



Control results for a model of resonant interaction between short and long capillary-gravity waves

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Abstract. The purpose of this article is the investigation of the global control properties of a coupled nonlinear dispersive system posed in the periodic domain \mathbb{T} , a system with the structure of a nonlinear Schrödinger equation and a nonlinear Korteweg-de Vries equation. Combining estimates derived from Bourgain spaces and using microlocal analysis we show that this system has global control properties. The main novelty of this work is twofold. One is that the global results for the nonlinear system are presented for the first time thanks to the propagation of singularities. The second one is that these propagation results are shown to a coupled dispersive system with two equations defined by differential operators with principal symbols of different orders.

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1. Introduction

1.1. Setting of the problem

Our work is related to the global control properties of a system composed of a nonlinear Schrödinger equation and a nonlinear Korteweg-de Vries equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2 u, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = \operatorname{Re}(\partial_x u), & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.1)$$

where $u = u(x, t)$ is a complex valued function and $v = v(x, t)$ is a real-valued function. The nonlinear Schrödinger–Korteweg-de Vries system (NLS–KdV) appears in the study of resonant interaction between short and long capillary-

gravity waves on water of uniform finite depth, in plasma physics and in a diatomic lattice system. Here, u represents the short wave, while v stands for the long wave, see e.g. [1, 2, 4, 14] and the references therein for more details about the physical motivation for this system.

The first goal of the manuscript is to establish well-posedness results for the system (1.1). To do so the main ingredient is a fixed point argument in the Bourgain spaces associated with the linear Schrödinger and linear Korteweg-de Vries equations. Once we have the global well-posedness of the NLS–KdV system, we can consider the system (1.1) from a control point of view with two forcing terms f and g added in each equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2 u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = \operatorname{Re}(\partial_x u) + g, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \quad (1.2)$$

where f and g are assumed to be supported in a non-empty subset $\omega \subset \mathbb{T}$. Here, the main point is to use microlocal analysis to prove results of propagation of the singularities, which is the key point to proving global control results. The main difficulty to prove these propagations is related to the fact that we have a coupled system defined by two differential operators with principal symbols of different orders.

To the authors' best knowledge, no global control results for the nonlinear NLS–KdV system (1.2) exist in the literature. However, for the linear system, control problems are considered in [3]. Precisely, the authors treated a linear Schrödinger–Korteweg-de Vries system

$$\begin{cases} i\partial_t w + \partial_x^2 w = a_1 w + a_2 y + h\mathbf{1}_\omega & \text{in } Q \\ \partial_t y + \partial_x^3 y + \partial_x(My) = \operatorname{Re}(a_3 w) + a_4 y + l\mathbf{1}_\omega & \text{in } Q \\ w(0, t) = w(1, t) = 0 & \text{in } (0, T), \\ y(0, t) = y(1, t) = \partial_x y(1, t) = 0 & \text{in } (0, T), \\ w(x, 0) = w_0(x), \quad y(x, 0) = y_0(x) & \text{in } (0, 1) \end{cases}$$

in a bounded domain $Q := (0, 1) \times (0, T)$ with a purely real or a purely imaginary control h acting in the Schrödinger equation and the control l acting in the KdV equation. Thanks to the Carleman estimates they proved an observability inequality that helps them to achieve the result. However, the following questions naturally arise:

Control problems: *What can be said about the global controllability of the system (1.2)? Could it be possible to find appropriate damping mechanisms to stabilize this system?*

In this way, our work will provide answers to these questions for the nonlinear NLS–KdV system (1.2). Although these issues are typical in control theory and have been the subject of study in several single equations over the past 30 years, the controllability problems involving nonlinear coupled dispersive systems are still not well understood.

1.2. Known results for single equations

Let us present a review of the control and stabilization results for the KdV and NLS equations. We caution that this is only a small sample of the extant works on these equations.

1.2.1. KdV equation. Russell and Zhang are the pioneers in the study of control problems for the KdV equation [25, 26]. They treated the following KdV system

$$u_t + uu_x + u_{xxx} = f, \quad (1.3)$$

with periodic boundary conditions and an internal control f . Since then, both controllability and stabilization problems have been intensively studied. We can cite, for instance, the exact boundary controllability of KdV on a bounded domain [12, 13, 16, 23, 28] and the internal control problem [10], among others.

It is well known that the KdV system (1.3) has at least the following conserved integral quantities

$$I_1(t) = \int_{\mathbb{T}} u(x, t) dx, \quad I_2(t) = \int_{\mathbb{T}} u^2(x, t) dx \quad \text{and} \\ I_3(t) = \int_{\mathbb{T}} \left(u_x^2(x, t) - \frac{1}{3} u^3(x, t) \right) dx.$$

From the historical origins of the KdV equation involving the behavior of water waves in a shallow channel, it is natural to think that I_1 and I_2 express conservation of *volume (or mass)* and *energy*, respectively.

The pioneer work in the periodic case is due to Russel and Zhang [25] and is purely linear. After some years and the discovery of a subtle smoothing property of solutions of the KdV equation due to Bourgain [7], the authors were able to extend their results to the nonlinear system [26]. Precisely, the authors studied the equation (1.3) assuming f supported in a given open set $\omega \subset \mathbb{T}$ and taking the control input $f(x, t)$ as follows

$$f(x, t) = [Gh](x, t) := g(x) \left(h(x, t) - \int_{\mathbb{T}} g(y) h(y, t) dy \right), \quad (1.4)$$

where h is considered as a new control input, and $g(x)$ is a given non-negative smooth function such that $\{g > 0\} = \omega$ and

$$2\pi [g] = \int_{\mathbb{T}} g(x) dx = 1.$$

For the chosen g , it is easy to see that

$$\frac{d}{dt} \int_{\mathbb{T}} u(x, t) dx = \int_{\mathbb{T}} f(x, t) dx = 0,$$

for any $t \in \mathbb{R}$ and for any solution $u = u(x, t)$ of the system

$$u_t + uu_x + u_{xxx} = Gh. \quad (1.5)$$

Thus, the mass of the system is indeed conserved. With this in hand, Russell and Zhang were able to show the local exact controllability and local exponential stabilizability for the system (1.5). Indeed, the results presented in [26]

are essentially linear; they are more or less small perturbations of the linear results. However, Laurent et al. in [19] showed global results for the system (1.5). The global control results are established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces.

1.2.2. NLS equation. Consider the following equation

$$i\partial_t u + \Delta u = \lambda|u|^2 u, \quad (x, t) \in \mathcal{M} \times \mathbb{R}. \quad (1.6)$$

The first results for the system (1.6), when \mathcal{M} is a compact Riemannian manifold of dimension two without boundary, is due to Dehman et al. in [15]. The authors considered the stabilization and exact controllability problem for NLS. Precisely, to prove the control properties, the authors were able to prove the propagation results in \mathcal{M} . However, these properties are shown considering ω be an open subset of \mathcal{M} and the following two assumptions:

- (A) ω geometrically controls \mathcal{M} ; i.e. there exists $T_0 > 0$, such that every geodesic of \mathcal{M} traveling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.
- (B) For every $T > 0$, the only solution lying in the space $C[0, T], H^1(\mathcal{M})$ of the system

$$\begin{cases} i\partial_t u + \Delta u + b_1(x, t)u + b_2(x, t)\bar{u}, & (x, t) \in \mathcal{M} \times (0, T), \\ u = 0, & (x, t) \in \omega \times (0, T), \end{cases}$$

where $b_1(t, x)$ and $b_2(t, x) \in L^\infty(0, T, L^p(\mathcal{M}))$ for some $p > 0$ large enough, is the trivial one $u \equiv 0$.

Considering the NLS on a periodic domain \mathbb{T} with Dirichlet or Neumann boundary conditions, Laurent [18] applied the method introduced by Dehman et al. to prove that this system is globally internally controllable.

When a compact Riemannian manifold of dimension $d \geq 3$ is considered, Strichartz estimates do not yield uniform well-posedness results at the energy level for the NLS equation, a property which seems to be very important to prove controllability results. In this way, Burq et al. in two works [8, 9] managed to introduce the Bourgain spaces $X^{s,b}$ on certain manifolds without boundary where the bilinear Strichartz estimates can be shown and, consequently, they get the uniform well-posedness for the NLS equation. Taking advantage of these results, Laurent [17] proved that the geometric control condition (\mathcal{A}) is sufficient to prove the exact controllability for the NLS in $X^{s,b}$ spaces on some three-dimensional Riemannian compact manifolds. Similarly in this work, we mention [24] and [18] where controllability results were studied for the NLS in Euclidean and periodic domains, respectively, relying on the properties of the Bourgain spaces.

1.3. Main results

Note that to deal with the nonlinearity associated with the KdV part in (1.2), we must use appropriated estimates which requires that the mean value of v

satisfying

$$[v] = \frac{1}{2\pi} \int_{\mathbb{T}} v(x, t) dx = 0, \quad \forall t \geq 0.$$

If we set $\tilde{\mu} := [v_0]$, we get from the second equation in (1.2) that

$$\tilde{\mu} := [v_0] = \frac{1}{2\pi} \int_{\mathbb{T}} v(x, 0) dx = \frac{1}{2\pi} \int_{\mathbb{T}} v(x, t) dx, \quad \forall t > 0.$$

Thus it is convenient to set $\tilde{v} = v - \tilde{\mu}$ and to study the following equivalent system

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x \tilde{v} + \beta|u|^2 u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t \tilde{v} + \partial_x^3 \tilde{v} + \frac{1}{2} \partial_x (\tilde{v}^2) + (\mu + \tilde{\mu}) \partial_x \tilde{v} = \operatorname{Re}(\partial_x u) + r, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = \tilde{v}_0(x), & x \in \mathbb{T}, \end{cases}$$

where $\mu \in \mathbb{R}$ is a constant.

Now consider $a \in L^\infty(\mathbb{T})$ a real valued function such that

$$a(x)^2 > \eta > 0, \tag{1.7}$$

in some non-empty open set $\omega \subset \mathbb{T}$ and the operator G defined as in [25], for some $g \in C^\infty(\mathbb{T})$ real valued function such that $g > 0$ in $\omega \subset \mathbb{T}$, as

$$Gh(x, t) := g(x) \left(h(x, t) - \int_{\mathbb{T}} g(y) h(y, t) dy \right), \tag{1.8}$$

where h is any function considered as a control input. In order to stabilize our system we have chosen $f := -ia(x)^2 u$ and $r := Gh$, with $h = -G^* \tilde{v}$, so the following closed-loop system reads

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x \tilde{v} + \beta|u|^2 u - ia(x)^2 u, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t \tilde{v} + \partial_x^3 \tilde{v} + \frac{1}{2} \partial_x (\tilde{v}^2) + (\mu + \tilde{\mu}) \partial_x \tilde{v} = \operatorname{Re}(\partial_x u) - GG^* \tilde{v}, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad \tilde{v}(x, 0) = \tilde{v}_0(x), & x \in \mathbb{T}. \end{cases} \tag{1.9}$$

In this case we can define the energy of the system as

$$E(t) := \|u(t)\|_{L^2(\mathbb{T})}^2 + \|\tilde{v}(t)\|_{L^2_0(\mathbb{T})}^2,$$

and $L^2_0(\mathbb{T}) = \{w \in L^2(\mathbb{T}); [w] = 0\}$. So, multiplying the first equation of (1.9) by u , the second one by \tilde{v} and integrating by parts we can obtain¹

$$\frac{d}{dt} E(t) = -2 \left(\|a(\cdot)u(t)\|_{L^2(\mathbb{T})}^2 + \|G\tilde{v}(t)\|_{L^2_0(\mathbb{T})}^2 \right) \leq 0. \tag{1.10}$$

This indicates that the controls play the role of two damping mechanisms. So, our result establishes that the system (1.9) is asymptotically exponentially stable. Precisely, we provide a positive answer to the global stabilization question already mentioned at the beginning of the introduction.

¹ Actually, by (1.8) and Fubini's theorem, G is self-adjoint (i.e. $G^* = G$), so we shall keep the notation for the feedback $h = -G^* \tilde{v} = -G\tilde{v}$ throughout.

Theorem 1.1. *Assume that $a \in L^\infty(\mathbb{T})$ satisfies the conditions (1.7) and Gh given by (1.8). Then, for every $R_0 > 0$, there exist $C := C(R_0) > 0$ and $\gamma > 0$ such that the following inequality holds*

$$\|(u, \tilde{v})(t)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq Ce^{-\gamma t} \|(u_0, \tilde{v}_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}, \quad \forall t \geq 0, \quad (1.11)$$

for every solution (u, \tilde{v}) of the system (1.9) with initial data $(u_0, \tilde{v}_0) \in L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ satisfying $\|(u_0, \tilde{v}_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq R_0$.

Once we have established the global stabilization result, the answer for the global exact controllability problem will be a consequence of the following local null controllability theorem.

Theorem 1.2. *Let ω be any nonempty set of \mathbb{T} . Then there exist $\delta > 0$ and $T > 0$ such that for every $(u_0, \tilde{v}_0) \in L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ with $\|(u_0, \tilde{v}_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} < \delta$, we can find $f \in C([0, T]; L^2(\mathbb{T}))$ and $h \in C([0, T]; L^2(\mathbb{T}))$, f and h compactly supported in $]0, T[\times \omega$, such that, the unique solution $(u, \tilde{v}) \in C([0, T]; L^2(\mathbb{T}) \times L^2(\mathbb{T}))$ of the system (1.9) satisfies $(u, \tilde{v})(x, T) = (0, 0)$.*

Finally, with the previous local controllability result in hand, combining it with the global stabilization result we get the global controllability result, which can be read as follows.

Theorem 1.3. *Let $\omega \subset \mathbb{T}$ be a nonempty open set and $R_0 > 0$. Then, there exist $T := T(R_0) > 0$ such that, for every $(u_0, v_0), (u_1, v_1) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$ with*

$$\|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \leq R_0 \quad \text{and} \quad \|(u_1, v_1)\|_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \leq R_0,$$

and $[v_0] = [v_1]$ one can find control inputs $f \in C([0, T]; L^2(\mathbb{T}))$ and $h \in C([0, T]; L_0^2(\mathbb{T}))$, with f and h supported in $\omega \times (0, T)$, such that the unique solution $(u, v) \in C([0, T]; L^2(\mathbb{T}) \times L^2(\mathbb{T}))$ of the system (1.2) satisfies $(u, v)(x, T) = (u_1(x), v_1(x))$.

1.4. Heuristic and outline of the manuscript

In this work, global control results are proved by combining estimates derived from Bourgain spaces and using microlocal analysis. To our knowledge, this is the first time that this method is used in a coupled dispersive system with two equations defined by differential operators with principal symbols of different orders and more importantly, a key feature in this contribution is that we are able to prove global control results for the nonlinear system in a bounded domain. This represents an improvement with respect to the previous paper [3] where the control problem, for the linear system, is considered. The key ingredients of this work are:

- *Strichartz, bilinear and multilinear estimates* associated to the solution of the problem under consideration;
- *Microlocal analysis* to prove *propagation of the regularity and compactness* for two equations defined by differential operators with principal symbols of different orders;
- *Unique continuation property* which is a consequence of the Carleman estimates for each equation, KdV and Schrödinger equations.

The proof of the Theorem 1.1 is equivalent to proving an *observability inequality*, which one, by using *contradiction arguments*, relies on to prove a *unique continuation property* for the system (1.9). This property is achieved thanks to the *propagation results* using the smooth properties of the Bourgain spaces. The main difficulty to prove the propagation results arises from the fact that the system (1.9) is defined by two differential operators with principal symbols of different orders. To overcome this difficulty we employ the estimates, proved in Sect. 2, for the solution of our problem in Bourgain spaces.

The strategy to prove Theorem 1.2 is to consider the control operator for the nonlinear problem as a perturbation of the control operator for the linear system associated with (1.9), the perturbation argument is due to Zuazua [29]. Additionally, the control result for large data (Theorem 1.3) will be a combination of a global stabilization result (Theorem 1.1) and the local control result (Theorem 1.2), as is usual in control theory.

It is important to point out that the dissipation laws in (1.9) are intrinsically linked with the physical problems modeled by the Schrödinger and the KdV equations. For these equations, the same dissipation laws were already considered in [18,24], for the NLS equation, and in [19,26], for the KdV equation.

Lastly, since we are working with a coupled system with the structure of the nonlinear Schrödinger equation and nonlinear Korteweg-de Vries equation it is natural to adapt the approach introduced by [18,19], however, a direct application of these results is not enough to get the main results of our article. The first novelty of this manuscript is to deal with the coupled terms, here it is necessary estimates these terms in the Bourgain spaces (see Lemma 2.5 and Proposition 2.6) to be able to perform with the fixed point argument, which is necessary to prove the well-posedness and the controllability results for the nonlinear system. A second and important point is to treat the nonlinear case, instead of the linear one (see [3]), to deal with a nonlinear problem it is necessary to prove propagation of compactness and regularity results, and also, a unique continuation property for the solution of the system (1.2). This was achieved thanks to the estimates in the Bourgain spaces, which are paramount and the key to dealing with the nonlinear terms.

Our manuscript is outlined as follows. Section 2 is to establish estimates needed in our analysis, *Strichartz, bilinear and multilinear estimates*. With these estimates, we are can prove the existence of solutions for the nonlinear NLS–KdV system with source and damping terms in Sect. 3. Next, Sects. 4 and 5, are aimed to present the proof of the stabilization and controllability theorems, respectively. We also present in Sect. 6 concluding remarks and open problems. Finally, Appendixes A and B are devoted to proving the *propagation results* and the *unique continuation property*, respectively.

2. Fourier transform restriction

In this section, we present definitions, some properties of the Bourgain spaces associated with the system (1.1), and estimates which will be essential to

prove our main results. It is well known that Bourgain [6] discovered a subtle smoothing property of solutions of the Schrödinger and KdV equations in \mathbb{R} and \mathbb{T} .

2.1. Definitions and notations

Let us now define the Bourgain spaces $X^{k,b}$ and $Y^{s,b}$ associated with the linear operators of the Schrödinger and KdV equations, respectively. For given $k, b, s \in \mathbb{R}$, functions $u : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{C}$ and $v : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$, in $\mathcal{S}(\mathbb{T} \times \mathbb{R})$, and $\mu \in \mathbb{R}$ define the quantities

$$\|u\|_{X^{k,b}} = \left(\sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} \langle n \rangle^{2k} \langle \tau + n^2 \rangle^{2b} |\hat{f}(n, \tau)|^2 d\tau \right)^{1/2}$$

and

$$\|v\|_{Y^{s,b}} = \left(\sum_{n \in \mathbb{Z}} \int_{-\infty}^{+\infty} \langle n \rangle^{2s} \langle \tau - n^3 + \mu n \rangle^{2b} |\hat{g}(n, \tau)|^2 d\tau \right)^{1/2},$$

where \hat{f} is the Fourier transform of f with respect to the variables x and t , with $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. The Bourgain spaces $X^{k,b}$ and $Y^{s,b}$ associated with Schrödinger and KdV operators are the completion of the Schwartz space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X^{k,b}}$ and $\|v\|_{Y^{s,b}}$, respectively.

Considering the time interval $I \subset \mathbb{R}$, we define $X^{k,b}(I)$ and $Y^{s,b}(I)$ as the restriction spaces of $X^{k,b}$ and $Y^{s,b}$, respectively, to the interval I with the norm

$$\|f\|_{X^{k,b}(I)} = \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} = \inf_{\tilde{g}|_I = g} \|\tilde{g}\|_{Y^{s,b}}. \quad (2.1)$$

We shall also consider the smaller spaces \tilde{X}^k and \tilde{Y}^k defined by the norms

$$\begin{aligned} \|u\|_{\tilde{X}^k} &:= \|u\|_{X^{k,1/2}} + \|\langle n \rangle^k \hat{u}(n, \tau)\|_{L_n^2 L_\tau^1} \\ \text{and } \|v\|_{\tilde{Y}^s} &:= \|v\|_{Y^{s,1/2}} + \|\langle n \rangle^s \hat{v}(n, \tau)\|_{L_n^2 L_\tau^1} \end{aligned} \quad (2.2)$$

and the restriction spaces $\tilde{X}^k(I)$ and $\tilde{Y}^k(I)$ as in (2.1). The companion spaces will be defined as Z^k and W^s via the norms

$$\|u\|_{Z^k} = \|u\|_{X^{k,-1/2}} + \left\| \frac{\langle n \rangle^k \hat{u}(n, \tau)}{\langle \tau + n^2 \rangle} \right\|_{L_\tau^1 L_n^2}$$

and

$$\|v\|_{W^s} = \|v\|_{Y^{s,-1/2}} + \left\| \frac{\langle n \rangle^s \hat{v}(n, \tau)}{\langle \tau - n^3 + \mu n \rangle} \right\|_{L_\tau^1 L_n^2},$$

respectively.

2.1.1. Notations. If $I = (0, T)$, we denote $X^{k,b}(I)$ (resp. $Y^{s,b}(I)$, $\tilde{X}^{k,b}(I)$ and $\tilde{Y}^{s,b}(I)$) by $X_T^{k,b}$ (resp. $Y_T^{s,b}$, $\tilde{X}_T^{k,b}$ and $\tilde{Y}_T^{s,b}$). Denote from now on $\psi \in C_0^\infty(\mathbb{R})$ a non-negative function supported in $[-2, 2]$ with $\psi \equiv 1$ on $[-1, 1]$ and $\psi_\delta(t) := \psi(t/\delta)$, for any $\delta > 0$. We also denote

$$H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}); [u] = 0\} \quad \text{and} \quad L_0^2(\mathbb{T}) = \{u \in L^2(\mathbb{T}); [u] = 0\}.$$

Finally, for any $a \in \mathbb{R}$, $a+$ (resp. $a-$) is a number slightly larger (resp. smaller) than a , more precisely, $a+ = a + \epsilon$, for some $\epsilon > 0$ as small as we want. We shall also denote by D^r the operator defined on $\mathcal{D}'(\mathbb{T})$ by

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \hat{u}(k) & \text{if } k \neq 0, \\ \hat{u}(0) & \text{if } k = 0. \end{cases} \tag{2.3}$$

The following properties of $X_T^{k,b}$ and $Y_T^{s,b}$, $k, s, b \in \mathbb{R}$, can be verified.

- (i) $X_T^{k,b}$ and $Y_T^{s,b}$ are Hilbert spaces.
- (ii) Let $\delta, \delta_1 \geq 0$, then $X_T^{k+\delta, b+\delta_1}$ is continuously embedded in $X_T^{k,b}$ and if $\delta > 0$ and $\delta_1 > 0$ we have a compact embedding. Similar result holds for $Y_T^{s,b}$ spaces.
- (iii) The spaces \tilde{X}_T^k and \tilde{Y}_T^s are continuously embedded in $C([0, T]; H^k(\mathbb{T}))$ and $C([0, T]; H^s(\mathbb{T}))$, respectively.

2.2. Overview of the estimates

This part of the work is dedicated to present estimates for the linear group associated with Schrödinger and KdV equations. Additionally, we will present the trilinear and bilinear estimates associated with the nonlinearities involved in our problem, which are the key to proving the global control results in this manuscript. These results will be borrowed from [2, 18, 19].

Let $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t(\partial_x^3 + \mu\partial_x)}$ be the unitary groups associated to the linear Schrödinger and the Airy equations, respectively. So, the following linear estimates are verified and the proofs were made in [2, Lemma 4.1.], [19, Lemma 3.4.] and [18, Lemma 1.2.], respectively, so we will omit them.

Proposition 2.1. *Let $u_0 \in H^k(\mathbb{T})$ and $v_0 \in H^s(\mathbb{T})$, then*

(i) $\|\psi_1(t)e^{it\partial_x^2}u_0\|_{X^{k,b}} \leq C\|u_0\|_{H^k}$ and

$$\left\| \psi_T(t) \int_0^t e^{i(t-t')\partial_x^2} F(t') dt' \right\|_{X^{k, \frac{1}{2}}} \leq C\|F\|_{Z^k}.$$

(ii) $\|\psi_1(t)e^{-t(\partial_x^3 + \mu\partial_x)}v_0\|_{Y^{s,b}} \leq C\|v_0\|_{H^s}$ and

$$\left\| \psi_T(t) \int_0^t e^{(t-t')(\partial_x^3 + \mu\partial_x)} G(t') dt' \right\|_{Y^{s, \frac{1}{2}}} \leq C\|G\|_{W^s}.$$

- (iii) Let $-1 \leq b \leq 1$, $s \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T})$. Then, for every $u \in Y^{s,b}$, $\varphi(x)u \in Y^{s-|b|, b}$. Similarly, the multiplication φ maps $Y_T^{s,b}$ into $Y_T^{s-|b|, b}$.
- (iv) Let $-1 \leq b \leq 1$, $k \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T})$. Then, for every $u \in X^{k,b}$, $\varphi(x)u \in X^{k-|b|, b}$. Similarly, the multiplication φ maps $X_T^{k,b}$ into $X_T^{k-|b|, b}$.

The next proposition, shown in [2, Lemma 4.2.] and [19, Lemma 3.3.], concerns the trilinear estimates for the term $|u|^2u$ and bilinear estimates for the term $\partial_x v^2$, respectively.

Proposition 2.2. *The following estimates hold true.*

(i) For any $k \geq 0$,

$$\|\psi(t)uv\bar{w}\|_{Z^k} \lesssim \|u\|_{X^{k,3/8}} \|v\|_{X^{k,3/8}} \|w\|_{X^{k,3/8}}. \tag{2.4}$$

(ii) For any $s \geq 0$, $T \in (0, 1)$ and $v_1, v_2 \in Y^{s, \frac{1}{2}}$ x -periodic functions having zero x -mean for all t , that is $[v_1] = [v_2] = 0$, there exist two constants $\theta > 0$ and $C > 0$, such that

$$\|\psi_T(t)\partial_x(v_1v_2)\|_{W^s} \leq CT^\theta \|v_1\|_{Y^{s,1/2}} \|v_2\|_{Y^{s,1/2}}, \tag{2.5}$$

where C is independent of T , v_1 and v_2 .

At this point, we present an elementary lemma concerning the stability of Bourgain’s spaces with respect to time localization which was proved in [2, Lemma 4.4.].

Proposition 2.3. *Let $X_{\tau=h(\xi)}^{s,b} := \left\{ f : \langle \tau - h(\xi) \rangle^b \langle \xi \rangle^s | \hat{f}(\tau, \xi) \in L^2 \right\}$. Then,*

$$\|\psi(t)f\|_{X_{\tau=h(\xi)}^{s,b}} \lesssim_{\psi,b} \|f\|_{X_{\tau=h(\xi)}^{s,b}}$$

for any $x, b \in \mathbb{R}$. Furthermore, if $-\frac{1}{2} < b' \leq b < \frac{1}{2}$, then for any $0 < T < 1$ we have

$$\|\psi_T(t)f\|_{X_{\tau=h(\xi)}^{s,b'}} \lesssim_{\psi,b',b} T^{b-b'} \|f\|_{X_{\tau=h(\xi)}^{s,-b}}$$

Finally, to end this subsection, we invoke the following result related to the linear system associated with the KdV equation. Precisely, we present below the following Strichartz estimate for the KdV equation shown in [19, Lemma 3.2.].

Proposition 2.4. *Let $T > 0$. The following estimate is verified*

$$\|v\|_{L^4(0,T;L^4(\mathbb{T}))} \leq C \|v\|_{Y_T^{0, \frac{1}{3}}}.$$

In other words, $Y_T^{0, \frac{1}{3}}$ is continuously embedded in $L^4(0, T; L^4(\mathbb{T}))$.

2.3. Auxiliary estimates

The next auxiliary lemma helps us to prove the necessary estimates for the following derivative terms $\partial_x u$ and $\partial_x v$ in W_T^k and Z_T^s , $k, s \in \mathbb{R}$, respectively.

Lemma 2.5. *Let $\epsilon > 0$ and $H : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$H(n, \tau) = \frac{|n|}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon} \langle \tau + n^2 \rangle^{\frac{1}{2}-\epsilon}}, \quad (n, \tau) \in \mathbb{Z} \times \mathbb{R},$$

then there exists a constant $C > 0$ such that

$$H(n, \tau) \leq C, \quad \forall (n, \tau) \in \mathbb{Z} \times \mathbb{R}.$$

Proof. It is enough to prove that, for every $(n, \tau) \in \mathbb{Z} \times \mathbb{R}$, with $|n|$ large enough, there exist $\alpha > 0$ and $C_0 > 0$ such that

$$\langle \tau - n^3 + \mu n \rangle \langle \tau + n \rangle \geq C_0 |n|^\alpha, \quad (2.6)$$

with $(\frac{1}{2} - \epsilon) \alpha > 1$.

Indeed, let us assume that (2.6) is verified. We then split the analysis into two cases, $|n| \leq C$, for some $C > 0$ large enough, and $|n| > C$. For the first case, note that considering a fixed $n_0 \in \mathbb{Z}$, we get

$$H(n_0, \tau) = \frac{|n_0|}{\langle \tau - n_0^3 + \mu n_0 \rangle^{\frac{1}{2} - \epsilon} \langle \tau + n_0^2 \rangle^{\frac{1}{2} - \epsilon}} \leq |n_0|, \quad (2.7)$$

for all $\tau \in \mathbb{R}$. On the other hand, (2.6) asserts that there exist $C_0 > 0$ and $\alpha > 0$ such that

$$H(n, \tau) \leq \frac{|n|}{C_0 |n|^{(\frac{1}{2} - \epsilon)\alpha}} \leq \frac{1}{C_0},$$

for every $(n, \tau) \in \mathbb{Z} \times \mathbb{R}$, with $|n| > C$. So, by (2.6) and (2.7),

$$H(n, \tau) \leq \max \left\{ \frac{1}{C_0}, C \right\}, \quad \forall (n, \tau) \in \mathbb{Z} \times \mathbb{R},$$

and the lemma is proved.

Now, to prove (2.6) we start by observing that

$$\begin{aligned} \langle \tau - n^3 + \mu n \rangle \langle \tau + n^2 \rangle &= (1 + |\tau - n^3 + \mu n|^2)^{\frac{1}{2}} (1 + |\tau + n^2|^2)^{\frac{1}{2}} \\ &\geq \frac{1}{4} (1 + |\tau - n^3 + \mu n|) (1 + |\tau + n^2|) \\ &\geq \frac{1}{4} (|\tau - n^3 + \mu n| + |\tau + n^2|) \\ &\geq \frac{1}{4} |n^3 - n^2 - \mu n| \\ &\geq \frac{1}{4} (|n|^3 - |n^2 + \mu n|) \\ &= \frac{1}{2} |n|^3 + \frac{1}{2} |n|^3 - |n^2 + \mu n| \\ &\geq \frac{1}{8} |n|^3 + \frac{1}{8} |n|^3 - \frac{1}{4} |n|^2 - \frac{1}{4} |\mu| |n|. \end{aligned}$$

Since we have that $\lim_{|n| \rightarrow \infty} \left(\frac{1}{8} |n|^3 - \frac{1}{4} |n|^2 - \frac{1}{4} |\mu| |n| \right) = \infty$, (2.6) is verified. \square

We are in position to prove the last linear estimates.

Proposition 2.6. *Let $T > 0$. If $u \in X_T^{k, \frac{1}{2}}$ and $v \in Y_T^{s, \frac{1}{2}}$, for $k, s \in \mathbb{R}$, we have*

$$\|\partial_x u\|_{W_T^k} \lesssim \|u\|_{X_T^{k, \frac{1}{2}}} \quad (2.8)$$

and

$$\|\partial_x v\|_{Z_T^s} \lesssim \|v\|_{Y_T^{s, \frac{1}{2}-}}. \tag{2.9}$$

Proof. We start observing that

$$\|\partial_x u\|_{W^k} = \left\| \langle \tau - n^3 + \mu n \rangle^{-\frac{1}{2}} \langle n \rangle^k \widehat{\partial_x u} \right\|_{L_\tau^2 L_n^2} + \left\| \frac{\langle n \rangle^k \widehat{\partial_x u}(n, \tau)}{\langle \tau - n^3 + \mu n \rangle} \right\|_{L_\tau^1 L_n^2}.$$

Since $\widehat{\partial_x u}(n, \tau) = in\widehat{u}(n, \tau)$ and

$$\frac{1}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}}} \leq \frac{1}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}},$$

for any $\epsilon > 0$, we have that

$$\left\| \langle \tau - n^3 + \mu n \rangle^{-\frac{1}{2}} \langle n \rangle^k \widehat{\partial_x u} \right\|_{L_\tau^2 L_n^2} \leq \left\| \frac{n}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}} \langle n \rangle^k \widehat{u} \right\|_{L^2 L^2}.$$

Thanks to the Lemma 2.5 and using duality argument, follows that

$$\begin{aligned} & \left\| \frac{n}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}} \langle n \rangle^k \widehat{u} \right\|_{L^2 L^2} \\ &= \sup_{\|\varphi\|_{L^2 L^2} \leq 1} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{|n|}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}} \langle n \rangle^k \widehat{u}(n, \tau) \overline{\widehat{\varphi}(n, \tau)} d\tau \\ &= \sup_{\|\varphi\|_{L^2 L^2} \leq 1} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} H(n, \tau) \langle \tau + n^2 \rangle^{\frac{1}{2}-\epsilon} \langle n \rangle^k \widehat{u}(n, \tau) \overline{\widehat{\varphi}(n, \tau)} d\tau \tag{2.10} \\ &\leq C \sup_{\|\varphi\|_{L^2 L^2} \leq 1} \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} \langle \tau + n^2 \rangle^{\frac{1}{2}-\epsilon} \langle n \rangle^k \widehat{u}(n, \tau) \overline{\widehat{\varphi}(n, \tau)} d\tau \\ &\leq C \sup_{\|\varphi\|_{L^2 L^2} \leq 1} \|\langle \tau + n^2 \rangle^{\frac{1}{2}-\epsilon} \langle n \rangle^k \widehat{u}\|_{L_\tau^2 L_n^2} \|\varphi\|_{L_\tau^2 L_n^2} \\ &\leq C \|u\|_{X^{k, \frac{1}{2}-\epsilon}}. \end{aligned}$$

Finally, using the Cauchy–Schwarz inequality we get

$$\begin{aligned} \left\| \frac{\langle n \rangle^k \widehat{\partial_x u}(n, \tau)}{\langle \tau - n^3 + \mu n \rangle} \right\|_{L_\tau^1 L_n^2} &\leq \left\| \frac{1}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}+\epsilon} \langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}} \langle n \rangle^k \widehat{\partial_x u}(n, \tau) \right\|_{L_\tau^1 L_n^2} \\ &\leq \left\| \frac{1}{\langle \tau - n^3 + \mu n \rangle^{\frac{1}{2}-\epsilon}} \langle n \rangle^k \widehat{\partial_x u}(n, \tau) \right\|_{L_\tau^2 L_n^2} \\ &\leq \|u\|_{X^{k, \frac{1}{2}-\epsilon}}, \end{aligned}$$

where the last inequality holds using (2.10). So, (2.8) is verified.

Analogously, we can prove (2.9). In fact, observing that

$$\|\partial_x v\|_{Z^s} = \|\partial_x v\|_{X^{s, -\frac{1}{2}}} + \left\| \frac{\langle n \rangle^s \widehat{\partial_x v}(n, \tau)}{\langle \tau + n^2 \rangle} \right\|_{L_\tau^1 L_n^2}$$

and

$$\begin{aligned} \left\| \frac{\langle n \rangle^s \widehat{\partial_x v}(n, \tau)}{\langle \tau + n^2 \rangle} \right\|_{L^1_\tau L^2_n} &= \left\| \frac{|n|}{\langle \tau + n^2 \rangle^{\frac{1}{2} + \epsilon} \langle \tau + n^2 \rangle^{\frac{1}{2} - \epsilon}} \langle n \rangle^s \widehat{v}(n, \tau) \right\|_{L^1_\tau L^2_n} \\ &\leq \left\| \frac{|n|}{\langle \tau + n^2 \rangle^{\frac{1}{2} - \epsilon}} \langle n \rangle^s \widehat{v}(n, \tau) \right\|_{L^2_\tau L^2_n} \\ &= \left\| \frac{|n|}{\langle \tau + n^2 \rangle^{\frac{1}{2} - \epsilon} \langle \tau - n^3 + \mu n \rangle^{\frac{1}{2} - \epsilon}} \langle \tau - n^3 + \mu n \rangle^{\frac{1}{2} - \epsilon} \langle n \rangle^s \widehat{v}(n, \tau) \right\|_{L^2_\tau L^2_n}, \end{aligned}$$

we conclude the proof of (2.9) arguing by duality as in (2.10) and so the lemma is proved. \square

3. Existence of solutions for the NLS–KdV system

In this section, we can prove that the following system

$$\begin{cases} i\partial_t u + \partial_x^2 u + i\varphi(t)^2 a(x)^2 u = i\partial_x v + \beta|u|^2 u, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = Re(\partial_x u) - GG^*v, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{3.1}$$

where $a(x)$ satisfies (1.7) and Gv is defined in (1.8), is well-posed in H^s , for $s \geq 0$. Precisely, we prove the result for \widetilde{X}_T^s and \widetilde{Y}_T^s , defined by (2.2). So, the result of well-posedness can be read as follows.

Theorem 3.1. *Let $T > 0$, $s \in \mathbb{R}_+$, $\beta, \mu \in \mathbb{R}$, $a \in C^\infty(\mathbb{T})$ and $\varphi \in C_0^\infty(\mathbb{R})$ taking real values. For a given $(u_0, v_0) \in H^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ there exists a unique solution $(u, v) \in \widetilde{X}_T^s \times \left(\widetilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T}))\right)$ of (3.1) and the same result is valid for $s = 0$ if $a \in L^\infty(\mathbb{T})$. Additionally, the flow map*

$$\begin{aligned} F : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) &\rightarrow \widetilde{X}_T^0 \times \widetilde{Y}_T^0 \\ (u_0, v_0) &\mapsto F(u_0, v_0) = (u, v) \end{aligned} \tag{3.2}$$

is Lipschitz on every bounded set.

Proof. We split the proof into three parts. Precisely, first, we prove the existence and uniqueness of a local solution. After that, we prove that the solution of (3.1) is globally defined. Finally, we are able to derive that the flow map (3.2) is Lipschitz. The idea of the proof is to follow the approach of [18, 19] which was inspired by Bourgain [6, 7]. A main difference here is the necessity of Proposition 2.6 which is essential for the proof of the fixed point theorem. For the sake of completeness let us present the details of the proof.

(a) **Existence and uniqueness of local solution.**

Let $T > 0$ to be determined later and $t \in [0, T]$, system (3.1) is equivalent to the following Duhamel integral equations

$$u(t) = e^{it\partial_x^2} u_0 - i \int_0^t e^{i(t-t')\partial_x^2} \{i\partial_x v(t') + \beta|u|^2 u(t') - i\varphi(t')^2 a(x)^2 u(t')\} dt'$$

and

$$v(t) = e^{-t(\partial_x^3 + \mu \partial_x)} v_0 + \int_0^t e^{-(t-t')(\partial_x^3 + \mu \partial_x)} \left\{ Re(\partial_x u)(t') - GG^* v(t') - \frac{1}{2} \partial_x (v^2)(t') \right\} dt'.$$

Therefore, to find a solution to the system (3.1) in the class $\tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$ is equivalent to proving the existence of a fixed point to the map

$$\Phi := (\Phi_1, \Phi_2) : \tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right) \rightarrow \tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$$

where

$$\Phi_1(u, v) = e^{it\partial_x^2} u_0 - i \int_0^t e^{i(t-t')\partial_x^2} \{ i\partial_x v(t') + \beta |u|^2 u(t') - i\varphi(t')^2 a(x)^2 u(t) \} dt'$$

and

$$\Phi_2(u, v) = e^{-t(\partial_x^3 + \mu \partial_x)} v_0 + \int_0^t e^{-(t-t')(\partial_x^3 + \mu \partial_x)} \left\{ Re(\partial_x u)(t') - \frac{1}{2} \partial_x (v^2)(t') - GG^* v(t') \right\} dt'.$$

Our aim is to prove that Φ is a contraction in some ball of $\tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$. To do so, observe that item (i) of Proposition 2.1, inequalities (2.4) and (2.5), Proposition 2.3, and estimate (2.8) give us

$$\begin{aligned} \|\Phi_1(u, v)\|_{\tilde{X}_T^s} &\lesssim \|u_0\|_{H^s} + \|\partial_x v\|_{Z_T^s} + \| |u|^2 u \|_{Z_T^s} + \|i\varphi(t')^2 a(x)^2 u\|_{Z_T^s} \\ &\lesssim \|u_0\|_{H^s} + \|\partial_x v\|_{Y_T^{s, \frac{1}{2}-}} + \|u\|_{X_T^{s, \frac{3}{8}}}^3 + \|i\varphi(t')^2 a(x)^2 u\|_{X_T^{s, \frac{1}{2}}} \\ &\lesssim \|u_0\|_{H^s} + T^{0+} \left(\|v\|_{Y_T^{s, \frac{1}{2}}} + \|u\|_{X_T^{s, \frac{1}{2}}}^3 + \|u\|_{X_T^{s, \frac{1}{2}}} \right) \end{aligned} \quad (3.3)$$

and, similarly,

$$\begin{aligned} \|\Phi_1(u, v) - \Phi_1(\tilde{u}, \tilde{v})\|_{\tilde{X}_T^s} &\lesssim T^{0+} \|u - \tilde{u}\|_{\tilde{X}_T^{s, \frac{1}{2}}} \left(1 + \|u\|_{X_T^{s, \frac{1}{2}}}^2 + \|\tilde{u}\|_{X_T^{s, \frac{1}{2}}}^2 \right) \\ &\quad + T^{0+} \|v - \tilde{v}\|_{\tilde{Y}_T^s}, \end{aligned} \quad (3.4)$$

for every $(u, v), (\tilde{u}, \tilde{v}) \in \tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$. On the other hand, observe that

$$\|Re(\partial_x u)\|_{\tilde{Y}_T^s} = \left\| \frac{\partial_x u + \overline{\partial_x u}}{2} \right\|_{\tilde{Y}_T^s} \lesssim \|\partial_x u\|_{\tilde{Y}_T^s},$$

then using items (ii) and (iv) of Proposition 2.1, estimates (2.5) and (2.9), we have the following

$$\begin{aligned} \|\Phi_2(u, v)\|_{\tilde{Y}_T^s} &\lesssim \|v_0\|_{H^s} + \|\partial_x u\|_{W_T^s} + \|\partial_x(v^2)\|_{W_T^s} + \|GG^*v\|_{W_T^s} \\ &\lesssim \|v_0\|_{H^s} + T^{0+} \left(\|u\|_{X_T^{s, \frac{1}{2}}} + \|v\|_{Y_T^{s, \frac{1}{2}}} \right) \end{aligned} \quad (3.5)$$

and similarly we get that,

$$\begin{aligned} \|\Phi_2(u, v) - \Phi_2(\tilde{u}, \tilde{v})\|_{\tilde{Y}_T^s} &\lesssim T^{0+} \|u - \tilde{u}\|_{\tilde{X}_T^s} \\ &\quad + T^{0+} \left(\|v\|_{Y_T^{s, \frac{1}{2}}} + \|\tilde{v}\|_{Y_T^{s, \frac{1}{2}}} \right) \|v - \tilde{v}\|_{\tilde{Y}_T^s}, \end{aligned} \quad (3.6)$$

for every $(u, v), (\tilde{u}, \tilde{v}) \in \tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$. We conclude from these estimates that if we take $T > 0$ small enough, the map Φ is a contraction in some suitable ball of $\tilde{X}_T^s \times \left(\tilde{Y}_T^s \cap L^2(0, T; L_0^2(\mathbb{T})) \right)$, then it has a fixed point.

Now, we prove the uniqueness in the class $\tilde{X}_T^s \times \tilde{Y}_T^s$ for the integral equation Φ_1 and Φ_2 . Set

$$w(t) = e^{it\partial_x^2} u_0 - i \int_0^t e^{i(t-t')\partial_x^2} \left\{ i\partial_t v(t') + \beta|u|^2 u(t') - i\varphi(t')^2 a(x)^2 u(t') \right\} dt'$$

and

$$\begin{aligned} z(t) &= e^{-t(\partial_x^3 + \mu\partial_x)} v_0 \\ &\quad + \int_0^t e^{-(t-t')(\partial_x^3 + \mu\partial_x)} \left\{ \operatorname{Re}(\partial_x u)(t') - GG^*v(t') - \frac{1}{2}\partial_x(v^2)(t') \right\} dt'. \end{aligned}$$

Observe that, $\beta|u|^2 u, \partial_x v \in X_T^{s, -\frac{1}{2}}$ and $\frac{1}{2}\partial_x(v^2), \partial_x u \in Y_T^{s, -\frac{1}{2}}$, hence we infer that,

$$\begin{aligned} \partial_t \left(\int_0^t e^{-it'\partial_x^2} \left\{ i\partial_x v + \beta|u|^2 u - i\varphi^2 a(x)^2 u \right\} \right. \\ \left. (t') dt' \right) &= e^{-it\partial_x^2} (i\partial_x v + \beta|u|^2 u - i\varphi^2 a^2 u) \end{aligned}$$

and

$$\begin{aligned} \partial_t \left(\int_0^t e^{-t'(\partial_x^3 + \mu\partial_x)} \left\{ \operatorname{Re}(\partial_x u) - GG^*v - \frac{1}{2}\partial_x(v^2) \right\} (t') dt' \right) \\ = e^{-t(\partial_x^3 + \mu\partial_x)} \left(\operatorname{Re}(\partial_x u) - GG^*v - \frac{1}{2}\partial_x(v^2) \right), \end{aligned}$$

in the distributional sense. This implies that (w, z) is a solution of

$$\begin{cases} i\partial_t w + \partial_x^2 w = i\partial_x v + \beta|u|^2 u - i\varphi^2 a(\cdot)^2 u, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t z + \partial_x^3 z + \mu\partial_x z = \operatorname{Re}(\partial_x u) - GG^*v - \frac{1}{2}\partial_x(v^2), & (x, t) \in \mathbb{T} \times \mathbb{R}_+. \end{cases}$$

Therefore, it follows that, $r_1(t) = e^{-it\partial_x^2}(u - w)$ and $r_2(t) = e^{-t(\partial_x^3 + \mu\partial_x)}(v - z)$ is a solution of $\partial_t r_1 = \partial_t r_2 = 0$ and $r_1(t) = r_2(t) = 0$. Thus $r_1 = r_2 = 0$ and (u, v) is the unique solution of the integral equations.

(b) **The solution is globally defined.**

In order to prove that our solutions are global, we observe that,

$$\frac{d}{dt} \left(\|u(t)\|_{L^2(\mathbb{T})}^2 + \|v(t)\|_{L_0^2(\mathbb{T})}^2 \right) \leq 0.$$

Then, integrating over t and using the Gronwall Inequality we get that, the $L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ norm of (u, v) remains bounded by every $t \in [0, T]$, hence the local solution of (3.1) can be extended to a global one.

Now we prove that, if $(u_0, v_0) \in H^2(\mathbb{T}) \times H_0^3(\mathbb{T})$ then the solution $(u, v) \in C([0, T]; H^2(\mathbb{T}) \times H_0^2(\mathbb{T}))$ of (3.1) can be extended for any $T > 0$ and then, by nonlinear interpolation (see, e.g., [5] and [27]), we can get the global well-posedness for the solution $(u, v) \in C([0, T]; H^s(\mathbb{T}) \times H_0^s(\mathbb{T}))$ of (3.1) with $(u_0, v_0) \in H^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ with $0 \leq s \leq 2$.

Indeed, we start by considering a smooth solution (u, v) of (3.1). Let $(z, w) = (u_t, v_t)$ so,

$$\begin{cases} i\partial_t z + \partial_x^2 z + ia(x)^2 z = i\partial_x w + 2\beta|u|^2 z + \beta u^2 \bar{z}, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t w + \partial_x^3 w + \partial_x(vw) + \mu\partial_x w = Re(\partial_x z) - GG^* w, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ z(x, 0) = z_0(x), \quad w(x, 0) = w_0(x), & x \in \mathbb{T}, \end{cases} \tag{3.7}$$

where

$$z_0 = iu_{0xx} + v_{0x} - i\beta|u_0|^2 u_0 - a^2 u_0 \in L^2(\mathbb{T}) \in L^2(\mathbb{T})$$

and

$$w_0 = -v_{0xxx} - \mu v_{0x} - v_0 v_{0x} + Re(u_{0x}) - GG^* v_0 \in L_0^2(\mathbb{T}).$$

If we consider $T > 0$ such that Φ is a contraction in some suitable ball of $\tilde{X}_T^0 \times \tilde{Y}_T^0$ as in (3.4) and (3.6), we have that

$$\|(u, v)\|_{\tilde{X}_T^0 \times \tilde{Y}_T^0} \leq C_1,$$

with $C_1 = C_1(\|u_0\|_{L^2(\mathbb{T})}, \|v_0\|_{L_0^2(\mathbb{T})}) > 0$. The same computation as in (3.3) and (3.4) leads us to

$$\|z\|_{\tilde{X}_T^0} \lesssim \|z_0\|_{L^2(\mathbb{T})} + T^{0+} \left(\|w\|_{\tilde{Y}_T^0} + \|z\|_{\tilde{X}_T^0} + 2C_1^2 \|z\|_{\tilde{X}_T^0} \right)$$

and

$$\|w\|_{\tilde{Y}_T^0} \lesssim \|w_0\|_{L_0^2(\mathbb{T})} + T^{0+} \left(\|z\|_{\tilde{X}_T^0} + C_1 \|w\|_{\tilde{Y}_T^0} \right),$$

or equivalently,

$$\|(z, w)\|_{\tilde{X}_T^0 \times \tilde{Y}_T^0} \leq C \|(z_0, w_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}.$$

Therefore, using the previous inequalities, we have

$$\|(z, w)\|_{L^\infty(0, T; L^2(\mathbb{T}) \times L_0^2(\mathbb{T}))} \leq \|(z, w)\|_{\tilde{X}_T^0 \times \tilde{Y}_T^0} \leq C \|(z_0, w_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}.$$

Thanks to the system (3.1), we get that

$$\begin{aligned}
\|\partial_x^3 v\|_{L_0^2(\mathbb{T})} &\leq \|w\|_{L_0^2(\mathbb{T})} + \|\operatorname{Re}(\partial_x u)\|_{L^2(\mathbb{T})} + \frac{1}{2}\|v\partial_x v\|_{L_0^2(\mathbb{T})} \\
&\quad + |\mu|\|\partial_x v\|_{L_0^2(\mathbb{T})} + \|GG^*v\|_{L_0^2(\mathbb{T})} \\
&\leq \|w\|_{L_0^2(\mathbb{T})} + C\|\partial_x u\|_{L^\infty(\mathbb{T})} + \|v\|_{L_0^2(\mathbb{T})}\|\partial_x v\|_{L^\infty(\mathbb{T})} + \|v\|_{L_0^2(\mathbb{T})} \\
&\quad + |\mu|\|\partial_x v\|_{L_0^2(\mathbb{T})} \\
&\leq \|w\|_{L_0^2(\mathbb{T})} + C\|u\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x^2 u\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \\
&\quad + C(1 + \|v\|_{L_0^2(\mathbb{T})})\|v\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x^3 v\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}} + |\mu|\|\partial_x v\|_{L_0^2(\mathbb{T})} \\
&\leq \frac{1}{4}\|(\partial_x^2 u, \partial_x^3 v)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} + \|w\|_{L_0^2(\mathbb{T})} + C(\|v\|_{L_0^2(\mathbb{T})} \\
&\quad + \|v\|_{L_0^2(\mathbb{T})}^3 + \|u\|_{L^2(\mathbb{T})}),
\end{aligned} \tag{3.8}$$

where we have used the Young inequality in this last inequality and

$$\begin{aligned}
\|\partial_x^2 u\|_{L^2(\mathbb{T})} &\leq \|z\|_{L^2(\mathbb{T})} + \|\partial_x v\|_{L_0^2(\mathbb{T})} + |\beta|\|u\|_{L^2(\mathbb{T})}\|u\|_{L^\infty} + \|a\|_{L^\infty}\|u\|_{L^2(\mathbb{T})} \\
&\leq \frac{1}{4}\|(\partial_x^2 u, \partial_x^3 v)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} + \|z\|_{L^2(\mathbb{T})} \\
&\quad + C|\beta|\|u\|_{L^2(\mathbb{T})}^2 + \|a\|_{L^\infty(\mathbb{T})}\|u\|_{L^2(\mathbb{T})},
\end{aligned} \tag{3.9}$$

where, for this last inequality, we have used that $H^2(\mathbb{T})$ is embedded in $L^\infty(\mathbb{T})$, the interpolation between $H^2(\mathbb{T})$ and $L^2(\mathbb{T})$ and Young inequality. So adding up the estimates (3.8) and (3.9) we can conclude that

$$\|(\partial_x^2 u, \partial_x^3 v)(t)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq C(\|u_0\|_{L^2(\mathbb{T})}, \|v_0\|_{L_0^2(\mathbb{T})})\|(u_0, v_0)\|_{H^2(\mathbb{T}) \times H_0^3(\mathbb{T})},$$

from which follows that $(u, v) \in C(\mathbb{R}_+; H^2(\mathbb{T}) \times H_0^2(\mathbb{T}))$. A similar argumentation can be done for $(u_0, v_0) \in H^{2k}(\mathbb{T}) \times H_0^{3k}(\mathbb{T})$, $k \in \mathbb{N}$, and for other values of $s \in \mathbb{R}_+$ using again nonlinear interpolation.

(c) **The flow map is Lipschitz.**

Finally, we prove that the map (3.2) is Lipschitz on bounded sets. To do so, consider (u, v) and (\tilde{u}, \tilde{v}) solutions of (3.1) with initial data $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0)$, respectively. Arguing as in (3.4) and (3.6) we have that,

$$\begin{aligned}
\|u - \tilde{u}\|_{\widetilde{X}_T^0} + \|v - \tilde{v}\|_{\widetilde{Y}_T^0} &\leq C\|u_0 - \tilde{u}_0\|_{L^2(\mathbb{T})} + C\|v_0 - \tilde{v}_0\|_{L_0^2(\mathbb{T})} \\
&\quad + T^{0+} \left(1 + \|u\|_{\widetilde{X}_T^0} + \|\tilde{u}\|_{\widetilde{X}_T^0}\right) \|u - \tilde{u}\|_{\widetilde{X}_T^0} \\
&\quad + CT^{0+} \left(1 + \|v\|_{\widetilde{Y}_T^0} + \|\tilde{v}\|_{\widetilde{Y}_T^0}\right) \|v - \tilde{v}\|_{\widetilde{Y}_T^0}.
\end{aligned}$$

Then, for $T > 0$ small enough, depending of the size of $(u_0, v_0), (\tilde{u}_0, \tilde{v}_0)$, it follows that

$$\|u - \tilde{u}\|_{\widetilde{X}_T^0} + \|v - \tilde{v}\|_{\widetilde{Y}_T^0} \leq C\|u_0 - \tilde{u}_0\|_{L^2(\mathbb{T})} + C\|v_0 - \tilde{v}_0\|_{L_0^2(\mathbb{T})}. \tag{3.10}$$

By iterating process we get that (3.10) is valid for every T , concluding the proof of Theorem 3.1. \square

4. Stabilization result

In this section, we are able to prove one of the main results of the article, precisely, we prove that the following system

$$\begin{cases} i\partial_t u + \partial_x^2 u + ia(x)^2 u = i\partial_x v + \beta|u|^2 u, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = Re(\partial_x u) - GG^*v, & (x, t) \in \mathbb{T} \times \mathbb{R}_+, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.1}$$

where $\beta, \mu \in \mathbb{R}$, is asymptotically stable for $(u_0, v_0) \in L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$, when two control inputs are acting in both equation in $\omega \subset \mathbb{T}$.

4.1. Proof of Theorem 1.1

As usual in the literature using the “Compactness–Uniqueness Argument” due to Lions [21], under the hypothesis of the Theorem 1.1, the global stabilization property is equivalent to show the following observability inequality:

For any $T > 0$ there exists $C = C(T) > 0$ such that

$$\|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 \leq C \left(\int_0^T \|au(t)\|_{L^2(\mathbb{T})}^2 dt + \int_0^T \|Gv(t)\|_{L_0^2(\mathbb{T})}^2 dt \right), \tag{4.2}$$

for any solution (u, v) of the system (4.1) with initial data $(u_0, v_0) \in L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ such that $\|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq R_0$.

Indeed, it follows from the energy estimate (1.10) that

$$\begin{aligned} \|(u, v)(T)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 &= \|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 \\ &\quad - \int_0^T \|au(t)\|_{L^2(\mathbb{T})}^2 dt - \int_0^T \|Gv(t)\|_{L_0^2(\mathbb{T})}^2 dt. \end{aligned}$$

Thus, from (4.2), we get that

$$\|(u, v)(T)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 \leq (1 - C^{-1})\|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2,$$

and since the solution (u, v) of the system (4.1) satisfies the semigroup property we have, for every $m \in \mathbb{N}$, that

$$\|(u, v)(mT)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 \leq (1 - C^{-1})^m \|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2,$$

which yields the desired result. □

4.2. Proof of the observability inequality

We argue by contradiction. If (4.2) does not occur, there exist $T > 0$ and a sequence $\{(u_{0n}, v_{0n})\}_{n \in \mathbb{N}}$ such that

$$\|(u_{0n}, v_{0n})\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq R_0 \tag{4.3}$$

and

$$\int_0^T \|au_n(t)\|_{L^2(\mathbb{T})}^2 dt + \int_0^T \|Gv_n(t)\|_{L_0^2(\mathbb{T})}^2 dt \leq \frac{1}{n} \|(u_{0n}, v_{0n})\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2. \tag{4.4}$$

Hence if we define $\alpha_n := \|(u_{0n}, v_{0n})\|_{L^2(\mathbb{T}) \times L^2_0(\mathbb{T})}^2$ we get

$$\alpha_n \leq R_0^2,$$

so we can extract a subsequence, still denoted by the same index, such that

$$\alpha_n \rightarrow \alpha,$$

with $\alpha \geq 0$. We split the analysis into two cases:

$$(i) \alpha > 0 \quad \text{and} \quad (ii) \alpha = 0.$$

(i) $\alpha > 0$.

It follows from Theorem 3.1 that the corresponding sequence of solutions $\{(u_n, v_n)\}$ associated with the initial data $\{(u_{0n}, v_{0n})\}$ is bounded in both spaces $L^\infty(0, T; L^2(\mathbb{T}) \times L^2_0(\mathbb{T}))$ and $X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$ then as $X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$ is a separable Hilbert space compactly embedded in $X_T^{-k, \frac{1}{2}-\delta} \times Y_T^{-s, \frac{1}{2}-\delta_1}$, for every $k, s > 0$ and $\delta, \delta_1 > 0$, we can get a subsequence, which will be denoted with the same index, such that

$$(u_n, v_n) \rightarrow (u, v) \text{ weakly in } X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$$

and

$$(u_n, v_n) \rightarrow (u, v) \text{ strongly in } X_T^{-k, \frac{1}{2}-\delta} \times Y_T^{-s, \frac{1}{2}-\delta_1}.$$

Moreover, by (4.4) we deduce that

$$au_n \rightarrow 0 \quad \text{in } L^2(0, T; L^2(\mathbb{T}))$$

and

$$Gv_n \rightarrow 0 = Gv \quad \text{in } L^2(0, T; L^2_0(\mathbb{T})),$$

from which follows that $u \equiv 0$ in ω and $v = c(t)$ in ω .

On the other hand, by Proposition 2.2, we infer that $\beta|u_n|^2 u_n$ is bounded in $X_T^{0, -\frac{1}{2}}$ and $\frac{1}{2}\partial_x(v_n^2)$ is bounded in $Y_T^{0, -\frac{1}{2}}$. Moreover, note that $(\beta|u_n|^2 u_n)$ is bounded in $X_T^{0, -b'}$ for $\frac{3}{8} < b' < \frac{1}{2}$, which is compactly embedded in $X_T^{-1, -\frac{1}{2}}$, then we can extract a subsequence, still denoted by the same index, such that, for some $f \in X_T^{-1, -\frac{1}{2}}$,

$$\beta|u_n|^2 u_n \rightarrow f \quad \text{strongly in } X_T^{-1, -\frac{1}{2}}.$$

As $Y_T^{0, \frac{1}{2}}$ is continuously embedded in $L^4(\mathbb{T} \times (0, T))$, thanks to the Proposition 2.4, $(\partial_x(v_n^2))$ is bounded in $L^2(0, T; H^{-1}(\mathbb{T})) = Y_T^{-1, 0}$, so, by interpolation between those spaces, we get that $(\partial_x(v_n^2))$ is bounded in $Y_T^{-\theta, -\frac{1}{2} + \frac{\theta}{2}}$, $\theta \in (0, 1)$, which is compactly embedded in $Y_T^{-1, -\frac{1}{2}}$. Therefore, we can extract a subsequence of $(\frac{1}{2}\partial_x(v_n^2))$ in $Y_T^{-1, -\frac{1}{2}}$, which will be denoted by the same index, such that

$$\frac{1}{2}\partial_x(v_n^2) \rightarrow \tilde{g} \quad \text{strongly in } Y_T^{-1, -\frac{1}{2}}.$$

With these convergences in hand we can pass to the limit in n , to obtain that (u, v) satisfies

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v = Re(\partial_x u) + \tilde{g}, & (x, t) \in \mathbb{T} \times (0, T). \end{cases}$$

Now, consider $w_n = u_n - u$, $z_n = v_n - v$,

$$f_n = -ia^2 u_n + \beta|u_n|^2 u_n - f$$

and

$$g_n = -GG^* v_n - \frac{1}{2}\partial_x(v_n^2) - \tilde{g}.$$

Observe that

$$\begin{aligned} \int_0^T \|Gz_n\|_{L^2_0(\mathbb{T})}^2 dt &= \int_0^T \|Gv_n\|_{L^2_0(\mathbb{T})}^2 dt \\ &+ \int_0^T \|Gv\|_{L^2_0(\mathbb{T})}^2 dt - 2 \int_0^T (Gv_n, Gv)_0 dt \rightarrow 0. \end{aligned}$$

On the other hand, we have

$$Gz_n(x, t) = g(x)z_n(x, t) + \int_{\mathbb{T}} g(y)z_n(y, t) dy.$$

Since $z_n \rightarrow 0$ weakly in $Y_T^{0, \frac{1}{2}}$, using Rellich theorem we get that $\int_{\mathbb{T}} g(y)z_n(y, \cdot) dy$ strongly converges to 0 in $L^2(0, T)$, from which follows that

$$\int_0^T \int_{\mathbb{T}} g^2(x)z_n(x, t)^2 dx dt \rightarrow 0.$$

Thus,

$$\begin{cases} i\partial_t w_n + \partial_x^2 w_n = i\partial_x z_n + f_n, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t z_n + \partial_x^3 z_n + \mu\partial_x z_n = Re(\partial_x w_n) + g_n, & (x, t) \in \mathbb{T} \times (0, T) \end{cases}$$

with

$$(w_n, z_n) \rightarrow (0, 0) \quad \text{in } L^2(0, T; L^2(\tilde{\omega}) \times L^2_0(\tilde{\omega}))$$

and

$$(f_n, g_n) \rightarrow (0, 0) \quad \text{strongly in } X_T^{-1, -\frac{1}{2}} \times Y_T^{-1, -\frac{1}{2}},$$

where $\tilde{\omega} = \text{supp}(g) \cap \text{supp}(a)$.

Now, we are in a position to use the results of the Appendix A and B. First, using the propagation of compactness, given in Proposition A.1, we have

$$(z_n, w_n) \rightarrow (0, 0) \quad \text{in } L^2_{loc}(0, T; L^2(\mathbb{T}) \times L^2_0(\mathbb{T})).$$

Then we can pick $t_0 \in [0, T]$ such that $(z_n(t_0), w_n(t_0)) \rightarrow (0, 0)$ strongly in $L^2(\mathbb{T}) \times L^2_0(\mathbb{T})$. Denote by (\tilde{u}, \tilde{v}) the solution of the problem

$$\begin{cases} i\partial_t \tilde{u} + \partial_x^2 \tilde{u} = i\partial_x \tilde{v} + \beta|\tilde{u}|^2 \tilde{u}, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t \tilde{v} + \partial_x^3 \tilde{v} + \mu\partial_x \tilde{v} + \partial_x(\tilde{v}^2) = Re(\partial_x \tilde{u}), & (x, t) \in \mathbb{T} \times (0, T), \\ (\tilde{u}, \tilde{v})(t_0) = (u(t_0), v(t_0)), & t_0 \in (0, T). \end{cases}$$

Theorem 3.1 gives us that the flow map is Lipschitz on bounded sets, hence as

$$\begin{aligned} (u_n(t_0), v_n(t_0)) &\rightarrow (u(t_0), v(t_0)) \quad \text{in } L^2(\mathbb{T}) \times L^2_0(\mathbb{T}), \\ ia^2 u_n &\rightarrow 0 \quad \text{in } L^2(0, T; L^2(\mathbb{T})) \end{aligned}$$

and

$$Gv_n \rightarrow 0 \quad \text{in } L^2(0, T; L^2_0(\mathbb{T})),$$

follows that

$$(u_n, v_n) \rightarrow (u, v) \quad \text{in } X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}.$$

Therefore, putting together all these convergences and passing to the limit, we can conclude that (u, v) solves

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2 u, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = Re(\partial_x u), & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, t) = 0 \quad v(x, t) = c(t), & (x, t) \in \omega \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}. \end{cases} \quad (4.5)$$

Note that, using the second equation of (4.5) we have that $\partial_t v = 0$ on $\omega \times (0, T)$, hence $c(t) \equiv c$, with $c \in \mathbb{R}$, for all $t \in (0, T)$. Thus thanks to the unique continuation property given by Corollary B.4, we ensure that $(u, v) = (0, 0)$. From this, we conclude that

$$\|(u_n(0), v_n(0))\|_{L^2(\mathbb{T}) \times L^2_0(\mathbb{T})} \rightarrow 0,$$

which is a contradiction with our hypothesis $\alpha > 0$.

(ii) $\alpha = 0$.

Observe that we can assume $\alpha_n > 0$ for all $n \in \mathbb{N}$. Pick the function as follows $w_n = \frac{u_n}{\alpha_n}$ and $z_n = \frac{v_n}{\alpha_n}$, for all $n \geq 1$. Thus,

$$\|(w_{0n}, z_{0n})\|_{L^2(\mathbb{T}) \times L^2_0(\mathbb{T})}^2 = 1$$

and (w_n, z_n) satisfies the system

$$\begin{cases} i\partial_t w_n + \partial_x^2 w_n = i\partial_x z_n + \alpha_n^2 \beta |v_n|^2 v_n - ia(x)^2 w_n, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t z_n + \partial_x^3 z_n + \frac{\alpha_n^2}{2} \partial_x(z_n^2) = Re(\partial_x w_n) - GG^* z_n, & (x, t) \in \mathbb{T} \times (0, T) \end{cases}$$

with

$$\int_0^T \|a(x)w_n(t)\|_{L^2(\mathbb{T})}^2 dt + \int_0^T \|Gz_n(t)\|_{L^2_0(\mathbb{T})}^2 dt \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

By a boot-strap argument we conclude that (w_n, z_n) is bounded in $X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$. Hence we can extract a subsequence of $\{(w_n, z_n)\}$, still denoted by the same index, such that

$$(w_n, z_n) \rightarrow (w, z) \quad \text{weakly in } X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$$

and

$$(w_n, z_n) \rightarrow (w, z) \quad \text{strongly in } X_T^{0-, -\frac{1}{2}} \times Y_T^{0-, -\frac{1}{2}}.$$

By Proposition 2.2 we have that $\beta|w_n|^2w_n$ is bounded in $X_T^{0,-\frac{1}{2}}$ then

$$\alpha_n^2\beta|w_n|^2w_n \rightarrow 0 \quad \text{strongly in } X_T^{0,-\frac{1}{2}},$$

as $\alpha_n \rightarrow 0$. Similarly, due to the fact that $\{\partial_x(v_n^2)\}$ is bounded in $Y_T^{0,-\frac{1}{2}}$ we have that

$$\alpha_n\partial_x(v_n^2) \rightarrow 0 \quad \text{strongly in } Y_T^{0,-\frac{1}{2}}.$$

Hence, (w, z) solves the following system

$$\begin{cases} i\partial_t w + \partial_x^2 w = i\partial_x z, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t z + \partial_x^3 z + \mu\partial_x z = \text{Re}(\partial_x w), & (x, t) \in \mathbb{T} \times (0, T), \\ w(x, t) = 0, \quad z(x, t) = c, & (x, t) \in \omega \times (0, T). \end{cases}$$

Again using Corollary B.4 we have that $w = z = 0$ in $(0, T) \times \mathbb{T}$. Then, by an application of Proposition A.1

$$(w_n, z_n) \rightarrow (0, 0) \quad \text{in } L^2_{loc}(0, T; L^2(\mathbb{T}) \times L^2_0(\mathbb{T})).$$

So we can conclude the proof as in the first case, showing the observability inequality (4.2). □

5. Controllability results

In this section, we are interested to prove the exact controllability of the following nonlinear system

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = \text{Re}(\partial_x u) + Gh, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.1}$$

where f and h are control functions. Before presenting the result for this system, we first need to prove the result for the linear system associated with (5.1),

$$\begin{cases} i\partial_t u + \partial_x^2 u = f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v = Gh, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, T) = 0, \quad v(x, T) = 0, & x \in \mathbb{T}. \end{cases} \tag{5.2}$$

In both cases, we take f and h with a special form, that is,

$$f := a^2\varphi^2h_1 \quad \text{and} \quad h := G^*h_2 = Gh_2, \tag{5.3}$$

with a and G satisfying (1.7) and (1.8), respectively, and h_1, h_2 in some appropriated space.

It is classical in the literature that the observability inequality for the NLS equation holds, precisely, the following proposition related to the observability inequality for the adjoint system associated with the Schrödinger equation is verified and for details see, e.g. [20] and [22].

Proposition 5.1. *Let $\omega \subset \mathbb{T}$. For any $a = a(x) \in C^\infty(\mathbb{T})$ and $\varphi = \varphi(t) \in C^\infty(]0, T[)$ real valued such that $a \equiv 1$ on $\omega \times]0, T[$ and $\varphi \equiv 1$ on $[\frac{T}{3}, \frac{2T}{3}]$ there exists $C := C(T) > 0$ such that*

$$\|\phi_0\|_{L^2(\mathbb{T})}^2 \leq C \int_0^T \|a(x)\varphi(t)e^{it\partial_x^2}\phi_0\|_{L^2(\mathbb{T})}^2 dt. \tag{5.4}$$

Additionally, an observability inequality for the single KdV equation can be verified, and we cite [25] for more details.

Proposition 5.2. *Let $T > 0$ be given. There exists $C = C(T) > 0$ such that,*

$$\|\psi_0\|_{L_0^2(\mathbb{T})}^2 \leq C \int_0^T \|Ge^{-t(\partial_x^3 + \mu\partial_x)}\psi_0\|_{L_0^2(\mathbb{T})}^2 dt. \tag{5.5}$$

As a consequence of the previous observability inequality, using the HUM method introduced by Lions [21], the exact controllability in $L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ for the linear system (5.2) holds. Indeed, just observe that by combining both observability estimates (5.4) and (5.5), we get

$$\begin{aligned} & \|(\phi_0, \psi_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}^2 \\ & \leq C \left(\int_0^T \|a(\cdot)\varphi(t)u(t)\|_{L^2(\mathbb{T})}^2 dt + \int_0^T \|Ge^{-t(\partial_x^3 + \mu\partial_x)}\psi(t)\|_{L_0^2(\mathbb{T})}^2 dt \right), \end{aligned}$$

which means that the linear system (5.2) is null controllable, that is, the map

$$\begin{aligned} S : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) & \rightarrow L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) \\ S(\varphi_0, \psi_0) & = (u(0), v(0)), \end{aligned}$$

where $(u(0), v(0))$ is the initial data associated with (5.2) with $f(x, t)$ and $h(x, t)$ defined by (5.3), is an isomorphism, and the following linear result is verified.

Theorem 5.3. *Let $\omega \subset \mathbb{T}$ be a nonempty open set and $T > 0$. Then for every $(u_0, v_0), (u_1, v_1) \in L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ one can find two control inputs $f \in C([0, T]; L^2(\mathbb{T}))$ and $h \in C([0, T]; L_0^2(\mathbb{T}))$, such that, the unique solution $(u, v) \in C([0, T]; L^2(\mathbb{T}) \times L_0^2(\mathbb{T}))$ of (5.2) satisfies $(u, v)(x, T) = (u_1(x), v_1(x))$.*

Now, we are in a position to prove the Theorem 1.2.

Proof of Theorem 1.2. For $T > 0$ to be determined, we consider the systems

$$\begin{cases} i\partial_t\phi + \partial_x^2\phi = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t\psi + \partial_x^3\psi + \mu\partial_x\psi = 0, & (x, t) \in \mathbb{T} \times (0, T), \\ \phi(x, 0) = \phi_0(x), \psi(x, 0) = \psi_0(x), & x \in \mathbb{T} \end{cases}$$

and

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2 u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v + \frac{1}{2}\partial_x(v^2) = \text{Re}(\partial_x u) + Gh, & (x, t) \in \mathbb{T} \times (0, T), \\ (u(\cdot, T), v(\cdot, T)) = (0, 0), & x \in \mathbb{T}, \end{cases}$$

with $f := a^2\varphi^2\phi$ and $h := G^*\psi = G\psi$. Let us define the operator

$$\begin{aligned} L : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) & \rightarrow L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) \\ (\phi_0, \psi_0) & \mapsto L(\phi_0, \psi_0) = (u|_{t=0}, v|_{t=0}) = (u_0, v_0) \end{aligned}$$

We split (u, v) into

$$(u, v) = (u_L, v_L) + (u_{NL}, v_{NL})$$

where (u_L, v_L) is solution of

$$\begin{cases} i\partial_t u_L + \partial_x^2 u_L = a^2 \varphi^2 \phi, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v_L + \partial_x^3 v_L + \mu \partial_x v_L = GG^* \psi, & (x, t) \in \mathbb{T} \times (0, T), \\ (u_L, v_L)(\cdot, T) = (0, 0), & x \in \mathbb{T}. \end{cases}$$

and (u_{NL}, v_{NL}) is solution of

$$\begin{cases} i\partial_t u_{NL} + \partial_x^2 u_{NL} = i\partial_x v + \beta |u|^2 u, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v_{NL} + \partial_x^3 v_{NL} + \mu \partial_x v_{NL} = \operatorname{Re}(\partial_x u) - \frac{1}{2} \partial_x (v^2), & (x, t) \in \mathbb{T} \times (0, T), \\ (u_{NL}, v_{NL})(\cdot, T) = (0, 0), & x \in \mathbb{T}. \end{cases}$$

Observe that we can follow [15] to construct an isomorphism

$$S : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$$

such that $(u_L(0), v_L(0)) = S_L(\phi_0, \psi_0)$. Additionally, we can also construct another application (see e.g. [19, Lemma 2.4])

$$K : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$$

which satisfies $K(\phi_0, \psi_0) = (u_{NL}(0), v_{NL}(0))$. With these information in hand, we have that if $(u, v), (u_L, v_L), (u_{NL}, v_{NL}) \in \widetilde{X}_T^0 \times \widetilde{Y}_T^0$ and

$$(u, v)(0) = (u_L, v_L)(0) + (u_{NL}, v_{NL})(0),$$

then we can rewrite the previous equality as follows

$$L(\phi_0, \psi_0) = S_L(\phi_0, \psi_0) + K(\phi_0, \psi_0),$$

where $K(\phi_0, \psi_0) = (u_{NL}(0), v_{NL}(0))$. So, we have that $L(\phi_0, \psi_0) = (u_0, v_0)$ is equivalent to

$$(\phi_0, \psi_0) = S_L^{-1}(u_0, v_0) - S_L^{-1}K(\phi_0, \psi_0).$$

Thus, let us define the following map

$$B : L^2(\mathbb{T}) \times L_0^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$$

by

$$B(\phi_0, \psi_0) = S_L^{-1}(u_0, v_0) - S_L^{-1}K(\phi_0, \psi_0).$$

Therefore, our null controllability problem is reduced to prove that B has a fixed point, so let us prove it now.

From now on we may fix $T < 1$. Since S_L is an isomorphism of $L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$, we have,

$$\|B(\phi_0, \psi_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq C (\|K(\phi_0, \psi_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} + \|(u_0, v_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}).$$

We are interested to estimate $\|K(\phi_0, \psi_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} = \|(u_{NL}(0), v_{NL}(0))\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}$. To make it we use the Duhamel formula and the $X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}-}$ estimates, to conclude that

$$\begin{aligned}
& \| (u_{NL}(0), v_{NL}(0)) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \\
& \leq \| (u_{NL}, v_{NL}) \|_{\widetilde{X}_T^0 \times \widetilde{Y}_T^0} \\
& \leq C \| |u|^2 u \|_{X_T^{0, -\frac{1}{2}}} + C \| \partial_x v \|_{X_T^{0, -\frac{1}{2}}} \\
& \quad + C \| \operatorname{Re}(\partial_x u) \|_{Y_T^{0, -\frac{1}{2}}} + C \| \partial_x(v^2) \|_{Y_T^{0, -\frac{1}{2}}} \\
& \leq CT^{0+} \| u \|_{X_T^{0, \frac{1}{2}}}^3 + CT^{0+} \| u \|_{X_T^{0, \frac{1}{2}}} + CT^{0+} \| v \|_{Y_T^{0, \frac{1}{2}}} + CT^{0+} \| v \|_{Y_T^{0+}}^2,
\end{aligned}$$

thanks to the estimates of the Sect. 2. Now, we notice that,

$$\begin{aligned}
\| (u, v) \|_{X_T^{0, \frac{1}{2}}} & \leq C \| (\varphi^2 a^2 \phi, GG^* \psi) \|_{L^2(0, T; L^2(\mathbb{T}) \times L_0^2(\mathbb{T}))} \\
& \leq C \| (\phi_0, \psi_0) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} < C\eta
\end{aligned}$$

and then,

$$\| B(\phi_0, \psi_0) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \leq C (T^{0+} \eta + T^{0+} \eta^2 + \| (u_0, v_0) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}).$$

Therefore, taking $\| (u_0, v_0) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}$ small enough, we can conclude that B maps the closed ball B_η of $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ into itself.

It remains to prove that B is a contraction. We start by considering the following two systems:

$$\begin{cases}
i \partial_t (u - \tilde{u}) + \partial_x^2 (u - \tilde{u}) = i \partial_x (v - \tilde{v}) + \beta (|u|^2 u - |\tilde{u}|^2 \tilde{u}) + a^2 \varphi^2 (\phi - \tilde{\phi}) \\
\partial_t (v - \tilde{v}) + \partial_x^3 (v - \tilde{v}) + \mu \partial_x (v - \tilde{v}) + \frac{1}{2} \partial_x (v^2 - \tilde{v}^2) = \operatorname{Re}(\partial_x u - \partial_x \tilde{u}) \\
+ GG^* (\psi - \tilde{\psi})(u - \tilde{u})(\cdot, T) = 0, \quad (v - \tilde{v})(\cdot, T) = 0.
\end{cases}$$

and

$$\begin{cases}
i \partial_t (u_{NL} - \tilde{u}_{NL}) + \partial_x^2 (u_{NL} - \tilde{u}_{NL}) = \beta (|u|^2 u - |\tilde{u}|^2 \tilde{u}) + i \partial_x (v - \tilde{v}) \\
\partial_t (v_{NL} - \tilde{v}_{NL}) + \partial_x^3 (v_{NL} - \tilde{v}_{NL}) + \mu \partial_x (v_{NL} - \tilde{v}_{NL}) \\
= \operatorname{Re}(\partial_x u - \partial_x \tilde{u}) + \frac{1}{2} (\partial_x (\tilde{v}^2 - v^2)) \\
(u_{NL} - \tilde{u}_{NL})(\cdot, T) = 0, \quad (v_{NL} - \tilde{v}_{NL})(\cdot, T) = 0.
\end{cases}$$

Again, due to the estimates provided in the Sect. 2, we get

$$\begin{aligned}
& \| B(\phi_0, \psi_0) - B(\tilde{\phi}_0, \tilde{\psi}_0) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \\
& \leq \| ((u_{NL} - \tilde{u}_{NL})(0), (v_{NL} - \tilde{v}_{NL})(0)) \|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \\
& \leq C \| |u|^2 u - |\tilde{u}|^2 \tilde{u} \|_{X_T^{0, -\frac{1}{2}}} + C \| \partial_x v - \partial_x \tilde{v} \|_{X_T^{0, -\frac{1}{2}}} \\
& \quad + C \| \partial_x (v^2) - \partial_x (\tilde{v}^2) \|_{Y_T^{0, -\frac{1}{2}}} + C \| \operatorname{Re}(\partial_x u - \partial_x \tilde{u}) \|_{Y_T^{0, -\frac{1}{2}}} \\
& \leq CT^{0+} \left(1 + \| u \|_{X_T^{0, \frac{1}{2}}}^2 + \| \tilde{u} \|_{X_T^{0, \frac{1}{2}}}^2 \right) \| u - \tilde{u} \|_{X_T^{0, \frac{1}{2}}} \\
& \quad + CT^{0+} \left(1 + \| v \|_{Y_T^{0, \frac{1}{2}}} + \| \tilde{v} \|_{Y_T^{0, \frac{1}{2}}} \right) \| v - \tilde{v} \|_{Y_T^{0, \frac{1}{2}}} \\
& \leq C (T^{0+} \eta^2 + T^{0+}) \| u - \tilde{u} \|_{X_T^{0, \frac{1}{2}}} \\
& \quad + (T^{0+} + T^{0+} \eta) \| v - \tilde{v} \|_{Y_T^{0, \frac{1}{2}}}.
\end{aligned}$$

Considering the system (5.1), we deduce that

$$\begin{aligned}
 & \|u - \tilde{u}\|_{X_T^{0, \frac{1}{2}}} + \|v - \tilde{v}\|_{Y_T^{0, \frac{1}{2}}} \\
 & \leq C \| |u|^2 u - |\tilde{u}|^2 \tilde{u} \|_{X_T^{0, \frac{1}{2}}} + C \| \partial_x v - \partial_x \tilde{v} \|_{X_T^{0, \frac{1}{2}}} \\
 & \quad + C \| \partial_x (v^2) - \partial_x (\tilde{v}^2) \|_{Y_T^{0, \frac{1}{2}}} + C \| \operatorname{Re}(\partial_x u) - \operatorname{Re}(\partial_x \tilde{u}) \|_{Y_T^{0, \frac{1}{2}}} \\
 & \quad + C \| \varphi^2 a^2 (\phi - \tilde{\phi}) \|_{L^2(\mathbb{T})} + \| GG^*(\psi - \tilde{\psi}) \|_{L_0^2(\mathbb{T})} \\
 & \leq C (T^{0+} \eta^2 + T^{0+}) \|u - \tilde{u}\|_{X_T^{0, \frac{1}{2}}} + (T^{0+} + T^{0+} \eta^2) \|v - \tilde{v}\|_{Y_T^{0, \frac{1}{2}}} \\
 & \quad + C \| \phi_0 - \tilde{\phi}_0 \|_{L^2(\mathbb{T})} + C \| \psi_0 - \tilde{\psi}_0 \|_{L_0^2(\mathbb{T})}.
 \end{aligned} \tag{5.6}$$

Then, for $\eta < 1$ fixed, we can get a small $T > 0$ such that,

$$\|u - \tilde{u}\|_{X_T^{0, \frac{1}{2}}} + \|v - \tilde{v}\|_{Y_T^{0, \frac{1}{2}}} \leq C \| \phi_0 - \tilde{\phi}_0 \|_{L^2(\mathbb{T})} + C \| \psi_0 - \tilde{\psi}_0 \|_{L_0^2(\mathbb{T})}. \tag{5.7}$$

Thanks to the inequalities (5.6) and (5.7), we get that

$$\begin{aligned}
 & \|B(\phi_0, \psi_0) - B(\tilde{\phi}_0, \tilde{\psi}_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})} \\
 & \leq C (T^{0+} + \eta T^{0+}) \|(\phi_0, \psi_0) - (\tilde{\phi}_0, \tilde{\psi}_0)\|_{L^2(\mathbb{T}) \times L_0^2(\mathbb{T})}.
 \end{aligned}$$

To finish, just choosing

$$T^{0+} + \eta T^{0+} < \frac{1}{2C},$$

we conclude that B is a contraction in the closed ball of $L^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ and the local controllability result is verified, and Theorem 1.2 is achieved. \square

6. Further comments and open issues

In this work, we showed the stabilization and control issues for the system

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta |u|^2 u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) + \mu \partial_x v = \operatorname{Re}(\partial_x u) + g, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}. \end{cases} \tag{6.1}$$

When we introduce two damping mechanisms $f := -ia^2(x)u$ and $g := -GG^*h$, where a satisfies (1.7) and G is defined by (1.8), we are able to prove that the solutions associated with (6.1) decay exponentially for t large enough. Moreover, together with a local exact controllability, we are also able to prove a global exact controllability property, that means, for any initial and final data, without restrictions on its norms, we can find two control inputs f and r such that we can drive the initial data to the final data by using these controls.

The strategy employed in the proof of the main results in this manuscript uses the contradiction argument, which aims to prove an observability inequality. Observability inequality follows by a combination of propagation of compactness and regularity, together with a unique continuation property for the operator associated with the NLS–KdV system, which is proved using the defect measure.

6.1. Only one control acting

Control problems of coupled dispersive systems are still not well understood, most of the results consider single equations, like the KdV equation, Schrödinger equation and others. In our case, we can combine two dispersive equations with nonlinearities and principal symbols of a different order. One of the main difficulties in our case arises in the proof of the propagation results, and was overcome using the smoothing properties of the solutions in Bourgain space.

This method presented here seems to work well however it has a drawback. When we decided to study the nonlinear problem directly, instead of first the linear problem, we were unable to remove controls, that is, with this method, the best possible result is to show the control results with a control in each equation.

It is important to mention that in [3], the authors proved that the following linear Schrödinger–KdV system

$$\begin{cases} i\partial_t w + \partial_x^2 w = a_1 w + a_2 y + h\mathbf{1}_\omega & \text{in } Q \\ \partial_t y + \partial_x^3 y + \partial_x(My) = \operatorname{Re}(a_3 w) + a_4 y + l\mathbf{1}_\omega & \text{in } Q \\ w(0, t) = w(1, t) = 0 & \text{in } (0, T), \\ y(0, t) = y(1, t) = \partial_x y(1, t) = 0 & \text{in } (0, T), \\ w(x, 0) = w_0(x), \quad y(x, 0) = y_0(x) & \text{in } (0, 1), \end{cases} \quad (6.2)$$

in a bounded domain $Q := (0, 1) \times (0, T)$, is controllable. They are not able to produce any nonlinear results due to the lack of regularity to treat the nonlinear problem in Sobolev spaces, in this way, our work extends, for the nonlinear system, the control problems for a more general system. The main point of dealing with nonlinear structure in this manuscript is to work in the Bourgain spaces, which helps us to treat the nonlinearities appropriately. However, we mention that the linear case helps to deal with the system (6.2) with only a control $h\mathbf{1}_\omega$. This was possible by using a Carleman estimate which involves both operators, Schrödinger and KdV operators. However, it is important to make it clear that in our case we can not use this inequality to remove one control, since the regularity used in [3, Theorem 1.3.] is not enough to deal with the nonlinearities involved in the system (6.1).

6.2. Others nonlinearities

Another interesting problem is considering the full physical model of the NLS–KdV system that appears in [2],

$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha uv + \beta|u|^2 u + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma\partial_x(|u|^2) + g, & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \mathbb{T}, \end{cases} \quad (6.3)$$

to prove controllability results. In this way, we propose the following natural issue.

Question. *A: Is it possible to prove control results for the system (6.3)?*

Note that, in this case, to reach the result we need to study some conservation laws associated with the solution of the system. In [2, Lemma 5.1.]

the authors proved that the evolution system (6.3) preserves the following quantities

$$M(t) := \int_{\mathbb{T}} |u(t)|^2 dx, \quad Q(t) := \int_{\mathbb{T}} \left\{ \alpha v(t)^2 + 2\gamma \Im \left(u(t) \overline{\partial_x u(t)} \right) \right\} dx$$

and

$$E(t) := \int_{\mathbb{T}} \left\{ \alpha \gamma v(t) |u(t)|^2 - \frac{\alpha}{6} v(t)^3 + \frac{\beta \gamma}{2} |u(t)|^4 + \frac{\alpha}{2} |\partial_x v(t)|^2 + \gamma |\partial_x u(t)|^2 \right\} dx.$$

These quantities give us only information of about “mass” $M(t)$ associated with the Schrödinger equation and there is no information about this conversation for the part of the KdV equation. So, it is necessary, before answering Question \mathcal{A} , to solve the following problem.

Question. \mathcal{B} : *How to deal with the KdV part presented in (6.3) to present some information for the “mass”?*

This problem is an open issue, and it seems that our work opens, from now on, some possibilities to deal with these types of systems.

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Appendix A: Propagation results in Bourgain spaces

This appendix is dedicated to presenting properties of propagation in Bourgain spaces for the linear differential operator associated with the NLS–KdV system. These results of propagation are the key to proving the global control results, that some parts were borrowed from [18, 19]. For self-contentedness, we will also give rigorous proof of them. The main ingredient is basically pseudo-differential analysis. Let us begin with the result of the propagation of compactness which will ensure strong convergence in appropriate spaces.

Proposition A.1. [Propagation of compactness] *Let $T > 0$, suppose that $(u_n, v_n) \in X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$ and $(f_n, g_n) \in X_T^{-1, -\frac{1}{2}} \times Y_T^{-1, -\frac{1}{2}}$, $n \in \mathbb{N}$, satisfies*

$$\begin{cases} i\partial_t u_n + \partial_x^2 u_n = i\partial_x v_n + f_n, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v_n + \partial_x^3 v_n + \mu\partial_x v_n = \operatorname{Re}(\partial_x u_n) + g_n, & (x, t) \in \mathbb{T} \times (0, T). \end{cases}$$

Assume that there exists a constant $C > 0$ such that

$$\|(u_n, v_n)\|_{X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}} \leq C, \quad \text{for } n \geq 1. \tag{A.1}$$

Additionally, suppose that

$$\begin{aligned} (u_n, v_n) &\rightarrow (0, 0) \quad \text{weakly in } X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}, \\ (f_n, g_n) &\rightarrow (0, 0) \quad \text{strongly in } X_T^{-1, -\frac{1}{2}} \times Y_T^{-1, -\frac{1}{2}}. \end{aligned}$$

and that for some nonempty set $\omega \subset \mathbb{T}$ we have

$$(u_n, v_n) \rightarrow (0, 0) \text{ in } L^2(0, T; L^2(\omega) \times L^2_0(\omega)).$$

Then,

$$(u_n, v_n) \rightarrow (0, 0) \text{ strongly in } L^2_{loc}(0, T; L^2(\mathbb{T}) \times L^2_0(\mathbb{T})). \tag{A.2}$$

Proof. We shall start by proving the following convergence

$$u_n \rightarrow 0 \text{ strongly in } L^2_{loc}(0, T; L^2(\mathbb{T})). \tag{A.3}$$

Denote $L_1 := i\partial_t + \partial_x^2$ the linear Schrödinger operator and consider $\varphi \in C^\infty(\mathbb{T})$ and $\psi \in C^\infty_0(]0, T[)$ taking real values to be determined. Define $Bu := \varphi(x)D^{-1}$ and $A = \psi(t)B$ and for $\epsilon > 0$, $A_\epsilon = Ae^{\epsilon\partial_x^2} = \psi(t)B_\epsilon$, with D^{-1} defined by (2.3). We consider,

$$\alpha_{n,\epsilon} := ([A_\epsilon, L_1]u_n, u_n)_{L^2(0, T; L^2(\mathbb{T}))} = ([A_\epsilon, \partial_x^2]u_n, u_n) - i(\psi'(t)Bu_n, u_n).$$

On the other hand, we also have

$$\alpha_{n,\epsilon} = (i\partial_x v_n, A_\epsilon^* u_n) - (A_\epsilon u_n, i\partial_x v_n) + (f_n, A_\epsilon^* u_n) - (A_\epsilon u_n, f_n). \tag{A.4}$$

Now, observe that using Lemma 2.5 we have

$$\begin{aligned} |(i\partial_x v_n, A_\epsilon^* u_n)_{L^2_{t,x}}| &\leq \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} |\partial_x \widehat{v_n}(m, \tau)| \left| \overline{\widehat{A_\epsilon^* u_n}(n, \tau)} \right| d\tau \\ &\leq \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} H(\tau, n) \langle \tau - n^3 + \mu n \rangle^{b'} \langle n \rangle^{-1} |\widehat{v_n}(n, \tau)| \langle \tau + n^2 \rangle^{b'} \langle n \rangle^1 \left| \overline{\widehat{A_\epsilon^* u_n}(n, \tau)} \right| d\tau \\ &\leq C \left(\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \langle \tau - n^3 + \mu n \rangle^{2b'} \langle n \rangle^{-2} |\widehat{v_n}(m, \tau)|^2 d\tau \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \langle \tau + n^2 \rangle^{2b'} \langle n \rangle^2 \left| \overline{\widehat{A_\epsilon^* u_n}(n, \tau)} \right|^2 d\tau \right)^{\frac{1}{2}} \\ &\leq C \|v_n\|_{Y_T^{-1, b'}} \|A_\epsilon^* u_n\|_{X_T^{1, b'}} \\ &\leq C \|v_n\|_{Y_T^{-1, b'}} \|u_n\|_{X_T^{0, b'}}. \end{aligned} \tag{A.5}$$

Noting that, for $b' = \frac{1}{2}-$, we get

$$v_n \rightarrow 0 \quad \text{strongly in } Y_T^{-1,b'}$$

and thanks to (A.1) we ensure that $\|u_n\|_{X_T^{0,b'}}$ is bounded. Since

$$v_n \rightarrow 0 \quad \text{in } Y_T^{-1,b'},$$

yields that

$$\sup_{0 < \epsilon \leq 1} |(i\partial_x v_n, A_\epsilon^* u_n)_{L^2_{t,x}}| \rightarrow 0,$$

due to (A.5). Additionally, holds that

$$|(f_n, A_\epsilon^* u_n)| \leq \|f_n\|_{X_T^{-1,-\frac{1}{2}}} \|u_n\|_{X_T^{0,\frac{1}{2}}},$$

which yields that

$$\sup_{1 < \epsilon \leq 0} |(f_n, A_\epsilon^* u_n)| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The same kind of estimates for the other terms in (A.4) give us the following limit

$$\sup_{1 < \epsilon \leq 0} |\alpha_{n,\epsilon}| \rightarrow 0, \quad \text{when } n \rightarrow \infty.$$

Noting that

$$[A, \partial_x^2] = -2\psi(t)(\partial_x \varphi) \partial_x D^{-1} - \psi(t)(\partial_x^2) D^{-1},$$

since D^{-1} commutes with ∂_x , and taking the supremum on ϵ tending to 0, we conclude that

$$([A, \partial_x^2] u_n, u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Putting together these convergences, above mentioned, we may have that

$$(\psi(t)(\partial_x^2 \varphi) D^{-1} u_n, u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note that, $-i\partial_x D^{-1}$ is the orthogonal projection on the subspace of functions with $\widehat{u}(0) = 0$, and $\widehat{u}_n(0)(t)$ tends to 0 in $L^2([0, T])$ we have

$$(\psi(t)(\partial_x \varphi) \widehat{u}_n(0)(t), u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus we have proved that for any $\varphi \in C^\infty(\mathbb{T})$ and $\psi \in C^\infty([0, T])$,

$$(\psi(t)(\partial_x \varphi) u_n, u_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We conclude the first part of this proof by observing that a function $\phi \in C^\infty(\mathbb{T})$ can be written as $\partial_x \varphi$ for some $\varphi \in C^\infty(\mathbb{T})$ if and only if $\int_{\mathbb{T}} \varphi dx = 0$. Thus for any $\chi \in C_0^\infty(\omega)$ and any $x_0 \in \mathbb{T}$, $\phi(x) = \chi(x) - \chi(\cdot - x_0)$ can be written as $\phi = \partial_x \varphi$ for some $\varphi \in C^\infty(\mathbb{T})$.

Now we use the hypothesis that $u_n \rightarrow 0$ strongly in $L^2(0, T; L^2(\omega))$ to conclude that

$$\lim_{n \rightarrow \infty} (\psi(t) \chi u_n, u_n) = 0.$$

Therefore, for any $x_0 \in \mathbb{T}$,

$$\lim_{n \rightarrow \infty} (\psi(t) \chi(\cdot - x_0) u_n, u_n) = 0$$

and yields that $u_n \rightarrow 0$ in $L^2_{loc}(0, T; L^2(\mathbb{T}))$ by constructing a partition of the unity of \mathbb{T} involving functions of the form $\chi_i(\cdot - x_0^i)$ with $\chi_i \in C^\infty(\omega)$ and $x_0^i \in \mathbb{T}$.

To prove that $v_n \rightarrow 0$ strongly in $L^2_{loc}(0, T; L^2_0(\mathbb{T}))$ we argue analogously, but in this case considering $L_2 = \partial_t + \partial_x^3 + \mu\partial_x$, $B_1 = \varphi(x)D^{-2}$ and $A_2 = \psi(t)B_2$ and for $\epsilon > 0$, $A_{2\epsilon} = A_2e^{\epsilon\partial_x^2}$. In this situation, we deal with

$$\widetilde{\alpha}_{n,\epsilon} = ([A_{2\epsilon}, L_2]v_n, v_n)_{L^2_{t,x}}$$

the novelty in this case is that

$$\widetilde{\alpha}_{n,\epsilon} = (g_n, A_{2\epsilon}^*v_n) + (A_{2\epsilon}v_n, g_n) + (Re(\partial_x u_n), v_n) + (v_n, Re(\partial_x u_n))$$

and the terms $(Re(\partial_x u_n), v_n) + (v_n, Re(\partial_x u_n))$ can be dealt as in (A.5). The rest of the proof is then completed following the same steps as the first case and the proof is complete. \square

We now prove the propagation of regularity for the linear operator associated with the NLS–KdV system.

Proposition A.2. [Propagation of regularity] *Let $T > 0$, $r \in \mathbb{R}_+$ and $(u, v) \in X_T^{r, \frac{1}{2}} \times Y_T^{r, \frac{1}{2}}$ solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + f, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v = Re(\partial_x u) + g, & (x, t) \in \mathbb{T} \times (0, T), \end{cases}$$

with $(f, g) \in X_T^{r, -\frac{1}{2}} \times Y_T^{r, -\frac{1}{2}}$. Assume that there exists a non-empty open set $\omega \subset \mathbb{T}$ such that

$$(u, v) \in L^2_{loc}(\]0, T[; H^{r+\rho}(\omega) \times H_0^{r+\rho}(\omega)),$$

for some $0 < \rho < \frac{1}{4}$. Then

$$(u, v) \in L^2_{loc}(\]0, T[; H^{r+\rho}(\mathbb{T}) \times H_0^{r+\rho}(\mathbb{T})).$$

Proof. We first regularize (u, v) by introducing, for each $n \in \mathbb{N}$,

$$u_n := e^{\frac{1}{n}\partial_x^2} u, \quad v_n := e^{\frac{1}{n}\partial_x^2} v,$$

and

$$f_n := e^{\frac{1}{n}\partial_x^2} f, \quad g_n := e^{\frac{1}{n}\partial_x^2} g.$$

In this case, by hypothesis over u and v , we have

$$\|(u_n, v_n)\|_{X_T^{r, \frac{1}{2}} \times Y_T^{r, \frac{1}{2}}} \leq C \quad \text{and} \quad \|(f_n, g_n)\|_{X_T^{r, -\frac{1}{2}} \times Y_T^{r, -\frac{1}{2}}} \leq C,$$

for some constant $C > 0$.

Pick $s = r + \rho$, $\varphi \in C^\infty(\mathbb{T})$ and $\psi \in C^\infty(\]0, T[)$ taking real values. As in the proof of Proposition A.1 we define $B = D^{2s-1}\varphi(x)$ and $A = \psi(t)B$, with D defined in (2.3). Set $L_1 = i\partial_t + \partial_x^2$ the Schrödinger operator and

$$\begin{aligned} \alpha_n &:= (L_1 u_n, A^* u_n)_{L^2_{t,x}} - (A u_n, L_1 u_n)_{L^2_{t,x}} \\ &= ([A, \partial_x^2] u_n, u_n)_{L^2_{t,x}} - i(\psi'(t) B u_n, u_n)_{L^2_{t,x}} \\ &= i(\partial_x v_n, A^* u_n)_{L^2_{t,x}} + (f_n, A^* u_n)_{L^2_{t,x}} - (A u_n, i\partial_x v_n)_{L^2_{t,x}} - (A u_n, f_n)_{L^2_{t,x}}. \end{aligned}$$

Notice that, as we choose $\rho > 0$ small enough, we have $r + 2\rho - \frac{1}{2} \leq r$. Hence, we obtain

$$\left| (Au_n, f_n)_{L^2_{t,x}} \right| \leq \|Au_n\|_{X_T^{-r, \frac{1}{2}}} \|f_n\|_{X_T^{r, -\frac{1}{2}}} \leq C, \forall n \in \mathbb{N}.$$

Arguing as in (A.4) we can similarly get that

$$\left| (\partial_x v_n, Au_n)_{L^2_{t,x}} \right| \leq C, \forall n \in \mathbb{N}$$

and also estimate all the other terms in order to get that $([A, \partial_x^2]u_n, u_n)$ is uniformly bounded in $n \in \mathbb{N}$.

On the other hand, observe that

$$[A, \partial_x^2] = -2\Psi(t)D^{2s-1}(\partial_x \varphi)\partial_x - \Psi(t)D^{2s-1}(\partial_x^2 \varphi)$$

and

$$\left| (\psi(t)D^{2s-1}(\partial_x \varphi)u_n, u_n, u_n) \right| \leq C\|u_n\|_{X_T^{r, \frac{1}{2}}} \|v_n\|_{X_T^{r, -\frac{1}{2}}} \leq C.$$

Analogously,

$$\left| (\psi(t)D^{2s-1}(\partial_x \varphi)\partial_x u_n, u_n) \right| \leq C.$$

Since, $fu \in L^2_{loc}(0, T; H^s(\mathbb{T}))$ and $f\partial_x u \in L^2_{loc}(0, T; H^{s-1}(\mathbb{T}))$ we can conclude that

$$fu_n = e^{\frac{1}{n}\partial_x^2} fu + \left[f, e^{\frac{1}{n}\partial_x^2} \right]$$

is uniformly bounded in $L^2_{loc}(0, T; H^s(\mathbb{T}))$, since $s \leq r + 1$. The same reasoning we get that $f\partial_x u$ is uniformly bounded in $L^2_{loc}(0, T; H^{s-1}(\mathbb{T}))$ thus,

$$\left| (\psi(t)D^{2s-1}f\partial_x u, D^s fu_n) \right| \leq C.$$

Due to the fact that $[D^s, f]$ is a pseudo-differential operator of order $s - 1$ and $u \in L^2(0, T; H^r(\mathbb{T}))$, we have

$$\begin{aligned} & \left| (\psi(t)D^{s-1}f\partial_x u_n, [D^s, f]u_n) \right| \\ & \leq \|D^{s-1}fu_n\|_{L^2(0, T; L^2(\mathbb{T}))} \|D^\rho[D^\rho, f]u_n\|_{L^2(0, T; L^2(\mathbb{T}))} \\ & \leq \|u_n\|_{L^2 H^r} \|u_n\|_{L^2 H^{s-1+\rho}} \leq C \end{aligned}$$

and finally,

$$\left| (\psi(t)D^{2s-1}(\partial_x \varphi)\partial_x u_n, u_n) \right| \leq C.$$

As in proposition A.1, we finish this proof by writing $\partial \varphi = f_2(\cdot) - f_2(\cdot - x_0)$, with $x_0 \in \mathbb{T}$, thus

$$\left| (\psi(t)D^{2s-1}f_2(\cdot - x_0)\partial_x u_n, u_n) \right| \leq C.$$

From this calculation we conclude that $u \in L^2_{loc}(0, T; H^{r+\rho}(\mathbb{T}))$.

To conclude the proof, we argue analogously for the KdV operator $L_2 = \partial_t + \partial_x^3 + \mu\partial_x$, and then we consider

$$\begin{aligned} \tilde{\alpha}_n &= ([A, L_2]v_n, v_n)_{L^2(0, T; L^2_0(\mathbb{T}))} \\ &= (Lv_n, A^*v_n)_{L^2_{t,x}} + (Av_n, L_2v_n)_{L^2_{t,x}} \end{aligned}$$

and we also have,

$$\tilde{\alpha}_n = (\operatorname{Re}(\partial_x u_n), A^* v_n)_{L^2_{t,x}} + (Av_n, g_n)_{L^2_{t,x}} + (g_n, A^* v_n)_{L^2_{t,x}} + (Av_n, \operatorname{Re}(\partial_x u_n))_{L^2_{t,x}}$$

and the main difference from the first equation is the estimate of

$$(\operatorname{Re}(\partial_x u_n), A^* v_n)_{L^2_{t,x}} \quad \text{and} \quad (Av_n, \operatorname{Re}(\partial_x u_n))_{L^2_{t,x}},$$

which can be dealt as in (A.5), and so the propagation of regularity is proved. \square

Remark A.3. We mention that the main ingredient in the proof of Propositions A.1 and A.2 is related into estimating the terms

$$(i\partial_x v_n, A^*_\epsilon u_n)_{L^2_{t,x}}, \\ (\operatorname{Re}(\partial_x u_n), v_n) + (v_n, \operatorname{Re}(\partial_x u_n)),$$

and

$$(\operatorname{Re}(\partial_x u_n), A^* v_n)_{L^2_{t,x}} + (Av_n, \operatorname{Re}(\partial_x u_n))_{L^2_{t,x}},$$

that is, where the coupled terms are analyzed. In this sense Lemma 2.5 is essential to prove the propagation results for the system under consideration in this work.

The next result is a consequence of Proposition A.2.

Corollary A.4. *Let $(u, v) \in X_T^{r, \frac{1}{2}} \times Y_T^{r, \frac{1}{2}}$ be a solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta|u|^2 u, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v + \frac{1}{2}\partial_x(v^2) + \mu\partial_x v = \operatorname{Re}(\partial_x u), & (x, t) \in \mathbb{T} \times (0, T). \end{cases} \tag{A.6}$$

Assume that, for some nonempty open set $\omega \subset \mathbb{T}$, $(u, v) \in [C^\infty(\omega \times (0, T))]^2$. So $(u, v) \in [C^\infty(\mathbb{T} \times (0, T))]^2$.

Proof. Note that, we may assume that $[v] = 0$, thus we can apply the estimate (2.5) to conclude that $\partial_x(v^2) \in Y_T^{r, -\frac{1}{2}}$ and we have that $\beta|u|^2 u \in X_T^{r, -\frac{1}{2}}$ by estimate (2.4). Then we can apply Proposition A.2 to conclude that $(u, v) \in L^2_{loc}(0, T; H^{r+\rho}(\mathbb{T}) \times H^{r+\rho}(\mathbb{T}))$. Hence we can get $t_0 \in (0, T)$ such that $(u(t_0), v(t_0)) \in H^{r+\rho}(\mathbb{T}) \times H^{r+\rho}(\mathbb{T})$, then solving the equation with $(u(t_0), v(t_0))$ as initial data, it follows that $(u, v) \in L^2(0, T; H^{r+\rho}(\mathbb{T}) \times H^{r+\rho}(\mathbb{T}))$ and iterating this process we get that if $(u, v) \in [C^\infty(\mathbb{T} \times (0, T))]^2$, showing the result. \square

Appendix B: Unique continuation for the NLS–KdV system

We present the unique continuation property for the NLS–KdV system. This will be proved by taking advantage of Carleman estimates already proved for the Korteweg-de Vries and Schrödinger operators in [3, 11]. For self-contentedness, we will present below the estimates necessary to demonstrate the result.

B.1: Carleman estimates

Consider the KdV equation on 2π -periodic conditions

$$\begin{cases} \partial_t q + \partial_x^3 q = f, & (x, t) \in (0, 2\pi) \times (0, T), \\ q(0, t) = q(2\pi, t) & t \in (0, T), \\ \partial_x q(0, t) = \partial_x q(2\pi, t) & t \in (0, T), \\ \partial_x^2 q(0, t) = \partial_x^2 q(2\pi, t) & t \in (0, T), \\ q(x, 0) = q_0(x), & x \in (0, 2\pi). \end{cases} \quad (\text{B.1})$$

Then the following Carleman inequality is a consequence of the result proved in [11, Proposition 3.1].

Proposition B.1. *Pick any $T > 0$. There exist two constants $C > 0$ and $s_0 > 0$ such that any $f \in L^2(0, T; L^2(0, 2\pi))$, any $q_0 \in L_0^2(0, 2\pi)$ and any $s \geq s_0$, the solution q of (B.1) fulfills*

$$\begin{aligned} \int_0^T \int_0^{2\pi} [s\varphi |q_{xx}|^2 + (s\varphi)^3 |q_x|^2 + (s\varphi)^5 |q|^2] e^{-2s\varphi} dx dt &\leq C \int_0^T \int_0^{2\pi} |f|^2 e^{-2s\varphi} dx dt \\ &+ C \left(\int_0^T \int_\omega [s\varphi |q_{xx}|^2 + (s\varphi)^3 |q_x|^2 + (s\varphi)^5 |q|^2] e^{-2s\varphi} dx dt \right). \end{aligned} \quad (\text{B.2})$$

Let us now borrow the results obtained in [3, Theorem 3.2.]. Consider the Schrödinger equation on 2π -periodic conditions

$$\begin{cases} i\partial_t p + \partial_x^2 p = g & (x, t) \in (0, 2\pi) \times (0, T), \\ p(0, t) = p(2\pi, t) = 0 & t \in (0, T), \\ \partial_x p(0, t) = \partial_x p(2\pi, t) = 0 & t \in (0, T), \\ p(x, 0) = p_0(x) & x \in (0, 2\pi). \end{cases}$$

Thus, the following estimate holds.

Theorem B.2. *There exist constants $C > 0$ and $s_0 \geq 1$ such that*

$$\begin{aligned} \int_0^T \int_0^{2\pi} [s\varphi |p_x|^2 + (s\varphi)^3 |p|^2] e^{-2s\varphi} dx dt &\leq \int_0^T \int_0^{2\pi} |g|^2 e^{-2s\varphi} dx dt \\ &+ C \int_0^T \int_\omega [(s\varphi)^3 |p|^2 + (s\varphi) |\operatorname{Re}(p_x)|^2] e^{-2s\varphi} dx dt \end{aligned} \quad (\text{B.3})$$

for all $s > s_0$ and $\omega \subset (0, 2\pi)$.

B.2: Unique continuation property

Carleman inequalities with an internal observation for the KdV and Schrödinger operators ensure the result of the unique continuation.

Theorem B.3. *Let $T > 0$ and $V_1, V_2, V_3 \in C^\infty((0, T) \times \mathbb{T})$, then the solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + V_1(x, t)u + V_2(x, t)\bar{u}, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \mu\partial_x v = V_3(t, x)\partial_x u + \operatorname{Re}(\partial_x u), & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, t) = v(x, t) = 0, & (x, t) \in \omega \times (0, T), \end{cases}$$

with $(u, v) \in C([0, T]; L^2(\mathbb{T}) \times L_0^2(\mathbb{T}))$, is the trivial one $u \equiv v \equiv 0$ in $\mathbb{T} \times (0, T)$.

Proof. Putting together (B.2) and (B.3), we have that

$$\begin{aligned}
 & \int_0^T \int_0^{2\pi} [s\tilde{\varphi} |u_x|^2 + (s\tilde{\varphi})^3 |u|^2] e^{-2s\hat{\varphi}} dxdt \\
 & \quad + \int_0^T \int_0^{2\pi} [s\tilde{\varphi} |v_{xx}|^2 + (s\tilde{\varphi})^3 |v_x|^2 + (s\tilde{\varphi})^5 |v|^2] e^{-2s\tilde{\varphi}} dxdt \\
 & \leq \int_0^T \int_0^{2\pi} (|g|^2 + |f|^2) e^{-2s\hat{\varphi}} dxdt \\
 & \quad + C \int_0^T \int_\omega [(s\varphi)^3 |u|^2 + (s\varphi) |\operatorname{Re}(u_x)|^2] e^{-2s\varphi} dxdt \\
 & \quad + C \left(\int_0^T \int_\omega [s\varphi |v_{xx}|^2 + (s\varphi)^3 |v_x|^2 + (s\varphi)^5 |v|^2] e^{-2s\varphi} dxdt \right),
 \end{aligned} \tag{B.4}$$

where $\tilde{\varphi} = \min \varphi$, $\hat{\varphi} = \max \varphi$, $g := i\partial_x v + V_1(x, t)u + V_2(x, t)\bar{u}$ and $f := V_3(t, x)\partial_x u + \operatorname{Re}(\partial_x u)$. The properties of the $\tilde{\varphi}$ and $\hat{\varphi}$ can be seen in [3, 11].

Now, note that the zero and first-order terms of the right-hand side of (B.4) can be absorbed by the left-hand side. Moreover, the second-order term v_{xx} of the right-hand side (B.4) can be treated as in [11, Lemma 3.7], resulting in the following estimate

$$\begin{aligned}
 & \int_0^T \int_0^{2\pi} [(s\tilde{\varphi})^3 |u|^2 + (s\tilde{\varphi})^5 |v|^2] e^{-2s\tilde{\varphi}} dxdt \\
 & \leq C \int_0^T \int_\omega [(s\varphi)^3 |u|^2 + (s\varphi)^5 |v|^2] e^{-2s\varphi} dxdt.
 \end{aligned} \tag{B.5}$$

Therefore, thanks to the hypothesis $u \equiv v \equiv 0$ in $\omega \times (0, T)$, the right-hand side of (B.5) yields that $u \equiv v \equiv 0$ in $\mathbb{T} \times (0, T)$, and so the result follows. \square

As a consequence of this unique continuation property, we have the following result.

Corollary B.4. *Let $(u, v) \in X_T^{0, \frac{1}{2}} \times Y_T^{0, \frac{1}{2}}$ be a solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x v + \beta |u|^2 u, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x (v^2) + \mu \partial_x v = -\operatorname{Re}(\partial_x u), & (x, t) \in \mathbb{T} \times (0, T), \\ u(x, t) = 0, \quad v(x, t) = c, & (x, t) \in \omega \times (0, T), \end{cases} \tag{B.6}$$

where $\omega \subset \mathbb{T}$ is a nonempty open set and c is some real constant. Then, there exists a small $T > 0$, such that $u = 0$ on $\mathbb{T} \times (0, T)$ and $v = c$ on $\mathbb{T} \times (0, T)$.

Proof. First note that using Corollary A.4 we have that $u, v \in C^\infty(\mathbb{T} \times (0, T))$ then, considering $(w, z) = (\partial_t u, \partial_t v)$ we get

$$\begin{cases} i\partial_t w + \partial_x^2 w = i\partial_x z + V_1(t, x)w + V_2(t, x)\bar{w}, & (x, t) \in \mathbb{T} \times (0, T), \\ \partial_t z + \partial_x^3 z + \mu \partial_x z + V_3(t, x)z = \operatorname{Re}(\partial_x w), & (x, t) \in \mathbb{T} \times (0, T), \\ z(x, t) = w(x, t) = 0, & (x, t) \in \omega \times (0, T). \end{cases}$$

Then the unique continuation property given by Theorem B.3 implies that $w = z = 0$. In analogous way, we can consider $(\tilde{w}, \tilde{z}) = (\partial_x u, \partial_x v)$ and then

we get that $\partial_x u = \partial_x v = 0$ in $\mathbb{T} \times (0, T)$. Hence we can conclude that both functions u , and v are constant and the result follows. \square

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