2|}\int_{\Omega_0}\int_{\mathcal{M}}\phi|\nu_2|\sigma(s)s\left(z_0(-\phi s)\right)^2\ dsd\phi \end{aligned}$$

This proves the estimate (5.17). As a consequence of (5.16), (5.17) and the hypothesis of $\tau_1 \leq s \leq \tau_2$ and $\phi \leq 1$, we also have

$$\int_0^T\int_{\mathcal{M}} s\sigma(s)\left(z(t,1,s)\right)^2\ dsdt \leq \frac{\tau_2}{|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\left(\|\omega_0\|^2 + \|z_0(-\phi s)\|^2_{L^2(\mathcal{Q})}\right). \tag{5.18}$$

Now, we are in a position to prove (5.9). Multiplying the first equation of (5.8) by ω, integrating over $[0,T]\times[0,\ell]$, and using the boundary conditions, it follows that

$$\begin{aligned}\|\partial_x^2\omega(0)\|^2_{L^2(0,T)} =&\|\omega_0\|^2+\int_0^T\left(\partial_x^2\omega(\ell)\right)^2\ dt-\|\omega(T)\|^2\\ \leq&\|\omega_0\|^2+\int_0^T\left(\nu_1\partial_x^2\omega(0)+\nu_2\int_{\mathcal{M}}\sigma(s)z(\cdot,1,s)ds\right)^2\ dt:=I_1+I_2.\end{aligned} \tag{5.19}$$

To estimate the integral I_2 on the right-hand side of (5.19), we use Young's inequality together with the Cauchy-Schwartz inequality, to obtain

$$\begin{aligned}I_2\leq&\nu_1^2\left(\partial_x^2\omega(t,0)\right)^2\\ &+2|\nu_1||\nu_2|\left(\partial_x^2\omega(t,0)\right)\left(\int_{\mathcal{M}}\sigma(s)ds\right)^{\frac{1}{2}}\left(\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)ds\right)^{\frac{1}{2}}\\ &+\nu_2^2\left(\left(\int_{\mathcal{M}}\sigma(s)ds\right)^{\frac{1}{2}}\left(\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)ds\right)^{\frac{1}{2}}\right)^2\\ \leq&\left[\nu_1^2+\frac{\nu_2^2}{2\theta}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\left(\partial_x^2\omega(t,0)\right)^2\\ &+\left[2\theta\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\left(\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)ds\right).\end{aligned} \tag{5.20}$$

Thereafter, inserting (5.20) into (5.19), we find

$$\begin{aligned}&\left[1-\nu_1^2-\frac{\nu_2^2}{2\theta}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}\leq\|\omega_0\|^2\\ &+\left[2\theta\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\left(\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)dsdt\right).\end{aligned} \tag{5.21}$$

Thanks to (5.5), one can choose $\theta>0$ large enough so that

$$1-\nu_1^2-\frac{\nu_2^2}{2\theta}\left(\int_{\mathcal{M}}\sigma(s)ds\right)>0. \tag{5.22}$$

This, together with (5.21) and (5.18), yields

$$\begin{aligned}\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}\leq\leq&C\left(\|\omega_0\|^2+\frac{1}{\tau_1}\int_0^T\int_{\mathcal{M}}s\sigma(s)z^2(\cdot,1,s)dsdt\right)\\ \leq&C\left(1+\frac{\tau_2}{\tau_1|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\right)\|\omega_0\|^2+\frac{C\tau_2}{\tau_1|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\|z_0(-\phi s)\|^2_{L^2(\mathcal{Q})}\\ \leq&C\left(\|\omega_0\|^2+\|z_0(-\phi s)\|^2_{L^2(\mathcal{Q})}\right).\end{aligned} \tag{5.23}$$

Clearly, combining (5.18) and (5.23), we get (5.9).

Now, let us prove (5.10). Multiplying the equation (5.8) by xu, integrating by parts over $(0,T)\times\Omega$, and isolating the term $\|\partial_x^2\omega\|^2_{L^2(0,T;L^2(\Omega))}$, we obtain

$$\begin{aligned}\|\partial_x^2\omega\|^2_{L^2(0,T;L^2(\Omega))} \leq & \int_\Omega \frac{x}{5}\omega_0^2(x)dx + \frac{\alpha}{5}\|\omega\|^2_{L^2(0,T;L^2(\Omega))}\\ &+\frac{\ell}{5}\left[\nu_1^2+\frac{\nu_2^2}{2\epsilon}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\int_0^T(\partial_x^2\omega(t,0))^2\\ &+\frac{\ell}{5}\left[2\epsilon\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(t,1,s)dsdt\\ \leq&\frac{\ell}{5}\|\omega_0\|^2+\frac{\alpha}{5}\|\omega\|^2_{L^2(0,T;L^2(\Omega))}\\ &+C_1\left[\int_0^T(\partial_x^2\omega(t,0))^2+\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(t,1,s)dsdt\right],\end{aligned}$$

where (5.20) is used and

$$C_1=\max\left\{\frac{\ell}{5}\left[\nu_1^2+\frac{\nu_2^2}{2\epsilon}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right],\frac{\ell}{5}\left[2\epsilon\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\right\}.$$

Now, taking into account the fact that e^{At} is a semigroup of contractions and using (5.9), we obtain (5.10) with the constant $C=\max\left\{\frac{\ell}{5},\frac{\alpha}{5},C_1\right\}$.

Finally, let us show (5.11) and (5.12), respectively. For (5.11), multiply the second equation in (5.8) by $\sigma(s)z$ and integrates by parts over $(0,T)\times\mathcal{Q}$, to obtain

$$\int_{\Omega_0}\int_{\mathcal{M}}s\sigma(s)z^2(0,\phi,s)\ dsd\phi\leq\int_{\Omega_0}\int_{\mathcal{M}}s\sigma(s)z^2(T,\phi,s)\ dsd\phi+\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(t,1,s)\ dsdt,$$

showing (5.11). To prove (5.12), we multiply the first equation in (5.8) by $2(T-t)\omega$ and integrating over $[0,T]\times[0,\ell]$, to find

$$T\|\omega_0\|^2\leq T\|\omega\|^2_{L^2(0,T;L^2(\Omega))}+T\int_0^T\left(\partial_x^2\omega(0)\right)^2dt,$$

giving (5.12). Last but not least, it is worth mentioning that the above estimates remain true for solutions stemming from $\Lambda_0\in H$, giving item ii.

Proof of item iii. Follows directly from (5.10) and from (5.16). □

5.2.2 Non-homogeneous problem

Let us now consider the linear system (5.8) with a source term $\varphi\in L^1(0,T;L^2(\Omega))$ in the right-hand side of the first equation. As done in the previous subsection, the system can be rewritten as follows:

$$\begin{cases}\Lambda_t(t)=A\Lambda(t)+(\varphi(t,\cdot),0), & t>0,\\ \Lambda(0)=\Lambda_0\in H,\end{cases}\tag{5.24}$$

where $\Lambda=(\omega,z)$ and $\Lambda_0=(\omega_0,z_0(-\cdot))$. With this in hand, the following result will be proved.

Theorem 5.2.1. *Under the assumption (2), it follows that:*

(*a*) *If $\Lambda_0 = (\omega_0, z_0(-\cdot)) \in H$ and $\varphi \in L^1(0,T;L^2(\Omega))$, then there exists a unique mild solution*

$$\Lambda = (\omega, z) \in Y_T \times C([0,T];L^2(\mathcal{Q}))$$

of (5.24) such that

$$\|(\omega,z)\|^2_{C([0,T];H)} \leq C\left(\|(\omega_0, z_0(-\cdot))\|^2_H + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right), \tag{5.25}$$

and

$$\|\omega\|^2_{Y_T} \leq C\left(\|(\omega_0, z_0(-\cdot))\|^2_H + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right), \tag{5.26}$$

for some constant $C > 0$, which is independent of Λ_0 and φ.

(*b*) *Given*

$$\omega \in Y_T = C(0,T;L^2(\Omega)) \cap L^2(0,T;H^2_0(\Omega))$$

and $p \in \{1,2\}$, we have $\omega^p \partial_x \omega \in L^1(0,T;L^2(\Omega))$ and the map

$$\mathcal{F} : \omega \in Y_T \mapsto \omega^p \partial_x \omega \in L^1(0,T;L^2(\Omega))$$

is continuous.

Proof. **Proof of item (a).** Since A is the infinitesimal generator of a semigroup of contractions e^{tA} and $\varphi \in L^1(0,T;L^2(\Omega))$ it follows from semigroups theory that there is a unique mild solution $\Lambda = (\omega, z) \in C([0,T];H)$ of (5.24) such that

$$\Lambda(t) = e^{tA}\Lambda_0 + \int_0^t e^{(t-s)A}(\varphi, 0)ds$$

and hence, we get

$$\|(\omega,z)\|_{C([0,T];H)} \leq C\left(\|(\omega_0, z_0(-\cdot))\|_H + \|\varphi\|_{L^1(0,T;L^2(\Omega))}\right).$$

Young's inequality gives

$$\|(\omega,z)\|^2_{C([0,T];H)} \leq 2C^2\left(\|(\omega_0, z_0(-\cdot))\|^2_H + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right),$$

which proves (5.25). To complete the proof of item (a), we must verify the validity of (5.26). For this, observe that from (5.25), we have

$$\max_{t\in[0,T]} \|\omega\|^2 \leq 2C^2\left(\|(\omega_0, z_0(-\cdot))\|^2_H + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right). \tag{5.27}$$

In turn, if we multiply the second equation in (5.24) by $\phi\sigma(s)z$, integrating over $[0,T] \times [0,1] \times [\tau_1, \tau_2]$ and arguing as for the proof of (5.17), we obtain

$$\begin{aligned} &\int_0^T \int_{\mathcal{M}} s\sigma(s)\left(z(t,1,s)\right)^2 \, dsdt \\ &\leq \frac{\tau_2}{|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\left(\|\omega_0\|^2 + \|z_0(-\phi s)\|^2_{L^2(\mathcal{Q})} + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right). \end{aligned} \tag{5.28}$$

Now, multiplying the first equation in (5.24) by ω, integrating over $[0,T]\times[0,\ell]$, and thanks to (5.28), we get

$$\begin{aligned}\|\partial_x^2\omega(0)\|^2_{L^2(0,T)} \leq &\|\omega_0\|^2+\int_0^T\left(\nu_1\partial_x^2\omega(0)+\nu_2\int_{\mathcal{M}}\sigma(s)z(\cdot,1,s)ds\right)^2\,dt\\ &+2\left(\max_{t\in[0,T]}\|\omega(t,x)\|\right)\int_0^T\|\varphi(t,x)\|\,dt.\end{aligned}\tag{5.29}$$

Now, replacing (5.20) in (5.29), we find

$$\begin{aligned}\|\partial_x^2\omega(0)\|^2_{L^2(0,T)} \ \leq\ & \|\omega_0\|^2+\left[\nu_1^2+\frac{\nu_2^2}{2\theta}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\int_0^T\left(\partial_x^2\omega(t,0)\right)^2dt\\ &+\left[2\theta\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\left(\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)dsdt\right)\\ &+2\left(\max_{t\in[0,T]}\|\omega(t,x)\|\right)\int_0^T\|\varphi(t,x)\|\,dt.\end{aligned}$$

Isolating $\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}$ and using Young's inequality for the last term of the right-hand side, we reach

$$\begin{aligned}&\left[1-\nu_1^2-\frac{\nu_2^2}{2\theta}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}\\ &\leq\|\omega_0\|^2+\left[2\theta\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\left(\int_0^T\int_{\mathcal{M}}\sigma(s)z^2(\cdot,1,s)dsdt\right)\\ &\quad+\left(\max_{t\in[0,T]}\|\omega(t,x)\|\right)^2+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}.\end{aligned}\tag{5.30}$$

Thanks to (5.5), (5.22) and (5.30), the estimate (5.25) becomes

$$\begin{aligned}\|\partial_x^2\omega(0)\|^2_{L^2(0,T)}\leq &C_1\left(2+C_2+\frac{\tau_2}{\tau_1|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\right)\|\omega_0\|^2\\ &+C_1\left(\frac{\tau_2}{\tau_1|\nu_2|}\left(\frac{T}{\tau_1}+1\right)+1+C_2\right)\|z_0(-\phi s)\|^2_{L^2(\mathcal{Q})}\\ &+C_1(1+C_2)\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\\ \leq &C\left(\|(\omega_0,z_0(-\phi s))\|^2_H+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right).\end{aligned}\tag{5.31}$$

Now, multiply the equation (5.24) by xu and integrate by parts over $(0,T)\times(0,\ell)$ and then perform similar calculations to those of the previous item to get

$$\begin{aligned}&\frac{5}{2}\|\partial_x^2\omega\|^2_{L^2(0,T;L^2(\Omega))}\leq\frac{\ell}{2}\|\omega_0\|^2+\frac{aT}{2}C\left(\|(\omega_0,z_0(-\phi s))\|^2_H+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right)\\ &+\frac{\ell}{2}C\left(\|(\omega_0,z_0(-\phi s))\|^2_H+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right)+\frac{\ell}{2}\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\\ &+\frac{\ell}{2}\left[\nu_1^2+\frac{\nu_2^2}{2\epsilon}\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]C\left(\|(\omega_0,z_0(-\phi s))\|^2_H+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right)\\ &+\frac{\ell}{2\tau_1}\left[2\epsilon\nu_1^2+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)ds\right)\right]\frac{\tau_2}{|\nu_2|}\left(\frac{T}{\tau_1}+1\right)\left(\|(\omega_0,z_0(-\phi s))\|^2_H+\|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right),\end{aligned}\tag{5.32}$$

where we have used Cauchy-Schwarz inequality, Young inequality, estimates (5.20), (5.28), and (5.31). Therefore, taking any $\epsilon > 0$ in (5.32), there exists $C > 0$ such that

$$\|\omega\|^2_{L^2(0,T;H^2_0(\Omega))} = \|\partial^2_x\omega\|^2_{L^2(0,T;L^2(\Omega))} \leq C\left(\|(\omega_0, z_0(-\phi s))\|^2_H + \|\varphi\|^2_{L^1(0,T;L^2(\Omega))}\right). \quad (5.33)$$

The estimate (5.26) follows directly from the estimates (5.27) and (5.33), and item (a) is achieved.

Proof of item (b). Given $\omega, v \in Y_T$ we have, for $p = 1$, that

$$\|\omega\partial_x\omega\|_{L^1(0,T;L^2(\Omega))} \leq k\int_0^T \|\omega\|_{H^2(\Omega)}\|\partial_x\omega\|dt \leq k\int_0^T \|\omega\|^2_{H^2(\Omega)}dt \leq k\|\omega\|^2_{Y_T} < \infty, \quad (5.34)$$

where k is the positive constant of the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Therefore, $\omega\partial_x\omega \in L^1(0,T;L^2(\Omega))$, for each $\omega \in Y_T$. Thus, using the triangle inequality, together with the Cauchy-Schwarz inequality, we get the classical estimate

$$\|\mathcal{F}(\omega) - \mathcal{F}(v)\|_{L^1(0,T;L^2(\Omega))} \leq k\|\omega - v\|_{Y_T}\left(\|\omega\|_{Y_T} + \|v\|_{Y_T}\right), \quad \text{for any}\, u, v \in Y_T. \quad (5.35)$$

Therefore, the map $\mathcal{F}$ is continuous concerning the corresponding topologies. On the other hand, when $p = 2$, we have for $\omega, v \in Y_T$ that

$$\|\mathcal{F}(\omega)\|_{L^1(0,T;L^2(\Omega))} \leq k\|\omega\|_{C(0,T;L^2(\Omega))}\int_0^T \|\omega\|^2_{H^2(\Omega)}dt \leq k\|\omega\|^3_{Y_T} < +\infty. \quad (5.36)$$

Hence, $\mathcal{F}(\omega)$ is well-defined and for any u, v in Y_T, we have

$$\|\mathcal{F}(\omega) - \mathcal{F}(v)\|_{L^1(0,T;L^2(\Omega))} \leq \frac{3k}{2}\left(\|\omega\|^2_{Y_T} + \|v\|^2_{Y_T}\right)\|\omega - v\|_{Y_T}. \quad (5.37)$$

Thereby, the map $\mathcal{F}$ is continuous for the corresponding topologies. □

5.2.3 Nonlinear problem

We are now in a position to prove the main result of the section. Precisely, the next result gives the well-posedness for the full system (5.2).

Theorem 5.2.2. *Suppose that (5.5) holds. Then, there exist constants $r, C > 0$ such that, for every $\Lambda_0 = (\omega_0, z_0(-\cdot)) \in H$ with $\|\Lambda_0\|^2_H \leq r$, the problem (5.2) admits a unique global solution $\omega \in Y_T$, which satisfies $\|\omega\|_{Y_T} \leq C\|\Lambda_0\|_H$.*

Proof. Given $\Lambda_0 = (\omega_0, z_0(-\cdot)) \in H$ such that $\|\Lambda_0\|^2_H \leq r$, where r is a positive constant to be chosen, define a mapping $\Upsilon : Y_T \to Y_T$ as follows: $\Upsilon(\omega) = y$, where y is the solution of (5.24) with a source term $\varphi = \omega^p\partial_x\omega = \mathcal{F}(\omega)$, $p \in \{1, 2\}$. The mapping Υ is well defined because of item (a) of Theorem 5.2.1 from which we obtain from (5.26) that

$$\|\Upsilon(\omega)\|^2_{Y_T} \leq C\left(\|\Lambda_0\|^2_H + \|\mathcal{F}(\omega)\|^2_{L^1(0,T:L^2(\Omega))}\right).$$

Note that $\Upsilon(\omega) - \Upsilon(v)$ is a solution of (5.24) with initial condition $\Lambda_0 = (0,0) \in H$ and source term $\varphi = \mathcal{F}(\omega) - \mathcal{F}(v)$. It follows from (5.26) that

$$\|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} \leq C\|\mathcal{F}(\omega) - \mathcal{F}(v)\|^2_{L^1(0,T;L^2(\Omega))},$$

where the constant $C > 0$ above does not depend on Λ_0 and φ.

Now, considering $p = 1$, we have from (5.34) that

$$\|\Upsilon(\omega)\|^2_{Y_T} \leq C\left(r + k^2\|\omega\|^4_{Y_T}\right), \ \forall \omega \in Y_T,$$

while from (5.35), we have that

$$\|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} \leq Ck^2\left(\|\omega\|^2_{Y_T} + \|v\|^2_{Y_T}\right)^2 \|\omega - v\|^2_{Y_T}, \ \forall \omega, v \in Y_T.$$

Thus, when $\|\omega\|^2_{Y_T} \leq R$ we get

$$\begin{aligned} \|\Upsilon(\omega)\|^2_{Y_T} &\leq C\left(r + k^2R^2\right), \ \forall \omega \in \mathcal{B}, \\ \|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} &\leq 4Ck^2R^2\|\omega - v\|^2_{Y_T}, \ \forall \omega, v \in \mathcal{B}. \end{aligned}$$

Next, pick $R = \dfrac{1}{5k^2C}$ and $r = \dfrac{1}{25k^2C^2}$. For $\omega \in \mathcal{B} = \{\omega \in Y_T; \|\omega\|^2_{Y_T} \leq R\}$, we have that

$$\begin{aligned} \|\Upsilon(\omega)\|^2_{Y_T} &\leq R, \ \forall \omega \in \mathcal{B}, \\ \|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} &\leq \frac{4}{5}\|\omega - v\|^2_{Y_T}, \ \forall \omega, v \in \mathcal{B}. \end{aligned} \tag{5.38}$$

On the other hand, when $p = 2$, we have from (5.36) that

$$\|\Upsilon(\omega)\|^2_{Y_T} \leq C\left(r + k^2\|\omega\|^6_{Y_T}\right), \ \forall \omega \in Y_T$$

and from (5.37), we have that

$$\|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} \leq C\left(\frac{3k}{2}\right)^2\left(\|\omega\|^2_{Y_T} + \|v\|^2_{Y_T}\right)^2 \|\omega - v\|^2_{Y_T}, \ \forall \omega, v \in Y_T.$$

Thus, when $\|\omega\|^2_{Y_T} \leq R$, we get

$$\begin{aligned} \|\Upsilon(\omega)\|^2_{Y_T} &\leq C\left(r + k^2R^3\right), \ \forall \omega \in \mathcal{B}, \\ \|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} &\leq 9Ck^2R^2\|\omega - v\|^2_{Y_T}, \ \forall \omega, v \in \mathcal{B}. \end{aligned}$$

Therefore, just take $R = \dfrac{1}{4k\sqrt{C}}$ and $r = \dfrac{1}{16kC^{\frac{3}{2}}}$ and we will have that

$$\begin{aligned} \|\Upsilon(\omega)\|^2_{Y_T} &\leq R, \ \forall \omega \in \mathcal{B}, \\ \|\Upsilon(\omega) - \Upsilon(v)\|^2_{Y_T} &\leq \frac{9}{16}\|\omega - v\|^2_{Y_T}, \ \forall\forall \omega, v \in \mathcal{B}. \end{aligned} \tag{5.39}$$

Consequently, due to (5.38) and (5.39), the restriction of the map Υ to $\mathcal{B}$ is well-defined, and Υ is a contraction on the ball $\mathcal{B}$. As an application of Banach Fixed Point Theorem, the map Υ possesses a unique fixed element ω, which turns out to be the unique solution to problem (5.2). Finally, the solution is global thanks to the dissipation property. Indeed, the energy $\mathcal{E}(t)$ (see (5.4)) of the system (5.2) satisfies

$$\mathcal{E}'(t) \leq \frac{1}{2}\langle GX, X\rangle_{\mathbb{R}^2} \leq 0,$$

where G and X are given in Proposition 5.2.1. □

5.3 Exponential stability of solutions

In this section, we will prove the two main results of our work. The first stabilization result will be proved *via* the Lyapunov approach. The second one is obtained showing an *observability inequality* which will be proved by the compactness-uniqueness argument.

5.3.1 Stabilization results via Lyapunov approach

Proof. Proof of Theorem 5.1.1 Initially, let us remember that the energy of the system (5.24), for $\varphi = \omega^p \partial_x \omega$, with $p \in \{1, 2\}$, is defined by

$$\mathcal{E}(t) = \|\Lambda(t)\|_H^2 = \|\omega(t)\|^2 + \|z(t)\|^2_{L^2(\mathcal{Q})},$$

where $\|z(t)\|^2_{L^2(\mathcal{Q})} = |\nu_2| \int_{\mathcal{M}} s\sigma(s) \int_0^1 z^2(t, \phi, s) d\phi ds$. Thus, using (5.24), we get

$$\begin{aligned}
\mathcal{E}'(t) &= 2\langle \Lambda_t(t), \Lambda(t)\rangle_H = 2\langle A\Lambda(t), \Lambda(t)\rangle_H + 2\langle (\omega^p \partial_x \omega, 0), \Lambda(t)\rangle_H \\
&= \langle GX, X\rangle_{\mathbb{R}^2} + 2\int_\Omega u^{p+1}\partial_x \omega dx \\
&= \langle GX, X\rangle_{\mathbb{R}^2} + 2\frac{\omega^{p+2}(\ell)}{p+2} - 2\frac{\omega^{p+2}(0)}{p+2} = \langle GX, X\rangle_{\mathbb{R}^2} \leq 0,
\end{aligned}$$

where G and X were given in (5.14). Let us now define a Lyapunov function

$$\Phi(t) = \mathcal{E}(t) + \mu_1 E_1(t) + \mu_2 E_2(t), \ \ t \geq 0,$$

where $E_1(t)$ and $E_2(t)$ are given by

$$E_1(t) = \int_\Omega x u^2(x, t) dx \quad \text{and} \quad E_2(t) = |\nu_2| \int_{\Omega_0} \int_{\mathcal{M}} s e^{-\delta\phi s} \sigma(s) z^2(t, \phi, s) ds d\phi,$$

μ_1 and μ_2 are positive constants to be determined and $\delta > 0$ is arbitrary constant. Note that

$$\mu_1 E_1(t) = \mu_1 \int_\Omega x u^2(x, t) dx \leq \ell\mu_1 \int_\Omega \omega^2(x, t) dx = \ell\mu_1 \|\omega\|^2$$

and

$$\mu_2 E_2(t) \leq \mu_2 |\nu_2| \int_{\Omega_0} \int_{\mathcal{M}} s\sigma(s) z^2(t, \phi, s) ds d\phi = \mu_2 \|z(t)\|^2_{\ell^2(\mathcal{Q})}.$$

Consequently,

$$\mu_1 E_1(t) + \mu_2 E_2(t) \leq \max\{\ell\mu_1, \mu_2\}\mathcal{E}(t)$$

and, therefore

$$\mathcal{E}(t) \leq \Phi(t) \leq (1 + \max\{\ell\mu_1, \mu_2\})\,\mathcal{E}(t). \tag{5.40}$$

Differentiating $E_1(t)$ and $E_2(t)$ using integration by parts and the boundary conditions of (5.2) and (5.8), we get

$$\begin{aligned} E_1'(t) &= \alpha\|\omega\|^2 - 3\beta\|\partial_x\omega\|^2 - 5\|\partial_x^2\omega\|^2 + \frac{2}{p+2}\int_\Omega \omega^{p+2}dx \\ &+\ell\left[\nu_1^2\left(\partial_x^2\omega(t,0)\right)^2 + 2\nu_1\nu_2\left(\partial_x^2\omega(t,0)\right)\left(\int_{\mathcal{M}}\sigma(s)z(t,1,s)ds\right)\right. \\ &\left.+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)z(t,1,s)ds\right)^2\right] \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} E_2'(t) &= -|\nu_2|\int_{\mathcal{M}} e^{-\delta s}\sigma(s)\left(z(t,1,s)\right)^2 ds + |\nu_2|\left(\int_{\mathcal{M}}\sigma(s)ds\right)\left(\partial_x^2\omega(t,0)\right)^2 \\ &-|\nu_2|\int_{\mathcal{M}}\int_{\Omega_0}\delta s e^{-\delta\phi s}\sigma(s)z^2 d\phi ds. \end{aligned}$$

Thus, for $\Phi(t) = \mathcal{E}(t) + \mu_1 E_1(t) + \mu_2 E_2(t)$, we find that

$$\begin{aligned} \Phi'(t)+2\mu\Phi(t) &= \langle GX, X\rangle_{\mathbb{R}^2} + \alpha\mu_1\|\omega\|^2 - 3\beta\mu_1\|\partial_x\omega\|^2 - 5\mu_1\|\partial_x^2\omega\|^2 + \frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx \\ &+ \ell\mu_1\left[\nu_1^2\left(\partial_x^2\omega(t,0)\right)^2 + 2\nu_1\nu_2\left(\partial_x^2\omega(t,0)\right)\left(\int_{\mathcal{M}}\sigma(s)z(t,1,s)ds\right)\right. \\ &\left.+\nu_2^2\left(\int_{\mathcal{M}}\sigma(s)z(t,1,s)ds\right)^2\right] \\ &- \mu_2|\nu_2|\int_{\mathcal{M}} e^{-\delta s}\sigma(s)\left(z(t,1,s)\right)^2 ds + \mu_2|\nu_2|\left(\int_{\mathcal{M}}\sigma(s)ds\right)\left(\partial_x^2\omega(t,0)\right)^2 \\ &- \mu_2|\nu_2|\int_{\mathcal{M}}\int_{\Omega_0}\delta s e^{-\delta\phi s}\sigma(s)z^2 d\phi ds + 2\mu\|\omega(t)\|^2 + 2\mu\|z(t)\|^2_{L^2(\mathcal{Q})} + 2\mu\mu_1\int_\Omega x u^2(x,t)dx \\ &+ 2\mu\mu_1|\nu_2|\int_{\Omega_0}\int_{\mathcal{M}} s e^{-\delta\phi s}\sigma(s)z(t,\phi,s)dsd\phi. \end{aligned}$$

Next, let

$$G_{\mu_1} = \mu_1\ell\begin{pmatrix} \nu_1^2 & \nu_1\nu_2 \\ \nu_1\nu_2 & \nu_2^2 \end{pmatrix}, \quad G_{\mu_2} = \mu_2\begin{pmatrix} |\nu_2|\int_{\mathcal{M}}\sigma(s)ds & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$X = \begin{pmatrix} \partial_x^2\omega(t,0) \\ \int_{\mathcal{M}}\sigma(s)z(t,1,s)ds \end{pmatrix}.$$

Thus, we have that

$$\begin{aligned}
\Phi'(t)+2\mu\Phi(t) =& \langle (G+G_{\mu_1}+G_{\mu_2})X, X\rangle_{\mathbb{R}^2} + (\alpha\mu_1+2\mu)\|\omega\|^2 - 3\beta\mu_1\|\partial_x\omega\|^2 - 5\mu_1\|\partial_x^2\omega\|^2 \\
&+ \frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx - \mu_2|\nu_2|\int_{\mathcal{M}} e^{-\delta s}\sigma(s)\left(z(t,1,s)\right)^2 ds \\
&- \mu_2|\nu_2|\int_{\mathcal{M}}\int_{\Omega_0}\delta s e^{-\delta\phi s}\sigma(s)z^2 d\phi ds + 2\mu\|z(t)\|^2_{L^2(\mathcal{Q})} + 2\mu\mu_1\int_\Omega x u^2(x,t)dx \\
&+ 2\mu\mu_1|\nu_2|\int_{\Omega_0}\int_{\mathcal{M}} s e^{-\delta\phi s}\sigma(s)z(t,\phi,s)dsd\phi \\
\leq& \langle (G+G_{\mu_1}+G_{\mu_2})X, X\rangle_{\mathbb{R}^2} + (\alpha\mu_1+2\mu(1+\mu_1\ell))\|\omega\|^2 - 3\beta\mu_1\|\partial_x\omega\|^2 - 5\mu_1\|\partial_x^2\omega\|^2 \\
&+ \frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx - \mu_2|\nu_2|e^{-\delta\tau_2}\int_{\mathcal{M}} \sigma(s)\left(z(t,1,s)\right)^2 ds \\
&- \mu_2|\nu_2|e^{-\delta\tau_2}\delta\int_{\mathcal{M}}\int_{\Omega_0} s\sigma(s)z^2 d\phi ds \\
&+ 2\mu\|z(t)\|^2_{L^2(\mathcal{Q})} + 2\mu\mu_1|\nu_2|\int_{\Omega_0}\int_{\mathcal{M}} s\sigma(s)z(t,\phi,s)dsd\phi.
\end{aligned}$$

Now, observe that

$$T(\mu_1,\mu_2) := G+G_{\mu_1}+G_{\mu_2} = G + \mu_1\ell\begin{pmatrix} \nu_1^2 & \nu_1\nu_2 \\ \nu_1\nu_2 & \nu_2^2 \end{pmatrix} + \mu_2\begin{pmatrix} |\nu_2|\int_{\mathcal{M}}\sigma(s)ds & 0 \\ 0 & 0 \end{pmatrix}$$

is a continuous map of $\mathbb{R}^2$ on the vector space of square matrices $M_{2\times 2}(\mathbb{R})$ and that the determinant and trace are continuous functions of $M_{2\times 2}(\mathbb{R})$ over $\mathbb{R}$, we have that $h_1(\mu_1,\mu_2) = \det T(\mu_1,\mu_2)$ and $h_2(\mu_1,\mu_2) = \mathrm{tr}T(\mu_1,\mu_2)$ are continuous from $\mathbb{R}^2$ over $\mathbb{R}$. Therefore, knowing that $h_1(0,0) = \det G > 0$ and $h_2(0,0) = \mathrm{tr}G < 0$ for μ_1,μ_2 small enough, one can claim that $h_1(\mu_1,\mu_2) > 0$ and $h_2(\mu_1,\mu_2) < 0$. Thereby, $G+G_{\mu_1}+G_{\mu_2}$ is negative defined for μ_1,μ_2 small enough. Moreover, using the Poincaré inequality[1] we find

$$\begin{aligned}
\Phi'(t)+2\mu\Phi(t) \leq& \left[\frac{\ell^2}{\pi^2}\left(\alpha\mu_1+2\mu(1+\mu_1\ell)\right) - 3\beta\mu_1\right]\|\partial_x\omega\|^2 - 5\mu_1\|\partial_x^2\omega\|^2 \\
&+ \frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx - \mu_2|\nu_2|e^{-\delta\tau_2}\int_{\mathcal{M}}\sigma(s)\left(z(t,1,s)\right)^2 ds \\
&+ \left(2\mu(1+\mu_1|\nu_2|) - \mu_2|\nu_2|e^{-\delta\tau_2}\delta\right)\|z(t)\|^2_{L^2(\mathcal{Q})}.
\end{aligned} \tag{5.42}$$

Now, we are going to estimate the integral

$$\frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx.$$

For this, applying the Cauchy-Schwarz inequality and using the fact that the energy of the system $\mathcal{E}(t)$ is non-increasing, together with the embedding $H_0^1(\Omega)\hookrightarrow L^\infty(\Omega)$ we have,

[1] $\|\omega\|^2 \leq \frac{\ell^2}{\pi^2}\|\partial_x\omega\|^2$, for $\omega\in H_0^2(\Omega)$,

for $\|(\omega_0, z_0)\|_H < r$, that

$$\begin{aligned} \frac{2\mu_1}{p+2}\int_\Omega \omega^{p+2}dx &\leq \frac{2\mu_1}{p+2}\|\omega\|^2_{L^\infty(\Omega)}\int_\Omega \omega^p dx \leq \frac{2\ell\mu_1}{p+2}\|\partial_x\omega\|^2\int_\Omega \omega^p dx \\ &\leq \frac{2\ell\mu_1}{p+2}\|\partial_x\omega\|^2\ell^{1-\frac{p}{2}}\|\omega\|^p \leq \frac{2\ell^{2-\frac{p}{2}}\mu_1}{p+2}\|\partial_x\omega\|^2\|(\omega_0,z_0)\|^p_H \\ &\leq \frac{2\ell^{2-\frac{p}{2}}\mu_1 r^p}{p+2}\|\partial_x\omega\|^2. \end{aligned} \tag{5.43}$$

Combining (5.43) and (5.42) yields

$$\begin{aligned} \Phi'(t)+2\mu\Phi(t) \leq& \left[\frac{\ell^2}{\pi^2}\left(\alpha\mu_1+2\mu(1+\mu_1\ell)\right)-3\beta\mu_1\right]\|\partial_x\omega\|^2-5\mu_1\|\partial_x^2\omega\|^2 \\ &+\frac{2\ell^{2-\frac{p}{2}}\mu_1 r^p}{p+2}\|\partial_x\omega\|^2-\mu_2|\nu_2|e^{-\delta\tau_2}\int_{\mathcal{M}}\sigma(s)\left(z(t,1,s)\right)^2 ds \\ &+\left(2\mu(1+\mu_1|\nu_2|)-\mu_2|\nu_2|e^{-\delta\tau_2}\delta\right)\|z(t)\|^2_{L^2(\mathcal{Q})} \\ \leq& \left[\frac{\ell^2}{\pi^2}\left(\alpha\mu_1+2\mu(1+\mu_1\ell)\right)-3\beta\mu_1+\frac{2\ell^{2-\frac{p}{2}}\mu_1 r^p}{p+2}\right]\|\partial_x\omega\|^2 \\ &-5\mu_1\|\partial_x^2\omega\|^2+\left(2\mu(1+\mu_1|\nu_2|)-\mu_2|\nu_2|e^{-\delta\tau_2}\delta\right)\|z(t)\|^2_{L^2(\mathcal{Q})}. \end{aligned}$$

Note that $\Phi'(t)+2\mu\Phi(t)<0$ when

$$2\mu(1+\mu_1|\nu_2|)-\mu_2|\nu_2|e^{-\delta\tau_2}\delta<0$$

and

$$\frac{\ell^2}{\pi^2}\left(\alpha\mu_1+2\mu(1+\mu_1\ell)\right)-3\beta\mu_1+\frac{2\ell^{2-\frac{p}{2}}\mu_1 r^p}{p+2}<0,$$

which holds for $\mu>0$ satisfying, respectively

$$\mu<\frac{\mu_2|\nu_2|e^{-\delta\tau_2}\delta}{2(1+\mu_1|\nu_2|)}$$

and

$$0<\mu<\frac{\mu_1}{2\ell^2(1+\ell\mu_1)(p+2)}\left[(p+2)(3\pi^2\beta-\alpha\ell^2)-2\pi^2\ell^{2-\frac{p}{2}}r^p\right],$$

where we need to take $r>0$ satisfying

$$(p+2)(3\pi^2\beta-\alpha\ell^2)-2\pi^2\ell^{2-\frac{p}{2}}r^p>0$$

or, equivalently, $r>0$ must satisfy

$$r<\left(\frac{(p+2)(3\pi^2\beta-\alpha\ell^2)}{2\pi^2\ell^{2-\frac{p}{2}}}\right)^{\frac{1}{p}}.$$

Thus, for μ_1,μ_2 small enough and an arbitrary $\delta>0$, taking

$$r<\left(\frac{(p+2)(3\pi^2\beta-\alpha\ell^2)}{2\pi^2\ell^{2-\frac{p}{2}}}\right)^{\frac{1}{p}}$$

and

$$\mu < \min\left\{\frac{\mu_2|\nu_2|e^{-\delta\tau_2}\delta}{2(1+\mu_1|\nu_2|)}, \frac{\mu_1}{2\ell^2(1+\ell\mu_1)(p+2)}\left[(p+2)(3\pi^2\beta-\alpha\ell^2)-2\pi^2\ell^{2-\frac{p}{2}}r^p\right]\right\},$$

we get that

$$\Phi'(t)+2\mu\Phi(t)<0 \iff \Phi(t)\leq \Phi(0)e^{-2\mu t}.$$

Lastly, from (5.40), we get

$$\mathcal{E}(t)\leq\Phi(t)\leq\Phi(0)e^{-2\mu t}\leq(1+\max\{\ell\mu_1,\mu_2\})\mathcal{E}(0)e^{-2\mu t}\leq\kappa\mathcal{E}(0)e^{-2\mu t},$$

for $\kappa>1+\max\{\ell\mu_1,\mu_2\}$, proving the theorem. □

5.3.2 Stabilization results via compactness-uniqueness argument

Proof. Proof of Theorem 5.1.2 First, we deal with the linear system (5.8) and claim that the following observability inequality holds

$$\|\omega_0\|^2+\|z_0\|^2_{L^2(\mathcal{Q})}\leq C\int_0^T\left((\partial_x^2\omega(t,0))^2+\int_{\mathcal{M}}s\sigma(s)z^2(t,1,s)\,ds\right)dt, \tag{5.44}$$

where $(\omega_0,z_0)\in H$ and $(\omega,z)(t)=e^{tA}(\omega_0,z_0)$ is the unique solution of (5.8). This leads to the exponential stability in H of the solution (ω,z) to (5.8). The proof of this inequality can be obtained by a contradiction argument. Indeed, if (5.44) is not true, then there exists a sequence $\{(\omega_0^n,z_0^n)\}_n\subset H$ such that

$$\|\omega_0^n\|^2+\|z_0^n\|^2_{L^2(\mathcal{Q})}=1 \tag{5.45}$$

and

$$\left\|\partial_x^2\omega^n(\cdot,0)\right\|^2_{L^2(0,T)}+\int_{\mathcal{M}}s\sigma(s)z^2(t,1,s)\,ds\to 0 \text{ as } n\to+\infty, \tag{5.46}$$

where $(\omega^n,z^n)(t)=e^{tA}(\omega_0^n,z_0^n)$. Then, arguing as in [19], we can deduce from Proposition 5.2.1 that $\{\omega^n\}_n$ is convergent in $L^2(0,T,L^2(\Omega))$. Moreover, $\{\omega_0^n\}_n$ is a Cauchy sequence in $L^2(\Omega)$ and, consequently, $\omega_0^n\to\omega_0\in L^2(\Omega)$. As

$$\|z^n(T,\cdot,\cdot)\|_{L^2(\mathcal{Q})}\leq C\left\|\partial_x^2\omega^n(\cdot,0)\right\|^2_{L^2(0,T)},$$

we get that $\|z^n(T,\cdot,\cdot)\|_{L^2(\mathcal{Q})}\to 0$. Thus, from (5.11) and (5.46) we get $z_0^n\to 0$. Thereafter, by virtue of (5.45) we obtain $\|\omega_0\|^2=1$. Next, take $(\omega,z)=e^{\cdot A}(\omega_0,z_0)$, and assume, for the sake of simplicity and without loss of generality, that $\alpha=\beta=1$. This, together with Proposition 5.2.1 and (5.46), implies that ω is solution of the system

$$\begin{cases}\partial_t\omega+\partial_x\omega+\partial_x^3\omega-\partial_x^5\omega=0, & x\in\Omega, t>0,\\ \omega(0,t)=\omega(\ell,t)=\partial_x\omega(\ell,t)=\partial_x\omega(0,t)=\partial_x^2\omega(\ell,t)=\partial_x^2\omega(0,t)=0, & t>0,\\ \omega(x,0)=\omega_0(x), & x\in\Omega,\end{cases}$$

with $\|\omega_0\|_{L^2(\Omega)} = 1$. The latter contradicts the result obtained in [19, Lemma 4.2], which states that the above system has only the trivial solution (see also Lemma 5.1.1). This proves the observability inequality (5.44).

Now, let us go back to the original system (5.2) and use the same arguments as in [83]. First, we restrict ourselves to the case $p = 1$ as the case $p = 2$ is similar. Next, consider an initial condition $\|(\omega_0, z_0)\|_H \leq \varrho$, where ϱ will be fixed later. Then, the solution ω of (5.2) can be written as $\omega = \omega_1 + \omega_2$, where ω_1 is the solution of (5.8) with the initial data $(\omega_0, z_0) \in H$ and ω_2 is solution of (5.24) with null data and right-hand side $\varphi = \omega\partial_x\omega \in L^1(0,T;L^2(\Omega))$, as in Lemma 5.2.1. In other words, ω_1 is the solution of

$$\begin{cases} \partial_t\omega_1 - \partial_x^5\omega_1 + \partial_x^3\omega_1 + \partial_x\omega_1 = 0, & x \in \Omega, t > 0, \\ \omega_1(t,0) = \omega_1(t,\ell) = \partial_x\omega_1(t,0) = \partial_x\omega_1(t,\ell) = 0, & t > 0, \\ \partial_x^2\omega_1(t,\ell) = \nu_1\partial_x^2\omega_1(t,0) + \nu_2\displaystyle\int_{t-\tau_2}^{t-\tau_1}\sigma(t-s)\partial_x^2\omega(s,0)\,ds, & t > 0, \\ \partial_x^2\omega_1(t,0) = z_0(t), & t \in (-\tau_2, 0), \\ \omega_1(0,x) = \omega_0(x), & x \in \Omega, \end{cases}$$

and ω_2 is solution of

$$\begin{cases} \partial_t\omega_2 - \partial_x^5\omega_2 + \partial_x^3\omega_2 + \partial_x\omega_2 = -\omega\partial_x\omega, & x \in \Omega, t > 0, \\ \omega_2(t,0) = \omega_2(t,\ell) = \partial_x\omega_2(t,0) = \partial_x\omega_2(t,\ell) = 0, & t > 0, \\ \partial_x^2\omega_2(t,\ell) = \nu_1\partial_x^2\omega_2(t,0) + \nu_2\displaystyle\int_{t-\tau_2}^{t-\tau_1}\sigma(t-s)\partial_x^2\omega(s,0)\,ds, & t \in (-\tau_2, 0), \\ \partial_x^2\omega_2(t,0) = 0, & x \in \Omega, \\ \omega_2(0,x) = 0, & x \in \Omega. \end{cases}$$

In light of the exponential stability of the linear system (5.8) (see the beginning of this subsection) and Theorem 5.2.1, we have

$$\|(\omega(T), z(T))\|_H \leq \chi\,\|(\omega_0, z_0)\|_H + C\|\omega\|^2_{L^2(0,T,H^2(\Omega))}, \tag{5.47}$$

in which $\chi \in (0,1)$. Subsequently, multiply $(5.2)_1$ by xu and performing the same computations as for (5.41), we get

$$\begin{aligned} &\int_\Omega x\omega^2(T,x)dx + 3\int_0^T\int_\Omega(\partial_x\omega(t,x))^2\,dxdt + 5\int_0^T\int_\Omega\left(\partial_x^2u(t,x)\right)^2dxdt = \\ &\int_0^T\int_\Omega\omega^2(t,x)dxdt + \ell\int_0^T\left(\nu_1\partial_x^2\omega(t,0) + \nu_2\int_{\mathcal{M}}\sigma(s)z(t,1,s)\,ds\right)^2dt + \int_\Omega x\omega_0^2(x)dx \\ &+\frac{2}{3}\int_0^T\int_\Omega\omega^3(t,x)dxdt. \end{aligned} \tag{5.48}$$

On one hand, multiplying the first equation of (5.2) by ω and arguing as done for (5.9) (see (5.19)), we get

$$\int_0^T\left(\nu_1\partial_x^2\omega(t,0) + \nu_2\int_{\mathcal{M}}\sigma(s)z(t,1,s)\,ds\right)^2dt \leq C\|(\omega_0, z_0)\|^2_H. \tag{5.49}$$

On the other hand, using Gagliardo–Nirenberg and Cauchy-Schwarz inequalities, together with the dissipativity of the system (5.2), we deduce that

$$\int_0^T \int_\Omega \omega^3 dxdt \leq C(T) \left\| (\omega_0, z_0) \right\|_H^2 \left\| \omega \right\|_{L^2(0,T;H^2(\Omega))}.$$

Applying Young's inequality to the last estimate and combining the obtained result with (5.48)-(5.49), we reach

$$\left\| \omega \right\|_{L^2(0,T;H^2(\Omega))}^2 \leq C \left\| (\omega_0, z_0) \right\|_H^2 \left(1 + \left\| (\omega_0, z_0) \right\|_H^2 \right). \tag{5.50}$$

Finally, recalling that $\left\| (\omega_0, z_0) \right\|_H \leq \varrho$, and inserting (5.50) into (5.47), we get

$$\left\| (\omega(T), z(T)) \right\|_H \leq \left\| (\omega_0, z_0) \right\|_H \left(\chi + C\varrho + C\varrho^3 \right).$$

Given $\eta > 0$ sufficiently small so that $\chi + \eta < 1$, one can choose ϱ small such that $\varrho + \varrho^3 < \frac{\eta}{C}$, to obtain

$$\left\| (\omega(T), z(T)) \right\|_H \leq (\chi + \eta) \left\| (\omega_0, z_0) \right\|_H.$$

Lastly, using the semigroup property and the fact that $\chi + \eta < 1$, we conclude the exponential stability result of Theorem 5.1.2. □

5.4 Conclusion

This chapter presented a study on the stability of the Kawahara equation with a boundary-damping control of finite memory type. It is shown that such a control is good enough to obtain the desirable property, namely, the exponential decay of the system's energy. The proof is based on two different approaches. The first one invokes a Lyapunov functional and provides an estimate of the energy decay. In turn, the second one uses a compactness-uniqueness argument that reduces the issue to a spectral problem.

Finally, we would like to point out that our well-posedness result (see Theorem 5.2.2) is shown for the nonlinearity $\omega^p \partial_x \omega$, where $p \in \{1, 2\}$. Notwithstanding, we believe that using an interpolation argument, this finding should remain valid if $p \in (1, 2)$. The same remark applies to the second stability result (see Theorem 5.1.2). It is also noteworthy that our first stability outcome (see Theorem 5.1.1) is established for a more general nonlinearity $\omega^p \partial_x \omega$, $p \in [1, 2]$.

6 Infinite memory effects on the stabilization of a Biharmonic Schrödinger equation

6.1 Introduction

6.1.1 Problem setting

Fourth-order nonlinear Schrödinger equation (4NLS) or biharmonic cubic nonlinear Schrödinger equation

$$i\partial_t y + \Delta y - \Delta^2 y = \lambda |y|^2 y, \tag{6.1}$$

has been introduced by Karpman [60] and Karpman and Shagalov [59] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (6.1) arises in many scientific fields such as quantum mechanics, nonlinear optics, and plasma physics, and has been intensively studied with fruitful references (see [8, 60, 78] and references therein).

Over the past twenty years, equation (6.1) has been deeply studied from a different mathematical viewpoint, including linear settings which can be written generically as

$$i\partial_t y + \alpha\Delta y - \beta\Delta^2 y = F, \tag{6.2}$$

with $\alpha, \beta \geq 0$ and different types of boundary conditions. For example, considering the problem (6.2) several authors treated this equation, see, for instance, [3, 49, 90, 95, 96, 99] and the references therein. Inspired by these results for the linear problem associated with the 4NLS, a mathematical viewpoint problem is to study the well-posedness and stabilization for solutions of the system (6.2) in an appropriate framework.

So, consider the equation (6.2) when $\alpha = \beta = 1$ in a n-dimensional open bounded subset of $\mathbb{R}^n$. Our goal is to consider an initial boundary value problem (IBVP) associated with (6.2) when the source term f is viewed as an infinite memory term:

$$F = -(-1)^j i \int_0^\infty f(s)\Delta^j y(x, t-s)ds.$$

Thus, the goal of this chapter is to deal with the following system

$$\begin{cases} i\partial_t y(x,t) + \Delta y(x,t) - \Delta^2 y(x,t) & \\ +(-1)^j i \displaystyle\int_0^\infty f(s)\Delta^j y(x,t-s)ds = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ y(x,t) = \nabla y(x,t) = 0, & (x,t) \in \Gamma \times \mathbb{R}_+^*, \\ y(x,-t) = y_0(x,t), & (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{6.3}$$

where $j \in \{0, 1, 2\}$, $\Omega \subset \mathbb{R}^n$ is a n-dimensional open bounded domain with a smooth boundary Γ, and $f : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$ is the kernel (or relaxation) function. We point out that for each j the memory term present in (6.3) is modified.

In (6.3), the memory kernel f satisfies the following assumptions:

Assumption 3. *Consider $f \in C^2(\mathbb{R}_+)$. For some positive constant c_0, we have the following conditions*

$$f' < 0, \quad 0 \leq f'' \leq -c_0 f', \quad f(0) > 0 \quad \textit{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{6.4}$$

Under the Assumption 3, let us introduce the following energy functionals associated with the solutions of (6.3)

$$E_j(t) = \frac{1}{2}\left(\|y\|^2 + \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds\right), \tag{6.5}$$

with $j \in \{0,1,2\}$ and $g = -f'$, so $g \in C^1(\mathbb{R}_+)$, g is non-negative and

$$g_0 := \int_0^\infty g(s)ds = f(0) \in \mathbb{R}_+^*.$$

It is worth mentioning that the abuse of notation $\Delta^{\frac{j}{2}}$ in (6.5) means the identity operator for $j = 0$, the ∇ operator for $j = 1$ and the laplacian operator for $j = 2$.

Therefore, taking into account the action of the infinite memory term in (6.3), the following issue will be addressed in this chapter:

Problem 6.1. *Does $E(t) \longrightarrow 0$, as $t \to \infty$? If so, can we provide a decay rate?*

It should be noted that the answer to the above question is crucial in the understanding of the behavior of the solutions to the fourth-order Schrödinger system when it is subject to an infinite memory term. In other words:

Problem 6.2. *Are the solutions to our problem stable despite the action of the memory term? If yes, then how robust is the stabilization property of the solutions?*

6.1.2 Historical background

Distributed systems with memory have a long history and have been first introduced in viscoelasticity by Maxwell, Boltzmann, and Volterra [11,12,72,94]. In the context of heat processes with finite dimension speed, these systems have been introduced by Cattaneo [29] (a previous work of Maxwell had been forgotten).

In our context, to our knowledge, there is no result considering the system (6.3) in n–dimensional case. However, considering the fourth-order Schrödinger system

$$i\partial_t u + \Delta^2 u = 0, \tag{6.6}$$

there are interesting results in the sense of control problems in a bounded domain of $\mathbb{R}$ or $\mathbb{R}^n$ and, more recently, on a periodic domain $\mathbb{T}$ and manifolds, which we will summarize below.

The first result about the exact controllability of the linearized fourth order Schrödinger equation (6.6) on a bounded domain Ω of $\mathbb{R}^n$ is due to Zheng and Zhongcheng in [100]. In this work, using an L^2–Neumann boundary control, the authors proved that the solution is exactly controllable in $H^s(\Omega)$, $s=-2$, for an arbitrarily small time. They used Hilbert Uniqueness Method (HUM) (see, for instance, [43, 68]) combined with the multiplier techniques to get the main result of the article. More recently, in [99], Zheng proved a global Carleman estimate for the fourth-order Schrödinger equation posed on a finite domain. The Carleman estimate is used to prove the Lipschitz stability for an inverse problem associated with the fourth-order Schrödinger system.

Still, on control theory Wen *et al.* in two works [95, 96], studied well-posedness and control problems related to the equation (6.6) on a bounded domain of $\mathbb{R}^n$, for $n\geq 2$. In [95], they considered the Neumann boundary controllability with collocated observation. With this result in hand, the stabilization of the closed-loop system under proportional output feedback control holds. Recently, the same authors, in [96], gave positive answers when considering the equation with hinged boundary by either moment or Dirichlet boundary control and collocated observation, respectively.

To get a general outline of the control theory already done for the system (6.6), two interesting problems were studied recently by Aksas and Rebiai [3] and Gao [49]: Uniform stabilization and stochastic control problem, in a smooth bounded domain Ω of $\mathbb{R}^n$ and on the interval $I=(0,1)$ of $\mathbb{R}$, respectively. In the first work, by introducing suitable dissipative boundary conditions, the authors proved that the solution decays exponentially in $L^2(\Omega)$ when the damping term is effective on a neighborhood of a part of the boundary. The results are established by using multiplier techniques and compactness/uniqueness arguments. Regarding the second work, the author showed Carleman estimates for forward and backward stochastic fourth order Schrödinger equations which provided the proof of the observability inequality, unique continuation property, and, consequently, the exact controllability for the forward and backward stochastic system associated with (6.6).

Recently, the authors of [18] showed the global stabilization properties and exact controllability of 4NLS

$$\begin{cases} i\partial_t u+\partial_x^2 u-\partial_x^4 u=\lambda|u|^2u+f(x,t), & (x,t)\in\mathbb{T}\times\mathbb{R},\\ u(x,0)=u_0(x), & x\in\mathbb{T}, \end{cases} \tag{6.7}$$

on a periodic domain $\mathbb{T}$ with internal control supported on an arbitrary sub-domain of $\mathbb{T}$. More precisely, by certain properties of propagation of compactness and regularity in Bourgain spaces, for the solution of the associated linear system, the authors proved that system (6.7) is globally exponentially stabilizable, considering $f(x,t)=-ia^2(x)u$. This property together with the local exact controllability ensures that 4NLS is globally exactly controllable on $\mathbb{T}$.

Lastly, the authors of [26] showed the global controllability and stabilization properties for the fractional Schrödinger equation on d-dimensional compact Riemannian manifolds without boundary (M, g),

$$\begin{cases} i\partial_t u + \Lambda_g^\sigma u + P'(|u|^2)u - a(x)(1-\Delta_g)^{-\frac{\sigma}{2}} a(x)\partial_t u = 0, & \text{on } M \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in M. \end{cases} \tag{6.8}$$

Under the suitable assumption of the damping term $a(x)$ they proved their result using microlocal analysis, being precise, they can prove propagation of regularity which together with the so-called Geometric Control Condition and Unique Continuation Property, shows the main results of the article. Is important to mention that when $\sigma = 4$ they have the equation (6.6).

6.1.3 Notations

Before presenting the main result let us give some notations and definitions. In what follows, the variables x, t, and s will be suppressed, except when there is ambiguity and, throughout this chapter, C will denote a constant that can be different from one step to the next in the proofs presented here. We will use the notations $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote, respectively, the complex inner product in $L^2(\Omega)$ and its associated standard norm, namely

$$\langle u, v \rangle = \mathrm{Re}\left(\int_\Omega u(x)\overline{v}(x)dx\right) \text{ and } \|u\| = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Now, consider the following approach

$$\eta^t(x,s) = \int_{t-s}^{t} y(x,\tau)d\tau \text{ and } \eta^0(x,s) = \int_0^s y_0(x,\tau)d\tau,\ x \in \Omega, s, t \in \mathbb{R}_+.$$

This approach ensures that η^t satisfies

$$\begin{cases} \partial_t \eta^t(x,s) + \partial_s \eta^t(x,s) = y(x,t), & x \in \Omega,\ s,t \in \mathbb{R}_+, \\ \eta^t(x,s) = 0, & x \in \Gamma,\ s,t \in \mathbb{R}_+, \\ \eta^t(x,0) = 0, & x \in \Omega,\ t \in \mathbb{R}_+. \end{cases} \tag{6.9}$$

To express the memory integral in (6.3) in terms of η^t, we will denote $g := -f'$. Thus, according to (6.4), we have $g \in C^1(\mathbb{R}_+)$ and

$$g > 0, \quad 0 \le -g' \le c_0 g, \quad g_0 = \int_0^\infty g(s)ds = f(0) > 0 \tag{6.10}$$

and

$$\lim_{s\to\infty} g(s) = 0. \tag{6.11}$$

Now on, rewrite (6.3) into

$$i\partial_t y(x,t) + \Delta y(x,t) - \Delta^2 y(x,t) + i(-1)^j \int_0^\infty g(s)\Delta^j \eta^t(x,s)ds = 0. \tag{6.12}$$

Define the following sets

$$H_j = \begin{cases} L^2(\Omega), & \text{if } j=0, \\ H_0^1(\Omega), & \text{if } j=1, \\ H_0^2(\Omega), & \text{if } j=2, \end{cases}$$

with natural inner product

$$\langle v, w\rangle_{H_j} = \begin{cases} \langle v(s), w(s)\rangle & \text{if } j=0, \\ \langle \nabla v(s), \nabla w(s)\rangle & \text{if } j=1, \\ \langle \Delta v(s), \Delta w(s)\rangle & \text{if } j=2 \end{cases}$$

and norm

$$\|v\|_{H_j} = \begin{cases} \|v(s)\| & \text{if } j=0, \\ \|\nabla v(s)\| & \text{if } j=1, \\ \|\Delta v(s)\| & \text{if } j=2, \end{cases}$$

respectively[1]. Consider

$$U = (y, \eta^t)^T \text{ and } U_0(x,s) = (y_0(x,0), \eta^0(x,s))^T$$

where

$$y \in L^2(\Omega) \quad \text{and} \quad \eta^t \in L_j$$

with

$$L_j = L_g^2(\mathbb{R}_+; H_j) := \left\{ v : \mathbb{R}_+ \longrightarrow H_j;\, \int_0^\infty g(s)\|v(s)\|_{H_j}^2 ds < +\infty \right\}.$$

Define the energy space as follows

$$\mathcal{H}_j = L^2(\Omega) \times L_j,\ j \in \{0,1,2\},$$

with inner product and norm

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{\mathcal{H}_j} = \langle v_1, w_1\rangle + \langle v_2, w_2\rangle_{L_j}$$

and

$$\|(v(s), w(s))\|_{\mathcal{H}_j} = \left(\|v(s)\|^2 + \|w(s)\|_{L_j}^2\right)^{\frac{1}{2}},$$

respectively. Therefore, the systems (6.3) and (6.9) can be seen as the following initial value problem (IVP)

$$\begin{cases} \partial_t U(t) &= \mathcal{A}_j U \\ U(0) &= U_0. \end{cases} \tag{6.13}$$

Here, the operator $\mathcal{A}_j$ is defined by

$$\mathcal{A}_j(U) = \begin{pmatrix} i\Delta y - i\Delta^2 y + (-1)^{j+1}\int_0^\infty g(s)\Delta^j \eta^t(\cdot, s) ds \\ \\ y - \eta_s^t \end{pmatrix} \tag{6.14}$$

[1] Here $\langle \nabla v(s), \nabla w(s)\rangle := \sum_{k=1}^n \langle \partial_{x_k} v, \partial_{x_k} w\rangle$ and $\|\nabla v(s)\|^2 = \sum_{k=1}^n \|\partial_{x_k} v(s)\|^2$.

with domain

$$D(\mathcal{A}_j) = \{U \in \mathcal{H}_j; \mathcal{A}_j(U) \in \mathcal{H}_j, y \in H_0^2(\Omega), \eta^t(x,0) = 0\}. \tag{6.15}$$

Remark 6.1.1. *Observe that for the fourth-order Schrödinger equation, the natural domain to be considered is $H_0^2(\Omega) \cap H^4(\Omega)$. However, since we are working with a more general operator, namely operator defined in (6.14) and (6.15), we need to impose $\mathcal{A}_j(U) \in \mathcal{H}_j$. However, note that the inclusion below*

$$\left(H_0^2(\Omega) \cap H^4(\Omega)\right) \times \{\eta^t \in L_j : (-1)^{j+1} \int_0^\infty g(s)\Delta^j \eta^t(\cdot,s)ds \in L^2(\Omega), \eta^t(x,0) = 0\} \subset D(\mathcal{A}_j)$$

is verified. So, the operator $\mathcal{A}_j(U)$ is well-defined.

6.1.4 Main result

As mentioned, some valuable efforts in the last years focus on the well-posedness and stabilization problem for the fourth-order Schrödinger system. So, in this chapter, we present a new way to ensure that, in some sense, the Problems 6.1 and 6.2 can be solved for the system (6.3) in n-dimensional case. To do that, we use the ideas contained in [50], so additionally to the Assumption 3 we have also assumed the memory kernel satisfying the following:

Assumption 4. *Assume there is a positive constant α_0 and a strictly convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying*

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} G'(t) = \infty \tag{6.16}$$

such that

$$g' \leq -\alpha_0 g \tag{6.17}$$

or

$$\int_0^\infty \frac{s^2 g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{6.18}$$

Additionally, when (6.17) is not verified, we will assume that y_0 satisfies,

$$\sup_{t\in\mathbb{R}_+} \max_{k\in\{0,\dots,n+1\}} \int_t^\infty \frac{g(s)}{G^{-1}(-g'(s))} \left\| \int_0^{s-t} \Delta^{\frac{j}{2}} \partial_s^k y_0(\cdot,\tau)d\tau \right\|^2 ds < \infty. \tag{6.19}$$

for $j \in \{0,1,2\}$.

The next theorem is the main result of the chapter.

Theorem 6.1.1. *Assume (6.10) and that the Assumption 4 holds. Let $n \in \mathbb{N}^*$, $U_0 \in D(\mathcal{A}_j^{2n})$ when $j = 0$, and $U_0 \in D(\mathcal{A}_j^{2n+2})$ when $j \in \{1,2\}$. Thus, there exists positive constants $\alpha_{j,n}$ such that the energy (6.5) associated with (6.13) satisfies*

$$E_j(t) \leq \alpha_{j,n} G_n\left(\frac{\alpha_{j,n}}{t}\right), \quad t \in \mathbb{R}_+^*, \ j \in \{0,1,2\}. \tag{6.20}$$

Here, G_n is defined, recursively, as follows:

$$G_m(s) = G_1(sG_{m-1}(s)), m = 2, 3, \dots, n, \ G_1 = G_0^{-1}, \tag{6.21}$$

where $G_0(s) = s$ if (6.17) is verified, and $G_0(s) = sG'(s)$ if (6.18) holds.

Remark 6.1.2. *Let us give some remarks about the Assumption 4.*

i. *Thanks to the relation (6.18), we have that (6.19) is valid, for example, if*

$$\|\Delta^{\frac{j}{2}}\partial_s^k y_0\|^2, \ k = 0, 1, \dots, n+1,$$

is bounded with respect to s.

ii. *There are many class of function g satisfying (6.10), (6.11), (6.16), (6.17), (6.18), and (6.19). For example, those that converge exponentially to zero as*

$$g_1(s) := d_1 e^{-q_1 s} \tag{6.22}$$

or those that converge at a slower rate, like

$$g_2(s) := d_2(1+s)^{-q_2} \tag{6.23}$$

with d_1, $q_1, d_2 > 0$, and $q_2 > 3$. Additionally, we point out that conditions (6.10) and (6.17) are satisfied for g_1 defined by (6.22) with $c_0 = \alpha_0 = q_1$, since

$$g_1'(s) = -q_1 d_1 e^{-q_1 s} = -q_1 g_1(s).$$

However, the conditions (6.10) and (6.18) are satisfied for g_2 given by (6.23) with $c_0 = q_2$ and $G(s) = s^p$, for $p > \dfrac{q_2+1}{q_2-3}$.

Remark 6.1.3. *Now, we will present the following remarks related to the main result of the chapter.*

i. *When (6.17) is verified, note that $G_n(0) = 0$, so (6.20) implies*

$$\lim_{t\to\infty} E_j(t) \le \alpha_{j,1} G_1\left(\frac{\alpha_{j,1}}{t}\right) = 0. \tag{6.24}$$

Since we have that $D(\mathcal{A}_j^2)$ is dense in $\mathcal{H}_j$, when $j = 0$, and $D(\mathcal{A}_j^4)$ is dense in $\mathcal{H}_j$ when $j = 1, 2$ (see Lemma 6.4.1 in 6.4), we have that (6.24) is valid for any $U_0 \in \mathcal{H}_j$. Therefore, in this case, (6.21) gives $G_n(s) = s^n$ and from (6.20) we get

$$E_j(t) \le \alpha_{j,n}\left(\frac{\alpha_{j,n}}{t}\right)^n = \frac{(\alpha_{j,n})^{n+1}}{t^n} = \beta_{j,n} t^{-n},$$

showing that the energy (6.5) associated with the solutions of the system (6.13) have a polynomial decay rate.

ii. *Given* (6.18) *verified, the relation of* (6.20) *is weaker than the previous case. For example, when* $g = g_2$ *defined by* (6.23), *we see that* $G(s) = s^p$ *with* $p > \dfrac{q_2+1}{q_2-3}$ *satisfies the Assumption 4. Moreover,*

$$G_0(s) = sG'(s) = ps^p, \; G_1(s) = \sqrt[p]{\frac{s}{p}},$$

$$G_2(s) = G_1(sG_1(s)) = \sqrt[p]{\frac{s\sqrt[p]{\frac{s}{p}}}{p}} = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}},$$

$$G_3(s) = G_1(sG_2(s)) = \sqrt[p]{\frac{s}{p}\left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}}} = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}}$$

and so,

$$G_n(s) = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^n}} = \left(\frac{s}{p}\right)^{p_n},$$

where $p_n = \sum_{m=1}^{n} p^{-m} = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}$. *Therefore, the energy* (6.5) *associated with the solutions of the system* (6.13) *satisfies*

$$E_j(t) \leq \alpha_{j,n}\left(\frac{1}{p}\frac{\alpha_{j,n}}{t}\right)^{p_n} = \beta_{j,n}t^{-p_n},$$

with $\beta_{j,n} = \alpha_{j,n}\left(\dfrac{\alpha_{j,n}}{p}\right)^{p_n} > 0$, *showing that the decay rate of* (6.20) *is arbitrarily near of* t^{-n}, *when* $p \to 1$, *that is,* $p_n \to n$ *when* $q_2 \to \infty$.

6.1.5 Novelty and structure of the chapter

Among the main novelties introduced in this chapter, we give an affirmative answer to the Problems 6.1 and 6.2, providing a further step toward a better understanding of the stabilization problem for the linear system associated with (6.1) in the n-dimensional case. Here, we have used the multipliers method and some arguments devised in [50].

Since we are working with a mixed dispersion we can consider three different memory kernels acting as damping control to stabilize equation (6.3) in contrast to [18], for example, where interior damping is required and no memory is taken into consideration, in a one-dimensional case. Moreover, if we also compare with the linear Schrödinger equation (see e.g. [30]) we have more kernels acting to decay the solution of the equation (6.3) since we have more regularity with the mixed dispersion, which is a gain due the bi-laplacian operator.

Additionally of this, recently, using another approach, the authors in [26] showed that the system (6.8) is stable, however considering a damping mechanism and some important assumptions such as the Geometric Control Condition (GCC) and Unique

Continuation Property (UCP). Here, we are not able to prove that the solutions decay exponentially, however, with the approach of this chapter, the (GCC) and (UCP) are not required. The drawback is that we only provide that the energy of the system (6.3), with memory terms, decays in some sense as explained in the Remark 6.1.3.

A natural issue is how to deal with the 4NLS system given in (6.1). The main point is that we are not able to use Strichartz estimates or Bourgain spaces to obtain more regularity for the solution of the problem with memory terms, therefore, Theorem 6.1.1 for the system (6.1) with memory terms remains open.

Now, let us present the outline of this chapter. In Section 6.2 we prove a series of lemmas that are paramount to prove the main result of the chapter. With the previous section in hand, Theorem 6.1.1 is shown in Section 6.3. Finally, for the sake of completeness, in Appendix 6.4, we present the existence of a solution for the system (6.13) in the energy space $\mathcal{H}_j$.

6.2 Auxiliary results

In this section, we will give some auxiliary lemmas that help us to prove the main result of the chapter. In this way, the first result shows identities for the derivatives of E_j given by (6.5).

Lemma 6.2.1. *Suppose the Assumption 3. Then, the energy functional satisfies*

$$E_j'(t) = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds, \ \ j \in \{0,1,2\}. \tag{6.25}$$

Proof. Observe that (6.25) is a direct consequence of (6.62), and the result follows. □

Next, we will give a H^1-estimate for the solution of (6.12).

Lemma 6.2.2. *There exist positive constants $c_{k,j}$, $j \in \{0,1,2\}$ and $k \in \{1,2\}$ such that the following inequality*

$$\|\nabla y\|^2 \leq c_{1,j}\|\eta^t\|^2_{L_j} + c_{2,j}\int_\Omega [Re(y_t)Im(y) - Im(y_t)Re(y)]\, dx, \tag{6.26}$$

holds.

Proof. We use the multipliers method to prove (6.26). First, multiplying the equation (6.12) by $\overline{y}$, integrating over Ω and taking the real part we get

$$-\mathrm{Im}\left(\int_\Omega y_t\overline{y}dx\right) - \|\nabla y\|^2 - \|\Delta y\|^2 + \mathrm{Re}\left((-1)^j i\int_0^\infty g(s)\int_\Omega \Delta^j\eta^t\overline{y}dxds\right) = 0, \tag{6.27}$$

taking into account the boundary conditions in (6.3) and (6.9), for $y(t,\cdot) \in H_0^2(\Omega)$, for all $t \in \mathbb{R}^+$.

Note that the last term of the left-hand side of (6.27) can be bounded using the generalized Young's Inequality giving

$$\begin{aligned}\left|(-1)^j i \int_0^\infty g(s) \int_\Omega \Delta^j \eta^t \overline{y} dx ds\right| &= \left|i\langle \eta^t, y\rangle_{L_j}\right| \\ &\le \|\eta^t\|_{L_j}\|y\|_{L_j} \\ &\le \epsilon\|y\|^2_{L_j} + C(\epsilon)\|\eta^t\|^2_{L_j} \\ &= \underbrace{g_0\epsilon}_{=:\delta}\|\Delta^{\frac{j}{2}} y\|^2 + C(\epsilon)\|\eta^t\|^2_{L_j} \\ &= \delta\|\Delta^{\frac{j}{2}} y\|^2 + C(\delta)\|\eta^t\|^2_{L_j},\end{aligned} \tag{6.28}$$

for any $\delta > 0$. Additionally of that, the first term of the left-hand side of (6.27) can be viewed as

$$\operatorname{Im}\left(\int_\Omega y_t \overline{y} dx\right) = \int_\Omega \left(\operatorname{Re}(y)\operatorname{Im}(y_t) - \operatorname{Re}(y_t)\operatorname{Im}(y)\right) dx. \tag{6.29}$$

So, replacing (6.28) and (6.29) in (6.27), yields

$$\begin{aligned}\|\nabla y\|^2 &\le \int_\Omega \left(\operatorname{Re}(y_t)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(y_t)\right) dx - \|\Delta y\|^2 + \delta\|\Delta^{\frac{j}{2}} y\|^2 + C(\delta)\|\eta^t\|^2_{L_j} \\ &\le \int_\Omega \left(\operatorname{Re}(y_t)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(y_t)\right) dx + \delta\|\Delta^{\frac{j}{2}} y\|^2 + C(\delta)\|\eta^t\|^2_{L_j}.\end{aligned} \tag{6.30}$$

We now split the remainder of the proof into three cases.

Case 1. $j = 0$

Poincaré inequality in (6.30) gives

$$\|\nabla y\|^2 \le \int_\Omega \left(\operatorname{Re}(y_t)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(y_t)\right) dx + \delta c_*\|\nabla y\|^2 + C(\delta)\|\eta^t\|^2_{L_j}. \tag{6.31}$$

Picking $\delta = \dfrac{1}{2c_*} > 0$ in (6.31) yields

$$\frac{1}{2}\|\nabla y\|^2 \le \int_\Omega \left(\operatorname{Re}(y_t)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(y_t)\right) dx + C(\delta)\|\eta^t\|^2_{L_j},$$

showing (6.26) with $c_{1,0} = 2C(\delta)$ and $c_{2,0} = 2$.

Case 2. $j = 1$

In this case (6.30) is giving by

$$\|\nabla y\|^2 \le \int_\Omega \left(\operatorname{Re}(y_t)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(y_t)\right) dx + \delta\|\nabla y\|^2 + C(\delta)\|\eta^t\|^2_{L_j}$$

and taking $\delta = \frac{1}{2} > 0$, the inequality (6.26) holds with $c_{1,1} = 2C(\delta)$ and $c_{2,1} = 2$.

Case 1. $j = 2$

Finally, just take any $\delta > 0$ such that $\delta < 1$. Therefore, using (6.30) we get (6.26) for $c_{1,2} = C(\delta)$ and $c_{2,2} = 1$, achieving the result. □

We need now define the following higher-order energy functionals

$$E_{j,k}(t) = \frac{1}{2}\left\|\partial_t^k U\right\|_{\mathcal{H}_j}^2, \tag{6.32}$$

for $U_0 \in D(\mathcal{A}_j^{2n+2})$ in the case when $j = 1, 2$, and $U_0 \in D(\mathcal{A}_0^{2n})$ when $j = 0$ with $n \in \mathbb{N}^*$. This is possible thanks to the Theorem 6.4.1 in 6.4 that guarantees $U \in C^k(\mathbb{R}_+; D(\mathcal{A}_j^{4-k}))$ for $k \in \{1, 2, 3, 4\}$ when $j \in \{1, 2\}$, and that $U \in C^k(\mathbb{R}_+; D(\mathcal{A}_j^{2-k}))$ for $k \in \{1, 2\}$ when $j = 0$. Additionally of that, the linearity of the operator $\mathcal{A}_j$ together with (6.25) gives

$$E'_{j,k}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds. \tag{6.33}$$

With this in hand, let us control the last term of the right-hand side of (6.26) in terms of the $E'_{j,1}$ and the L_j-norms of the $\Delta^{\frac{j}{2}}\eta_{tt}^t$.

Lemma 6.2.3. *The following estimate is valid*

$$\int_\Omega (Re(y_t)Im(y) - Re(y)Im(y_t))\,dx \le \epsilon\|\nabla y\|^2 + c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta_{tt}^t\|^2 ds - c_\epsilon E'_{j,1}(t), \tag{6.34}$$

for any $\epsilon > 0$.

Proof. Differentiating (6.9) with respect to t, multiplying the result by $g(s)$, and integrating on $[0, \infty)$ we have

$$y_t = \frac{1}{g_0}\int_0^\infty g(s)\left(\eta_{tt}^t(s,x) + \eta_{st}^t(s,x)\right) ds,$$

taking into account the third relation in (6.10). So, we get

$$\begin{aligned}
\mathcal{I} &:= \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\,dx \\
&= \int_\Omega \mathrm{Re}\left(\frac{1}{g_0}\int_0^\infty g(s)\left(\eta_{tt}^t + \eta_{st}^t\right) ds\right)\mathrm{Im}(y)dx \\
&\quad - \int_\Omega \mathrm{Re}(y)\mathrm{Im}\left(\frac{1}{g_0}\int_0^\infty g(s)\left(\eta_{tt}^t + \eta_{st}^t\right) ds\right) dx.
\end{aligned} \tag{6.35}$$

Now, let us bound the right-hand side of (6.35). To do that, reorganize the terms of the (RHS) and note that

$$\begin{aligned}
(RHS) =& \frac{1}{g_0}\int_0^\infty g(s)\int_\Omega \left(\mathrm{Re}\left(\eta_{tt}^t\right)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}\left(\eta_{tt}^t\right)\right) dxds \\
&+ \frac{1}{g_0}\int_0^\infty (-g'(s))\int_\Omega \left(\mathrm{Re}\left(\eta_t^t\right)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}\left(\eta_t^t\right)\right) dxds \\
\le& \frac{1}{g_0}\int_0^\infty g(s)\int_\Omega |y||\eta_{tt}^t|dxds + \frac{1}{g_0}\int_0^\infty (-g'(s))\int_\Omega |y||\eta_t^t|dxds \\
\le& \frac{1}{g_0}\int_0^\infty g(s)\|y\|\|\eta_{tt}^t\|ds + \frac{1}{g_0}\int_0^\infty (-g'(s))\|y\|\|\eta_t^t\|ds.
\end{aligned} \tag{6.36}$$

The generalized Young inequality gives for any $\delta > 0$ that

$$\|y\|\|\eta_t^t\| \leq \delta\|y\|^2 + C_\delta\|\eta_t^t\|^2$$

and

$$\|y\|\|\eta_{tt}^t\| \leq \delta\|y\|^2 + C_\delta\|\eta_{tt}^t\|^2.$$

Substituting both inequalities into (6.36) yields

$$\begin{aligned}(RHS) \leq & \delta\frac{1}{g_0}\int_0^\infty g(s)\|y\|^2 ds + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2 ds \\ & + \delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|y\|^2 ds + C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2 ds.\end{aligned} \tag{6.37}$$

Now replacing (6.37) into (6.35) we have

$$\begin{aligned}\mathcal{I} \leq & \delta\frac{1}{g_0}\int_0^\infty g(s)\|y\|^2 ds + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2 ds \\ & + \delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|y\|^2 ds + C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2 ds \\ = & \delta\left(1 + \frac{1}{g_0}\left(\int_0^\infty(-g'(s))ds\right)\right)\|y\|^2 + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2 ds \\ & + C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2 ds \\ \leq & c_*\delta\left(1 + \frac{1}{g_0}\left(\int_0^\infty(-g'(s))ds\right)\right)\|\nabla y\|^2 + c_{**}C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta_{tt}^t\|^2 ds \\ & + c_{**}C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\Delta^{\frac{j}{2}}\eta_t^t\|^2 ds,\end{aligned} \tag{6.38}$$

thanks to Poincaré inequality. Here,

$$c_{**} = \begin{cases} 1, & \text{if } j = 0, \\ c_*, & \text{if } j = 1, \\ c_*^2, & \text{if } j = 2, \end{cases} \tag{6.39}$$

and $c_* > 0$ is the Poincaré constant. Finally, taking $k = 1$ in (6.33), we see that (6.38) leads to (6.34) with $\epsilon = c_*\delta\left(1 + \frac{1}{g_0}\left(\int_0^\infty(-g'(s))ds\right)\right)$ and $c_\epsilon = c_{**}C_\delta\frac{1}{g_0}$. □

Now, just in the case $j = 2$, we need an estimate H^2-for the solution of (6.12) similar to the estimate (6.26). This estimate is reported in the following lemma.

Lemma 6.2.4. *When $j = 2$, there exist positive constants $c_{k,2}$, $k \in \{1, 2\}$, such that the following inequality*

$$\|\Delta y\|^2 \leq c_{1,2}\|\eta^t\|_{L_2}^2 + c_{2,2}\int_\Omega \left[Re(y_t)Im(y) - Im(y_t)Re(y)\right] dx \tag{6.40}$$

holds.

Proof. Multiplying equation (6.12) by $\overline{y}$, integrating over we have

$$0 = i \int_{\Omega} y_t \overline{y} dx - \|\nabla y\|^2 - \|\Delta y\|^2 + i \int_0^{\infty} g(s) \int_{\Omega} \Delta^2 \eta^t \overline{y} dx ds,$$

since the boundary conditions (6.3) and (6.9) are verified and $y(t,\cdot) \in H_0^2(\Omega)$ for all $t \in \mathbb{R}^+$. Now, taking the real part in the previous equality give us

$$-\mathrm{Im}\left(\int_{\Omega} y_t \overline{y} dx\right) - \|\nabla y\|^2 - \|\Delta y\|^2 + \mathrm{Re}\left(i \int_0^{\infty} g(s) \int_{\Omega} \Delta^2 \eta^t \overline{y} dx ds\right) = 0. \tag{6.41}$$

Taking into account that

$$\mathrm{Im}\left(\int_{\Omega} y_t \overline{y} dx\right) = \int_{\Omega} \left(\mathrm{Re}(y)\mathrm{Im}(y_t) - \mathrm{Re}(y_t)\mathrm{Im}(y)\right) dx \tag{6.42}$$

and, thanks to the generalized Young inequality, we have that

$$\begin{aligned} \left| i \int_0^{\infty} g(s) \int_{\Omega} \Delta^2 \eta^t \overline{y} dx ds \right| &= |i \langle \eta^t, y \rangle_{L_2}| \leq \|y\|_{L_2} \|\eta^t\|_{L_2} \\ &\leq \underbrace{g_1 \epsilon}_{=:\delta} \|\Delta y\|^2 + C(\epsilon) \|\eta^t\|_{L_2}^2 = \delta \|\Delta y\|^2 + C(\delta) \|\eta^t\|_{L_2}^2. \end{aligned} \tag{6.43}$$

We get, putting (6.42) and (6.43) into (6.41), that

$$\|\Delta y\|^2 \leq \int_{\Omega} \left(\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t)\right) dx + \delta \|\Delta y\|^2 + C(\delta) \|\eta^t\|_{L_2}^2. \tag{6.44}$$

Finally, pick $\delta = \frac{1}{2} > 0$ in (6.44) to get

$$\frac{1}{2} \|\Delta y\|^2 \leq \int_{\Omega} \left(\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t)\right) dx + C(\delta) \|\eta^t\|_{L_2}^2,$$

showing (6.40) with $c_{1,2} = 2C(\delta)$ and $c_{2,2} = 2$. □

As a consequence of (6.34), the last term of the right-hand side of (6.40) can be bounded as follows.

Lemma 6.2.5. *For any $\epsilon > 0$, we have the following inequality*

$$\int_{\Omega} \left(Re(y_t)Im(y) - Re(y)Im(y_t)\right) dx \leq \epsilon \|\Delta y\|^2 + c_{\epsilon} \int_0^{\infty} g(s) \|\Delta \eta_{tt}^t\|^2 ds - c_{\epsilon} E_{2,1}'(t). \tag{6.45}$$

Proof. Using Poncaré inequality in the first term of the right-hand side of (6.34), and taking $\epsilon = c^* \epsilon$, where c^* is the Poincaré constant, the result follows. □

The next lemma combines the previous one to get an estimate in H_j for solutions of (6.12).

Lemma 6.2.6. *There exist a positive constant $c = c(j) > 0$, with $j \in \{1, 2\}$ such that*

$$\|\Delta^{\frac{j}{2}} y\|^2 \leq c(E_j(0)) + E_{j,1}(0) + E_{j,2}(0)). \tag{6.46}$$

Proof. Pick $\epsilon = \dfrac{1}{2c_{2,j}}$ in (6.34) and (6.45) when $j=1$ and $j=2$, respectively. So we have

$$\int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\,dx \leq \frac{1}{2c_{2,j}}\|\Delta^{\frac{j}{2}} y\|^2 + c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - c_\epsilon E'_{j,1}(t).$$

Replacing the previous inequality in (6.26) and in (6.40) for $j=1$ and $j=2$, respectively, we get that

$$\|\Delta^{\frac{j}{2}} y\|^2 \leq c_{1,j}\|\eta^t\|^2_{L_j} + \frac{1}{2}\|\Delta^{\frac{j}{2}} y\|^2 + c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - c_{2,j}c_\epsilon E'_{j,1}(t). \tag{6.47}$$

Therefore, the properties (6.10) for the function g, together to the fact that $E_{j,k}$, given in (6.32), is non-increasing and (6.33) give us

$$\begin{aligned}
\|\Delta^{\frac{j}{2}} y\|^2 \leq & 2c_{1,j}\|\eta^t\|^2_{L_j} + 2c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t) \\
\leq & c_{j,4}\left(E_j(t) + E_{j,1}(t) + E_{j,2}(t)\right) \\
\leq & c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right)
\end{aligned}$$

where $c = c(j) := c_{j,4} = \max\{4c_{1,j}, 4c_{2,j}c_\epsilon, 2c_0c_{2,j}c_\epsilon\}$, for $j \in \{1,2\}$, proving the lemma. $\square$

Before presenting the main result of this section, the next result ensures that the following norms $\|\Delta^{\frac{j}{2}}\eta^t\|$, $\|\eta^t\|$, and $\|\eta^t_{tt}\|$ can be controlled by the generalized energies $E_{k,j}(0)$ and the initial condition y_0, for $t \geq s \geq 0$. The result is the following one.

Lemma 6.2.7. *Considering the hypothesis of the Lemma 6.2.6, the following inequality holds*

$$\|\Delta^{\frac{j}{2}}\eta^t\|^2 \leq M_{j,0}(t,s), \tag{6.48}$$

where

$$M_{j,0}(t,s) := \begin{cases} c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right), & \text{if } 0 \leq s \leq t, \\[2ex] \left\|\displaystyle\int_0^{s-t} \Delta^{\frac{j}{2}} y_0(\cdot,\tau)d\tau\right\|^2 \\ \quad +2s^2 c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right), & \text{if } s > t \geq 0. \end{cases} \tag{6.49}$$

Additionally, for $j=0$, we have

$$\|\eta^t\|^2 \leq M_{0,0}(t,s) := \begin{cases} 2s^2E_0(0), & \text{if } 0 \leq s \leq t, \\[2ex] 2\left\|\displaystyle\int_0^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 4s^2E_0(0), & \text{if } s > t \geq 0 \end{cases} \tag{6.50}$$

and

$$\|\eta^t_{tt}\|^2 \leq M_{0,2}(t,s) := \begin{cases} 2s^2E_{0,2}(0), & \text{if } 0 \leq s \leq t, \\[2ex] 2\left\|\displaystyle\int_0^{s-t} \partial^2_\tau y_0(\cdot,\tau)d\tau\right\|^2 + 4s^2E_{0,2}(0), & \text{if } s > t \geq 0. \end{cases} \tag{6.51}$$

Proof. Let us first prove (6.48). Hölder inequality and (6.46), for$j \in \{1,2\}$, gives that

$$\begin{aligned}\|\Delta^{\frac{j}{2}}\eta^t\|^2 =& \left\|\int_{t-s}^t \Delta^{\frac{j}{2}} y(\cdot,\tau)d\tau\right\|^2 \leq \left(\int_{t-s}^t 1\cdot\|\Delta^{\frac{j}{2}} y(\cdot,\tau)\|d\tau\right)^2 \leq s\left(\int_{t-s}^t \|\Delta^{\frac{j}{2}} y(\cdot,\tau)\|^2 d\tau\right)\\ \leq& s^2 c\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right),\end{aligned}$$

for $t \geq s \geq 0$. Analogously,

$$\begin{aligned}\|\Delta^{\frac{j}{2}}\eta^t\|^2 =& \left\|\int_{t-s}^t \Delta^{\frac{j}{2}} y(\cdot,\tau)d\tau\right\|^2\\ &\leq 2\left\|\int_0^{s-t} \Delta^{\frac{j}{2}} y_0(\cdot,\tau)d\tau\right\|^2 + 2s^2 c\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right),\end{aligned}$$

when $s > t \geq 0$. Consequently, (6.48) is verified.

Now, for $j = 0$, since $\|y\|^2$ is part of E_0 (see (6.5)), and the energy E_0 is non-increasing, we observe, using Hölder inequality, that

$$\begin{aligned}\|\eta^t\|^2 =& \left\|\int_{t-s}^t y(\cdot,\tau)d\tau\right\|^2 \leq \left(\int_{t-s}^t 1\cdot\|y(\cdot,\tau)\|d\tau\right)^2 \leq s\int_{t-s}^t \|y(\cdot,\tau)\|^2 d\tau\\ \leq& s\int_{t-s}^t 2E_0(\tau)d\tau \leq s\int_{t-s}^t 2E_0(0)d\tau = 2s^2 E_0(0),\end{aligned}$$

for $t \geq s \geq 0$. On the other hand,

$$\begin{aligned}\|\eta^t\|^2 =& \left\|\int_0^{s-t} y_0(\cdot,\tau)d\tau + \int_0^t y(\cdot,\tau)d\tau\right\|^2 \leq 2\left\|\int_0^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 2\left\|\int_0^t y(\cdot,\tau)d\tau\right\|^2\\ \leq& 2\left\|\int_0^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 4E_0(0)s^2,\end{aligned}$$

for $s > t \geq 0$. Thus, (6.50) follows.

Finally, let us prove (6.51). To do that, observe that (6.13) is linear and $V = \partial_t^2 U$ is solution for (6.13) with initial condition $V(0)(x,s) = (\partial_t^2 y_0(x,0), \zeta^0(x,s))$, where $\zeta^0(x,s) = \int_0^s \partial_\tau^2 y_0(x,\tau)d\tau$. Thanks to relation (6.49), for $j \in \{1,2\}$, we get that

$$M_{j,2}(t,s) := \begin{cases} c_{j,5}\left(E_{j,2}(0)+E_{j,3}(0)+E_{j,4}(0)\right), \text{ if } 0 \leq s \leq t, \\ \\ 2\left\|\displaystyle\int_0^{s-t} \Delta^{\frac{j}{2}}\partial_\tau^2 y_0(\cdot,\tau)d\tau\right\|^2 \\ \qquad +2s^2 c_{j,5}\left(E_{j,2}(0)+E_{j,3}(0)+E_{j,4}(0)\right), \text{ if } s > t \geq 0, \end{cases} \tag{6.52}$$

and so,

$$\|\eta_{tt}^t\|^2 \leq M_{j,2}(t,s).$$

Therefore, inequality (6.51) follows using the previous inequality with $j = 0$, and thanks to the relation (6.50), the result is proved. □

The next result is the key lemma to establish the stabilization result for the Biharmonic Schrödinger system (6.3).

Lemma 6.2.8. *There exist positive constants $d_{j,k}$, for each $j \in \{0,1,2\}$ and each $k \in \{0,2\}$ such that the following inequality holds*

$$\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)} \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -d_{j,k}E'_{j,k}(t) + d_{j,k}G_0(\epsilon_0 E_j(t)), \tag{6.53}$$

for any $\epsilon_0 > 0$. Here, $E_{j,0} = E_j, E'_{j,0} = E'_j(0)$ and G_0 defined as in Theorem 6.1.1.

Proof. Suppose, first, that the relation (6.17) is satisfied. So, thanks to the relation (6.33), we have

$$E'_{j,k} = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -\frac{1}{2}\alpha_0 \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds,$$

for each $j \in \{0,1,2\}$ and each $k \in \{0,2\}$, that is,

$$\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -\frac{2}{\alpha_0}E'_{j,k},$$

showing (6.53) for each $d_{j,k} = \dfrac{2}{\alpha_0}$ and $G_0(s) = s$.

On the other hand, suppose now that (6.18) and (6.11) are verified. Let us assume, without loss of generality, that $E_j(t) > 0$ and $g' < 0$ in $\mathbb{R}_+$. Let $\tau_{j,k}(t,s), \theta_j(t,s), j \in \{0,1,2\}$, $k \in \{0,2\}$ and ϵ_0 be a positive real number which will be fixed later on, and $K(s) = \dfrac{s}{G^{-1}(s)}$, for $s > 0$. Assumption 4 implies that

$$\lim_{s\to 0^+} K(s) = \lim_{s\to 0^+} \frac{s}{G^{-1}(s)} = \lim_{\tau = G^{-1}(s)\to 0^+} \frac{G(\tau)}{\tau} = G'(0) = 0.$$

Additionally, thanks to the continuity of K we have $K(0) = 0$.

We claim that the function K is non-decreasing. Indeed, since G is convex we have that G^{-1} is concave and $G^{-1}(0) = 0$, implying that

$$K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2}s_2 + \left(1 - \frac{s_1}{s_2}\right)\cdot 0\right)} \leq \frac{s_1}{\frac{s_1}{s_2}G^{-1}(s_2)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2),$$

for $0 \leq s_1 < s_2$, proving the claim.

Now, note that thanks to the fact that K is non-decreasing and by (6.48), (6.50), (6.52), and (6.51), we get

$$K\left(-\theta_{j,k}(t,s)g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2\right) \leq K\left(-\theta_{j,k}(t,s)g'(s)M_{j,k}(t,s)\right). \tag{6.54}$$

Inequality (6.54) yields that

$$
\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds =& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)} G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\times \frac{\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{-\theta_{j,k}g'(s)} K\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds\\
\leq& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)} G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\times \frac{\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{-\theta_{j,k}g'(s)} K\left(-M_{j,k}\theta_{j,k}g'(s)\right) ds\\
\leq& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)} G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\times \frac{M_{j,k}(t,s)\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)} ds.
\end{aligned}
\tag{6.55}
$$

Denote the dual function of G by $G^*(s) = \sup_{\tau\in\mathbb{R}_+}\{s\tau - G(\tau)\}$, for $s \in \mathbb{R}_+$. From the Assumption 4 we have

$$
G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)),\ s \in \mathbb{R}_+.
$$

Observe also that

$$
s_1 s_2 \leq G(s_1) + G^*(s_2),\ \forall s_1, s_2 \in \mathbb{R}_+,
$$

in particular

$$
s_1 = G^{-1}\left(-\theta_{j,k}(t,s)g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)
$$

and

$$
s_2 = \frac{M_{j,k}\tau_{j,k}G'(\epsilon_0 E_j(t))g(s)}{-M_{j,k}(t,s)g'(s)\theta_{j,k}}.
$$

Therefore, we obtain, by using the previous equality in (6.55), that

$$
\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)}\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds\\
&+ \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}} G^*\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}\right) ds.
\end{aligned}
$$

Using that $G^*(s) \leq s(G')^{-1}(s)$, we get

$$
\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\left[\int_0^\infty \frac{1}{\tau_{j,k}}\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds\right.\\
&\left.+ \int_0^\infty \frac{1}{\tau_{j,k}} \frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}(G')^{-1}\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}\right) ds\right].
\end{aligned}
$$

Pick $\theta_{j,k} = \frac{1}{M_{j,k}}$, to ensure that

$$
\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq& \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}M_{j,k}}\left(-g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds\\
&+ \int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)}(G')^{-1}\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-g'(s)\right)}\right) ds.
\end{aligned}
$$

Thanks to the fact that $(G')^{-1}$ is non-decreasing we get,

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}M_{j,k}}\left(-g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds \\ & + \int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)}(G')^{-1}\left(m_0 M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)\right)ds,\end{aligned}$$

where $m_0 = \sup_{s\in\mathbb{R}_+}\dfrac{g(s)}{G^{-1}\left(-g'(s)\right)}$. Note that (6.18) and (6.19), yields that

$$m_1 = \sup_{s\in\mathbb{R}_+}\int_0^\infty \frac{M_{j,k}(s,t)g(s)}{G^{-1}\left(-g'(s)\right)}ds < \infty.$$

Thus, using that $\tau_{j,k}(t,s) = \frac{1}{m_0 M_{j,k}(t,s)}$ and relation (6.25), we have that

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & -\frac{m_0}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \left(g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds \\ & + \left(\epsilon_0 E_j(t)\right)\int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)}ds \\ = & -\frac{2m_0}{G'\left(\epsilon_0 E_j(t)\right)}E'_{j,k}(t) + \epsilon_0 m_1 E_j(t).\end{aligned}$$

Finally, multiplying the previous inequality by $G'\left(\epsilon_0 E_j(t)\right) = \dfrac{G_0\left(\epsilon_0 E_j(t)\right)}{\epsilon_0 E_j(t)}$ gives

$$\frac{G_0\left(\epsilon_0 E_j(t)\right)}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq -2m_0 E'_{j,k}(t) + m_1 G_0\left(\epsilon_0 E_j(t)\right),$$

which taking $d_{j,k} = \max\{2m_0, m_1\}$, ensures (6.53), showing the lemma. □

6.3 Proof of the stabilization result

Proof Theorem 6.1.1. Let us split the proof into two cases: a) $n = 1$ and b) $n > 1$.

a) $n = 1$

Poincaré inequality gives us

$$\|y\|^2 \leq c_*\|\nabla y\|^2 \leq c_*^2\|\Delta y\|^2,$$

where $c_* > 0$ is the Poincaré constant. Summarizing,

$$\|y\|^2 \leq c_{**}\|\Delta^{\frac{j}{2}}y\|^2,$$

for c_{**} defined by (6.39). From the definition of E_j given by (6.5) we found that

$$\frac{2}{\epsilon_0 c_{**}}G_0(\epsilon_0 E_j(t)) \leq \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\|\Delta^{\frac{j}{2}}y\|^2 + \frac{1}{c_{**}}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds.$$

Thanks to the inequality (6.47), we have

$$\begin{aligned}\frac{2}{\epsilon_0 c_{**}} G_0(\epsilon_0 E_j(t) \leq & \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\left(2c_{1,j}\|\eta^t\|^2_{L_j} + 2c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t)\right) \\ & + \frac{1}{c_{**}}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds \\ = & \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\left(2c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t)\right) \\ & + \left(2c_{1,j} + \frac{1}{c_{**}}\right)\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds.\end{aligned} \tag{6.56}$$

Combining (6.56) with (6.53), gives

$$\begin{aligned}\frac{2}{\epsilon_0 c_{**}} G_0(\epsilon_0 E_j(t)) \leq & -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t) + 2c_{2,j}c_\epsilon d_{j,2}G_0(\epsilon_0 E_j(t)) - 2c_{2,j}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}c_\epsilon E'_{j,1}(t) \\ & - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)E'_j(t) + d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)G_0(\epsilon_0 E_j(t)).\end{aligned}$$

So,

$$\begin{aligned}&\left(\frac{2}{\epsilon_0 c_{**}} - 2c_{2,j}c_\epsilon d_{j,2} - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)\right)G_0(\epsilon_0 E_j(t)) \\ &\leq -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t) - 2c_{2,j}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}c_\epsilon E'_{j,1}(t) - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)E'_j(t).\end{aligned} \tag{6.57}$$

Observe that $H_0(s) = \dfrac{G_0(s)}{s}$ is non-decreasing and E_j is non-increasing for each j, thus $\dfrac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}$ is non-increasing for each j, and therefore by (6.57) we get

$$\begin{aligned}&\left(\frac{2}{\epsilon_0 c_{**}} - 2c_{2,j}c_\epsilon d_{j,2} - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)\right)G_0(\epsilon_0 E_j(t)) \\ &\leq -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t) - 2c_{2,j}\frac{G_0(\epsilon_0 E_j(0))}{\epsilon_0 E_j(0)}c_\epsilon E'_{j,1}(t) - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)E'_j(t).\end{aligned} \tag{6.58}$$

For $\epsilon_0 > 0$ small enough we have

$$c_1 = \left(\frac{2}{\epsilon_0 c_{**}} - 2c_{2,j}c_\epsilon d_{j,2} - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)\right) > 0.$$

Thus, dividing (6.58) by $c_1 > 0$ yields that

$$G_0(\epsilon_0 E_j(t)) \leq -c_2\left(E'_j(t) + E'_{j,1}(t) + E'_{j,2}(t)\right), \tag{6.59}$$

where

$$c_2 = \max\left\{\frac{2c_{2,j}c_\epsilon d_{j,2}}{c_1}, \frac{2c_{2,j}\frac{G_0(\epsilon_0 E_j(0))}{\epsilon_0 E_j(0)}c_\epsilon}{c_1}, \frac{d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)}{c_1}\right\}.$$

Now, integrating (6.59) on $[0,t]$, $t \in \mathbb{R}_+^*$, and observing that $G_0(\epsilon_0 E_j(t))$ is non-increasing gives

$$\begin{aligned}
tG_0(\epsilon_0 E_j(t)) = &\int_0^t G_0(\epsilon_0 E_j(t))ds \leq \int_0^t G_0(\epsilon_0 E_j(s))ds \\
\leq &-c_2 \int_0^t \left(E_j'(s) + E_{j,1}'(s) + E_{j,2}'(s)\right) ds \\
= &-c_2 \left(E_j(t) + E_{j,1}(t) + E_{j,2}(t)\right) + c_2 \left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right) \\
\leq &c_2 \left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right) =: c_3.
\end{aligned}$$

Because G_0 is invertible and non-decreasing, we deduce that

$$E_j(t) \leq \frac{1}{\epsilon_0}(G_0)^{-1}\left(\frac{c_3}{t}\right) = \frac{1}{\epsilon_0} G_1\left(\frac{c_3}{t}\right) \leq \alpha_{j,1} G_1\left(\frac{\alpha_{j,1}}{t}\right),$$

for $\alpha_{j,1} = \max\left\{\frac{1}{\epsilon_0}, c_3\right\}$, showing (6.20) when $n = 1$.

b) $n > 1$

Suppose, for induction hypothesis, that for some $n \in \mathbb{N}^*$, we have that (6.20) is verified when $U_0 \in D(\mathcal{A}_j^{2n+2})$ for $j \in \{1,2\}$ and $U_0 \in D(\mathcal{A}_j^{2n})$ for $j = 0$. For $j \in \{1,2\}$, let us take $U_0 \in D(\mathcal{A}_j^{2(n+1)+2})$ and for $j = 0$, take $U_0 \in D(\mathcal{A}_j^{2(n+1)})$. So when $j \in \{1,2\}$ we have

$$U_0 \in D(\mathcal{A}_j^{2(n+1)+2}) \subset D(\mathcal{A}_j^{2n+2}), \quad U_t(0) \in D(\mathcal{A}_j^{2(n+1)+1}) \subset D(\mathcal{A}_j^{2n+2}),$$

and

$$U_{tt}(0) \in D(\mathcal{A}_j^{2n+2}).$$

Now, for $j = 0$, we found

$$U_0 \in D(\mathcal{A}_j^{2(n+1)}) \subset D(\mathcal{A}_j^{2n}), \quad U_t(0) \in D(\mathcal{A}_j^{2n+1}) \subset D(\mathcal{A}_j^{2n}), \quad \text{and } U_{tt}(0) \in D(\mathcal{A}_j^{2n}).$$

So, follows from the induction hypothesis that: there exists $\alpha_{j,n}$ such that

$$E_j(t) \leq \alpha_{j,n} G_n\left(\frac{\alpha_{j,n}}{t}\right), \forall t \in \mathbb{R}_+^*.$$

Now, since U_t and U_{tt} are solution of (6.13) with initial conditions

$$U_t(0) \in D(\mathcal{A}_j^{2n+2}) \text{ and } U_{tt}(0) \in D(\mathcal{A}_j^{2n+2}),$$

respectively, the induction hypothesis guarantees the existence of $\beta_{n,t} > 0$ and $\gamma_{n,t} > 0$, such that

$$E_{j,1}(t) \leq \beta_{j,n} G_n\left(\frac{\beta_{j,n}}{t}\right), \forall t \in \mathbb{R}_+^* \quad \text{and} \quad E_{j,2}(t) \leq \gamma_{j,n} G_n\left(\frac{\gamma_{j,n}}{t}\right), \forall t \in \mathbb{R}_+^*,$$

respectively. Thus, as $G_n' s$ are non-decreasing for $\tilde{d}_{j,n} = \max\{3\alpha_{j,n}, 3\beta_{j,n}, 3\gamma_{j,n}\}$, we get

$$E_j(t) + E_{j,1}(t) + E_{j,2}(t) \leq \tilde{d}_{j,n} G_n\left(\frac{\tilde{d}_{j,n}}{t}\right).$$

Finally, how $t \in [T, 2T]$, we have

$$G_0(\epsilon_0 E_j(2T)) \leq G_0(\epsilon_0 E_j(t))$$

and from (6.59) we found the following

$$\begin{aligned} TG_0(\epsilon_0 E_j(2T)) \leq & \int_T^{2T} G_0(\epsilon_0 E_j(t))dt \leq -c_2 \int_T^{2T} \left(E_j'(t) + E_{j,1}'(t) + E_{j,2}'(t)\right) dt \\ =& - c_2 \left(E_j(2T) + E_{j,1}(2T) + E_{j,2}(2T)\right) + c_2 \left(E_j(T) + E_{j,1}(T) + E_{j,2}(T)\right) \\ \leq & c_2 \left(E_j(T) + E_{j,1}(T) + E_{j,2}(T)\right) \leq c_2 \tilde{d}_{j,n} G_n\left(\frac{\tilde{d}_{j,n}}{T}\right) \leq d_{j,n} G_n\left(\frac{d_{j,n}}{T}\right), \end{aligned}$$

where $d_{j,n} = \max\{c_2\tilde{d}_{j,n}, \tilde{d}_{j,n}\}$. Moreover, as G_0 is non-decreasing, $G_1 = G_0^{-1}$ is also non-decreasing. Therefore,

$$\begin{aligned} E_j(2T) \;& \leq \frac{1}{\epsilon_0} G_0^{-1}\left(\frac{2d_{j,n}}{2T} G_n\left(\frac{2d_{j,n}}{2T}\right)\right) = \frac{1}{\epsilon_0} G_1\left(\tilde{s} G_n\left(\tilde{s}\right)\right) \\ & = \frac{1}{\epsilon_0} G_{n+1}\left(\tilde{s}\right) = \alpha_{j,n+1} G_{n+1}\left(\frac{\alpha_{j,n+1}}{2T}\right), \end{aligned}$$

where $\alpha_{j,n+1} := \max\left\{\frac{1}{\epsilon_0}, 2d_{j,n}\right\}$. In other words, there is $\alpha_{j,n+1} > 0$ such that (6.20) holds for $n+1$. By the principle of induction we have that (6.20) is verified for all $n \in \mathbb{N}^*$, showing Theorem 6.1.1. □

Appendix

6.4 Well-posedness *via* Semigroup theory

This section is devoted to proving that the system (6.13) is well-posed in the energy space $\mathcal{H}_j$. To do that, first, let us present some properties of $\mathcal{A}_j$, defined by (6.14)-(6.15) and its adjoin $\mathcal{A}_j^*$ defined by

$$\mathcal{A}_j^*(V) = \begin{pmatrix} -i\Delta v + i\Delta^2 v + (-1)^j \int_0^\infty g(s)\Delta^j \zeta^t(\cdot, s)ds \\ \\ \zeta_s^t + \dfrac{g'(s)}{g(s)}\zeta^t - v \end{pmatrix} \tag{6.60}$$

with

$$D(\mathcal{A}_j^*) = \{V = (v, \zeta^t) \in \mathcal{H}_j; \mathcal{A}^*{}_j(V) \in \mathcal{H}_j, v \in H_0^2(\Omega), \zeta^t(x, 0) = 0\}, \tag{6.61}$$

for $j \in \{0, 1, 2\}$. So, our first result in this section ensures that $\mathcal{A}_j$ (resp. $\mathcal{A}_j^*$) is dissipative, and $D(\mathcal{A}_j)$ (resp. $D(\mathcal{A}_j^*)$) is dense in the energy space[2].

Lemma 6.4.1. *$\mathcal{A}_j$ and $\mathcal{A}_j^*$ are dissipative. Moreover, $D(\mathcal{A}_j)$ and $D(\mathcal{A}_j^*)$ are dense in $\mathcal{H}_j$, for $j \in \{0, 1, 2\}$.*

Proof. Indeed, let $(y, \eta^t) \in D(\mathcal{A}_j)$ so

$$\langle \mathcal{A}_j(y, \eta^t), (y, \eta^t)\rangle = -\mathrm{Re}\left(\int_0^\infty g(s)\int_\Omega \Delta^{\frac{j}{2}}\eta_s^t \Delta^{\frac{j}{2}}\overline{\eta^t} dxds\right).$$

As

$$\Delta^{\frac{j}{2}}\eta_s^t \Delta^{\frac{j}{2}}\overline{\eta^t} = \frac{1}{2}(|\Delta^{\frac{j}{2}}\eta^t|^2)_s + i\mathrm{Im}(\Delta^{\frac{j}{2}}\eta_s^t \Delta^{\frac{j}{2}}\overline{\eta^t}),$$

integration by parts over variable s, ensures that

$$\begin{aligned} \langle \mathcal{A}_j(y, \eta^t), (y, \eta^t)\rangle =& - \mathrm{Re}\left(\int_0^\infty g(s)\int_\Omega \left(\frac{1}{2}(|\Delta^{\frac{j}{2}}\eta^t|^2)_s + i\mathrm{Im}(\Delta^{\frac{j}{2}}\eta_s^t \Delta^{\frac{j}{2}}\overline{\eta^t})\right) dxds\right) \\ =& \frac{1}{2}\mathrm{Re}\left(\int_0^\infty g'(s)\int_\Omega |\Delta^{\frac{j}{2}}\eta^t|^2 dxds\right) \\ =& \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds \leq 0, \end{aligned} \tag{6.62}$$

since (6.10) is verified. So, $\mathcal{A}_j$ is dissipative. Similarly, $\mathcal{A}_j^*$ defined by (6.60) is dissipative.

Now, let us prove that $D(\mathcal{A}_j)$ is dense on $\mathcal{H}_j$. Since we showed that $\mathcal{A}_j$ is dissipative, we need to prove that the image of $I - \mathcal{A}_j$ is $\mathcal{H}_j$, since $\mathcal{H}_j$ is reflexive. To do that, pick

2 Now on, we will use the following Poincaré inequality $\|y\|^2 \leq c_*\|\nabla y\|^2, \quad y \in H_0^1(\Omega)$, where $c_* > 0$ is the Poincaré constant.

$(f_1, f_2) \in \mathcal{H}_j = L^2(\Omega) \times L^2_g(\mathbb{R}_+; H_0^j(\Omega))$, we claim that there exists $(y, \eta^t) \in D(\mathcal{A}_j)$ such that

$$(y, \eta^t) - (i\Delta y - i\Delta^2 y + (-1)^{j+1}\int_0^\infty g(s)\Delta^j \eta^t(\cdot, s)ds, y - \eta_s^t) = (f_1, f_2).$$

Or equivalently, we claim that there exits $(y, \eta^t) \in D(\mathcal{A}_j)$ satisfying

$$\begin{cases} y - i\Delta y + i\Delta^2 y + (-1)^j \displaystyle\int_0^\infty g(s)\Delta^j \eta^t(\cdot, s)ds = f_1 \\ \eta^t - y + \eta_s^t = f_2. \end{cases} \tag{6.63}$$

Indeed, multiplying the second equation of (6.63) by e^s and integrating over s, we get

$$\eta^t(x, s) \quad = \quad (1 - e^{-s})y + \int_0^s e^{\tau - s} f_2(\tau)d\tau = (1 - e^{-s})y + f_3(s). \tag{6.64}$$

Since $f_2 \in L^2_g(\mathbb{R}_+; H_0^j(\Omega))$, taking $f_3 = \displaystyle\int_0^s e^{\tau-s} f_2(\tau)d\tau$ we have

$$\begin{aligned} \int_0^\infty g(s)\|\Delta^{\frac{j}{2}} f_3(s)\|^2 ds &= \int_0^\infty g(s)e^{-2s}\int_\Omega \left|\int_0^s e^\tau \Delta^{\frac{j}{2}} f_2(\tau)d\tau\right|^2 dxds \\ &\le \int_0^\infty g(s)e^{-s}\int_\Omega \int_0^s e^\tau |\Delta^{\frac{j}{2}} f_2(\tau)|^2 d\tau dxds \\ &\le \int_0^\infty \int_0^s g(s)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 d\tau ds \\ &= \int_0^\infty \int_\tau^\infty g(s)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 dsd\tau \\ &\le \int_0^\infty \int_\tau^\infty g(\tau)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 dsd\tau \\ &= \|f_2\|^2_{L^2_g(\mathbb{R}_+; H_0^j(\Omega))} < +\infty, \end{aligned}$$

that is, $f_3 \in L^2_g(\mathbb{R}_+; H_0^j(\Omega))$. Now, for $y \in H_0^2(\Omega)$ holds that

$$\int_0^\infty g(s)\|(1 - e^{-s})\Delta^{\frac{j}{2}} y\|^2 ds = \|\Delta^{\frac{j}{2}} y\|^2 \int_0^\infty g(s)(1 - e^{-s})^2 ds \le \|\Delta^{\frac{j}{2}} y\|^2 g_1 < +\infty,$$

since

$$g_1 := \int_0^\infty g(s)(1 - e^{-s})ds \le \int_0^\infty g(s)ds = g_0.$$

So $(1 - e^{-s})y \in L^2_g(\mathbb{R}_+; H_0^j(\Omega))$. Therefore, for $y \in H_0^2(\Omega)$, choosing η^t as in (6.64), follows that $\eta^t \in L^2_g(\mathbb{R}_+; H_0^j(\Omega))$ and, so $\eta^t(x, 0) = 0$. Thanks to (6.63) we get

$$\eta_s^t = f_2 - \eta^t + y \in L^2_g(\mathbb{R}_+; H_0^j(\Omega)).$$

Finally, let us prove that $y \in H_0^2(\Omega)$ satisfies

$$y - i\Delta y + i\Delta^2 y + (-1)^j \int_0^\infty g(s)\Delta^j \eta^t(\cdot, s)ds = f_1,$$

for $\eta^t = (1 - e^{-s})y + f_3$. This is equivalent to obtain $y \in H_0^2(\Omega)$ satisfying the following elliptic equation

$$y - i\Delta y + i\Delta^2 y + (-1)^j g_1 \Delta^j y = f_1 - (-1)^j \int_0^\infty g(s)\Delta^j f_3(\cdot, s)ds,$$

which is a direct consequence of the Lax-Milgram theorem. Therefore, $(y, \eta^t) \in D(\mathcal{A}_j)$ is a strong solution of $(I - \mathcal{A}_j)(y, \eta^t) = (f_1, f_2)$ and $I - \mathcal{A}_j$ is surjective, showing the result. Similarly, it is shown that $D(\mathcal{A}_j^*)$ defined by (6.61) is dense in $\mathcal{H}_j$. $\square$

The main result of this section is a consequence of the Lemma 6.4.1 and can be read as follows.

Theorem 6.4.1. *Suppose that Assumption 3 and* (6.9) *are verified. Thus, for each* $j \in \{0, 1, 2\}$, *the linear operator* $\mathcal{A}_j$ *defined by* (6.14) *is the infinitesimal generator of a semigroup of class* C_0 *and, for each* $n \in \mathbb{N}$ *and* $U_0 \in D(\mathcal{A}_j^n)$, *the system* (6.13) *has unique solution in the class* $U \in \bigcap_{k=0}^{n} C^k(\mathbb{R}_+; D(\mathcal{A}_j^{n-k}))$.

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