A QUALITATIVE STUDY OF THE GENERALIZED DISPERSIVE SYSTEMS WITH TIME–DELAY: THE UNBOUNDED CASE

ROBERTO DE A. CAPISTRANO–FILHO*

Departamento de Matemática, Universidade Federal de Pernambuco, S/N Cidade Universitária, 50740-545, Recife (PE), Brazil
Email address: roberto.capistranofilho@ufpe.br

FERNANDO A. GALLEGO

Departamento de Matemática y Estadística, Universidad Nacional de Colombia (UNAL), Cra 27 No. 64-60, 170003, Manizales, Colombia
Email address: fagallegor@unal.edu.co

VILMOS KOMORNIK

Département de Mathématique, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cedex, France
Email address: komornik@math.unistra.fr

Abstract. We study the asymptotic behavior of the solutions of the time-delayed higher-order dispersive nonlinear differential equation

$$u_t(x,t)+Au(x,t)+\lambda_0(x)u(x,t)+\lambda(x)u(x,t-\tau)=0$$

where

$$Au=(-1)^{j+1}\partial_x^{2j+1}u+(-1)^m\partial_x^{2m}u+\frac{1}{p+1}\partial_x u^{p+1}$$

with $m\leq j$ and $1\leq p<2j$. Under suitable assumptions on the time delay coefficients, we prove that the system is exponentially stable if the coefficient of the delay term is bounded from below by a suitable positive constant, without any assumption on the sign of the coefficient of the undelayed feedback. Additionally, in the absence of delay, general results of stabilization are established in $H^s(\mathbb{R})$ for $s\in[0,2j+1]$. Our results generalize several previous theorems for the Korteweg-de Vries type delayed systems in the literature.

Contents

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*Corresponding author.

1. Introduction

1.1. **Model description.** Under suitable assumptions on amplitude, wavelength, wave steepness, and so on, the study on asymptotic models for water waves has been extensively investigated to understand the full water wave system; see, for instance, [1, 6, 7] and references therein for a rigorous justification of various asymptotic models for surface and internal waves.

Formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces. The physical condition $\delta \ll 1$ characterizes the waves, called long waves or shallow water waves. In particular, considering the relations between ε and δ, we can have two well-known regimes:

1. Korteweg-de Vries (KdV): $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$. Under this regime, Korteweg and de Vries [24][1] derived the following well-known equation as a central equation among other dispersive or shallow water wave models called the KdV equation from the equations for capillary-gravity waves:
$$\pm 2\eta_t + 3\eta\eta_x + \left(\frac{1}{3} - \mu\right)\eta_{xxx} = 0.$$
2. Kawahara: $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$. In connection with the critical Bond number $\mu = \frac{1}{3}$, Hasimoto [16] derived a fifth-order KdV equation of the form
$$\pm 2\eta_t + 3\eta\eta_x - \nu\eta_{xxx} + \frac{1}{45}\eta_{xxxxx} = 0,$$
which is nowadays called the Kawahara equation.

In the last years, many authors have been interested to find the behavior of solutions for the time-delayed KdV equation, time-delayed Kawahara equation, and other time-delayed dispersive systems see, for instance, [2, 3, 11, 12, 21, 32] and the reference therein. In this article, our goal is to study general results for a higher-order dispersive equation which in some sense recovers the

[1]This equation was first introduced by Boussinesq [9], and Korteweg and de Vries rediscovered it twenty years later.

equations mentioned in the cited articles. Due to this advance for this type of dispersive equation, our main focus is to investigate the stabilization of the higher-order extension, for example, of KdV and Kawahara equations.

To be precise, we will consider the Cauchy problem for the following higher-order KdV-type equation posed in $\mathbb{R}$:

$$\begin{cases} u_t(x,t) + (-1)^{j+1}\partial_x^{2j+1}u(x,t) + \frac{1}{2}\partial_x(u^2) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Specifically, (1.1) is called KdV and fifth-order KdV–type equation when $j = 1$ and $j = 2$, respectively. More generally, we aim to prove the stabilization results of the solutions for a time-delayed higher-order dispersive system with a strong dissipative term, namely

$$\begin{cases} u_t(x,t) + (-1)^{j+1}\partial_x^{2j+1}u(x,t) + (-1)^m\partial_x^{2m}u(x,t) + \lambda_0(x)u(x,t) & \\ \qquad\qquad + \lambda(x)u(x,t-\tau) + \frac{1}{p+1}\partial_x u^{p+1}(x,t) = 0, & \text{in} \quad \mathbb{R}\times(0,\infty), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}, \\ u(x,s) = u_0(x,s), & \text{in} \quad \mathbb{R}\times[-\tau,0], \end{cases} \tag{1.2}$$

with $m \leq j$, $j, m \in \mathbb{N}$, and $1 \leq p < 2j$. Here, the constant $\tau > 0$ is the time delay and the coefficients are considered with the following regularity

$$\lambda_0(x), \lambda(x) \in L^\infty(\mathbb{R}).$$

Thus, our main intention is to furnish sufficient conditions on the coefficients λ, λ_0 to have well-posedness and exponential decay estimates for the model (1.2).

1.2. **Review of the results in the literature.** Let us briefly discuss the preceding works [2, 3, 11, 12, 21, 32] and the results concerning the well-posedness of (1.1).

The local and global well-posedness of (1.1) has been widely studied. The local well-posedness result was first proved by Gorsky and Himonas [15] for $s \geq -\frac{1}{2}$, and Hirayama [17] improved this to $s \geq -\frac{j}{2}$. Both works were based on the standard Fourier restriction norm method. Hirayama improved the bilinear estimate by using the factorization of the resonant function. The global well-posedness of (1.1) was established for $j = 1, 2$ by Colliander et al. [13] and Kato [19], respectively, via the “I-method”. In [18] the authors extended the results of [13] and [19] to $j \geq 3$. Their method follows the argument in [13] for the periodic KdV equation, while some estimates are slightly different. They proved that the IVP (1.1) is globally well-posed in $H^s(\mathbb{T})$ for $j \geq 3$ and $s \geq -\frac{j}{2}$.

To our knowledge, the only work concerning the control and stabilization properties for the system (1.1) was done by the first author with two collaborators in [10]. The authors studied the local and global control results for (1.1) posed on the unit circle. More precisely, they considered the system

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = f(t,x), & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x), & x \in H^s(\mathbb{T}), \end{cases}$$

posed on a periodic domain $\mathbb{T}$. They showed the globally controllability in $H^s(\mathbb{T})$, for $s \geq 0$ by the forcing term

$$f(t,x) := g(x)\left(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)\,dy\right),$$

supported in a given open set $\omega \subset \mathbb{T}$, where g is a given nonnegative smooth function satisfying

$$\int_{\mathbb{T}} g(x)\,dx = 1 \quad \text{and} \quad \omega := \{g > 0\},$$

and h is the control input.

Considering the particular cases of the system (1.2), without delay and damping terms, we recall some results from the literature concerning the asymptotic properties. The well-known Korteweg–de Vries–Burgers equation

$$u_t + u_{xxx} - u_{xx} + uu_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \tag{1.3}$$

corresponding to the case $j = m = 1$, was extensively studied. For example, in [4] Amick *et. al* proved that the L^2-norm of the solutions to (1.3) tends to zero as $t \to \infty$ in a polynomial way, namely

$$\|u(\cdot, t)\|_2 \leq Ct^{-\frac{1}{2}} \quad \text{for all} \quad t > 0,$$

with a positive constant C. More recently, Cavalcanti *et al.* [12] studied the following damped KdV–Burgers equation:

$$\begin{cases} u_t(x,t) + u_{xxx}(x,t) - u_{xx}(x,t) + \lambda_0(x)u(x,t) + u(x,t)u_x(x,t) = 0, & \text{in} \quad \mathbb{R} \times (0, \infty), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}. \end{cases} \tag{1.4}$$

Under appropriate conditions on the damping coefficient λ_0, the authors established its well-posedness and exponential stability for indefinite damping $\lambda_0(x)$, giving exponential decay estimates on the L^2-norm of solutions. In [14], the results of [12] were extended by generalizing the nonlinear term of (1.4), and proving the global well-posedness and exponential stabilization of the generalized KdV-Burger equation under the presence of the localized and indefinite damping. In [21] the authors consider the KdV-Burgers equation (1.4) in the presence of a delayed feedback $\lambda(x)u(x, t - \tau)$. They considered the system with damping and delay feedback, showing the exponential decay estimates under appropriate conditions on the damping coefficients.

It is important to point out the exponential decay estimates obtained in [26, 29] for the KdV equation posed in an interval with localized damping. Also, periodic conditions have been considered in [20, 23] while more general nonlinearities have been considered in [31]. Additionally, the robustness concerning the delay of the boundary stability of the nonlinear KdV equation has been studied in [5]. The authors obtain, under an appropriate condition on the feedback, with and without delay, the local stabilization results for the KdV equation with noncritical length. Moreover, in [32], the authors extended this result for the nonlinear Korteweg-de Vries equation in the presence of an internal delayed term.

In two recent articles [2, 3] the authors studied the qualitative and numerical analysis of the following nonlinear fourth-order delayed dispersive equation in a bounded domain $I = [0, L]$ with boundary and initial conditions:

$$\begin{cases} u_t(x,t) - \sigma u_{xx}(x,t) + \mu u_{xxxx}(x,t) + u(x, t - \tau)u_x(x,t) = 0, & (x,t) \in I \times (0, \infty), \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0, & t > 0, \\ u(x,h) = f(x,h), & (x,h) \in I \times [-\tau, 0]. \end{cases} \tag{1.5}$$

The well-posedness, as well as the exponential stability of the zero solution of (1.5), was established in [2]. The main ingredient of the proof was the exploitation of the Schauder Fixed Point Theorem. This improved an earlier result [3] in the sense that no interior damping control was required. Additionally, numerical simulations were also presented in this article to illustrate the theoretical result.

Finally, in a recent paper [11] the authors considered the Kawahara equation posed on a bounded interval under the presence of localized damping $(a\,(x)\,u(x,t))$ and delay $(b(x)u(x, t - h))$ terms. They proved its exponential stabilization under suitable assumptions. First, they showed the Kawahara system is exponentially stable under some restriction of the spatial length of the domain. Next, they introduced a more general delayed system and suitable energy functions, they proved, via the Lyapunov approach, the exponential stability for small initial data small, under a restriction on the spatial length of the domain. Then they used a compactness-uniqueness argument to remove these hypotheses, thereby obtaining a semi-global stabilization result.

We end our review with a recent paper [22] where abstract linear and nonlinear evolutionary systems with feedback were studied. Specifically, the authors considered the system

$$\left\{\begin{array}{ll} U'(t) = AU(t) + k(t)BU(t-\tau) + F(U(t)), & \text{in } (0,+\infty). \\ U(0) = U_0, & \text{for } t \in (0,\tau), \\ BU(t-\tau) = f(t), & \end{array}\right. \tag{1.6}$$

where A generates an exponentially stable semigroup in a Hilbert space H, B is a continuous linear operator of H into itself, $k(t) \in L^1_{loc}(0,+\infty)$ and $\tau > 0$ is a delay parameter, with $F : H \to H$ is a Lipschitz function. The main purpose of [22] was to give a well-posedness result and an exponential decay estimate for the model (1.6) with a damping coefficient $k(t)$ belonging only to L^1_{loc}. By using the semigroup approach combined with Gronwall's inequality, under some mild assumptions on the involved functions and parameters, the well-posedness of the problem was established and an exponential decay estimate was obtained.

1.3. **Main results.** With this state of the art, let us now present our main results which give a necessary next step to understanding the asymptotic behavior for generalized dispersive systems.

From now on, for the sake of simplicity, the norms in the spaces $L^p(\mathbb{R})$ and $L^\infty(\mathbb{R})$ will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_\infty$, respectively. Furthermore, we introduce the Banach space

$$\mathcal{B}_{s,T} := C([0,T];H^s(\mathbb{R})) \cap L^2(0,T;H^{s+j}(\mathbb{R}))$$

with the natural norm

$$\|u\|_{\mathcal{B}_{s,T}} = \|u\|_{C([0,T];H^s(\mathbb{R}))} + \|\partial_x^{s+j}u\|_{L^2(0,T;L^2(\mathbb{R}))}.$$

For $s = 0$ we omit the subscript s, so that $\mathcal{B}_T = \mathcal{B}_{0,T}$. Our first result ensures that (1.2) is well-posed in the space $H^s(\mathbb{R})$ for $0 \le s \le 2j+1$ and $1 \le p < 2j$.

Theorem 1.1. *Let $T > 0$. Consider $1 \le p < 2j$, $j \ge 1$, and $0 \le s \le 2j+1$ be given. In addition, assume that $\lambda_0, \lambda \in L^\infty(\mathbb{R})$ when $s = 0$ and $\lambda_0, \lambda \in H^j(\mathbb{R})$ when $s > 0$. Then, for any $u_0 \in C([-\tau,0];H^s(\mathbb{R}))$, the IVP (1.2) admits a unique solution $u \in \mathcal{B}_{s,T}$. Moreover, there exists a nondecreasing continuous function $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+$, such that*

$$\|u\|_{\mathcal{B}_{s,T}} \le \beta_s(\|u_0\|_2)\|u_0\|_{H^s(\mathbb{R})}.$$

Remark 1.2. The same result as stated in Theorem 1.1 is obtained when considering the Cauchy problem described in equation (1.2) without the presence of a time delay, i.e., when the parameter τ is set to be zero.

Consequently, the subsequent result demonstrates that every mild solution of equation (1.2) qualifies as a regular solution when the origin is not taken into account.

Corollary 1.3. *Under the assumptions of Theorem 1.1, for any $u_0 \in C([-\tau,0];L^2(\mathbb{R}))$, the corresponding solution u of (1.2) belongs to*

$$\mathcal{B}_{2j+1,[\varepsilon,T]} := C([\varepsilon,T];H^{2j+1}(\mathbb{R})) \cap L^2(\varepsilon,T;H^{3j+1}(\mathbb{R}))$$

for every $T > 0$ and $0 < \varepsilon < T$.

The second main result is related to the exponential stabilization of (1.2). For that, consider initially the following assumption

$$\lambda_0(x) \ge \gamma_0 \quad \text{for a.e.} \quad x \in \mathbb{R} \tag{1.7}$$

with some positive constant γ_0. If the coefficient of the delay term λ satisfies the estimate $\|\lambda\|_\infty < \gamma_0$, we can obtain the exponential decay estimates for the $\mathcal{E}(t)$ associated to the solution of the system (1.2), where

$$\mathcal{E}(t) := \mathcal{E}(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx + \frac{1}{2}\int_{t-\tau}^{t}\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)\,dx\,ds.$$

In that case, the delay effect is compensated by the undelayed damping term (cf. [27]), and the following result holds:

Theorem 1.4. *Let $\lambda_0, \lambda \in L^\infty(\mathbb{R})$ satisfying (1.7) and*

$$\frac{e^\tau + 1}{2}|\lambda(x)| \leq \gamma + \beta(x) \quad \text{for a.e.} \quad x \in \mathbb{R} \tag{1.8}$$

with

$$0 \leq \gamma < \gamma_0 \quad \text{and} \quad \|\beta\|_q < \left(\frac{\gamma_0 - \gamma}{c_q}\right)^{1-\frac{1}{2q}}. \tag{1.9}$$

Here,

$$c_q := \left(1 - \frac{1}{2q}\right)\left(\frac{2}{q}\right)^{\frac{1}{2q-1}} \quad \text{for} \quad 1 \leq q < \infty.$$

Thus, the solution of (1.2) is exponentially stable. Moreover, the solution u of (1.2) satisfies

$$\mathcal{E}(t) \leq C(u_0, \tau)e^{-\nu t}.$$

Here, ν is defined by

$$\nu = \min\left\{2\left(\gamma_0 - \gamma - \frac{2q-1}{2q}\left(\frac{2}{q}\right)^{\frac{1}{2q-1}}\|\beta\|_q^{\frac{2q}{2q-1}}\right), 1\right\} \tag{1.10}$$

and

$$C(u_0, \tau) = \frac{1}{2}\|u(0)\|_2^2 + \int_{-\tau}^{0} e^s \|\lambda\|_\infty \|u(s)\|_2^2 ds. \tag{1.11}$$

After having restricted ourselves to the case where λ_0 is bounded from below by a positive constant, the next issue is to extend our results to the case where the coefficient of the undelayed feedback λ_0 is indefinite. We have the following result:

Theorem 1.5. *Consider $\lambda_0, \lambda \in L^\infty(\mathbb{R})$ satisfying (1.8) and*

$$\lambda_0(x) \geq \gamma_0 - \beta_0(x) \quad \text{for a.e.} \quad x \in \mathbb{R}, \tag{1.12}$$

with β_0 satisfying

$$\|\beta_0\|_q < \left(\frac{\gamma_0}{c_q}\right)^{1-\frac{1}{2q}} \tag{1.13}$$

and

$$0 \leq \gamma < \gamma_0 \quad \text{and} \quad \|\beta_0 + \beta\|_q < \left(\frac{\gamma_0 - \gamma}{c_q}\right)^{1-\frac{1}{2q}}. \tag{1.14}$$

For every $u_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$, the system (1.2) has a unique global mild solution. It is exponentially stable, that is,

$$\mathcal{E}(t) \leq C(u_0, \tau)e^{-\tilde{\nu} t}$$

with $C(u_0, \tau) > 0$ as in (1.11), and $\tilde{\nu}$ defined by

$$\tilde{\nu} = \min\left\{2\left(\gamma_0 - \gamma - \frac{2q-1}{2q}\left(\frac{2}{q}\right)^{\frac{1}{2q-1}}\|\beta + \beta_0\|_q^{\frac{2q}{2q-1}}\right), 1\right\}. \tag{1.15}$$

As a consequence of Theorem 1.5, we can find information on the solution of the system (1.2) for any interval $[t, t+T]$, for $T > 0$. The next result will be crucial in getting the exponential stabilization in the space $H^{2j+1}(\mathbb{R})$.

Corollary 1.6. *Let $T > 0$ and $u_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$. Consider λ_0 and λ satisfying (1.8) and (1.12), respectively. Then, there exists a nondecreasing continuous function $\alpha_0 : \mathbb{R}^+ \to \mathbb{R}^+$ such that the corresponding solution u of problem (1.2) with $1 \leq p < 2j$ satisfies*

$$\|u\|_{\mathcal{B}_{0,[t,t+T]}} \leq C_{2T}\left\{2C(u_0, \tau)e^{-\nu t} + \|\lambda\|_\infty \|u\|_{L^1(t-\tau,t;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|u\|_{L^2(t-\tau,t;L^2(\mathbb{R}))}\right\}$$

where C_T is given by (2.10) below and $C(u_0, \tau)$ defined by (1.11).

Considering the general equation (1.2) in the H^s-norm, with $s \in [0, 2j+1]$ without the presence of the time-delay term, that is, $\lambda = \tau = 0$, the previous corollary helps us to prove the following general theorem about the exponential decay:

Theorem 1.7. *Consider $\lambda_0 \in L^\infty(\mathbb{R})$ satisfying (1.7) an (1.8). For $\lambda = \tau = 0$, the system (1.2) is exponentially stable. In other words, there exist a time $T_0 > 0$ and a positive function $\eta(s) > 0$ such that for every $u_0 \in H^s(\mathbb{R})$ with $s \in [0, 2j+1]$, $1 \leq p < 2j$, (1.2) has a unique global mild solution satisfying*

$$\|u(t)\|_{H^s(\mathbb{R})} \leq \gamma(\|u_0\|_2, T_0)\|u_0\|_{H^s(\mathbb{R})} e^{-\eta(s)t}, \quad \text{for } t \geq T_0. \tag{1.16}$$

Here, $\gamma : (0, \infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

1.4. Comments and article's structure.

1. It is important to point out that to prove Theorem 1.1 we use a method introduced by Tartar [33] and adapted by Bona and Scott [8, Theorem 4.3]. This method was first used to prove the global well-posedness of IVP for the KdV equation on the whole line, and here, we adapt the spaces for our purpose.
2. Theorems 1.4 and 1.5 generalize the results in [21] for the general differential operator
$$Au := (-1)^{j+1}\partial_x^{2j+1}u(x,t) + (-1)^m\partial_x^{2m}u(x,t),$$
that is, considering appropriate values of j, we can recover the results in [21]. Additionally to that, considering the real line, the results proved here can be seen as extensions of [2, 3, 11, 12, 32].
3. Note that Theorem 1.7 extends (and recovers) the previous results of [12, 14] by proving the stabilization in $\mathcal{B}_{s,T}$ for $0 \leq s \leq 2j+1$ with only a damping term, without the presence of the time-delay term.
4. The mains results of the paper are summarized in the following table:

Type of the feedback law	Well-posedness	Exponential stabilizition
Damping (indefined and localized) term + Time Delay term	$\mathcal{B}_{s,T}$ $0 \leq s \leq 2j+1$	$\|u(t)\|_2 \leq C(u_0,\tau)e^{-\eta t}$ in L^2-norm
Damping term (indefined and localized)	$\mathcal{B}_{s,T}$ $0 \leq s \leq 2j+1$	$\|u(t)\|_{H^s(\mathbb{R})} \leq e^{-\eta t}\|u_0\|_{H^s(\mathbb{R})}$ $0 \leq s \leq 2j+1$

This paper consists of five parts including the introduction. In Section 2 we analyze the well-posedness of the system (1.2) when the initial data belongs to space $C([-\tau, 0]; H^s(\mathbb{R}))$, where $s \in [0, 2j+1]$ under some assumptions of λ_0 and λ. Section 3 is devoted to the exponential stabilization of the system (1.2) in $L^2(\mathbb{R})$, that is, to the proof of Theorems 1.4 and 1.5. In Section 4 we prove Theorem 1.7, establishing the exponential stabilization of the system (1.2) without the presence of time delay term ($\lambda = \tau = 0$) in the space $H^s(\mathbb{R})$, for $s \in [0, 2j+1]$. Finally, in Section 5, we conclude the work with some further considerations.

2. Well-posedness theory

In this section, we establish the well-posedness theory in $H^s(\mathbb{R})$, for $s \in [0, 2j+1]$. The strategy is establish the well-posedness results in $L^2(\mathbb{R})$ and $H^{2j+1}(\mathbb{R})$, respectively. After that, we will use interpolation arguments, due to Tartar [33] and adapted by Bona and Scott [8, Theorem 4.3], to achieve Theorem 1.1.

2.1. Linear system.
For simplicity, consider $m = j \in \mathbb{N}$ in the system (1.2). Let us start studying the following linear model associated with (1.2), namely

$$\begin{cases} u_t(x,t) + (-1)^{j+1}\partial_x^{2j+1}u(x,t) + (-1)^j\partial_x^{2j}u(x,t) + \lambda_0 u(x,t) & \\ \qquad\qquad + \lambda u(x,t-\tau) = 0, & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,s) = u_0(x,s), & \text{in } \mathbb{R} \times [-\tau, 0]. \end{cases} \tag{2.1}$$

This section is devoted to proving the well-posedness via *semigroup theory*.We first take a look at the properties of the following operator

$$A_{\lambda_0}u := -(-1)^{j+1}\partial_x^{2j+1}u(x,t)-(-1)^j\partial_x^{2j}u(x,t) - \lambda_0 u. \tag{2.2}$$

The following well-posedness result can be proved.

Proposition 2.1. *If $\lambda_0 \in L^\infty(\mathbb{R})$, then the operator A_{λ_0} defined by (2.2) on $\mathcal{D}(A_{\lambda_0}) := H^{2j+1}(\mathbb{R})$ generates a strongly continuous semigroup in the Hilbert space $H := L^2(\mathbb{R})$.*

Proof. Note that $A_{\lambda_0} = A + \bar{A}_{\lambda_0}$, where

$$Au = -(-1)^{j+1}\partial_x^{2j+1}u(x,t) - (-1)^j\partial_x^{2j}u(x,t)$$

and

$$\bar{A}_{\lambda_0} = -\lambda_0 u.$$

It suffices to observe that $\bar{A}_{\lambda_0}$ is a bounded perturbation of the operator A and to prove that A generates a strongly continuous semigroup in $L^2(\mathbb{R})$. Indeed, according to the Lumer-Phillips theorem it is sufficient to check that A is dissipative and that $I - A$ is onto. The dissipativity follows by a direct computation: if $u_0 \in H^{2j+1}(\mathbb{R})$ is real-valued, then u is also real-valued and

$$(Au,u)_H = \int_{-\infty}^{\infty}\left(-(-1)^{j+1}\partial_x^{2j+1}u-(-1)^j\partial_x^{2j}u\right)u\,dx = -\int_{-\infty}^{\infty}(\partial_x^j u)^2dx \leqslant 0,$$

since we have

$$\int_{-\infty}^{\infty}(-1)^{j+1}(\partial_x^{2j+1}u)u\,dx = 0.$$

Thanks to the fact that $\Re Av = A(\Re v)$ for all $v \in H^{2j+1}(\mathbb{R})$, it follows that

$$\Re(Au,u)_H = -\int_{-\infty}^{\infty}(\partial_x^j u)^2dx \leqslant 0$$

for all $u_0 \in H^{2j+1}(\mathbb{R})$.

It remains to show that for every $f \in L^2(\mathbb{R})$ there exists $u \in H^{2j+1}(\mathbb{R})$ satisfying the equality

$$-(-1)^{j+1}\partial_x^{2j+1}u(x,t)-(-1)^j\partial_x^{2j}u(x,t) + u = f.$$

Taking the Fourier transform in the previous equation, it is equivalent to

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{1-((-1)^{j+1}(i\xi)^{2j+1})-(-1)^j(i\xi)^{2j}}.$$

This is possible by the fact that the denominator

$$h(\xi) := 1-((-1)^{j+1}(i\xi)^{2j+1})-(-1)^j(i\xi)^{2j}$$

never vanishes. Since, moreover, $h(\xi)$ is a continuous function satisfying $|h(\xi)| \to \infty$ as $|\xi| \to \infty$, $1/h$ is bounded, and therefore the last equation has a unique solution $\hat{u} \in L^2(\mathbb{R})$. Finally, since the function

$$\frac{1+|\xi|+|\xi|^2+|\xi|^3+\cdots+|\xi|^{2j+1}}{|1-((-1)^{j+1}(i\xi)^{2j+1})-(-1)^j(i\xi)^{2j}|}$$

tends to 1 as $|\xi| \to \infty$ and hence it is bounded by some constant M on $\mathbb{R}$, we conclude that

$$\left|(i\xi)^j\hat{u}(\xi)\right| \leqslant M|\hat{f}(\xi)|, \quad j \in \mathbb{N}.$$

Since $\hat{f} \in L^2(\mathbb{R})$, this implies the regularity property $u \in H^{2j+1}(\mathbb{R})$. □

Let us now use an iterative procedure (see e.g. [28]) and semigroup theory (see e.g. [30]), to prove that (2.1) is well-posed.

Theorem 2.2. *If $\lambda_0, \lambda \in L^\infty(\mathbb{R})$ and $u_0 \in C([-\tau,0];H)$, then there exists a unique solution $u \in C([-\tau,+\infty);H)$ of the problem (2.1).*

Proof. Consider the interval $[0,\tau]$. Note that (2.1) can be seen as an inhomogeneous Cauchy problem of the form

$$\begin{cases} u_t(t) - A_{\lambda_0}u(t) = g_0(t), & \text{in} \quad (0,\tau), \\ u(0) = u_0, \end{cases}$$

with $g_0(t) = -\lambda u_0(t-\tau)$, for $t \in [0,\tau]$, which have unique solution $u(\cdot) \in C([0,\tau), H)$. Now, considering the interval $t \in [\tau, 2\tau]$, problem (2.1) can be rewritten as

$$\begin{cases} u_t(t) - A_{\lambda_0}u(t) = g_1(t), & \text{in} \quad (\tau,2\tau), \\ u(\tau) = u(\tau_-), \end{cases}$$

where $g_1(t) = -\lambda u(t-\tau)$. Thanks to the first step of the proof the function $u(t)$, for $t \in [0,\tau]$, its known, thus $g_1(t)$ can be considered as a known function for $t \in [\tau, 2\tau]$. Therefore, this analysis yields the existence of a solution $u(\cdot) \in C([0,2\tau], H)$. By a bootstrap argument we get a solution $u \in C([0,\infty), H)$. □

2.2. **Non-homogeneous system.** We are interested in extending the previous results for the nonlinear system (1.2). In this way, we first consider the corresponding linear inhomogeneous initial value problem

$$\begin{cases} u_t(x,t) + (-1)^{j+1}\partial_x^{2j+1}u(x,t) + (-1)^j\partial_x^{2j}u(x,t) + \lambda_0 u(x,t) & \\ \qquad\qquad\qquad +\lambda u(x,t-\tau) = f(x,t), & \text{in} \quad \mathbb{R}\times(0,T), \\ u(x,s) = u_0(x,s), & \text{in} \quad \mathbb{R}\times[-\tau,0], \end{cases} \tag{2.3}$$

for some $T>0$. Consider the operator A_{λ_0} defined by Proposition 2.1. So, we may rewrite (2.3) in the following way

$$\begin{cases} u_t(x,t) + \lambda u(x,t-\tau) = A_{\lambda_0}u(x,t) + f(x,t), & \text{in} \quad \mathbb{R}\times(0,T), \\ u(x,s) = u_0(x,s), & \text{in} \quad \mathbb{R}\times[-\tau,0]. \end{cases} \tag{2.4}$$

Since A_{λ_0} generates a strongly continuous semigroup of contractions in $L^2(\mathbb{R})$ by Proposition 2.1, for any given data $u_0 \in C([-\tau,0], H)$ and $f \in L^1(0,T;L^2(\mathbb{R}))$, problem (2.4) has a unique mild solution $u \in C([-\tau,T];L^2(\mathbb{R}))$, satisfying the following Duhamel's formula

$$u(t) = S(t)u_0(0) - \int_0^t S(t-s)\lambda u(s-\tau)\,ds + \int_0^t S(t-s)f(s)\,ds, \quad t\in[0,T]. \tag{2.5}$$

Therefore, we can prove that the mild solution of (2.4) depends continuously on the initial data.

Proposition 2.3. *If $u_0 \in C([-\tau,0],H)$ and $f \in L^1(0,T;L^2(\mathbb{R}))$, then the solution of (2.4) satisfies the following estimate:*

$$\|u(t)\|_{C([0,T];L^2(\mathbb{R}))} \le e^{\|\lambda\|_\infty T}\left(\|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \int_{-\tau}^0 \|u(s)\|_2\,ds\right) \tag{2.6}$$

and

$$\|u(t)\|_{C([-\tau,T];L^2(\mathbb{R}))} \le C\left(\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))} + \|f\|_{L^1(0,T;L^2(\mathbb{R}))}\right), \tag{2.7}$$

where the positive constant C is given by $C = C(\|\lambda\|_\infty, T, \tau)$.

Proof. The Duhamel's formula (2.5) give us that

$$\begin{aligned} \|u(t)\|_2 \le& \|u(0)\|_2 + \|\lambda\|_\infty \int_0^t \|u(s-\tau)\|_2\,ds + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} \\ \le& \|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \int_{-\tau}^0 \|u(s)\|_2\,ds + \|\lambda\|_\infty \int_0^t \|u(s)\|_2\,ds \\ =& \|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \int_{-\tau}^t \|u(s)\|_2\,ds. \end{aligned}$$

Thus, the result follows as a direct application of Gronwall's lemma. □

As a consequence of the previous inequality, we have the following proposition.

Proposition 2.4. *Let $u_0 \in C([-\tau,0],H)$ and $f \in L^1(0,T;L^2(\mathbb{R}))$, then the solution of (2.4) belongs to $\mathcal{B}_T$ and the following estimates holds true*

$$\|u(t)\|_{C([-\tau,T];L^2(\mathbb{R}))} \leq C(\|\lambda\|_\infty,T,\tau)\left(\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}\right) \tag{2.8}$$

where $C=C(\|\lambda\|_\infty,T,\tau)$ is a positive constant and

$$\begin{aligned}\|u\|_{\mathcal{B}_T} \leq C_T\{&\|u(0)\|_2+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}+\|\lambda\|_\infty\|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))}\\ &+\|\lambda\|_\infty^{1/2}\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\}\end{aligned} \tag{2.9}$$

where

$$C_T=\sqrt{\frac{3}{2}}\left(1+e^{2\|\lambda\|_\infty T}\right)^{1/2}e^{(\|\lambda\|_\infty+\|\lambda_0\|_\infty)T}. \tag{2.10}$$

Moreover, we have that

$$\begin{aligned}\frac{1}{2}\|u(t)\|_2^2+\int_0^t\|\partial_x^j u\|_2^2ds+\int_0^t\int_{\mathbb{R}}\lambda_0u^2(x,s)dxds+\int_0^t\int_{\mathbb{R}}\lambda u(x,s-\tau)u(x,s)dxds\\ =\frac{1}{2}\|u(0)\|_2^2+\int_0^t\int_{\mathbb{R}}f(x,s)u(x,s)dxds,\end{aligned} \tag{2.11}$$

for all $t\in[0,T]$,

Proof. Multiplying the equation (2.3) by u and integrating by parts, taking into account that

$$\int_{\mathbb{R}}(-1)^{j+1}(\partial_x^{2j+1}u)udx=0 \quad\text{and}\quad \int_{\mathbb{R}}(-1)^j\partial_x^{2j}u(x,t)udx=\int_{\mathbb{R}}(\partial_x^ju)^2dx,$$

the relation (2.11) holds. Now, estimate (2.8) follows by relation (2.7).

Let us now prove (2.9). Thanks to (2.6) we infer that

$$\begin{aligned}\|u(t)\|_2^2+2\int_0^t\|\partial_x^ju\|_2^2ds\leq{}&\|u(0)\|_2^2\\ &+2\|f\|_{L^1(0,T;L^2(\mathbb{R}))}e^{\|\lambda\|_\infty T}\left(\|u(0)\|_2+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}+\|\lambda\|_\infty\int_{-\tau}^0\|u(s)\|_2\,ds\right)\\ &+2\|\lambda_0\|_\infty\int_0^t\|u(s)\|_2^2ds+\|\lambda\|_\infty\int_0^t\|u(s-\tau)\|_2^2ds+\|\lambda\|_\infty\int_0^t\|u(s)\|_2^2ds.\end{aligned}$$

The previous inequality together with the following inequality

$$\int_0^t\|u(s-\tau)\|_2^2\,ds\leq\int_{-\tau}^0\|u(s)\|_2^2\,ds+\int_0^t\|u(s)\|_2^2\,ds,$$

ensures that

$$\begin{aligned}\|u(t)\|_2^2+2\int_0^t\|\partial_x^ju\|_2^2ds\leq{}&\|u(0)\|_2^2+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}^2\\ &+e^{2\|\lambda\|_\infty T}\left(\|u(0)\|_2+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}+\|\lambda\|_\infty\|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))}\right)^2\\ &+\|\lambda\|_\infty\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}^2+2(\|\lambda\|_\infty+\|\lambda_0\|_\infty)\int_0^t\|u\|_2^2ds.\end{aligned} \tag{2.12}$$

From (2.12) we have

$$\begin{aligned}\|u(t)\|_2^2+2\int_0^t\|\partial_x^ju\|_2^2ds\leq{}&2(\|\lambda\|_\infty+\|\lambda_0\|_\infty)\int_0^t\|u\|_2^2ds\\ &+\left(1+e^{2\|\lambda\|_\infty T}\right)\{\|u(0)\|_2+\|f\|_{L^1(0,T;L^2(\mathbb{R}))}\\ &+\|\lambda\|_\infty\|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))}+\|\lambda\|_\infty^{1/2}\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\}^2.\end{aligned} \tag{2.13}$$

An application of Gronwall's Lemma gives the following

$$\|u(t)\|_2^2 + 2\int_0^t \|\partial_x^j u\|_2^2\, ds \leq \left(1+e^{2\|\lambda\|_\infty T}\right) e^{2(\|\lambda\|_\infty + \|\lambda_0\|_\infty)T} \times$$
$$\times \left\{\|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}^2,$$

thus

$$\|u\|_{\mathcal{B}_T}^2 \leq \frac{3}{2}\left(1+e^{2\|\lambda\|_\infty T}\right) e^{2(\|\lambda\|_\infty + \|\lambda_0\|_\infty)T} \times$$
$$\times \left\{\|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}^2,$$

and so

$$\|u\|_{\mathcal{B}_T} \leq \sqrt{\frac{3}{2}}\left(1+e^{2\|\lambda\|_\infty T}\right)^{1/2} e^{(\|\lambda\|_\infty + \|\lambda_0\|_\infty)T} \times$$
$$\times \left\{\|u(0)\|_2 + \|f\|_{L^1(0,T;L^2(\mathbb{R}))} + \|\lambda\|_\infty \|u\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|u\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}$$

showing (2.9) with C_T defined by (2.10). □

2.3. **Nonlinear estimates.** In this subsection, we present some nonlinear estimates that will be used to prove Theorem 1.1.

Lemma 2.5. *Let $1 \leq p < 2j$ with $j \geq 1$. Then, there exists a positive constant C, such that, for any $T > 0$ and $u, v \in B_T$, we have*

$$\|u^p v_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{p}{2}} C T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T}^p \|v\|_{\mathcal{B}_T}.$$

Proof. Recall that $H^j(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, for $j \geq 1$ and that (3.6) holds. On the other hand,

$$\begin{aligned}
\|u^p v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leq C\int_0^T \|u(t)\|_\infty^p \|v_x(t)\|_2 dt \\
&\leq 2^{\frac{p}{2}} C \int_0^T \|u(t)\|_2^{\frac{p}{2}} \|u_x(t)\|_2^{\frac{p}{2}} \|v_x(t)\|_2 dt \\
&\leq 2^{\frac{p}{2}} C \|u\|_{C([0,T];L^2)}^{\frac{p}{2}} \int_0^T \|u_x(t)\|_2^{\frac{p}{2}} \|v_x(t)\|_2 dt.
\end{aligned}$$

Gagliardo-Nirenberg inequality says the following

$$\|\partial_x^m u\|_2 \leq C\|\partial_x^j u\|_2^{\frac{m}{j}} \|u\|_2^{1-\frac{m}{j}}, \quad m \leq j. \tag{2.14}$$

Thus, by using (2.14) and Hölder inequality, we obtain

$$\begin{aligned}
\|u^p v_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leq 2^{\frac{p}{2}} C \|u\|_{C([0,T];L^2)}^{\frac{p}{2}} \int_0^T \|\partial_x^j u(t)\|_2^{\frac{p}{2j}} \|u(t)\|_2^{\frac{p}{2}\left(1-\frac{1}{j}\right)} \|v_x(t)\|_2 dt \\
&\leq 2^{\frac{p}{2}} C \|u\|_{C([0,T];L^2)}^{\frac{p}{2}\left(2-\frac{1}{j}\right)} \left(\int_0^T \|\partial_x^j u(t)\|_2^{\frac{p}{2j}\cdot\frac{4j}{p}}\right)^{\frac{p}{4j}} \left(\int_0^T \|v_x(t)\|_2^2 dt\right)^{\frac{1}{2}} \left(\int_0^T dt\right)^{\frac{2j-p}{4j}} \\
&\leq 2^{\frac{p}{2}} C T^{\frac{2j-p}{4j}} \|u\|_{C([0,T];L^2)}^{\frac{p}{2}\left(2-\frac{1}{j}\right)} \|\partial_x^j u\|_{L^2([0,T];L^2)}^{\frac{p}{2j}} \|\partial_x v\|_{L^2([0,T];L^2)} \\
&\leq 2^{\frac{p}{2}} C T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T}^{\frac{p}{2}\left(2-\frac{1}{j}\right)} \|u\|_{\mathcal{B}_T}^{\frac{p}{2j}} \|v\|_{\mathcal{B}_T},
\end{aligned}$$

proving the result. □

The next lemma gives the estimates for the nonlinear terms.

Lemma 2.6. *For any $T > 0$, $1 \leq p < 2j$, $\lambda_0 \in L^\infty(\mathbb{R})$ and $u, v, w \in \mathcal{B}_T$, we have*

(i) $\|\lambda_0 u\|_{L^1(0,T;L^2(\mathbb{R}))} \leq T^{\frac{1}{2}}\|\lambda_0\|_\infty \|u\|_{\mathcal{B}_T}$;

(ii) $\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{1}{2}} T^{\frac{1}{4}} \|u\|_{\mathcal{B}_T} \|w\|_{\mathcal{B}_T}$;

(iii) $\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T} \|w\|_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T}^{p-1}$;

(iv) *The map*

$$M : \mathcal{B}_T \to L^1(0,T;L^2(\mathbb{R})),$$

defined by $Mu := u^p u_x$*, is locally Lipschitz continuous and satisfies*

$$\begin{aligned}\|Mu - Mv\|\|_{L^1(0,T;L^2(\mathbb{R}))} \leq & 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} C \left(\|u\|_{\mathcal{B}_T}^p + \|u\|_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T}^{p-1} + \|v\|_{\mathcal{B}_T}^p \right) \|u-v\|_{\mathcal{B}_T} \\ & + 2^{\frac{1}{2}} T^{\frac{1}{4}} C \|u\|_{\mathcal{B}_T} \|u-v\|_{\mathcal{B}_T},\end{aligned}$$

for a positive constant C*.*

Proof. Note that (i) follows using Hölder inequality. For (ii), observe that thanks to (3.6), we have

$$\|uw_x\|_{L^1(0,T;L^2(\mathbb{R}))} \leq 2^{\frac{1}{2}} T^{\frac{1}{4}} \|u\|_{\mathcal{B}_T} \|w\|_{\mathcal{B}_T}.$$

Let us now prove (iii). Note that

$$\begin{aligned}&\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \\ &\leq 2^{\frac{p}{2}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}} \int_0^T \|\partial_x^j u(t)\|_2^{\frac{1}{2j}} \|u(t)\|_2^{\frac{1}{2}\left(1-\frac{1}{j}\right)} \|\partial_x^j v(t)\|_2^{\frac{p-1}{2j}} \|v(t)\|_2^{\frac{p-1}{2}\left(1-\frac{1}{j}\right)} \|w_x(t)\|_2 dt \\ &\leq 2^{\frac{p}{2}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}\left(1-\frac{1}{j}\right)} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}\left(1-\frac{1}{j}\right)} \int_0^T \|\partial_x^j u(t)\|_2^{\frac{1}{2j}} \|\partial_x^j v(t)\|_2^{\frac{p-1}{2j}} \|w_x(t)\|_2 dt \\ &= 2^{\frac{p}{2}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}\left(2-\frac{1}{j}\right)} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}\left(2-\frac{1}{j}\right)} \int_0^T \|\partial_x^j u(t)\|_2^{\frac{1}{2j}} \|\partial_x^j v(t)\|_2^{\frac{p-1}{2j}} \|w_x(t)\|_2 dt.\end{aligned}$$

Thus, Hölder inequality ensures that

$$\begin{aligned}&\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} \\ &\leq 2^{\frac{p}{2}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}\left(2-\frac{1}{j}\right)} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}\left(2-\frac{1}{j}\right)} \left(\int_0^T \|\partial_x^j u\|_2^2 dt \right)^{\frac{1}{4j}} \left(\int_0^T \|\partial_x^j v\|_2^2 dt \right)^{\frac{p-1}{4j}} \left(\int_0^T \|w_x\|_2^2 dt \right)^{\frac{1}{2}} T^{\frac{2j-p}{4j}}.\end{aligned}$$

The previous inequality gives

$$\begin{aligned}\|u|v|^{p-1}w_x\|_{L^1(0,T;L^2(\mathbb{R}))} &\leq 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T}^{\frac{1}{2}\left(2-\frac{1}{j}\right)} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2}\left(2-\frac{1}{j}\right)} \|u\|_{\mathcal{B}_T}^{\frac{1}{2j}} \|v\|_{\mathcal{B}_T}^{\frac{p-1}{2j}} \left(\int_0^T \|\partial_x^j w\|_2^2 dt \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T}^{p-1} \|w\|_{\mathcal{B}_T},\end{aligned}$$

which allows us to get (iii). Finally, using the Mean Valued Theorem, (ii), (iii) and Lemma 2.5, we have

$$\begin{aligned}\|Mu - Mv\|_{L^1(0,T;L^2)} \leq & C \|(1 + |u|^{p-1} + |v|^{p-1})|u-v|u_x\|_{L^1(0,T;L^2)} + \|v^p(u-v)_x\|_{L^1(0,T;L^2)} \\ \leq & C \left\{ 2^{\frac{1}{2}} T^{\frac{1}{4}} \|u-v\|_{\mathcal{B}_T} \|u\|_{\mathcal{B}_T} + 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \|u-v\|_{\mathcal{B}_T} \|u\|_{\mathcal{B}_T}^p \right. \\ & \left. + 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \|u-v\|_{\mathcal{B}_T} \|u\|_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T}^{p-1} + 2^{\frac{p}{2}} T^{\frac{2-p}{4}} \|u-v\|_{\mathcal{B}_T} \|v\|_{\mathcal{B}_T}^p \right\},\end{aligned}$$

and (iv) holds. □

2.4. **Nonlinear system.** We are in a position to considering the nonlinear model (1.2), with $u_0 \in C([-\tau,0];L^2(\mathbb{R}))$. Before presenting it, let us introduce the following definitions:

Definition 2.7. *A mild solution of* (1.2) *will be a function* $u \in \mathcal{B}_T$, $T > 0$, *which satisfies*

$$u(t) = S(t)u_0(0) - \int_0^t S(t-s)\lambda u_0(s-\tau)ds - \int_0^t S(t-s)u^p(s)\partial_x u(s)ds, \quad t \in [0,T].$$

A global mild solution of (1.2) *will be a function* $u : [0,\infty) \to H^1(\mathbb{R})$ *whose restriction to every bounded interval* $[0,T]$ *is a mild solution of* (1.2).

2.4.1. *Well-posedness theory in $L^2(\mathbb{R})$.* With these definitions in hand and the previous estimates, the following result gives the local well-posedness for the higher order dispersive equation and a priori estimate for the solutions of (1.2).

Proposition 2.8. *Let $1 \leq p < 2j$ with $j \geq 1$, and $\lambda_0, \lambda \in L^\infty(\mathbb{R})$. For $u_0 \in L^2(\mathbb{R})$, there exist $T > 0$ and a unique mild solution $u \in \mathcal{B}_T$ of (1.2), such that*

$$\|u\|_{\mathcal{B}_T} \leq C_T \left\{ \|u(0)\|_2 + \|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \right\}.$$

Here, C_T is given by (2.10).

Proof. Let $T > 0$ be determined later. For each $u \in \mathcal{B}_T$ consider the problem

$$\begin{cases} v_t = A_{\lambda_0} v - \lambda v(x, t-\tau) - Mu, & \text{in} \quad \mathbb{R} \times (0,T), \\ v(x,s) = u_0(x,s), & \text{in} \quad \mathbb{R} \times [-\tau, 0]. \end{cases} \tag{2.15}$$

Since A_{λ_0} generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $L^2(\mathbb{R})$. Proposition 2.4 allows us to conclude that (2.15) has a unique mild solution $v \in B_{0,T}$, such that

$$\begin{aligned} \|v\|_{\mathcal{B}_T} \leq & C_T \left\{ \|u_0\|_2 + \|Mu\|_{L^1(0,T;L^2(\mathbb{R}))} \right. \\ & \left. + \|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \right\} \end{aligned} \tag{2.16}$$

where C_T is given by (2.10). Moreover, we have that

$$\begin{aligned} \frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\partial_x^j v\|_2^2 ds + \int_0^t \int_{\mathbb{R}} \lambda_0 v^2(x,s) dx ds + \int_0^t \int_{\mathbb{R}} \lambda v(x, s-\tau) v(x,s) dx ds \\ = \frac{1}{2}\|u_0\|_2^2 + \int_0^t \int_{\mathbb{R}} Mu(x,s) v(x,s) dx ds. \end{aligned}$$

Thus, we can define the operator $\Gamma : \mathcal{B}_T \longrightarrow \mathcal{B}_T$ given by $\Gamma(u) = v$. Thanks to the Lemma 2.5 and (2.16), we have

$$\|\Gamma u\|_{\mathcal{B}_T} \leq C_T \{ \|u_0\|_2 + 2^{p/2} C T^{\frac{2j-p}{4j}} \|u\|_{\mathcal{B}_T}^{p+1} + \|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \},$$

and, for $u \in B_R(0) := \{u \in \mathcal{B}_T : \|u\|_{\mathcal{B}_T} \leq R\}$, it follows that

$$\|\Gamma u\|_{\mathcal{B}_T} \leq C_T \{ \|u_0\|_2 + 2^{p/2} C T^{\frac{2j-p}{4j}} R^{p+1} + \|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \}.$$

Choosing

$$R = 2C_T \left(\|u_0\|_2 + \left(\|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \right) \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \right),$$

we obtain the following estimate

$$\|\Gamma u\|_{0,T} \leq \left(K_1 + \frac{1}{2} \right) R,$$

where $K_1 = K_1(T) = 2^{p/2} C_T C T^{\frac{2j-p}{4j}} R^p$. On the other hand, note that $\Gamma u - \Gamma w$ is solutions of

$$\begin{cases} v_t = A_{\lambda_0} v - (Mu - Mv), & \text{in} \quad \mathbb{R} \times (0,T), \\ v(x,s) = 0, & \text{in} \quad \mathbb{R} \times [-\tau, 0]. \end{cases}$$

We will now prove that Γ has a unique fixed point. To do that, note that thanks to Proposition 2.4, we have

$$\|\Gamma u - \Gamma w\|_{\mathcal{B}_T} \leq C_T \|Mu - Mw\|_{L^1(0,T;L^2)}.$$

Lemma 2.6, precisely estimate (iv), allows us to conclude that

$$\|\Gamma u - \Gamma w\|_{\mathcal{B}_T} \leq C_T C \left\{ 2^{\frac{p}{2}} T^{\frac{2j-p}{4j}} \left(\|u\|_{\mathcal{B}_T}^p + \|u\|_{\mathcal{B}_T} \|w\|_{\mathcal{B}_T}^{p-1} + \|w\|_{\mathcal{B}_T}^p \right) + 2^{\frac{1}{2}} T^{\frac{1}{4}} \|u\|_{\mathcal{B}_T} \right\} \|u - w\|_{\mathcal{B}_T}.$$

Suppose that $u, w \in B_R(0)$ defined above. Then,

$$\|\Gamma u - \Gamma w\|_{\mathcal{B}_T} \leq K_2 \|u - w\|_{\mathcal{B}_T},$$

where

$$K_2 = K_2(T) = C_T C\{2^{\frac{1}{2}} T^{\frac{1}{4}} R + 3(2^{\frac{p}{2}}) T^{\frac{2j-p}{4j}} R^p\}.$$

Since $K_1 \leq K_2$, we can choose $T > 0$ to obtain $K_2 < \frac{1}{2}$,

$$\|\Gamma u\|_{\mathcal{B}_T} \leq R \quad \text{and} \quad \|\Gamma u - \Gamma w\|_{\mathcal{B}_T} < \frac{1}{2}\|u - w\|_{\mathcal{B}_T},$$

for all $u, w \in B_R(0) \subset B_{0,T}$. Hence, $\Gamma : B_R(0) \to B_R(0)$ is a contraction and, by Banach fixed point theorem, we obtain a unique $u \in B_R(0)$, such that $\Gamma(u) = u$ and consequently, the local well-posedness result for $0 < T \leq \tau$ small enough to the system. Thus, u is a unique local mild solution to the problem, and estimate (2.8) holds. □

We are now able to present this subsection's main result.

Theorem 2.9. *Consider $\lambda_0, \lambda \in L^\infty(\mathbb{R})$. Therefore, the system (1.2) admits a unique global mild solution for every initial data $u_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$ satisfying*

$$\begin{aligned} \frac{1}{2}\|u(0)\|_2^2 =& \frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|u_x\|_2^2 \, ds + \int_0^t \int_{\mathbb{R}} \lambda_0 u^2(x, s) dx ds \\ &+ \int_0^t \int_{\mathbb{R}} \lambda u(x, s - \tau) u(x, s) dx ds, \end{aligned} \tag{2.17}$$

for all $t \geq 0$. Moreover, there exists a non-decreasing continuous function $\beta_0 : \mathbb{R}_+ \to \mathbb{R}_+$, such that the solution u (1.2) satisfies

$$\|u\|_{\mathcal{B}_T} \leq C_T \left\{ \|u(0)\|_2 + \|\lambda\|_\infty \|u_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2} \|u_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))} \right\}$$

and

$$\|u\|_{\mathcal{B}_T} \leq \beta_0(\|u_0\|_2) \|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))},$$

with $\beta_0 = C_T \left(1 + \left(\|\lambda\|_\infty \tau^{1/2} + \|\lambda\|_\infty^{1/2}\right)\right)$ and C_T given by (2.10).

Proof. To prove the global well-posedness we need to prove that the norms of the solutions of (1.2) remain bounded in the existence time interval. To do that, let us consider the functional

$$\mathcal{E}(t) := \mathcal{E}(u(t)) = \frac{1}{2} \int_{\mathbb{R}} u^2(x, t) dx + \frac{1}{2} \int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)} |\lambda(x)| u^2(x, s) \, dx \, ds. \tag{2.18}$$

Taking the time derivative in time of this function, we have that

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) =& \int_{\mathbb{R}} u(t)((-1)^j \partial_x^{2j} u(t) - \lambda_0 u(t) - \lambda u(t - \tau) + u^p(t) u_x(t)) dx + \frac{1}{2} \int_{\mathbb{R}} |\lambda| u^2(t) dx \\ &- \frac{1}{2} e^{-\tau} \int_{\mathbb{R}} |\lambda| u^2(t - \tau) dx - \frac{1}{2} \int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)} |\lambda| u^2(x, s) dx ds. \end{aligned}$$

Integrating by parts, using the Young inequality and taking into account the hypothesis (1.7), (1.8), and (1.9) we get that

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) \leq& - \int_{\mathbb{R}} (\partial_x^j u)^2(t) dx - \gamma_0 \int_{\mathbb{R}} u^2(t) dx + \frac{e^\tau + 1}{2} \int_{\mathbb{R}} |\lambda(x)| u^2(t) dx \\ &- \frac{1}{2} \int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)} |\lambda| u^2(x, s) dx ds \\ \leq& - \int_{\mathbb{R}} (\partial_x^j u)(t) dx - (\gamma_0 - \gamma) \int_{\mathbb{R}} u^2(t) dx + \int_{\mathbb{R}} \beta(x) u^2(t) dx \leq 0. \end{aligned}$$

Here, we have used that the third integral in the previous inequality can be handled as in the proof of Theorem 3.1, so, here, we omitted the details. Therefore, this inequality ensures that $\|u(t)\|_2$ remains bounded for $t \in [0, T]$.

Finally, thanks to the estimate (2.17), we deduce that $\|u\|_{\mathcal{B}_T}$ remains bounded for $t \in [0,T]$, and so, the local solution u given by Proposition 2.8 can be extended on $[0,\tau]$. Now, once we have a solution $u \in \mathcal{B}_\tau$ we can apply the same arguments as we did on Theorem 2.2 to prove the existence of a global mild solution of (1.2). Finally, note that the proof of Proposition 2.8 guarantees the function β_0 is given by

$$\beta_0(s) = C_T\left(1 + \left(\|\lambda\|_\infty \tau^{1/2} + \|\lambda\|_\infty^{1/2}\right)\right),$$

showing the result. □

2.4.2. *Well-posedness theory in $H^{2j+1}(\mathbb{R})$.* We will analyze the well-posedness in $\mathcal{B}_{2j+1,T}$, with $1 \le p < 2j$ with $j \ge 1$. To do that, let us first consider the following linearized problem

$$\begin{cases} v_t + (-1)^{j+1}\partial_x^{2j+1}v + (-1)^m\partial_x^{2m}v + \partial_x(u^p v) \\ \qquad\qquad +\lambda_0(x)v + \lambda(x)v(x,t-\tau) = 0, & \text{in } \mathbb{R}\times(0,\infty), \\ v(x,s) = v_0(x,s), & \text{in } \mathbb{R}\times[-\tau,0]. \end{cases} \tag{2.19}$$

Then, we can establish the following proposition:

Proposition 2.10. *For $T > 0$, $\lambda_0, \lambda \in L^\infty(\mathbb{R})$, and $u \in \mathcal{B}_T$, if $v_0 \in C([-\tau,0];L^2(\mathbb{R}))$, then system (2.19) admits a unique solution $v \in B_{0,T}$, such that*

$$\|v(t)\|_{C([-\tau,T];L^2(\mathbb{R}))} \le C(\|\lambda\|_\infty, T, \tau)\left(\|v_0\|_{C([-\tau,0];L^2(\mathbb{R}))} + (p+1)\|u\|_{\mathcal{B}_T}^p\|v\|_{\mathcal{B}_T}\right), \tag{2.20}$$

where $C(\|\lambda\|_\infty, T, \tau)$ is a positive constant and

$$\|v\|_{\mathcal{B}_T} \le C_T\left\{\|v(0)\|_2 + \|u\|_{\mathcal{B}_T}^p + \|\lambda\|_\infty\|v_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|v_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\} \tag{2.21}$$

and

$$\|v\|_{\mathcal{B}_T} \le \sigma(\|u\|_{\mathcal{B}_T}, T)\left\{\|v(0)\|_2 + \|\lambda\|_\infty\|v_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|v_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}. \tag{2.22}$$

Here, $1 \le p < 2j$, with $j \le 1$, and $\sigma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function.

Proof. The existence of a solution follows the same steps as done in Proposition 2.8 and Theorem 2.9, so we will omit it. Let us prove inequality (2.22). Thanks to (2.8) and (2.9), we deduce (2.20) and

$$\begin{aligned}\|v\|_{\mathcal{B}_T} \le & C_T\left\{\|v(0)\|_2 + \|(u^p v)_x\|_{L^1(0,T;L^2(\mathbb{R}))}\right. \\ & \left. + \|\lambda\|_\infty\|v_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|v_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}.\end{aligned}$$

Lemma 2.6 and Young inequality imply that there exist a constant $C > 0$ such that

$$\|v(t)\|_{C([-\tau,T];L^2(\mathbb{R}))} \le e^{\|\lambda\|_\infty T}\left(\|v(0)\|_2 + (p+1)\|u\|_{\mathcal{B}_T}^p\|v\|_{\mathcal{B}_T}\right)$$

and

$$\|v\|_{\mathcal{B}_T} \le CC_T\left\{\|v(0)\|_2 + \|u\|_{\mathcal{B}_T}^p + \|\lambda\|_\infty\|v_0\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|v_0\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\},$$

giving (2.21). On the other hand, following the same ideas as done in Proposition 2.4, from (2.13) we infer that

$$\begin{aligned}\|v(t)\|_2^2 + 2\int_0^t \|\partial_x^j v\|_2^2 ds \le & 2(\|\lambda\|_\infty + \|\lambda_0\|_\infty)\int_0^t \|v\|_2^2 ds \\ & + \left(1 + e^{2\|\lambda\|_\infty T}\right)\left\{\|v(0)\|_2 + \|\partial_x(u^p v)v\|_{L^1(0,t;L^2(\mathbb{R}))}\right. \\ & \left. + \|\lambda\|_\infty\|v\|_{L^1(-\tau,0;L^2(\mathbb{R}))} + \|\lambda\|_\infty^{1/2}\|v\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\right\}^2.\end{aligned}$$

Gagliardo-Nerinberg inequality ensures that

$$\begin{aligned}\|\partial_x(u^pv)v\|_{L^1(0,t;L^2(\mathbb{R}))} = \|u^pv_xv\|_{L^1(0,t;L^2(\mathbb{R}))} &\leq \int_0^t \|u^p(s)v_x(s)v(s)\|_2 ds\\ &\leq \|u\|^p_{\mathcal{B}_T}\int_0^t \|v_x(s)\|_2\|v(s)\|_\infty ds \leq 2^{\frac{1}{2}}\|u\|^p_{\mathcal{B}_T}\int_0^t \|v_x(s)\|_2^{\frac{3}{2}}\|v(s)\|_2^{\frac{1}{2}}ds\\ &\leq 2^{\frac{1}{2}}C\|u\|^p_{\mathcal{B}_T}\int_0^t \|v(s)\|_2^{\frac{3}{2}\left(1-\frac{1}{j}\right)}\|\partial_x^jv(s)\|_2^{\frac{3}{2j}}\|v(s)\|_2^{\frac{1}{2}}ds\\ &= 2^{\frac{1}{2}}C\|u\|^p_{\mathcal{B}_T}\int_0^t \|v(s)\|_2^{\frac{4j-3}{2j}}\|\partial_x^jv(s)\|_2^{\frac{3}{2j}}ds\end{aligned}$$

and using young inequality, we have that

$$\int_0^t \|v(s)\|_2^{\frac{4j-3}{2j}}\|\partial_x^jv(s)\|_2^{\frac{3}{2j}}ds \leq \frac{(4j-3)}{4j\eta^{\frac{4j}{4j-3}}}\int_0^t\|v(s)\|_2^2ds + \frac{3\eta^{\frac{4j}{3}}}{4j}\int_0^t\|\partial_x^jv(s)\|_2^2ds$$

where $\eta = \left[\frac{4j}{3\sqrt{2}C(1+e^{2\|\lambda\|_\infty T})\|u\|^p_{\mathcal{B}_T}}\right]^{\frac{3}{4j}}$. Hence, it yields that

$$\begin{aligned}\|v(t)\|_2^2 + \int_0^t\|\partial_x^jv\|_2^2ds \leq& \rho(\|u\|_{\mathcal{B}_T})\int_0^t\|v(s)\|_2^2ds + \left(1+e^{2\|\lambda\|_\infty T}\right)\Big\{\|v(0)\|_2\\ &+\|\lambda\|_\infty\|v\|_{L^1(-\tau,0;L^2(\mathbb{R}))}+\|\lambda\|_\infty^{1/2}\|v\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\Big\}^2,\end{aligned}$$

where

$$\begin{aligned}\rho(\|u\|_{\mathcal{B}_T}) =& 2(\|\lambda\|_\infty+\|\lambda_0\|_\infty)\\ &+(1+e^{2\|\lambda\|_\infty T})2^{\frac{1}{2}}C\|u\|^p_{\mathcal{B}_T}\frac{(4j-3)}{4j}\left[\frac{4j}{3\sqrt{2}C(1+e^{2\|\lambda\|_\infty T})\|u\|^p_{\mathcal{B}_T}}\right]^{-\frac{3}{4j-3}}\\ =& 2(\|\lambda\|_\infty+\|\lambda_0\|_\infty)\\ &+(4j-3)\left[\frac{3\sqrt{2}C(1+e^{2\|\lambda\|_\infty T})\|u\|^p_{\mathcal{B}_T}}{4j}\right]^{\frac{4j}{4j-3}}.\end{aligned}$$

Finally, we have that

$$\begin{aligned}\|v(t)\|_2^2 + \int_0^t\|\partial_x^jv\|_2^2ds \leq& \rho(\|u\|_{\mathcal{B}_T})\int_0^t\left(\|v(s)\|_2^2+\int_0^s\|\partial_x^jv(t)\|_2^2dr\right)ds\\ &+\left(1+e^{2\|\lambda\|_\infty T}\right)\Big\{\|v(0)\|_2+\|\lambda\|_\infty\|v\|_{L^1(-\tau,0;L^2(\mathbb{R}))}\\ &+\|\lambda\|_\infty^{1/2}\|v\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\Big\}^2.\end{aligned}$$

Employing Gronwall's inequality, we conclude

$$\begin{aligned}\|v(t)\|_2^2 + \int_0^t\|\partial_x^jv\|_2^2ds \leq& \left(1+e^{2\|\lambda\|_\infty T}\right)e^{\rho(\|u\|_{\mathcal{B}_T})t}\Big\{\|v(0)\|_2+\|\lambda\|_\infty\|v\|_{L^1(-\tau,0;L^2(\mathbb{R}))}\\ &+\|\lambda\|_\infty^{1/2}\|v\|_{L^2(-\tau,0;L^2(\mathbb{R}))}\Big\}^2.\end{aligned}$$

Estimate (2.22) follows directly from the above inequality. □

We now give the existence of a solution to the problem (1.2).

Theorem 2.11. *Let $T>0$, $\lambda_0,\lambda\in H^j(\mathbb{R})$ and $1\leq p<2j$, with $j\geq 1$. For $u_0(x,t-\tau)\in C([-\tau,0];H^{2j+1}(\mathbb{R}))$, there exists a unique mild solution $u\in\mathcal{B}_{2j+1,T}$ of (1.2) such that*

$$\|u\|_{\mathcal{B}_{2j+1,T}} \leq \beta_{2j+1}(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))},$$

where $\beta_{2j+1} : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing continuous function.

Proof. We will split the proof into several steps.
Step 1: $u \in L^2(0,T;H^{2j+1}(\mathbb{R}))$.

Since $u_0 \in H^{2j+1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, by Theorem 2.9, there exists a unique solution $u \in \mathcal{B}_T$, such that

$$\|u\|_{\mathcal{B}_T} \le \beta_0(\|u_0\|_2)\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}. \tag{2.23}$$

We will show that $u \in \mathcal{B}_{2j+1,T}$. Let $v = u_t$. Then, v solves the problem

$$\begin{cases} v_t + (-1)^{j+1}\partial_x^{2j+1}v + (-1)^m\partial_x^{2m}v + \partial_x(u^pv) + \lambda_0(x)v + \lambda(x)v(x,t-\tau) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}, \\ v(x,s) = v_0(x,s) & \text{in } \mathbb{R}\times(-\tau,0), \end{cases}$$

where $v_0(x) = -(-1)^{j+1}\partial_x^{2j+1}u_0 + (-1)^m\partial_x^{2m}u_0 - \frac{1}{p+1}\partial_x(u_0^{p+1}) - \lambda_0(x)u_0 - \lambda(x)u_0(x,-\tau)$. We can bound v_0 as follows:

$$\begin{aligned}\|v_0\|_2 &\le \|\partial_x^{2j+1}u_0\|_2 + \|\partial_x^{2m}u_0\|_2 + \|u_0^p\partial_xu_0\|_2 + \|\lambda_0u_0\|_2 + \|\lambda(x)u_0(-\tau)\|_2 \\ &\le (1+\|\lambda_0\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^{2j+1}(\mathbb{R})} + \|u_0\|_\infty^{\frac{p}{2}}\|\partial_xu_0\|_2 + \|\lambda\|_\infty\|u_0(\cdot,-\tau)\|_2 \\ &\le (1+\|\lambda_0\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^{2j+1}(\mathbb{R})} + 2^{\frac{p}{2}}\|u_0\|_2^{\frac{p}{2}}\|\partial_xu_0\|_2^{\frac{p+2}{2}} + \|\lambda\|_\infty\|u_0(\cdot,-\tau)\|_2.\end{aligned}$$

Using the Gagliardo-Nirenberg inequality, we have

$$\|\partial_xu_0\|_2 \le C\|\partial_x^{2j+1}u_0\|_2^{\frac{1}{2j+1}}\|u_0\|_2^{1-\frac{1}{2j+1}}. \tag{2.24}$$

Applying (2.24), we ensures that

$$\begin{aligned}\|v_0\|_2 \le&(1+\|\lambda_0\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^{2j+1}(\mathbb{R})} \\ &+ 2^{\frac{p}{2}}C\|\partial_x^{2j+1}u_0\|_2^{\frac{p+2}{2(2j+1)}}\|u_0\|_2^{\frac{p+2}{2}\left(1-\frac{1}{2j+1}\right)+\frac{p}{2}} + \|\lambda\|_\infty\|u_0(\cdot,-\tau)\|_2.\end{aligned}$$

Then, Young inequality guarantees the following:

$$\begin{aligned}\|v_0\|_2 \le&(1+\|\lambda_0\|_{L^\infty(\mathbb{R})})\|u_0\|_{H^{2j+1}(\mathbb{R})} + 2^{\frac{p(2j+1)}{4j-p}}\|u_0\|_2^{\left(\frac{p+2}{2}\left(1-\frac{1}{2j+1}\right)+\frac{p-2}{2}\right)\frac{2(2j+1)}{4j-p}}\|u_0\|_2 \\ &+ \|\partial_x^{2j+1}u_0\|_2 + \|\lambda\|_\infty\|u_0(\cdot,-\tau)\|_2\end{aligned}$$

and, consequently, leader to

$$\begin{aligned}\|v_0\|_2 &\le C(\|u_0\|_2)\|u_0\|_{H^{2j+1}(\mathbb{R})} + \|\lambda\|_\infty\|u_0(\cdot,-\tau)\|_2 \\ &\le C(\|u_0\|_2)\|u_0\|_{H^{2j+1}(\mathbb{R})} + \|\lambda\|_\infty\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))} \\ &\le (C(\|u_0\|_2) + \|\lambda\|_\infty)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))},\end{aligned} \tag{2.25}$$

where

$$C(s) = 2 + \|\lambda_0\|_{L^\infty(\mathbb{R})} + 2^{\frac{p(2j+1)}{4j-p}}s^{\left(\frac{p+2}{2}\left(1-\frac{1}{2j+1}\right)+\frac{p-2}{2}\right)\frac{2(2j+1)}{4j-p}}.$$

Thanks to Proposition 2.10, we see that $v \in B_{0,T}$ and

$$\|v\|_{\mathcal{B}_T} \le \sigma(\|u\|_{\mathcal{B}_T})\|v_0\|_{C([-\tau,0];L^2(\mathbb{R}))}.$$

Combining (2.23) and (2.25), we get

$$\|v\|_{\mathcal{B}_T} \le \sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)\left(C(\|u_0\|_2) + \|\lambda\|_\infty\|\right)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))}. \tag{2.26}$$

which means

$$u, u_t \in L^2(0,T;H^j(\mathbb{R})). \tag{2.27}$$

Therefore,

$$u \in C([0,T];H^j(\mathbb{R})) \hookrightarrow C([0,T];C(\mathbb{R})). \tag{2.28}$$

On the other hand, note that $u^pu_x, \lambda_0 u$ and $\lambda u(\cdot, t-\tau)$ belong to $L^2(0,T;L^2(\mathbb{R}))$. Moreover,

$$(-1)^{j+1}\partial_x^{2j+1}u + (-1)^m\partial_x^{2m}u = -u_t - u^pu_x - \lambda_0 u - \lambda u(\cdot, t-\tau) \text{ in } D'(0,T,\mathbb{R}).$$

Hence,

$$(-1)^{j+1}\partial_x^{2j+1}u + (-1)^m\partial_x^{2m}u = f \in L^2(0,T;L^2(\mathbb{R})).$$

Taking Fourier transform, we have

$$\widehat{u} = \frac{\widehat{f} + \widehat{u}}{[(1 - ((-1)^{j+1}(i\xi)^{2j+1}) - (-1)^j(i\xi)^{2j})]}$$

and,

$$\|u(t)\|^2_{H^{2j+1}(\mathbb{R})} \le C\left\{\|f(t)\|_2^2 + \|u(t)\|_2^2\right\} \tag{2.29}$$

where $C = \sup_{\xi\in\mathbb{R}} \frac{1+|\xi|+|\xi|^2+|\xi|^3+\cdots+|\xi|^{2j+1}}{|1-((-1)^{j+1}(i\xi)^{2j+1})-(-1)^j(i\xi)^{2j}|}$. Integrating (2.29) over $[0,T]$, we deduce that

$$u \in L^2(0,T;H^{2j+1}(\mathbb{R})), \tag{2.30}$$

achieving step 1.

Step 2: $u \in C(0,T;H^{2j+1}(\mathbb{R})$

Observe that, according to (2.27), $u_t \in L^2(0,T;H^{-(2j+1)}(\mathbb{R}))$. Then, considering the Hilbert triple $H^{2j+1}(\mathbb{R}) \hookrightarrow H^{2m}(\mathbb{R}) \hookrightarrow H^{-(2j+1)}(\mathbb{R})$, by [34, Chapter III - Lemma 1.2], we have $u \in C([0,T];H^{2m}(\mathbb{R}))$. This implies further

$$\partial_x^{2m}u, \lambda_0 u, \lambda u(\cdot, \cdot - \tau) \in C([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R})). \tag{2.31}$$

On the other hand, note that

$$\begin{aligned}\|u(t)^pu_x(t) - u(t_0)^pu_x(t_0)\|_2 \le& \|[u(t)^p - u(t_0)^p]u_x(t)\|_2 + \|u(t_0)^p[u_x(t) - u_x(t_0)]\|_2\\ \le& C\left\{\|(1+|u(t)|^{p-1} + |u(t_0)|^{p-1})|u(t) - u(t_0)|u_x(t)\|_2\right.\\ &\left.+\|(1+|u(t_0)|^p)|u_x(t) - u_x(t_0)|\|_2\right\}\\ \le& C\left\{(1+\|u(t)\|_\infty^{p-1} + \|u(t_0)\|_\infty^{p-1})\|u(t) - u(t_0)\|_\infty\|u_x(t)\|_2\right.\\ &\left.+(1+\|u(t_0)\|_\infty^p)\|u_x(t) - u_x(t_0)\|_2\right\}.\end{aligned}$$

Then, the regularity given in (2.28) ensures that

$$\lim_{t\to t_0}\|u(t)^pu_x(t) - u(t_0)^pu_x(t_0)\|_2 = 0$$

and, therefore $u^pu_x \in C([0,T];L^2(\mathbb{R}))$. The results above also guarantee that

$$u^pu_x \in C([0,T];L^2(\mathbb{R})) \cap L^2(0,T;H^1(\mathbb{R})). \tag{2.32}$$

Indeed, note that $(u^pu_x)_x = pu^{p-1}u_x^2 + u^pu_{xx}$, so it is sufficient to combine (2.28), (2.30) and the following two estimates

$$\|pu^{p-1}u_x^2\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\left\{(1+\|u\|^{p-1}_{C([0,T];C(\mathbb{R}))})\|u_x\|_{C([0,T];C(\mathbb{R}))}\|u_x\|_{L^2([0,T];L^2(\mathbb{R}))}\right\}$$

and

$$\|u^pu_{xx}\|_{L^2(0,T;L^2(\mathbb{R}))} \le C\left\{(1+\|u\|^p_{C([0,T];C(\mathbb{R}))})\|u_{xx}\|_{L^2([0,T];L^2(\mathbb{R}))}\right\},$$

to ensures (2.32).

Now, since

$$(-1)^{j+1}\partial_x^{2j+1}u = -(-1)^m\partial_x^{2m}u - u_t - (u^pu_x) - \lambda_0 u - \lambda, u(\cdot, t-\tau), \tag{2.33}$$

from (2.27), (2.31), (2.32), we obtain

$$u \in L^2(0,T;H^{3j+1}(\mathbb{R})) \hookrightarrow L^2(0,T;H^{2j+2}(\mathbb{R})). \tag{2.34}$$

Finally, considering the Hilbert triple $H^{2j+2}(\mathbb{R}) \hookrightarrow H^{2j+1}(\mathbb{R}) \hookrightarrow H^{-(2j+2)}(\mathbb{R})$, [34, Chapter III - Lemma 1.2] gives

$$u \in C([0,T]; H^{2j+1}(\mathbb{R})), \tag{2.35}$$

and step 2 is proved. Note that (2.34) and (2.35) imply that $u \in \mathcal{B}_{2j+1,T}$.

Step 3: The following estimate holds

$$\|u\|_{C([0,T];H^{2j+1}(\mathbb{R}))} \leq \sigma_1(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))}.$$

Indeed, according to (2.29), we get

$$\|u(t)\|_{H^{2j+1}(\mathbb{R})} \leq C\left\{\|u_t(t)\|_2 + \|u^p(t)u_x(t)\|_2 + \|\lambda_0 u(t)\|_2 + \|\lambda u(t-\tau)\|_2 + \|u(t)\|_2\right\}. \tag{2.36}$$

Now, by using Gagliardo-Nirenberg inequality, we obtain

$$\|u(t)^p u_x(t)\|_2 \leq 2^{\frac{1}{p}}\|u(t)\|_2^{\frac{p}{2}}\|u_x(t)\|_2^{\frac{p+2}{2}} \leq 2^{\frac{1}{p}}C\|u(t)\|_2^{\frac{4j(p+1)+p}{2(2j+1)}}\|u(t)\|_{H^{2j+1}(\mathbb{R})}^{\frac{p+2}{2(2j+1)}}.$$

Moreover, Young inequality gives

$$\|u(t)^p u_x(t)\|_2 \leq C\|u(t)\|_2^{\frac{4j(p+1)+p}{4j-p}} + \frac{1}{2C}\|u(t)\|_{H^{2j+1}(\mathbb{R})}.$$

Replacing the estimate above into the inequality (2.36), yields that

$$\|u(t)\|_{H^{2j+1}(\mathbb{R}))} \leq C\left\{\|u_t\|_{\mathcal{B}_T} + (1+\|\lambda_0\|_\infty)\|u\|_{\mathcal{B}_T} + \|u\|_{\mathcal{B}_T}^{\frac{4j(p+1)+p}{4j-p}} + \|\lambda\|_\infty\|u(t-\tau)\|_2\right\}.$$

Then, using (2.23) and (2.26) it follows that

$$\begin{aligned}\|u(t)\|_{H^{2j+1}(\mathbb{R}))} \leq & C\left\{\sigma(\beta_0(\|u_0\|_2)\|u_0\|_2)C(\|u_0\|_2)\|u_0\|_{H^{2j+1}(\mathbb{R})} + (C+\|\lambda_0\|_\infty)\beta_0(\|u_0\|_2)\|u_0\|_2\right.\\ & \left.+\beta_0^{\frac{4j(p+1)+p}{4j-p}}(\|u_0\|_2)\|u_0\|_2^{\frac{4j(p+1)+p}{4j-p}-1}\|u_0\|_2 + \|\lambda\|_\infty\|u(t-\tau)\|_2\right\}\\ \leq & \bar{\sigma}_1(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))} + \|\lambda\|_\infty\|u(t-\tau)\|_2,\end{aligned}$$

where

$$\bar{\sigma}_1(s) = C\left\{\sigma(\beta_0(s)s)C(s) + (C+\|b\|_\infty)\beta_0(s) + (\beta_0(s))^{\frac{4j(p+1)+p}{4j-p}}s^{\frac{4j(p+1)+p}{4j-p}-1}\right\}.$$

Therefore, we obtain that

$$\begin{aligned}\|u\|_{C([0,T];H^{2j+1}(\mathbb{R})))} \leq & \bar{\sigma}_1(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))} + \|\lambda\|_\infty\|u\|_{C([-\tau,T-\tau];L^2(\mathbb{R}))}\\ \leq & \bar{\sigma}_1(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))} + \|\lambda\|_\infty\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}\\ & + \|\lambda\|_\infty\|u\|_{C([0,T-\tau];L^2(\mathbb{R}))}\\ \leq & \left(\bar{\sigma}_1(\|u_0\|_2) + \|\lambda\|_\infty\right)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))} + \|\lambda\|_\infty\|u\|_{\mathcal{B}_T}.\end{aligned}$$

Finally, from Theorem 2.9 and the previous inequality, we deduce that

$$\|u\|_{C([0,T];H^{2j+1}(\mathbb{R})))} \leq \sigma_1(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))},$$

where $\sigma_1(s) = \bar{\sigma}_1(s) + \|\lambda\|_\infty(\beta_0(s)+1)$, finishing the step 3.

Step 4: The following estimate is verified

$$\|\partial_x^{3j+1}u\|_{L^2(0,T;L^2(\mathbb{R}))} \leq \sigma_3(\|u_0\|_2)\|u_0\|_{C([-\tau,0];H^{2j+1}(\mathbb{R}))}.$$

In fact, from the equation (2.33), we deduce that

$$\begin{aligned}\int_0^T \|u(t)\|_{H^{3j+1}}^2 dt =& \int_0^T \|\partial_x^{3j+1} u(t)\|_2^2 dt + \int_0^T \|u(t)\|_2^2 dt\\ \leq& \int_0^T \|u(t)\|_2^2 dt + \int_0^T \|\partial_x^{2m+j} u(t)\|_2^2 dt + \int_0^T \|\partial_x^j u_t(t)\|_2^2 dt\\ &+ \int_0^T \|\partial_x^j (u^p u_x)\|_2^2 dt + \int_0^T \|\partial_x^j (\lambda_0 u(t))\|_2^2 dt + \int_0^T \|\partial_x^j (\lambda_0 u(t-\tau))\|_2^2 dt\\ \leq& CT \left(\|u\|_{C([0,T];H^{2j+1})} + \|u\|_{C([0,T];H^{2m+j})}\right) + \|u_t\|_{B_{0,T}}\\ &+ \int_0^T \|\partial_x^j (u^p u_x)\|_2^2 dt + \int_0^T \|\partial_x^j (\lambda_0 u(t))\|_2^2 dt + \int_0^T \|\partial_x^j (\lambda_0 u(t-\tau))\|_2^2 dt.\end{aligned}$$

Note that $H^{2m+j}(\mathbb{R}) \subset H^{3j+1}(\mathbb{R})$, thus using step 4 and Proposition 2.10, there exists a function σ_2 such that

$$\begin{aligned}\int_0^T \|u(t)\|_{H^{3j+1}}^2 dt \leq& \sigma_2 \left(\|u_0\|_2\right) \|u_0\|_{C([-\tau,0];H^{2j+1})} + \int_0^T \|\partial_x^j (u^p u_x)\|_2^2 dt\\ &+ \int_0^T \|\partial_x^j (\lambda_0 u(t))\|_2^2 dt + \int_0^T \|\partial_x^j (\lambda_0 u(t-\tau))\|_2^2 dt.\end{aligned} \tag{2.37}$$

We have to estimate the last three integrals in the RHS of (2.37). For the second and third terms, observe that

$$\partial_x^j (\lambda_0 u) = \sum_{k=0}^{j} \binom{j}{k} \partial_x^k (\lambda_0) \partial_x^{j-k} (u)$$

and

$$\partial_x^j (\lambda u(\cdot, t-\tau)) = \sum_{k=0}^{j} \binom{j}{k} \partial_x^k (\lambda) \partial_x^{j-k} (u(\cdot, t-\tau)),$$

thus we have that

$$\begin{aligned}\int_0^T \|\partial_x^j (\lambda_0 u)(t)\|_2^2 dt \leq& \sum_{k=0}^{j} \int_0^T \binom{j}{k} \|\partial_x^k (\lambda_0) \partial_x^{j-k} (u)(t)\|_2^2 dt\\ =& \int_0^T \|\lambda_0\|_\infty^2 \|\partial_x^j (u)(t)\|_2^2 dt\\ &+ \sum_{k=1}^{j} \int_0^T \binom{j}{k} \|\partial_x^k (\lambda_0)\|_2^2 \|\partial_x^{j-k} (u)(t)\|_\infty^2 dt\\ \leq& C \|\lambda_0\|_{H^1(\mathbb{R})}^2 \int_0^T \|\partial_x^j (u)(t)\|_2^2 dt\\ &+ \sum_{k=1}^{j} \int_0^T \binom{j}{k} \|\lambda_0\|_{H^k(\mathbb{R})}^2 \|u(t)\|_{H^{1+j-k}(\mathbb{R})}^2 dt\\ \leq& CT \|\lambda_0\|_{H^j(\mathbb{R})}^2 \|u\|_{C([0,T];H^{2j+1}(\mathbb{R})}^2.\end{aligned} \tag{2.38}$$

Similarly, it follows that

$$\begin{aligned}\int_0^T \|\partial_x^j(\lambda_0 u)(t-\tau)\|_2^2 dt \leq & C\|\lambda_0\|_{H^1(\mathbb{R})}^2 \int_0^T \|\partial_x^j(u)(t-\tau)\|_2^2 dt \\ & + \sum_{k=1}^{j}\int_0^T \left(\begin{array}{c} j \\ k \end{array}\right) \|\lambda_0\|_{H^k(\mathbb{R})}^2 \|u(t-\tau)\|_{H^{1+j-k}(\mathbb{R})}^2 dt \\ = & C\|\lambda_0\|_{H^1(\mathbb{R})}^2 \int_{-\tau}^{T-\tau} \|\partial_x^j(u)(t)\|_2^2 dt \\ & + \sum_{k=1}^{j}\int_{-\tau}^{T-\tau} \left(\begin{array}{c} j \\ k \end{array}\right) \|\lambda_0\|_{H^k(\mathbb{R})}^2 \|u(t)\|_{H^{1+j-k}(\mathbb{R})}^2 dt \\ \leq & CT\|\lambda_0\|_{H^j(\mathbb{R})}^2 \left(\|u\|_{C([0,T];H^{2j+1}(\mathbb{R})}^2 + \int_{-\tau}^0 \|u_0(t)\|_{H^j(\mathbb{R})}^2 dt \right).\end{aligned} \tag{2.39}$$

Now, we concentrate in estimate the nonlinear term, that is, the first term in the RHS of (2.37). Observing that

$$\partial_x^j(u^p u_x) = \sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array}\right) \partial_x^k(u^p)\partial_x^{j-k}(u_x),$$

we get

$$\begin{aligned}\int_0^T \|\partial_x^j(u^p u_x)(t)\|_2^2 dt \leq & \sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array}\right) \int_0^T \|\partial_x^k(u^p)(t)\|_\infty^2 \|\partial_x^{j-k}(u_x)(t)\|_2^2 dt \\ = & \sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array}\right) \int_0^T \|\partial_x^k(u^p)(t)\|_\infty^2 \|u(t)\|_{H^{1+j-k}(\mathbb{R})}^2 dt \\ \leq & C\sum_{k=0}^{j} \left(\begin{array}{c} j \\ k \end{array}\right) \int_0^T \|\partial_x^k(u^p)(t)\|_\infty^2 \|u(t)\|_{H^{2j+1}(\mathbb{R})}^2 dt.\end{aligned}$$

On the other hand, note that the k-order derivative of the function u^p can be written in the following way

$$\partial_x^k(u^p) = C_{0,p}u^{p-1}\partial_x^k u + \sum_{n=1}^{k-1} C_{n,p}u^{p-n}F_n(u) + C_{k,p}u^{p-k}u_x^k,$$

where $C_{n,p}$ is a constant given by

$$C_{n,p} = M_n \prod_{i=0}^{n}(p-i)$$

with $M_n \in \mathbb{N}$ and $F_n(u)$ is a differential operator involving sums and products of derivatives of u with order less than $n+1$. Thanks to this fact, we have the following estimate:

$$\begin{aligned}\|\partial_x^k(u^p)\|_\infty \leq & |C_{0,p}|\|u\|_\infty^{p-1}\|\partial_x^k u\|_\infty + \sum_{n=1}^{k-1}|C_{n,p}|\|u\|_\infty^{p-n}\|F_n(u)\|_\infty + |C_{k,p}|\|u\|_\infty^{p-k}\|u_x\|_\infty^k \\ \leq & C\Bigg(|C_{0,p}|\|u\|_{H^{2j+1}(\mathbb{R})}^{p-1}\|u\|_{H^{k+1}(\mathbb{R})} + \sum_{n=1}^{k-1}|C_{n,p}|\|u\|_{H^{2j+1}(\mathbb{R})}^{p-n}\|u\|_{H^{n+2}(\mathbb{R})}^{m(k)} \\ & + |C_{k,p}|\|u\|_{H^{2j+1}(\mathbb{R})}^{p-k}\|u\|_{H^2(\mathbb{R})}^k \Bigg),\end{aligned}$$

where $m(n) \in \mathbb{N}$ and $C > 0$ is a constant. Since $H^{2j+1}(\mathbb{R}) \subset H^{k+1}(\mathbb{R})$, $H^{2j+1}(\mathbb{R}) \subset H^{n+1}(\mathbb{R})$ and $H^{2j+1}(\mathbb{R}) \subset H^{2}(\mathbb{R})$, it follows that

$$\|\partial_x^k(u^p)\|_\infty \leq C\left(|C_{0,p}|\|u\|^p_{H^{2j+1}(\mathbb{R})} + \sum_{n=1}^{k-1}|C_{n,p}|\|u\|^{p-n+m(k)}_{H^{2j+1}(\mathbb{R})} + |C_{k,p}|\|u\|^p_{H^{2j+1}(\mathbb{R})}\right).$$

Hence, it yields that

$$\begin{aligned}\int_0^T \|\partial_x^j(u^pu_x)(t)\|_2^2dt \leq & C|C_{0,p}|^2\int_0^T \|u(t)\|^{2(p+1)}_{H^{2j+1}(\mathbb{R})}dt \\ & + C\sum_{k=0}^{j}\sum_{n=0}^{k}\begin{pmatrix} j \\ k\end{pmatrix}|C_{n,p}|^2\int_0^T \|u(t)\|^{2(p-n+m(k)+1)}_{H^{2j+1}(\mathbb{R})}dt \\ & + C\sum_{k=0}^{j}\begin{pmatrix} j \\ k\end{pmatrix}|C_{k,p}|^2\int_0^T \|u(t)\|^{2(p+1)}_{H^{2j+1}(\mathbb{R})}dt.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\int_0^T \|\partial_x^j(u^pu_x)(t)\|_2^2dt \leq & CT\left(|C_{0,p}|^2\|u\|^{2(p+1)}_{C([0,T];H^{2j+1}(\mathbb{R}))}\right. \\ & + \sum_{k=0}^{j}\sum_{n=0}^{k}\begin{pmatrix} j \\ k\end{pmatrix}|C_{n,p}|^2\|u(t)\|^{2(p-n+m(k)+1)}_{C([0,T];H^{2j+1}(\mathbb{R}))} \\ & \left.+ \sum_{k=0}^{j}\begin{pmatrix} j \\ k\end{pmatrix}|C_{k,p}|^2\|u(t)\|^{2(p+1)}_{C([0,T];H^{2j+1}(\mathbb{R}))}\right).\end{aligned} \tag{2.40}$$

Therefore, putting the estimates (2.38), (2.39), and (2.40) together in (2.37), step 3 holds. Consequently, by the previous steps, the theorem is shown. ☐

2.5. **Interpolation arguments.** In this part of the work, we present the proof of the well-posedness result for the system (1.2). To do that, let us introduce an interpolation argument due to Tartar [33] and adapted by Bona and Scott [8, Theorem 4.3].

Let B_0 and B_1 be two Banach spaces, where $B_1 \subset B_0$ with the inclusion map continuous. Consider $f \in B_0$ and define

$$K(f,t) = \inf_{g\in B_1}\left\{\|f-g\|_{B_0} + t\|g\|_{B_1}\right\},$$

for $t \geq 0$. For $0 < \theta < 1$ and $1 \leq p \leq +\infty$, we introduce the set

$$\mathcal{B}_{\theta,p} := [B_0, B_1]_{\theta,p} = \left\{f \in B_0 : \|f\|_{\theta,p} : \left(\int_0^\infty K(f,t)t^{-\theta p-1}dt\right)^{\frac{1}{p}} < \infty\right\},$$

with the usual modification for the case $p = \infty$. Then, $B_{\theta,p}$ is a Banach space with norm $\|\cdot\|_{\theta,p}$. Given two pairs (θ_1, p_1) and (θ_2, p_2) as above, $(\theta_1, p_1) \prec (\theta_2, p_2)$ will denote

$$\begin{cases}\theta_1 < \theta_2, & \text{or} \\ \theta_1 = \theta_3, & \text{and } p_1 > p_2.\end{cases}$$

If $(\theta_1, p_1) \prec (\theta_2, p_2)$, then $\mathcal{B}_{\theta_2,p_2} \subset \mathcal{B}_{\theta_1,p_1}$ with the inclusion map continuous. Then, the following result, proved by [8, Theorem 4.3], holds.

Theorem 2.12. *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$ with continuous inclusion mappings, for $j = 1, 2$. Let α and q lie in the ranges $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Suppose that $\mathcal{A}$ is a mapping satisfying:*

(i) $\mathcal{A} : \mathcal{B}^1_{\alpha,q} \to B_0^2$ *and, for* $f, g \in \mathcal{B}^1_{\alpha,q}$,

$$\|\mathcal{A}f - \mathcal{A}g\|_{B_0^2} \leq C_0\left(\|f\|_{\mathcal{B}^1_{\alpha,q}} + \|g\|_{\mathcal{B}^1_{\alpha,q}}\right)\|f-g\|_{B_0^1}.$$

(ii) $\mathcal{A} : B_1^1 \to B_1^2$ *and, for* $h \in \mathcal{B}_1^1$,

$$\|\mathcal{A}h\|_{B_1^2} \le C_1\left(\|h\|_{\mathcal{B}^1_{\alpha,q}}\right)\|h\|_{B_1^1},$$

where $C_j : \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous nondecreasing functions, for* $j = 0, 1$. *Then, if* $(\theta, p) \ge (\alpha, q)$, $\mathcal{A}$ *maps* $\mathcal{B}^1_{\theta,p}$ *into* $\mathcal{B}^2_{\theta,p}$ *and, for* $f \in \mathcal{B}^1_{\theta,p}$, *we have*

$$\|\mathcal{A}f\|_{\mathcal{B}^2_{\theta,p}} \le C\left(\|f\|_{\mathcal{B}^1_{\alpha,q}}\right)\|f\|_{\mathcal{B}^1_{\theta,p}},$$

where $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^\theta$, *with* $r > 0$.

It follows from Theorem 2.9 that, for each fixed $T > 0$, the solution map

$$\mathcal{A} : C([-\tau, 0]; L^2(\mathbb{R})) \to \mathcal{B}_{0,T}, \quad \mathcal{A}u_0 = u \tag{2.41}$$

is well-defined. Moreover, we have the following result:

Proposition 2.13. *The solution map* (2.41) *is locally Lipschitz continuous, that is, there exists a continuous function* $\mathcal{C}_0 : \mathbb{R}^+ \times (0, \infty) \to \mathbb{R}^+$, *nondecreasing in its first variable, such that, for all* $u_0, v_0 \in C([-\tau, 0]; L^2(\mathbb{R}))$, *we have*

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,T} \le \mathcal{C}_0\left(\|u_0\|_2 + \|v_0\|_2, T\right)\|u_0 - v_0\|_{C([-\tau,0];L^2(\mathbb{R}))}.$$

Proof. Let $0 < \theta \le T$ and $n = \left[\frac{T}{\theta}\right]$. Theorem 2.9 ensures that

$$\|\mathcal{A}u_0\|_{\mathcal{B}_{0,\theta}} \le \beta_0(\|u_0\|_2)\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}, \tag{2.42}$$

where $\beta_0(s)$ is a constant function given by $\beta_0 = C_T\left(1 + \left(\|\lambda\|_\infty\tau^{1/2} + \|\lambda\|_\infty^{1/2}\right)\right)$ and

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \le C_\theta\left\{\|u_0 - v_0\|_2 + \|M(\mathcal{A}u_0) - M(\mathcal{A}v_0)\|_{L^1(0,\theta;L^2(\mathbb{R}))}\right\},$$

where $C_\theta = 2e^{\theta\|b\|_\infty}$. Moreover, thanks to the Lemma 2.6 we have

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,\theta}} \le& C_\theta\|u_0 - v_0\|_2 + C_\theta C\left\{2^{\frac{p}{2}}\theta^{\frac{2j-p}{4j}}\left(\|\mathcal{A}u_0\|^p_{\mathcal{B}_{0,\theta}} + \|\mathcal{A}u_0\|_{\mathcal{B}_{0,\theta}}\|\mathcal{A}v_0\|^{p-1}_{\mathcal{B}_{0,\theta}}\right.\right.\\ &\left.\left.+\|\mathcal{A}v_0\|^p_{\mathcal{B}_{0,\theta}}\right) + 2^{\frac{1}{2}}\theta^{\frac{1}{4}}\|\mathcal{A}u_0\|_{\mathcal{B}_{0,\theta}}\right\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,\theta}}.\end{aligned}$$

Now, inequality (2.42) together with the following estimate

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{0,\theta} \le& C_\theta\|u_0 - v_0\|_2 + C_\theta C\left\{2^{\frac{p}{2}}\theta^{\frac{2j-p}{4j}}\beta_0^p\left(\|u_0\|^p_{C([-\tau,0];L^2(\mathbb{R}))}\right.\right.\\ &\left.+\|\ u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}\|\ v_0\|^{p-1}_{C([-\tau,0];L^2(\mathbb{R}))} + \|\ v_0\|^p_{C([-\tau,0];L^2(\mathbb{R}))}\right)\\ &\left.+2^{\frac{1}{2}}\beta_0\theta^{\frac{1}{4}}\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}\right\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,\theta}},\end{aligned}$$

yields that

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,\theta}} \le 2C_T\|u_0 - v_0\|_2, \tag{2.43}$$

choosing θ small enough. Observe that denoting

$$B_{0,[k\theta,(k+1)\theta]} := C\left([k\theta, (k+1)\theta]; L^2(\mathbb{R})\right) \cap L^2(k\theta, (k+1)\theta; H^j(\mathbb{R})).$$

with norm $\|\cdot\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}}$, in a analogously way as done to (2.43), we can deduce thanks to estimate (2.42) that

$$\begin{aligned}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}} \le& C_\theta\|u(k\theta) - v(k\theta)\|_2 + C_\theta C\beta_0\left\{2^{\frac{p}{2}}\theta^{\frac{2j-p}{4j}}\beta_0^p\left(\|u_0\|^p_{C([-\tau,0];L^2(\mathbb{R}))}\right.\right.\\ &\left.+\|\ u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}\|\ v_0\|^{p-1}_{C([-\tau,0];L^2(\mathbb{R}))} + \|\ v_0\|^p_{C([-\tau,0];L^2(\mathbb{R}))}\right)\\ &\left.+2^{\frac{1}{2}}\beta_0\theta^{\frac{1}{4}}\|u_0\|_{C([-\tau,0];L^2(\mathbb{R}))}\right\}\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}}.\end{aligned}$$

Lastly, we get

$$\|\mathcal{A}u_0 - \mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}} \le 2C_T\|u(k\theta) - v(k\theta)\|_2, \quad k = 0, 1, ..., n-1. \tag{2.44}$$

On the other hand, note that (2.43) and (2.44) imply that

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}}\leq 2^kC_T^k\|u_0-v_0\|_2,\quad k=0,1,...,n-1,$$

and, therefore,

$$\|\mathcal{A}u_0-\mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}}\leq 2^nC_T^n\|u_0-v_0\|_2.$$

Finally,

$$\begin{aligned}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{\mathcal{B}_{0,T}}&\leq\sum_{k=0}^{n-1}\|\mathcal{A}u_0-\mathcal{A}v_0\|_{\mathcal{B}_{0,[k\theta,(k+1)\theta]}}\leq\sum_{k=0}^{n-1}2^nC_T^n\|u_0-v_0\|_2\\&\leq 2^nC_T^nn\|u_0-v_0\|_2\leq\mathcal{C}_0(\|u_0\|_2+\|v_0\|_2)\|u_0-v_0\|_{C([-\tau,0];L^2(\mathbb{R}))},\end{aligned}$$

where $\mathcal{C}_0(s)=\frac{T}{\theta(s)}\left[2C_T\right]^{\frac{T}{\theta(s)}}$, getting the proposition. □

We are in a position to prove the main result of this section.

Proof of Theorem 1.1. We define

$$B_0^1=L^2(\mathbb{R}),\quad B_0^2=\mathcal{B}_{0,T},\quad B_1^1=H^{2j+1}(\mathbb{R})\quad\text{and}\quad B_1^2=\mathcal{B}_{2j+1,T}.$$

Thus,

$$\mathcal{B}^1_{\frac{s}{2j+1},2}=[L^2(\mathbb{R}),H^{2j+1}(\mathbb{R})]_{\frac{s}{2j+1},2}=H^s(\mathbb{R})\quad\text{and}\quad\mathcal{B}^2_{\frac{s}{2j+1},2}=[B_{0,T},B_{2j+1,T}]_{\frac{s}{2j+1},2}=B_{s,T}.$$

Combining Proposition 2.13 and Theorem 2.11 we obtain (i) and (ii) in the Theorem 2.12. Then, Theorem 2.12 gives the existence of the solution to the equation (1.2), and Theorem 1.1 follows. □

Finally, we present the proof of the Corollary 1.3, which establishes that every mild solution of system (1.2) is indeed a regular solution when the origin is not considered.

Proof of Corollary 1.3. The result is shown by using the bootstrap argument. Consider $T>0$ and $0<\varepsilon<T$. So, for $u_0\in C([-\tau,0];L^2(R))$, it follows from Theorem 1.1 that the problem (1.2) has a unique solution

$$u\in\mathcal{B}_{0,T}=C([0,T];L^2(\mathbb{R}))\cap L^2(0,T;H^j(\mathbb{R})).$$

So, we have that $u(t)\in H^j(\mathbb{R})$ for almost every $t\in[0,T]$. Let $t_0\in(0,\varepsilon)$ such that $u(t_0)\in H^j(\mathbb{R})$. Applying Theorem 1.1 and Remark 1.2, with $u_0=u(t_0)$, we conclude that the restriction of u to $[t_0,T]$ is the solution of (1.2) with the initial data $u(t_0)$, and belongs to the following class

$$\mathcal{B}_{j,[t_0,T]}=C([t_0,T];H^j(\mathbb{R}))\cap L^2(t_0,T;H^{2j}(\mathbb{R})).$$

So, we have that $u(t)\in H^{2j}(\mathbb{R})$ for almost every $t\in[t_0,T]$. Let $t_1\in(t_0,\varepsilon)$ such that $u(t_1)\in H^{2j}(\mathbb{R})$. Again, it follows from Theorem 1.1 and Remark 1.2 that the restriction of u to $[t_1,T]$ is the solution of (1.2), with respect to the initial data $u(t_1)$ belongs to

$$\mathcal{B}_{2j,[t_1,T]}=C([t_1,T];H^{2j}(\mathbb{R}))\cap L^2(t_1,T;H^{3j}(\mathbb{R})).$$

Finally, since $u(t)\in H^{3j}(\mathbb{R})$ for almost every $t\in[t_1,T]$ and $H^{3j}(\mathbb{R})\subset H^{2j+1}(\mathbb{R})$, it follows the same way that for $t_2\in(t_1,\varepsilon)$ such that $u(t_2)\in H^{2j+1}(\mathbb{R})$, the restriction of u to $[t_2,T]$ is the solution of (1.2), with respect to the initial data $u(t_2)$ belonging to the following set

$$\mathcal{B}_{2j+1,[t_2,T]}:=C([t_2,T];H^{2j+1}(\mathbb{R}))\cap L^2(t_2,T;H^{3j+1}(\mathbb{R})),$$

and the corollary holds. □

3. Exponential stabilization: Damping and delayed system

This section is devoted to proving the exponential stabilization for the system (2.1) in $L^2(\mathbb{R})$. In this way, first, we prove the result for the linear system and, after that, we extend to the nonlinear one. To do that consider the following Lyapunov functional

$$E(t) := E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx \tag{3.1}$$

and, for $\lambda \in L^\infty(\mathbb{R})$, remember the definition of $\mathcal{E}(\cdot)$ which is given by (2.18).

3.1. Linear system.

The first result ensures the exponential stability for the linear system (2.1) with localized damping and delay terms and can be read as follows.

Theorem 3.1. *Let $\lambda, \lambda_0 \in L^\infty(\mathbb{R})$ and λ_0 be constants satisfying (1.7). If there is a constant $\gamma > 0$ and a function $\beta \in L^p(\mathbb{R})$, for $1 \leq p < \infty$, such that the function λ verifies (1.8) and (1.9), then the system (2.1) is exponentially stable. Particularly, solutions u of (2.1) satisfies the following inequality*

$$\mathcal{E}(t) \leq C(u_0)e^{-\nu t}, \tag{3.2}$$

where ν and $C(u_0)$ are defined by (1.10) and (1.11), respectively.

Proof. Using the same argument as done in [12, Theorem 4.7] we consider $u_0 \in H^{2j+1}$, then $u \in H^{2j+1}$. Taking the derivative in t of $\mathcal{E}(t)$ we get

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) =& \int_{\mathbb{R}} u(t)(-(-1)^j\partial_x^{2j}u(t) - \lambda_0 u(t) - \lambda u(t-\tau))dx + \frac{1}{2}\int_{\mathbb{R}}|\lambda|u^2(t)dx\\ &-\frac{1}{2}e^{-\tau}\int_{\mathbb{R}}|\lambda|u^2(t-\tau)dx - \frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dx\,ds,\end{aligned}$$

since we have

$$\int_{\mathbb{R}}(-1)^{j+1}(\partial_x^{2j+1}u)udx = 0. \quad \text{for all} \quad u \in H^{2j+1}.$$

So, integrating by parts, and using the Young inequality, remembering that (1.7) and (1.8) are satisfied, we have

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq& -\int_{\mathbb{R}}(\partial_x^j u)^2(t)dx - \gamma_0\int_{\mathbb{R}} u^2(t)dx + \frac{e^\tau+1}{2}\int_{\mathbb{R}}|\lambda(x)|u^2(t)dx\\ &-\frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds\\ \leq& -\int_{\mathbb{R}}(\partial_x^j u)^2(t)dx - (\gamma_0-\gamma)\int_{\mathbb{R}} u^2(t)dx + \int_{\mathbb{R}}\beta(x)u^2(t)dx\\ &-\frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)\ dx\ ds,\end{aligned}$$

thanks to the fact that

$$(-1)^j\int_{\mathbb{R}} u\partial_x^{2j}udx = \int_{\mathbb{R}}(\partial_x^j u)^2dx.$$

The Hölder inequality ensures that

$$\frac{d\mathcal{E}}{dt}(t) \leq -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + \|\beta\|_q\|u\|_{2q'}^2 - \frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dx\,ds, \tag{3.3}$$

with $q' = \frac{q}{q-1}$. Observing that

$$\|u\|_{2q'}^2 = \left(\int_{\mathbb{R}}(u(t))^{2q'}dx\right)^{\frac{1}{q'}} = \left(\int_{\mathbb{R}} u^2(t)(u(t))^{\frac{2}{q-1}}dx\right)^{\frac{1}{q'}} \leq \|u\|_2^{\frac{2}{q'}}\|u\|_\infty^{\frac{2}{q'(q-1)}} = \|u\|_2^{\frac{2}{q'}}\|u\|_\infty^{\frac{2}{q}} \tag{3.4}$$

and using (3.4) in (3.3), yields that

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + \|\beta\|_q \|u\|_2^{\frac{2}{q'}} \|u(t)\|_\infty^{\frac{2}{q}} \\ & - \frac{1}{2}\int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dx\,ds.\end{aligned} \tag{3.5}$$

Finally, note that the following inequality holds true

$$\|v\|_\infty^2 \leq 2\|v\|_2\|\partial_x^j v\|_2, \tag{3.6}$$

for all $v \in H^j(\mathbb{R})$. Indeed, if $v \in C_c^\infty(\mathbb{R})$ and $y \in \mathbb{R}$, then we have the following inequality:

$$\left|v(y)^2\right| = \left|\int_{-\infty}^y 2vv_x dx\right| \leqslant 2\int_{-\infty}^\infty |v|\cdot|v_x|\,dx \leqslant 2\|v\|_2\,\|v_x\|_2\,,$$

since for $j \geq 1$ we have the following embedding $H^j(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$, proving our estimate for smooth functions. The general case follows by density. So, by Young inequality, from (3.5) and using (3.6), we have for every fixed $\delta > 0$ the following

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + 2^{1/q}\|\beta\|_q \|u\|_2^{\frac{2q-1}{q}}\, \|u_x\|_2^{\frac{1}{q}} \\ & - \frac{1}{2}\int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds \\ \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + \left(\frac{1}{\delta}\|\beta\|_q\|u\|_2^{\frac{2q-1}{q}}\right)\left(\delta 2^{\frac{1}{q}}\|\partial_x^j u_x\|_2^{\frac{1}{q}}\right) \\ & - \frac{1}{2}\int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds \\ \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + \frac{\left(\frac{1}{\delta}\|\beta\|_q\|u\|_2^{\frac{2q-1}{q}}\right)^{\frac{2q}{2q-1}}}{\frac{2q}{2q-1}} + \frac{\left(\delta 2^{\frac{1}{q}}\|\partial_x^j u_x(t)\|_2^{\frac{1}{q}}\right)^{2q}}{2q} \\ & - \frac{1}{2}\int_{t-\tau}^t \int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds.\end{aligned}$$

Pick $\delta > 0$ such that $4\delta^{2q} = 2q$, which gives

$$\frac{d\mathcal{E}}{dt}(t) \leq -\left(\gamma_0 - \gamma - \frac{2q-1}{2q}\left(\frac{2}{q}\right)^{\frac{1}{2q-1}}\|\beta\|_p^{\frac{2q}{2q-1}}\right)\|u(t)\|_2^2 - \frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds.$$

Thus, taking into account (1.9) we have

$$\frac{d\mathcal{E}}{dt}(t) \leq -\nu\mathcal{E}(t)$$

where ν is as in (1.10). To finish, estimate (3.2) is a direct consequence of the Gronwall's Lemma considering $C(u_0) = \mathcal{E}(0)$. □

3.2. **Nonlinear system.** In this section, we are interested to prove that the higher-order dispersive system in an unbounded domain is asymptotically stable when we introduce a localized damping mechanism and a delay term. Precisely, we will use the Lyapunov approach to prove that the energy $\mathcal{E}(t)$ defined by (2.18) tends to 0 as t goes to ∞.

Observe that we can consider the function $\lambda_0, \lambda \in L^\infty(\mathbb{R})$ satisfying (1.7), (1.8) and (1.9). Thus, in this case, we have a simple situation that can be easily proved using the arguments done in the previous section.

Proof of Theorem 1.4. The proof of this result is a consequence of the Lyapunov functional defined in (3.1) and is analogous to what was done in Theorem 3.1, so we will omit it. □

3.3. **Indefinite damping case.** In this section, our issue is to see what happens with general dissipative damping. To be precise, consider the coefficient λ_0 changing sign. Let us assume, that there exist a number $\gamma > 0$ and a function $\beta \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$, such that (1.12) and (1.13) are satisfied. In this spirit, we can prove the following asymptotic result for the solutions of the linearized system associated with (1.2), for $m = j$.

Theorem 3.2. *Consider $\lambda, \lambda_0 \in L^\infty(\mathbb{R})$, with λ_0 satisfying (1.12) and (1.13). If there exist a constant $\gamma > 0$ and a function $\beta \in L^p(\mathbb{R})$, with the same p as in (1.13), such that the function λ satisfies (1.8) and (1.14), so the system*

$$\begin{cases} u_t(x,t) + (-1)^{j+1}\partial_x^{2j+1}u(x,t) + (-1)^j\partial_x^{2j}u(x,t) + \lambda_0 u(x,t) & \\ \qquad\qquad +\lambda u(x,t-\tau) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,s) = u_0(x,s) & \text{in } \mathbb{R}\times[-\tau,0] \end{cases} \tag{3.7}$$

is exponentially stable. Moreover, the solution of (3.7) satisfy

$$\mathcal{E}(t) \leq C(u_0,\tau)e^{-\tilde{\nu}t}, \tag{3.8}$$

with $\tilde{\nu}$ defined by (1.15).

Proof. Taking the derivative in time of $\mathcal{E}(t)$, using the equation (3.7), after that, integrating by parts and using the Young inequality we get

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) \leq & -\int_{\mathbb{R}}(\partial_x^j u)^2(t)dx - \gamma_0\int_{\mathbb{R}}u^2(t)dx + \int_{\mathbb{R}}(\beta_0(x) + \frac{e^\tau+1}{2}|\lambda(x)|)u^2(t)dx \\ & -\frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}}e^{-(t-s)}|\lambda(x)|u^2(x,s)dxds \\ \leq & -\int_{\mathbb{R}}(\partial_x^j u)^2(t)dx - (\gamma_0-\gamma)\int_{\mathbb{R}}u^2(t)dx + \int_{\mathbb{R}}(\beta_0(x)+\beta(x))u^2(t)dx \\ & -\frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}}e^{-(t-s)}|\lambda(x)|u^2(x,s)\ dx\ ds, \end{aligned}$$

Hölder inequality ensures that

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 \\ & + \|\beta_0+\beta\|_p\|u\|_{2q}^2 - \frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}}e^{-(t-s)}|\lambda|u^2(x,s)dxds, \end{aligned} \tag{3.9}$$

with $q = \frac{p}{p-1}$. Thanks to the inequality (3.9), we deduce that

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 \\ & + \|\beta+\beta_0\|_p\|u(t)\|_\infty^{\frac{2}{p}}\|u\|_2^{\frac{2}{q}} - \frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}}e^{-(t-s)}|\lambda|u^2(x,s)dxds. \end{aligned}$$

Remember that (3.6) is still valid, so thanks to the Young inequality, we obtain

$$\begin{aligned} \frac{d\mathcal{E}}{dt}(t) \leq & -\|\partial_x^j u(t)\|_2^2 - (\gamma_0-\gamma)\|u(t)\|_2^2 + \frac{\left(\frac{1}{\delta}\|\beta_0+\beta\|_p\|u\|_2^{\frac{2p-1}{p}}\right)^{\frac{2p}{2p-1}}}{\frac{2p}{2p-1}} + \frac{\left(\delta 2^{\frac{1}{p}}\|\partial_x^j u\|_2^{\frac{1}{p}}\right)^{2p}}{2p} \\ & -\frac{1}{2}\int_{t-\tau}^t\int_{\mathbb{R}}e^{-(t-s)}|\lambda|u^2(x,s)dxds, \end{aligned}$$

for every fixed $\delta > 0$. Now, taking $\delta > 0$ such that $4\delta^{2p} = 2p$, yields that

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(t) \leq & -\left(\gamma-\gamma_0-\frac{2p-1}{2p}\left(\frac{2}{p}\right)^{\frac{1}{2p-1}}\|\beta_0+\beta\|_p^{\frac{2p}{2p-1}}\right)\|u(t)\|_2^2 \\ & -\frac{1}{2}\int_{t-\tau}^{t}\int_{\mathbb{R}} e^{-(t-s)}|\lambda| u^2(x,s)dxds. \end{aligned} \tag{3.10}$$

Finally, under the assumption (1.14) and taking in consideration (3.10), we get

$$\frac{d\mathcal{E}}{dt}(t) \leq -\tilde{\nu}\mathcal{E}(t)$$

where

$$\tilde{\nu} = 2\Big(\gamma_0-\gamma-\frac{2p-1}{2p}\left(\frac{2}{p}\right)^{\frac{1}{2p-1}}\|\beta+\beta_0\|_p^{\frac{2p}{2p-1}}\Big).$$

This give us the exponential estimate (3.8) for the solution of (3.7), with $C(u_0,\tau)$ defined as in (1.11). □

Directly, we can extend the well-posedness and the stability results for the nonlinear setting showing Theorem 1.5. This is a consequence of the previous theorem and the results presented in Subsection 2.4.1. With this in hand, let us prove the Corollary 1.6.

Proof of Corollary 1.6. Note that, after a change of variable, the restriction of u to $[t, t+T]$ is a solution to the problem (1.2) concerning the initial data $u(t)$. Observe also that $u \in C([\tau_1, t]; L^2(\mathbb{R}))$ for all $\tau_1 \in [-\tau, t)$. Thus, by Theorem 2.9, we have

$$\|u\|_{\mathcal{B}_{0,[t,t+T]}} \leq C_{t+T}\left\{\|u(t)\|_2+\|\lambda\|_\infty\|u\|_{L^1(t-\tau,t;L^2(\mathbb{R}))}+\|\lambda\|_\infty^{1/2}\|u\|_{L^2(t-\tau,t;L^2(\mathbb{R}))}\right\},$$

where C_s is given by

$$C_s = \sqrt{\frac{3}{2}}\left(1+e^{2\|\lambda\|_\infty s}\right)^{1/2} e^{(\|\lambda\|_\infty+\|\lambda_0\|_\infty))s}.$$

Thus, from Theorem 1.5 it follows that

$$\|u\|_{\mathcal{B}_{0,[t,t+T]}} \leq C_{2T}\left\{2C(u_0,\tau)e^{-\nu t}+\|\lambda\|_\infty\|u\|_{L^1(t-\tau,t;L^2(\mathbb{R}))}+\|\lambda\|_\infty^{1/2}\|u\|_{L^2(t-\tau,t;L^2(\mathbb{R}))}\right\}$$

where $C(u_0,\tau)$ is given by (1.11), giving the proof of the corollary. □

4. Exponential stabilization: Damping system

In this section, we establish the exponential stability in the space $H^s(\mathbb{R})$, for $s \in [0, 2j+1]$, for the general dispersive system (1.2) without the time delay term ($\tau = \lambda = 0$):

$$\begin{cases} u_t(x,t)+(-1)^{j+1}\partial_x^{2j+1}u(x,t)+(-1)^m\partial_x^{2m}u(x,t)+\lambda_0(x)u(x,t) & \\ \qquad\qquad +\frac{1}{p+1}\partial_x u^{p+1}(x,t)=0, & \text{in} \quad \mathbb{R}\times(0,\infty), \\ u(x,0)=u_0(x), & \text{in} \quad \mathbb{R}, \end{cases} \tag{4.1}$$

with $m \leq j$, $j, m \in \mathbb{N}$, and $1 \leq p < 2j$.

Remarks 4.1. It is important to point out that:

i. The strategy to obtain the stabilization results in $H^s(\mathbb{R})$, for any $s \in [0, 2j+1]$, will be to prove the result in $L^2(\mathbb{R})$, and after that, to prove the results in the domain of the operator, that is, $H^{2j+1}(\mathbb{R})$. So, with these two results in hand, we employ the interpolation results due to J.-L. Lions [25] to get the exponential decay for any $s \in [0, 2j+1]$.
ii. Note that when $\tau = \lambda = 0$, the stabilization for the system (4.1) in $L^2(\mathbb{R})$ holds thanks to the Theorem 1.4 with localized damping λ_0 (see Section 3).

4.1. **Stabilization in $H^{2j+1}(\mathbb{R})$.** Note that due to the Remarks 4.1, we will just consider the stabilization problem associated with the solutions of (4.1) in the space $H^{2j+1}(\mathbb{R})$.

Proposition 4.2. *Let $T>0$. For $1\leq p<2j$, with $j\geq 1$ and λ_0 satisfying (1.7) or (1.12), there exist $\gamma>0$, $T_0>0$ and a nonnegative continuous function $\alpha_3:\mathbb{R}^+\to\mathbb{R}^+$, such that, for every $u_0\in H^{2j+1}(\mathbb{R})$, the corresponding solution u of (4.1) satisfies*

$$\|u(t)\|_{H^{2j+1}(\mathbb{R})}\leq \alpha_{2j+1}(\|u_0\|_2,T_0)\|u_0\|_{H^{2j+1}(\mathbb{R})}e^{-\gamma t},\qquad \forall t\geq T_0. \tag{4.2}$$

Proof. Firstly, note that there exists a positive constant c such that the following estimate holds

$$\frac{1}{c}\|u(t)\|_{H^{2j+1}(\mathbb{R})}\leq \|u(t)\|_2+\|\partial_x^{2j+1}u(t)\|_2,$$

thus, from Theorem 1.5, it follows that

$$\frac{1}{c}\|u(t)\|_{H^{2j+1}(\mathbb{R})}\leq 2C(u_0,\tau)e^{-\tilde{\nu}t}+\|\partial_x^{2j+1}u(t)\|_2,\quad \text{for all } t>0, \tag{4.3}$$

with $\tilde{\nu}$ defined by (1.15) and $C(u_0,\tau)>0$ as in (1.11). Since $\tau=0$, then $C(u_0,0)=\frac{1}{2}\|u_0\|_2^2$. The inequality (4.3) shows that we just need to establish an exponential estimate for the $2j+1$ derivative in space of $u(t)$. To do that, observe that

$$\|\partial_x^{2j+1}u(t)\|_2\leq\|u_t(t)\|+\|\partial_x^{2m}u(t)\|_2+\|u^pu_x(t)\|_2+\|\lambda_0u(t)\|_2,\forall t>0.$$

From Gagliardo-Neirenberg inequality, it yields that

$$\|\partial_x^{2j+1}u(t)\|_2\leq\|u_t(t)\|+\|\partial_x^{2j+1}u(t)\|_2^{\frac{2m}{2j+1}}\|u(t)\|_2^{1-\frac{2m}{2j+1}}+\|u(t)\|_\infty^p\|u_x(t)\|_2+\|\lambda_0\|_\infty\|u(t)\|_2.$$

Young inequality together with Theorem 1.5, when $\lambda=\tau=0$, give that

$$\begin{aligned}\frac{1}{2}\|\partial_x^{2j+1}u(t)\|_2&\leq\|u_t(t)\|+C\|u(t)\|_2+2^{\frac{p}{2}}\|u\|_2^{\frac{p}{2}}\|\partial_xu\|_2^{\frac{p+2}{2}}+\|\lambda_0\|_\infty\|u(t)\|_2\\&\leq\|u_t(t)\|+(C+\|\lambda_0\|_\infty)\|u(t)\|_2+2^{\frac{p}{2}}C\|\partial_x^{2j+1}u\|_2^{\frac{p+2}{2(2j+1)}}\|u\|_2^{\frac{p+2}{2}\left(1-\frac{1}{2j+1}\right)+\frac{p}{2}}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{1}{4}\|\partial_x^{2j+1}u(t)\|_2&\leq\|u_t(t)\|+(C+\|\lambda_0\|_\infty)\|u(t)\|_2+C\|u\|_2^{\left(\frac{p+2}{2}\left(1-\frac{1}{2j+1}\right)+\frac{p}{2}\right)\left(\frac{2(2j+1)}{4j-p}\right)}\\&\leq\|u_t(t)\|+2^{\frac{1}{2}}C(u_0,0)^{\frac{1}{2}}(C+\|\lambda_0\|_\infty)e^{-\frac{\tilde{\nu}}{2}t}+2^{\frac{1}{2}}C(u_0,0)^{\frac{1}{2}}e^{-\frac{\tilde{\nu}}{2}\left(\frac{4j(p+1)+p}{4j-p}\right)t}.\end{aligned}\tag{4.4}$$

Let $v=u_t$. Then, by Proposition 2.10, with $\lambda=\tau=0$, v solves linearized equation (2.19) with initial data $v_0(x)=-(-1)^{j+1}\partial_x^{2j+1}u_0+(-1)^m\partial_x^{2m}u_0-\frac{1}{p+1}\partial_x(u_0^{p+1})-\lambda_0(x)u_0$ such that

$$\|v\|_{\mathcal{B}_T}\leq\sigma(\|u\|_{\mathcal{B}_T},T)\|v(0)\|_2. \tag{4.5}$$

Now, after a change of variable, the restriction of v to $[t,t+T]$ is a solution of the system (2.19), considering $\lambda=\tau=0$, concerning the initial data $v(t)$, thus

$$\|v\|_{\mathcal{B}_{[t,t+T]}}\leq\sigma(\|u\|_{\mathcal{B}_{[t,t+T]}},T)\|v(t)\|_2. \tag{4.6}$$

Applying Corollary 1.6, it follows that

$$\|v\|_{\mathcal{B}_{[t,t+T]}}\leq\sigma\Big(C_{2T}\left\{2C(u_0,0)e^{-\nu t}\right\},T\Big)\|v(t)\|_2.$$

Therefore, we obtain

$$\|v\|_{\mathcal{B}_{[t,t+T]}}\leq\gamma(u_0,t,T)\|v(t)\|_2,$$

where $\gamma(s,t,T)=\sigma\left(2C_{2T}C(s,0)\right),T)$. On the other hand, the solution v may be written as

$$v(t)=S(t)v_0-\int_0^tS(t-s)[u^p(s)v(s)]_xds$$

where $S(t)$ is a C_0-semigroup of contraction in $L^2(\mathbb{R})$ generated by the operator associated to system (2.19). Note that $v_1(t) = S(t)v_0$ is solution of the problem (2.19) with $u^p = 0$. Then, proceeding as in the proof of Theorem 3.2 with $\lambda = \tau = 0$, we have

$$\|v_1(t)\|_2 \leq 2^{1/2}C(v_0,0)^{1/2}e^{-\frac{\nu}{2}t}, \qquad \forall t \geq 0, \tag{4.7}$$

with ν defined by (1.15). Denote

$$v_2(t) = \int_0^t S(t-s)[u^p(s)v(s)]_x ds.$$

Note that

$$\|v_2(T)\|_2 \leq \|pu^{p-1}u_x v\|_{L^1(0,T;L^2(\mathbb{R}))} + \|u^p v_x\|_{L^1(0,T;L^2(\mathbb{R}))}.$$

So, thanks to the Lemma 2.6, the following holds

$$\|v_2(T)\|_2 \leq 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u\|^p_{\mathcal{B}_{0,T}}\|v\|_{\mathcal{B}_{0,T}}. \tag{4.8}$$

Using (4.5), (4.7) and (4.8), we obtain

$$\|v(T)\|_2 \leq \|v_1(T)\|_2 + \|v_1(T)\|_2 \leq 2^{1/2}C(v_0,0)^{1/2}e^{-\frac{\nu}{2}t} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u\|^p_{\mathcal{B}_{0,T}}\|v\|_{\mathcal{B}_{0,T}}.$$

Let us now consider a sequence $y_n(\cdot) = v(\cdot, nT)$ and introduce $w_n(\cdot,t) = v(\cdot, t+nT)$. For $t \in [0,T]$, w_n solves the problem

$$\begin{cases} \partial_t w_n + (-1)^{j+1}\partial_x^{2j+1}w_n + (-1)^m\partial_x^{2m}w_n + [u(\cdot+nT)^p w_n]_x + \lambda_0 w_n = 0, & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ w_n(0) = y_n, & \text{in } \mathbb{R}. \end{cases}$$

Observe that we can obtain for y_n an estimate similar to the one obtained for $v(T)$, namely

$$\begin{aligned} \|y_{n+1}\|_2 = \|w_n(T)\|_2 &\leq \|S(T)y_n\|_2 + \left\|\int_0^T S(T-s)[[u(s+nT)^p w_n(s)]_x ds\right\|_2 \\ &\leq 2^{1/2}C(v(\cdot+nT),0)^{1/2}e^{-\frac{\nu}{2}T} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u(\cdot+nT)\|^p_{\mathcal{B}_{0,T}}\|w_n\|_{\mathcal{B}_{0,T}} \\ &\leq 2^{1/2}C(v(\cdot+nT),0)^{1/2}e^{-\frac{\nu}{2}T} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u\|^p_{\mathcal{B}_{0,[nT,(n+1)T]}}\|v\|_{\mathcal{B}_{0,[nT,(n+1)T]}}. \end{aligned}$$

Thus, (4.6) implies that

$$\|y_{n+1}\|_2 \leq 2^{1/2}C(v(\cdot+nT),0)^{1/2}e^{-\frac{\nu}{2}T} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u\|^p_{\mathcal{B}_{0,[nT,(n+1)T]}}\sigma(\|u\|_{\mathcal{B}_{[nT,(n+1)T]}},T)\|y_n\|_2$$

where

$$C(v(\cdot+nT),0) = \frac{1}{2}\|y_n\|_2^2,$$

getting that

$$\|y_{n+1}\|_2 \leq \left(e^{-\frac{\nu}{2}T} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\|u\|^p_{\mathcal{B}_{0,[nT,(n+1)T]}}\sigma(\|u\|_{\mathcal{B}_{[nT,(n+1)T]}},T)\right)\|y_n\|_2. \tag{4.9}$$

Moreover, we can choose $\beta > 0$, small enough, such that

$$e^{-\frac{\nu}{2}T} + 2^{\frac{p}{2}}T^{\frac{2j-p}{4j}}\beta^p\beta\sigma(\beta,T) < 1.$$

With this choice of β, Corollary 1.6 allows us to pick $N > 0$, large enough, satisfying

$$\|u\|_{0,[nT,(n+1)T]} \leq 2C_{2T}C(u_0,0)e^{-\nu nT} \leq 2C_{2T}C(u_0,0)e^{-\nu NT} \leq \beta, \qquad \forall n > N.$$

Thus, from (4.9) we obtain the following estimate

$$\|y_{n+1}\|_2 \leq r\|y_n\|_2, \qquad \forall n \geq N, \quad \text{where } 0 < r < 1,$$

which implies

$$\|v((n+k)T)\|_2 \leq r^k\|v(nT)\|_2, \qquad \forall n \geq N. \tag{4.10}$$

Now, pick $T_0 = NT$ and $t \geq T_0$. Then, there exists $k \in \mathbb{N}$ and $\theta \in [0,T]$, satisfying

$$t = (N+k)T + \theta.$$

Therefore, from (4.6) and (4.10), it is follows that

$$\begin{aligned}\|v(t)\|_2 &\le \|v\|_{0,[(N+k)T,(N+k+1)T]} \le \gamma(u_0,t,T)\|v((N+k)T)\|_2\\ &\le \gamma(u_0,t,T)r^{\frac{t-NT-\theta}{T}}\|v(T_0)\|_2\\ &\le \gamma(u_0,t,T)r^{\frac{t-NT-\theta}{T}}\sigma((\|u\|_{\mathcal{B}_{T_0}},T_0))\|v(0)\|_2\\ &\le \eta_1(\|u_0\|)e^{-\delta_1 t}\|v_0\|_2,\end{aligned}$$

where $\delta_1=\frac{1}{T}\ln\left(\frac{1}{r}\right)$ and $\eta_1(s)=\gamma(u_0,t,T)\sigma(\beta_0(s,T_0))r^{-(N+1)}$. So,

$$\|u_t(t)\|_2 \le C\eta_1(\|u_0\|_2)\|u_0\|_{H^{2j+1}(\mathbb{R})}e^{-\delta_1 t}, \qquad \forall t\ge T_0.$$

Finally, from (4.4) and since $C(u_0,0)^{1/2}\le C\|u_0\|_{H^{2j+1}}$, we have

$$\begin{aligned}\frac{1}{4}\|\partial_x^{2j+1}u(t)\|_2 \le& C\eta_1(\|u_0\|_2)\|u_0\|_{H^{2j+1}(\mathbb{R})}e^{-\delta_1 t}\\ &+(C+\|\lambda_0\|_\infty)\,e^{-\frac{\tilde\nu}{2}t}\|u_0\|_{H^{2j+1}}+e^{-\frac{\tilde\nu}{2}\left(\frac{4j(p+1)+p}{4j-p}\right)t}\|u_0\|_{H^{2j+1}},\end{aligned}$$

showing the inequality (4.2). □

4.2. **Stabilization in $H^s(\mathbb{R})$.** Lastly, let us present the proof of the third main result of this work.

Proof of Theorem 1.7. We already know that there exists a unique solution u of the system (1.2) in the class $\mathcal{B}_{s,T}$ for every $T>0$. Moreover, if $0<\varepsilon<T$ then it follows from Corollary 1.3, $u\in\mathcal{B}_{2j+1,[\varepsilon,T]}$. On the other hand, from the interpolation inequality [25, inequality (2.43)], we have

$$\|u(t)\|_{H^s(\mathbb{R})}=\|u(t)\|_{[L^2(\mathbb{R}),H^{2j+1}(\mathbb{R})]_{2,\frac{s}{2j+1}}}\le C\|u(t)\|_2^{1-\frac{s}{2j+1}}\|u(t)\|_{H^{2j+1}(\mathbb{R})}^{\frac{s}{2j+1}}, \quad \forall t\ge\varepsilon,$$

where $C>0$ is a constant that comes from the interpolation argument. Thus, Proposition 4.2 and Theorems 1.4 and 1.5, with $\lambda=\tau=0$, imply the existence of $T_0>0$, $\nu>0$ and $\gamma>0$ such that

$$\|u(t)\|_{H^s(\mathbb{R})}\le Ce^{-\left(1-\frac{s}{2j+1}\right)\nu t}\|u_0\|_2^{\left(1-\frac{s}{2j+1}\right)}\alpha_{2j+1}^{\frac{s}{2j+1}}(\|u(\varepsilon)\|_2,T_0)\|u(\varepsilon)\|_{H^{2j+1}(\mathbb{R})}^{\frac{s}{2j+1}}e^{-\frac{s}{2j+1}\gamma t}, \qquad \forall t\ge T_0.$$

Hence, (1.16) holds with

$$\eta(s)=\left(1-\frac{s}{2j+1}\right)\nu+\frac{s}{2j+1}\gamma>0$$

and

$$\gamma(s,T_0)=Cs^{-\frac{s}{2j+1}}\alpha_3^{\frac{s}{2j+1}}(\|u(\varepsilon)\|_2,T_0)\|u(\varepsilon)\|_{H^{2j+1}}^{\frac{s}{2j+1}},$$

and the result is shown. □

5. Concluding remarks

In this work, we gave a more general framework to treat stabilization problems for a general higher-order dispersive system, which extends several previous results that appear in the literature. So, in terms of generality, we can consider the nonlinear general differential operator

$$\mathcal{V}u:=(-1)^{j+1}\partial_x^{2j+1}u(x,t)+(-1)^m\partial_x^{2m}u(x,t)+\frac{1}{p+1}\partial_x u^{p+1}(x,t), \tag{5.1}$$

with $1\le p<2j$, instead of the typical KdV equation (or co-related systems) as is usual in the literature[2]. Thus, summarizing, we studied the asymptotic behavior of the equation (1.2) posed on

[2]One may generalize the linear operator associated to $\mathcal{V}$ as

$$\mathcal{V}u=\sum_{m=0}^{j}\alpha_m\partial_x^{2m+1}u+\sum_{m=0}^{j}\beta_m\partial_x^{2m}u,$$

where $\alpha_m,\beta_m\in\mathbb{R}$. However, the main analyses in the paper are almost analogous without additional difficulties, thus the operator (5.1) does not lose the generality in a sense of the aim of this paper.

an unbounded domain $\mathbb{R}$ with the constant $\tau > 0$ as a time delay, and with the coefficients

$$\lambda_0(x), \lambda(x) \in L^\infty(\mathbb{R}) \quad \text{and} \quad j, m \in \mathbb{N}.$$

Considering $p = 1$, when we have $j = m = 1$, in (5.1), we recover the result proved in [21] for the KdV-Burgers operator. Additionally, when $j = 2$ and $m = 1$, we have that the results of this manuscript are still valid for the fifth order KdV-Burgers type operator. Finally, we can take, without loss of generality, $j = m \in \mathbb{N}$ in (5.1). So, we can define Lyapunov functionals associated with the solution of (1.2)

$$E(t) := E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx$$

and, for $\lambda \in L^\infty(\mathbb{R})$,

$$\mathcal{E}(t) := \mathcal{E}(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)dx + \frac{1}{2}\int_{t-\tau}^{t}\int_{\mathbb{R}} e^{-(t-s)}|\lambda(x)|u^2(x,s)\ dx\ ds.$$

thanks to the damping mechanism and the delay term, we showed that solutions of (1.2) satisfy

$$\mathcal{E}(t) \leq C(u_0)e^{-\nu t}.$$

Additionally to that, it is important to note that Section 4 extends for the operator (5.1) and the space H^s, for $s \in [0, 2j+1]$ the results proposed in [12, 14]. As well as, extend for an unbounded domain the results showed in [11, 32] with appropriated choice of j in the operator (5.1). However, considering the full system (1.2), that is, the system with damping and delayed terms, is still an open problem to prove the stabilization result in $H^{2j+1}(\mathbb{R})$. Finally, let us give some further comments.

5.1. **Weak versus strong damping mechanism.** Observe that taking $\beta(x) = 0$, in the Theorems 1.4 and 1.5, the results are still valid. However, for a more general framework, we keep this term. Taking in mind that $\beta(x) \neq 0$, additionally, we can consider λ_0 and λ are constants such that $|\lambda| < \lambda_0$, and the delay τ is sufficiently small, so Theorem 1.4 gives us the exponential stability for the solution of (1.2). This is possible using the method introduced in [27] for wave equations. In fact, in this case, we choose a sufficiently small delay τ and a constant γ such that

$$\frac{e^\tau + 1}{2}|\lambda| < \gamma < \lambda_0.$$

Since now $\lambda_0 = \gamma_0$ (see (1.7)), the conditions (1.8) and (1.9) are satisfied, and the Theorem 1.4 follows. Note that the same remark applies if $\lambda, \lambda_0 \in L^\infty(\mathbb{R})$ with λ_0 satisfying (1.7), and $|\lambda| < \gamma_0$.

With respect to the Theorem 1.5, for a more general framework, if λ_0 satisfies (1.12) and (1.13) instead of $\lambda_0(x) \geq \gamma_0$ for a.e. $x \in \mathbb{R}$, then for the same function, β satisfying

$$\frac{e^\tau + 1}{2}|\lambda(x)| \leq \gamma + \beta(x) \quad \text{for a.e.} \quad x \in \mathbb{R},$$

we expect a smaller decay rate $\tilde{\nu}$ than ν in Theorem 1.5. The explanation for this is the fact that there is a "good" part γ of λ_0 that will compensate for the delay feedback and its indefinite component β_0.

5.2. **General framework.** One may generalize the system (1.2) as

$$\begin{cases} \partial_t u(x,t) + \sum\limits_{m=0}^{j} \alpha_m \partial_x^{2m+1} u(x,t) + \sum\limits_{k=1}^{n} \beta_k \partial_x^{2k} u(x,t) + \lambda_0(x) u(x,t) & \\ \qquad\qquad +\lambda(x)u(x,t-\tau) + \frac{1}{2}\partial_x u^2(x,t) = 0, & \text{in} \quad \mathbb{R}\times(0,\infty), \\ u(x,s) = u_0(x,s), & \text{in} \quad \mathbb{R}\times[-\tau, 0], \end{cases} \tag{5.2}$$

where $\alpha_m, \beta_k \in \mathbb{R}$ and $j, n \in \mathbb{N}$. The previous system, which depends on the parameters α_m and β_k, recovers various delayed dispersive equations. Note that depending on the choice of the constants, we have:

1. Burgers equation ($\alpha_m = 0$ and $n = 1$);
2. KdV equation ($j = 1$ and $\beta_k = 0$);

3. Kawahara equation ($j = 2$, $\alpha_0 = 1$, $\alpha_1 = -1$ and $\beta_k = 0$);
4. KdV–Burgers equation ($j = 1$ and $n = 1$);
5. Kawahara–Burgers equation ($j = 2$, $\alpha_0 = 1$, $\alpha_1 = -1$ and $n = 1$);
6. Fourth-order dispersive equation ($\alpha_m = 0$, $n = 2$, $\beta_1 = -1$ and $\beta_2 = 1$).

However, we point out that for the system (5.2) the main analyses in the paper are almost analogous without additional difficulties, thus the equation (1.2) does not lose the generality in a sense of the aim of this paper.

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