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Massera's Theorems for a Higher Order Dispersive System

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Abstract
This work is devoted to presenting Massera-type theorems for the Kawahara system, a higher-order dispersive equation, posed in a bounded domain. Precisely, thanks to some properties of the semigroup and the decay of the solutions of this equation, we can prove its solutions are periodic, quasi-periodic, and almost periodic.

Keywords Kawahara equation · Bounded solutions · Periodic solutions · Massera theorems · Damping mechanism

Mathematics Subject Classification Primary 35B10 · 35B15 · 70K43 · Secondary 93D15

1 Introduction

1.1 Problem Under Consideration

Results of the existence of periodic solutions for differential equations date back to the 50s, when in 1950, J.L. Massera published a remarkable paper [17] on the existence of periodic solutions to ordinary differential equations (ODE) with periodic right-hand sides. Precisely, the corresponding linear setup Massera's theorem is as follows: Consider the ODE of the form

$$\dot{x} = A(t)x + b(t), \quad x \in \mathbb{R}^m, \tag{1.1}$$

with the matrix $A(t)$ and the vector $b(t)$ continuous on $\mathbb{R}_+$ and periodic with the same period τ. Then, the system (1.1) has a periodic solution with period τ if and only if it has a bounded solution on $\mathbb{R}_+$.

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So, in this context, we are interested to prove some periodic properties of the following Kawahara equation in a bounded domain

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = \varphi(t), \ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{1.2}$$

with a boundary force $\varphi(t)$ in a bounded domain $I = (0,1)$ and a damping term $\alpha u_{xx}(0,t)$, where $|\alpha| < 1$. Precisely, we are interested to understand if the system (1.2) has good properties when we investigate its solutions, considering the context introduced to Massera. Roughly speaking, we are interested in the study of the existence and qualitative property of recurrent solutions. This kind of property may be reformulated in the following question.

Question $\mathcal{A}$: Are there periodic solutions for the system (1.2)?

1.2 Physical Motivation

Under suitable assumption on amplitude, wavelength, wave steepness, and so on, the properties of the asymptotic models for water waves has been extensively studied to understand the full water wave system.[1] In this spirit, formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces.

The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long-wave approximations according to relations between ε and δ. So, if we consider

$$\varepsilon = \delta^4 \ll 1 \quad \text{and} \quad \mu = \frac{1}{3} + \nu \varepsilon^{\frac{1}{2}},$$

and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation which is an equation derived by Hasimoto and Kawahara in [12, 14] that take the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0.$$

Rescaling this equation, we will study in this paper the following system

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0.$$

1.3 Historical Background

Before answering the Question $\mathcal{A}$, let us introduce a state of the art related to the Kawahara equation. As mentioned before, problems related to higher-order dispersive systems are extensively studied. Precisely, stabilization and control problems have been studied in recent years.

[1] See for instance [1, 3, 15] and references therein, for a rigorous justification of various asymptotic models for surface and internal waves.

A pioneer work is due to Silva and Vasconcellos [19]. The authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second work in this way is due Capistrano-Filho *et al.* [2]. In this work, the authors considered the Kawahara equation in a bounded domain $Q_T = (0, T) \times (0, L)$,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t, x), & \text{in } Q_T, \\ u(t, 0) = u(t, L) = u_x(t, 0) = u_x(t, L) = u_{xx}(t, L) = 0, & \text{on } [0, T], \\ u(0, x) = u_0(x), & \text{in } [0, L]. \end{cases} \tag{1.3}$$

In this article, the authors were able to introduce an internal feedback law in (1.3), considering general nonlinearity $u^p u_x$, $p \in [1, 4)$, instead of uu_x. They proved that under the effect of the damping mechanism the energy associated with the solutions of the system decays exponentially.

Related to internal control issues in a bounded domain, Chen [8] presented results considering the Kawahara equation (1.3) posed on a bounded interval with a distributed control $f(t, x)$ and homogeneous boundary conditions. She showed the result by taking advantage of a Carleman estimate associated with the linear operator of the Kawahara equation with an internal observation. With this in hand, she was able to get a null controllable result when f is effective in a $\omega \subset (0, L)$. In [4], considering the system (1.3) with an internal control $f(t, x)$ and homogeneous boundary conditions, the authors can show that the equation in consideration is exactly controllable in L^2-weighted Sobolev spaces and, additionally, the Kawahara equation is controllable by regions on L^2-Sobolev space.

Recently, a new tool to find control properties for the Kawahara operator was proposed in [5, 7]. First, in [5], the authors showed a new type of controllability for the Kawahara equation, what they called *overdetermination control problem*. They can find a control acting at the boundary that guarantees that the solution of the problem under consideration satisfies an integral condition. In addition, when the control acts internally in the system, instead of the boundary, the authors proved that this condition is also satisfied. After that, in [7], the authors extend this idea for the internal control problem for the Kawahara equation on unbounded domains. Precisely, under certain hypotheses over the initial and boundary data, they can prove that there exists an internal control input such that solutions of the Kawahara equation satisfies an integral overdetermination condition considering the Kawahara equation posed in the real line, left half-line, and right half-line.

We finish presenting the last works in control theory related to the Kawahara equation. In [6, 10], under suitable assumptions on the time delay coefficients, the authors can prove that solutions of the Kawahara system are exponentially stable. The results are obtained using the Lyapunov approach and compactness-uniqueness argument. We caution that this is only a small sample of the extant work on control theory for the Kawahara equation.

1.4 Notation and Auxiliary Results

Before presenting the main results of the article, let us introduce some notation and two auxiliary results. Denote by $C^1(\mathbb{R})$ a function space whose elements are continuously differentiable complex-valued functions on $\mathbb{R}$ and its norm

$$\|u\|_{C^1(\mathbb{R})} := \|u\|_{C(\mathbb{R})} + \|u_t\|_{C(\mathbb{R})}, \quad \forall u \in C^1(\mathbb{R}).$$

Set

$$C_b^1(\mathbb{R}) = \{u \in C^1(\mathbb{R}); \|u\|_{C^1(\mathbb{R})} < +\infty\}.$$

Consider by $L^2(I)$ the space of all Lebesgue square integrable complex-valued functions on I with the following inner product

$$(u, v) := \mathrm{Re}\left(\int_0^1 u\overline{v}dx\right), \quad \forall u, v \in L^2(I),$$

where $\overline{v}$ denotes the conjugate of v. With the previous inner product, we define in $L^2(I)$ the following norm

$$\|u\| = (u, u)^{\frac{1}{2}}.$$

Additionally, let $H^s(I)$, $s \geq 0$, be the classical Sobolev spaces of complex-valued functions on I with its classical inner product and norm, denoted by $\|\cdot\|_{H^s(I)}$. Finally, consider

$$\begin{aligned} H_\alpha^s(I) = \{u \in H^s(I) \mid & u^{(5i)}(0) = 0 = u^{(5i)}(1), u^{(5i+1)}(0) = 0 = u^{(5i+1)}(1), \\ & u^{(5i+2)}(1) = \alpha u^{(5i+2)}(0)\}, \end{aligned}$$

where the derivatives are of order less than or equal to $n-1$. The norm and inner product of $H_\alpha^s(I)$ are inherited from $H^s(I)$.

The first result is devoted to proving the well-posedness *via* semigroup theory, which is the key to proving the other main results of the article. Precisely, we first prove that the linear Kawahara operator generates $\{S(t)\}_{t\geq 0}$ the C_0–semigroup of contraction on $L^2(I)$.

Theorem 1.1 *There exists $\omega > 0$ such that for any $k = 0, 1, 2, 3, 4$ and 5, we can find a positive constant $C_k > 0$ which the semigroup associated to the linear Kawahara operator satisfies*

$$\|S(t)u_0\|_{H_\alpha^k(I)} \leq C_k e^{-\omega t}\|u_0\|_{H_\alpha^k(I)},$$

for all $t > 0$.

The previous theorem is the key to proving the existence of the bounded solution for the Kawahara equation (1.2). For that, pick

$$X := C_b(\mathbb{R}, H^2(I))$$

with a norm

$$\|u\|_X := \sup_{t\in\mathbb{R}} \|u(t)\|_{H^2(I)}$$

and define the following set

$$X_\rho := \{u \in X \mid \|u\|_X \leq \rho\}.$$

The next theorem, thanks to the previous one, ensures that the solutions of (1.2) are bounded.

Theorem 1.2 *There exists a constant $\epsilon > 0$ such that for all $\varphi \in C^1(\mathbb{R})$ satisfying $\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon$, the system (1.2) admits a unique solution u such that*

$$\|u\|_X \leq C\epsilon,$$

where $C > 0$ is a constant independent of ϵ.

1.5 Massera-Type Theorems and Structure of the Article

With the previous background in hand it is clear that no results concerning the recurrent solutions for the Kawahara system are presented in the literature. This manuscript is interesting to fill this gap by giving answers for the Question $\mathcal{A}$ before presented. Precisely, the next three theorems give us Massera-type theorems for a higher-order dispersive system that is, the result below ensures that the solution of (1.2) is T–periodic.

Theorem 1.3 *Let*

$$\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon,$$

where ϵ is the constant determined by Theorem 1.2. If φ is a function T–periodic, thus u solution of (1.2), given by Theorem 1.2, is also a function T–periodic.

Additionally, the next Massera-type theorem gives some property of the periodicity of the solution to (1.2). The result can be read as follows.

Theorem 1.4 *Let*

$$\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon,$$

where ϵ is the constant determined by Theorem 1.2. If φ is a quasi-periodic function, the solution u of (1.2), obtained in Theorem 1.2, is also a quasi-periodic function. Moreover, if φ is $\overline{\omega}$–quasi-periodic function in t, thus the solution u of (1.2), obtained in Theorem 1.2, is also $\overline{\omega}$–quasi-periodic function in t.

Finally, let us present the last result of this work. Precisely, we can prove that the solutions of (1.2) are almost periodic.

Theorem 1.5 *Let $\|\varphi\|_{C^1(\mathbb{R})} \leq \epsilon$, where $0 < \epsilon \ll 1$ is obtained via Theorem 1.2. If φ, φ' are functions almost periodic, the solution u of (1.2), given by Theorem 1.2, is also an almost periodic function.*

The remainder of the paper is organized as follows. In Sect. 2, we present the auxiliary results that are essential for the proof of the Massera-type theorems, precisely, we present the proof of Theorems 1.1 and 1.2. After that, in Sect. 3, we present the answer for the question $\mathcal{A}$ which is divided into three results, that is, we present the proof of Theorems 1.3, 1.4 and 1.5. Further comments are presented in Sect. 4. Finally, in the Appendix, we give some properties of the energy associated with (1.2).

2 Preliminaries

In this section, we are interested to prove some properties of the following linear Kawahara system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{2.1}$$

which are essential for the rest of the article.

2.1 Well-Posedness: Linear System

From now on C with or without subscripts denotes positive constants whose value may change on different occasions. We will write the dependence of constant on parameters explicitly if it is essential. Additionally, we denote $\lambda_* > 0$ is the smallest constant such that the following inequality holds

$$\|u_x\|^2 \geq \lambda_* \|u\|^2, \ u \in H_0^1(I).$$

Consider the following operator $A : D(A) \subset L^2(I) \longrightarrow L^2(I)$ defined by

$$Au := -u_{xxx} + u_{xxxxx},$$

where

$$D(A) = \{u \in H^5(I) : u(0) = u(1) = u_x(0) = u_x(1) = 0, u_{xx}(1) = \alpha u_{xx}(0)\},$$

with $|\alpha| < 1$, and its adjoint $A^*v = v_{xxx} - v_{xxxxx}$ with

$$D(A^*) = \{v \in H^5(I) : v(0) = v(1) = v_x(0) = v_x(1) = 0, v_{xx}(0) = \alpha v_{xx}(1)\}.$$

Thus, the following property holds.

Proposition 2.1 *A generates a C_0 semigroup of contractions on $L^2(I)$.*

Proof Since A is a continuous linear operator, using the closed graph theorem, A has the closed graph. Moreover, as $D(A)$ is dense in $L^2(I)$, if we prove that A and A^* are dissipative, thanks to [18, Corollary 4.4] we have that A generates a C_0 semigroup of contractions on $L^2(I)$. To do this, by using the definitions of A and A^*, we get, integrating by parts that

$$(Au, u) = \frac{1}{2}(\alpha^2 - 1)(u_{xx}(0))^2 \leq 0$$

and

$$(A^*v, v) = \frac{1}{2}(\alpha^2 - 1)(v_{xx}(1))^2 \leq 0,$$

that is, A and A^* are dissipative, and so the proof is finished. □

From now on, denote by $\{S(t)\}_{t\geq 0}$ the C_0-semigroup associated with A, so $u(t) = S(t)u_0$ is the mild solution of the linearized system (2.1). The next result ensures some properties of the solution of the linear Kawahara system.

Proposition 2.2 *Let u solution of (2.1). Then, we have for all $T > 0$ that*

(i) $\|u(\cdot, T)\| \leq \|u_0\|$;

(ii) $(1 - \alpha^2)\int_0^T u_{xx}^2(0, t)dt \leq \|u_0\|^2$;

(iii) $\|u\|_{L^2(0,T;H^2(I))} \leq \sqrt{\frac{1}{3}\left(\frac{1}{1-\alpha^2} + 4T\right)}\|u_0\|$.

Proof Since $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of contractions, item (i) follows. Now, observe that

$$\frac{d}{dt}\left(\|u(t)\|^2_{L^2(I)}\right) = 2(u_t, u) = 2(Au, u) = (\alpha^2 - 1)(u_{xx}(0))^2,$$

and integrating in $(0, T)$ we get (ii).

Now, multiplying (2.1) by xu, integrating by parts, and using the boundary condition we have that

$$\begin{aligned}\frac{d}{dt}\left(\int_I xu^2dx\right) =& 2\int_I xuu_tdx = -2\int_I xuu_{xxx}dx + 2\int_I xuu_{xxxxx}dx\\ &= \int_I u^2dx - 3\int_I (u_x)^2dx + \alpha^2(u_{xx}(0,t))^2 - 5\int_I (u_{xx})^2dx.\end{aligned}$$

Integrating both side in $(0, T)$, holds that

$$\begin{aligned}&3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt\\ &\leq \int_I x(u_0)^2dx + \int_0^T\int_I u^2dxdt + \int_0^T \alpha^2(u_{xx}(0,t))^2dt\\ &\leq \int_I (u_0)^2dx + \int_0^T \|u_0\|^2dt + \int_0^T \alpha^2(u_{xx}(0,t))^2dt,\end{aligned}$$

since $\|u(t)\| \leq \|u_0\|$ for al $t \geq 0$. So, using item (ii), we obtain

$$3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt \leq (1 + T + \frac{\alpha^2}{1-\alpha^2})\|u_0\|^2.$$

Therefore,

$$\begin{aligned}3\|u\|^2_{H^2(I)} &= 3\int_0^T\int_I u^2dxdt + 3\int_0^T\int_I (u_x)^2dxdt + 3\int_0^T\int_I (u_{xx})^2dxdt\\ &\leq 3T\|u_0\|^2 + (1 + T + \frac{\alpha^2}{1-\alpha^2})\|u_0\|^2 = \left(\frac{1}{1-\alpha^2} + 4T\right)\|u_0\|^2,\end{aligned}$$

showing the result. □

The next result ensures the decay of the semigroup associated with the Kawahara operator. This can be proved by using the results shown in the Appendix.

Proposition 2.3 *Existe $\omega > 0$ e $C > 0$ tal que*

$$\|S(t)u_0\| \leq Ce^{-\omega t}\|u_0\|, \ t \geq 0.$$

Proof Consider $E(t) = \frac{1}{2}\|u\|^2(t)$ the energy associated with (2.1). So, thanks to the Theorem A.2, we have that

$$\|S(t)u_0\|^2 = 2E(t) \leq C\|u_0\|^2e^{-\mu t},$$

and taking the square root of both sides in the previous inequality with $\omega = \frac{\mu}{2} > 0$ the results holds. $\square$

The next result shows that the solutions of (2.1) are bounded.

Proposition 2.4 *There exists $C > 0$ such that for any $t > 0$,*

$$\|S(t)u_0\|_{H^2(I)} \leq C\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t}+1}\|u_0\|,$$

holds for any $u_0 \in L^2(I)$.

Proof Define the following function

$$g(t) := \int_0^t \|S(s)u_0\|^2_{H^2(I)} ds.$$

Applying the mean value theorem we have the existence of $\tau \in \left(0, \frac{t}{2}\right)$ such that

$$\|S(\tau)u_0\|^2_{H^2(I)} \cdot \left(\frac{t}{2}\right) = \int_0^{\frac{t}{2}} \|S(s)u_0\|^2_{H^2(I)} ds.$$

Thanks to item (iii) of Proposition 2.2, we get

$$\|S(\tau)u_0\|^2_{H^2(I)} \cdot \left(\frac{t}{2}\right) = \int_0^{\frac{t}{2}} \|S(s)u_0\|^2_{H^2(I)} ds \leq \frac{1}{3}\left(\frac{1}{1-\alpha^2}+2t\right)\|u_0\|^2.$$

Thus,

$$\|S(\tau)u_0\|^2_{H^2(I)} \leq \frac{1}{3}\left(\frac{1}{1-\alpha^2}+2t\right)\left(\frac{2}{t}\right)\|u_0\|^2 = \frac{4}{3}\left(\frac{1}{1-\alpha^2}\frac{1}{2t}+1\right)\|u_0\|^2$$

and so,

$$\|S(\tau)u_0\|_{H^2(I)} \leq \frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{2t}+1}\|u_0\| \leq \frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t}+1}\|u_0\|.$$

Finally, semigroup properties ensure that

$$\|S(t)u_0\|_{H^2(I)} = \|S(t-\tau)S(\tau)u_0\|_{H^2(I)} \leq C_1\|S(\tau)u_0\|_{H^2(I)} \leq C_1\frac{2\sqrt{3}}{3}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t}+1}\|u_0\|,$$

and the proof is achieved. $\square$

2.2 Proof of Theorem 1.1

Considering $k = 0$, thus the result is a consequence of Proposition 2.3. Now, taking $u_0 \in D(A) = H^5_\alpha(I)$, semigroup theory ensures that $u = S(t)u_0 \in D(A)$ and $u_t = Au =$

$S(t)(Au_0)$, with $Au_0 \in L^2(I)$. Pick $v = u_t$, we have that v satisfies the following initial value problem

$$\begin{cases} v_t = Av, & (x,t) \in I \times (0,T), \\ v(x,0) = v_0(x) = (Au_0)(x) \in L^2(I), & x \in I. \end{cases} \tag{2.2}$$

Proposition 2.3 yields that

$$\|v(\cdot,t)\| \le Ce^{-\omega t}\|v_0\| = Ce^{-\omega t}\|Au_0\|.$$

Since the following norms $\|u\| + \|Au\|$ and $\|u\|_{D(A)}$ are equivalents in $D(A)$, we ensure the existence of two constants $M_1, M_2 > 0$ such that

$$M_1\|u\|_{D(A)} \le \|u\| + \|Au\| \le M_2\|u\|_{D(A)}.$$

Thus,

$$\begin{aligned} \|S(t)u_0\|_{H^5_\alpha(I)} &\le M_1^{-1}(\|S(t)u_0\| + \|A(S(t)u_0)\|) \\ &\le M_1^{-1}(Ce^{-\omega t}\|u_0\| + Ce^{-\omega t}\|Au_0\|) \\ &\le M_1^{-1}CM_2e^{-\omega t}\|u_0\|_{H^5_\alpha(I)}. \end{aligned}$$

The results for $k = 1, 2, 3$ and 4, are consequences of an interpolation argument. So, Theorem 1.1 is shown. □

2.3 Well-Posendess: Bounded Solutions for the Nonlinear System

Consider the following initial boundary value problem

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = \varphi(t),\ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{2.3}$$

Let us study the bounded solution to the system (2.3). To do this, let us consider

$$y(x,t) := u(x,t) + A(x)\varphi(t),$$

with the function A defined by

$$A(x) = \frac{\alpha - 1}{\alpha + 1}x^2 + \frac{2}{\alpha + 1}x - 1$$

and $\varphi \in C^1(\mathbb{R})$. If we suppose that u satisfies (2.3), so we have that y satisfies

$$\begin{cases} y_t + y_{xxx} - y_{xxxxx} + yy_x + ay_x + by = f, & (x,t) \in I \times \mathbb{R}, \\ y(0,t) = y(1,t) = 0, & t \in \mathbb{R}, \\ y_x(1,t) = \alpha y_x(0,t), & t \in \mathbb{R}, \\ y_{xx}(1,t) = \alpha y_{xx}(0,t) + \left(\dfrac{-2(\alpha-1)^2}{\alpha+1}\right)\varphi(t) & t \in \mathbb{R}, \end{cases} \tag{2.4}$$

with

$$y_x(0,t) = \frac{2}{\alpha+1}\varphi(t), \quad a(x,t) = -A(x)\varphi(t), \quad b(x,t) = -A'(x)\varphi(t) \tag{2.5}$$

and

$$f = A(x)\varphi'(t) - A(x)A'(x)\varphi^2(t). \tag{2.6}$$

Moreover, we consider that y is a mild solution of (2.4) if satisfies the integral equation

$$y(t) = S(t-r)y(r) + \int_r^t S(t-r)(-yy_x - ay_x - by + f)(s)ds, \tag{2.7}$$

for all $t \geq r$ and each $r \in \mathbb{R}$. Thus, as y is a mild solution of (2.4), we have that

$$u(x,t) = y(x,t) - A(x)\varphi(t)$$

is a mild solution of (2.3). With this in hand, we are in a position to prove our second auxiliary result.

Proof of Theorem 1.2 A straightforward calculation shows that by using integration by parts, we get

$$\int_0^\infty e^{-\frac{\omega}{2}\tau}\sqrt{\frac{1}{\tau}}d\tau = \int_0^\infty \omega e^{-\frac{\omega}{2}\tau}\sqrt{\tau}d\tau = \int_0^\infty \omega e^{-\frac{\omega}{4}\tau}e^{-\frac{\omega}{4}\tau}\sqrt{\tau}d\tau. \tag{2.8}$$

Pick a function $h(\tau) = e^{-\frac{\omega}{2}\tau}\tau$, with $\tau \in \mathbb{R}$. Since

$$h'(\tau) = \left(1 - \frac{\omega}{2}\tau\right)e^{-\frac{\omega}{2}\tau} = 0 \Leftrightarrow \tau = \frac{2}{\omega},$$

$h'(t) > 0$ for $\tau < \frac{2}{\omega}$ and $h'(t) < 0$ for $\tau > \frac{2}{\omega}$, yields that

$$h(\tau) \leq h\left(\frac{2}{\omega}\right) = \frac{2e^{-1}}{\omega}, \quad \forall \tau \in \mathbb{R}.$$

So we have

$$e^{-\frac{\omega}{4}\tau}\sqrt{\tau} \leq \sqrt{\frac{2e^{-1}}{\omega}}.$$

Therefore,

$$\begin{aligned}\int_0^\infty e^{-\frac{\omega}{2}\tau}\sqrt{\frac{1}{\tau}}d\tau &= \int_0^\infty \omega e^{-\frac{\omega}{4}\tau}e^{-\frac{\omega}{4}\tau}\sqrt{\tau}d\tau \\ &\leq \omega\sqrt{\frac{2e^{-1}}{\omega}}\int_0^\infty e^{-\frac{\omega}{4}\tau}d\tau \\ &= 4\sqrt{\frac{2e^{-1}}{\omega}}.\end{aligned} \tag{2.9}$$

On the other hand,

$$\int_0^{\infty} e^{-\frac{\omega}{2}\tau} d\tau = \frac{2}{\omega}. \tag{2.10}$$

Thus, Theorem 1.1, Proposition 2.4 and Agmon inequality,[2] ensure that

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} S(t-s) y(\cdot,s) y_x(\cdot,s) ds \right\|_{H^2(I)} \\
&\leq \int_{-\infty}^{t} \left\| S\left(\frac{t-s}{2}\right) S\left(\frac{t-s}{2}\right) y(\cdot,s) y_x(\cdot,s) \right\|_{H^2(I)} ds \\
&\leq C_2 \int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \left\| S\left(\frac{t-s}{2}\right) y(\cdot,s) y_x(\cdot,s) \right\|_{H^2(I)} ds \\
&\leq C C_2 \int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{2}{t-s}+1}\, \| y(\cdot,s) y_x(\cdot,s) \| ds \\
&\leq C \int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{2}{t-s}+1}\, \| y(\cdot,s) \|_{H^2(I)} \| y_x(\cdot,s) \| ds \\
&\leq C \| y \|_X^2 \int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)} \sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1} ds.
\end{aligned}$$

Thanks to the previous inequality, taking $\tau = t - s$, we get

$$\begin{aligned}
&\left\| \int_{-\infty}^{t} S(t-s) y(\cdot,s) y_x(\cdot,s) ds \right\|_{H^2(I)} \\
&\leq C \| y \|_X^2 \int_0^{\infty} e^{-\frac{\omega}{2}\tau} \left(\sqrt{\frac{1}{1-\alpha^2}} \sqrt{\frac{1}{\tau}} + \sqrt{1} \right) d\tau \\
&\leq C \| y \|_X^2 \int_0^{\infty} e^{-\frac{\omega}{2}\tau} \left(\sqrt{\frac{1}{\tau}} + 1 \right) d\tau \\
&\leq C \left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega} \right) \| y \|_X^2 .
\end{aligned} \tag{2.11}$$

Now, considering $\varphi \in C^1(\mathbb{R})$, we have $a, b \in X$. Therefore, we get, using the same computations as before, that

$$\begin{aligned}
\left\| \int_{-\infty}^{t} S(t-s) a(\cdot,s) y_x(\cdot,s) ds \right\|_{H^2(I)} &\leq C \| a \|_X \| y \|_X \int_0^{\infty} e^{-\frac{\omega}{2}\tau} \left(\sqrt{\frac{1}{\tau}} + 1 \right) d\tau \\
&\leq C \left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega} \right) \| a \|_X \| y \|_X
\end{aligned} \tag{2.12}$$

[2] Agmon inequality in one dimensional case: $\|u\|_{L^\infty(I)} \leq C \|u\|_{L^2(I)}^{\frac{3}{4}} \|u\|_{H^2(I)}^{\frac{1}{4}} \leq C \|u\|_{H^2(I)}$, $I = (0, 1)$.

and

$$\begin{aligned}\left\|\int_{-\infty}^{t} S(t-s)b(\cdot,s)y(\cdot,s)ds\right\|_{H^2(I)} \leq & C\,\|b\|_X\,\|y\|_X \int_0^\infty e^{-\frac{\omega}{2}\tau}\left(\sqrt{\frac{1}{\tau}}+1\right)d\tau \\ \leq & C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\,\|y\|_X\,.\end{aligned} \tag{2.13}$$

Using the change of variable y by $y-z$ in (2.12) and (2.13), respectively, yields that

$$\left\|\int_{-\infty}^{t} S(t-s)a(\cdot,s)\left[y(\cdot,s)-z(\cdot,s)\right]_x ds\right\|_{H^2(I)} \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\,\|y-z\|_X \tag{2.14}$$

and

$$\left\|\int_{-\infty}^{t} S(t-s)b(\cdot,s)\left[y(\cdot,s)-z(\cdot,s)\right]ds\right\|_{H^2(I)} \leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\,\|y-z\|_X\,. \tag{2.15}$$

Additionally, thanks to Theorem 1.1, we have

$$\begin{aligned}\left\|\int_{-\infty}^{t} S(t-s)f(\cdot,s)ds\right\|_{H^2(I)} \leq & \int_{-\infty}^{t}\|S(t-s)\,f(\cdot,s)\|_{H^2(I)}\,ds \\ \leq & C_2\int_{-\infty}^{t} e^{-\omega(t-s)}\,\|f(\cdot,s)\|_{H^2(I)}\,ds \\ \leq & C\,\|f\|_X\int_{-\infty}^{t} e^{-\omega(t-s)}ds \\ = & C\,\|f\|_X\int_0^\infty e^{-\omega\tau}ds=\frac{C}{\omega}\,\|f\|_X\,,\end{aligned} \tag{2.16}$$

where in the last line we have used the following change of variable $\tau=t-s$. Also, note that by analogous process yields that

$$\begin{aligned}&\left\|\int_{-\infty}^{t} S(t-s)\left[y(\cdot,s)y_x(\cdot,s)-z(\cdot,s)z_x(\cdot,s)\right]ds\right\|_{H^2(I)} \\ &\leq \int_{-\infty}^{t}\left\|S\left(\frac{t-s}{2}\right)S\left(\frac{t-s}{2}\right)\left[y(\cdot,s)y_x(\cdot,s)-z(\cdot,s)z_x(\cdot,s)\right]\right\|_{H^2(I)} ds \\ &\leq C\int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}\left\|\frac{1}{2}\frac{d}{dx}(y^2-z^2)(\cdot,s)\right\|ds \\ &\leq C\int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}\,\|(y-z)(\cdot,s)\|_{H^1(I)}\,\|(y+z)(\cdot,s)\|_{H^1(I)}\,ds \\ &\leq C\,\|y-z\|_X\,\|y+z\|_X\int_{-\infty}^{t} e^{-\frac{\omega}{2}(t-s)}\sqrt{\frac{1}{1-\alpha^2}\frac{1}{t-s}+1}ds \\ &\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y-z\|_X\,\|y+z\|_X\,.\end{aligned} \tag{2.17}$$

Set

$$(\Psi y)(t) := \int_{-\infty}^{t} S(t-s)(-yy_x - ay_x - by + f)(\cdot, s)ds. \tag{2.18}$$

Now, using (2.11), (2.12), (2.13) and (2.16), we get that

$$\begin{aligned}
\|\Psi y\|_X =& \sup_{t\in\mathbb{R}} \|(\Psi y)(t)\|_{H^2(I)} \le \|(\Psi y)(t)\|_{H^2(I)} \\
\le& \left\| \int_{-\infty}^{t} S(t-s)(yy_x)(\cdot, s)ds \right\|_{H^2(I)} + \left\| \int_{-\infty}^{t} S(t-s)(ay_x)(\cdot, s)ds \right\|_{H^2(I)} \\
&+ \left\| \int_{-\infty}^{t} S(t-s)(by)(\cdot, s)ds \right\|_{H^2(I)} + \left\| \int_{-\infty}^{t} S(t-s)f(\cdot, s)ds \right\|_{H^2(I)} \\
\le& C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|y\|_X^2 + C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|a\|_X \|y\|_X \\
&+ C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|b\|_X \|y\|_X + \frac{C}{\omega} \|f\|_X .
\end{aligned} \tag{2.19}$$

Thanks to (2.14), (2.15) and (2.17), we get that

$$\begin{aligned}
\|\Psi y - \Psi z\|_X =& \sup_{t\in\mathbb{R}} \|(\Psi y - \Psi z)(t)\|_{H^2(I)} \le \|(\Psi y - \Psi z)(t)\|_{H^2(I)} \\
\le& \left\| \int_{-\infty}^{t} S(t-s)(yy_x - zz_x)(\cdot, s)ds \right\|_{H^2(I)} \\
&+ \left\| \int_{-\infty}^{t} S(t-s)\left(a(y-z)_x\right)(\cdot, s)ds \right\|_{H^2(I)} \\
&+ \left\| \int_{-\infty}^{t} S(t-s)\left(b(y-z)\right)(\cdot, s)ds \right\|_{H^2(I)} \\
\le& C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|y-z\|_X \|y+z\|_X \\
&+ C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|a\|_X \|y-z\|_X \\
&+ C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right) \|b\|_X \|y-z\|_X .
\end{aligned} \tag{2.20}$$

Moreover, we have that

$$\|\Psi y\|_X \le C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)\rho^2 + C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)(\|a\|_X + \|b\|_X)\rho + \frac{C}{\omega}\|f\|_X \tag{2.21}$$

and

$$\|\Psi y - \Psi z\|_X \le C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)\rho \|y-z\|_X + C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)(\|a\|_X + \|b\|_X)\|y-z\|_X , \tag{2.22}$$

for $y, z \in X_\rho$. By hypothesis

$$\|\varphi\|_{C_1(\mathbb{R})} := \max\{\sup_{t\in\mathbb{R}} |\varphi(t)|, \sup_{t\in\mathbb{R}} |\varphi'(t)|\} \leq \epsilon,$$

since $a(x,t) = -A(x)\varphi(t)$ we get

$$\|a(\cdot,t)\|^2_{H^2(I)} = (\varphi(t))^2 \int_0^1 (A^2 + A_x^2 + A_{xx}^2)dx = (C\varphi(t))^2,$$

with $C^2 := \int_0^1 (A^2 + A_x^2 + A_{xx}^2)dx$. So

$$\|a\|_X = \sup_{t\in\mathbb{R}}\{|\varphi(t)|C\} \leq C\|\varphi\|_{C_1(\mathbb{R})} \leq C\epsilon.$$

Analogously, taking $b(x,t) = -A_x(x)\varphi(t)$, ensures that

$$\|b(\cdot,t)\|^2_{H^2(I)} = (\varphi(t))^2 \int_0^1 (A_x^2 + A_{xx}^2 + A_{xxx}^2)dx = (C\varphi(t))^2.$$

Thus,

$$\|b\|_X = \sup_{t\in\mathbb{R}}\{|\varphi(t)|C\} \leq C\|\varphi\|_{C_1(\mathbb{R})} \leq C\epsilon.$$

As $f(x,t) = A(x)\varphi'(t) + A'(x)\varphi(t) - A(x)A'(x)\varphi^2(t)$ and $0 < \epsilon \leq 1$, follows that

$$\|f\|_X \leq C\epsilon,$$

where $C > 0$ independent of ϵ.

Finally, thanks to the previous inequality, let us consider

$$\rho = \frac{3C}{\omega}\|f\|_X \leq \frac{3C^2}{\omega}\epsilon.$$

For $\epsilon \ll 1$ small enough we have

$$\rho \ll 1,$$

$$C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)\rho \leq \frac{1}{3},$$

and

$$C\left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega}\right)(\|a\|_X + \|b\|_X) < \frac{1}{3}.$$

Therefore, thanks to (2.21) and (2.22), the following holds true

$$\|\Psi y\|_X \leq \rho$$

and

$$\|\Psi y - \Psi z\|_X \leq \frac{2}{3}\|y - z\|_X,$$

respectively. Therefore, using the Banach fixed-point theorem, there exists a unique $y \in X_\rho$ such that $\Psi y = y$. Thus, for such y yields that

$$\|y\|_X = \|\Psi y\|_X \leq \rho$$

and y is a mild solution for (2.4). As $y(x,t) = u(x,t) + A(x)\varphi(t)$, we get

$$\|u\|_X \leq C\epsilon,$$

showing the result. □

3 Massera's Theorems for the Kawahara Operator

In this section, our goal is to present several Massera's type theorems associated with the system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} + uu_x = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = \varphi(t),\ u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{3.1}$$

These theorems ensure that this higher-order dispersive equation has recurrent solutions. Let us start proving the first result in this way.

3.1 Proof of Theorem 1.3

We have that $v(x,t) = u(x,t+T)$ is the unique solution of

$$\begin{cases} v_t + v_{xxx} - v_{xxxxx} + vv_x = 0, & (x,t) \in I \times \mathbb{R}, \\ v(0,t) = \varphi(t+T) = \varphi(t), & t \in \mathbb{R}, \\ v(1,t) = v_x(1,t) = v_x(0,t) = 0, & t \in \mathbb{R}, \\ v_{xx}(1,t) = \alpha v_{xx}(0,t), & t \in \mathbb{R}. \end{cases}$$

The system above is exactly (3.1), so the uniqueness of solutions gives us that

$$u(x,t+T) = v(x,t) = u(x,t),$$

for all $(x,t) \in I \times \mathbb{R}$, showing the result. □

3.2 Quasi-Periodic Solution

In this section, we are interested in analyzing the quasi-periodic solutions of (3.1). Before it, we present some definitions necessary for this study.

Definition 3.1 We say that the real numbers $\omega_1, \omega_2, \ldots, \omega_k$ are rationally independent when

$$m_1\omega_1 + \cdots + m_k\omega_k = 0,$$

only happens when $m_1 = \cdots m_k = 0$, with $m_1, \ldots, m_k \in Q$, where Q is the set of all rational numbers.

Let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and e_i a unitary vector of $\mathbb{R}^k$ such that the i-th component is 1 and the others are zero. We have the following definition.

Definition 3.2 A function $f(x,t)$ is denoted by $\overline{\omega}$–quasi-periodic in t uniformly with respect to $x \in I$, if there are $\omega_1, \dots, \omega_k \in \mathbb{R}$ rationally independent and a function $F(x,u) \in C(I \times \mathbb{R}^k, \mathbb{R})$ such that

$$f(x,t) = F(x, t\omega_1, t\omega_2, \dots, t\omega_k) = F(x, t\overline{\omega}), \quad \forall t \in \mathbb{R} \text{ and } x \in I,$$

where $\overline{\omega} = (\omega_1, \dots, \omega_k)$ and $F(x, u + 2\pi e_i) = F(x,u)$, for all $u \in \mathbb{R}^k$ and $x \in I$, $i = 1, 2, \dots, k$. The numbers $\omega_1, \dots, \omega_k$ are called basic frequencies of f.

Definition 3.3 We say that $\varphi(t)$ is $\overline{\omega}$–quasi-periodic in t if there is $\Phi(u) \in C(\mathbb{R}^k, \mathbb{R})$ such that

$$\Phi(u + 2\pi e_i) = \Phi(u), \quad \forall u \in \mathbb{R}^k, i = 1, \dots, k$$

and

$$\varphi(t) = \Phi(t\overline{\omega}) = \Phi(t\omega_1, \dots, t\omega_k) \quad \forall t \in \mathbb{R}.$$

With these definitions in hand, we prove now the second main result of the work.

Proof of Theorem 1.4 Since $\varphi(t)$ is $\overline{\omega}$–quasi-periodic function in t, by definition, there exists $\Phi(u) \in C(\mathbb{R}^k, \mathbb{R})$ such that

$$\Phi(u + 2\pi e_i) = \Phi(u), \forall u \in \mathbb{R}^k, \ i = 1, \dots, k$$

and

$$\varphi(t) = \Phi(t\overline{\omega}) = \Phi(t\omega_1, \dots, t\omega_k), \quad \forall t \in \mathbb{R}.$$

Set $\overline{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ and $\varphi_{\overline{\alpha}}(t) = \Phi(\overline{\alpha} + t\overline{\omega})$. Thanks to the Theorem 1.2, for each boundary force $\varphi_{\overline{\alpha}}(t)$ the equation (3.1) has unique solution $u_{\overline{\alpha}} \in X_{C_\epsilon}$.

Pick now

$$U(x, \overline{\alpha}) := u_{\overline{\alpha}}(x, 0). \tag{3.2}$$

Thus, U is well-defined due to the uniqueness of the solutions. We prove the result by several claims.

Claim 1 $u_{\overline{\alpha}}(x, t+h) = u_{\overline{\alpha}+h\overline{\omega}}(x,t)$.

Indeed, noting that

$$\varphi_{h\overline{\omega}+\overline{\alpha}}(t) = \Phi(t\overline{\omega} + h\overline{\omega} + \overline{\alpha}) = \Phi((t+h)\overline{\omega} + \overline{\alpha}) = \varphi_{\overline{\alpha}}(t+h),$$

we have that $u_{\overline{\alpha}}(x, t+h)$ and $u_{\overline{\alpha}+h\overline{\omega}}(x,t)$ are solutions of (3.1) with boundary force $\varphi_{h\overline{\omega}+\overline{\alpha}}(t)$. The uniqueness of solutions ensures that

$$u_{\overline{\alpha}}(x, t+h) = u_{\overline{\alpha}+h\overline{\omega}}(x,t),$$

and Claim 1 is proved.

Claim 2 *$U(x, \overline{\alpha}) = U(x, \alpha_1, \ldots, \alpha_k)$ has period 2π with respect to each argument α_i.*

In fact, taking $t = 0$, in Claim 1, we get

$$u_{\overline{\alpha}}(x, h) = u_{\overline{\alpha}+h\overline{\omega}}(x, 0) = U(x, \overline{\alpha} + h\overline{\omega}), \quad \forall h \in \mathbb{R}.$$

As h is arbitrary, we have that

$$u_{\overline{\alpha}}(x, t) = U(x, \overline{\alpha} + t\overline{\omega}), \quad \forall t \in \mathbb{R}.$$

So, (3.2) help us to ensure that

$$U(x, \overline{\alpha} + 2\pi e_i) = u_{\overline{\alpha}+2\pi e_i}(x, 0),$$

where $\{e_1, \ldots, e_k\}$ is the standard basis in $\mathbb{R}^k$. Since

$$\varphi_{\overline{\alpha}+2\pi e_i}(t) = \Phi(\overline{\omega} + \overline{\alpha} + 2\pi e_i) = \Phi(\overline{\omega} + \overline{\alpha}) = \varphi_{\overline{\alpha}}(t),$$

holds true, the uniqueness of solution guaranteed by Theorem 1.2, gives

$$u_{\overline{\alpha}+2\pi e_i} = u_{\overline{\alpha}},$$

therefore

$$U(x, \overline{\alpha} + 2\pi e_i) = u_{\overline{\alpha}+2\pi e_i}(x, 0) = u_{\overline{\alpha}}(x, 0) = U(x, \overline{\alpha}),$$

and Claim 2 is shown.

Finally, taking $\overline{\alpha} = (0, \ldots, 0) \in \mathbb{R}^k$, we have that the external force $\varphi_{\overline{\alpha}}(t) = \varphi(t)$ and $u(x, t) = U(x, t\overline{\omega})$. Thus, we get u is a $\overline{\omega}$–quasi-periodic solution in t. $\square$

3.3 Almost Periodic Solution

In this section the goal is to prove that (3.1) have almost periodic solutions. To do that, let us begin this subsection with the following definition.

Definition 3.4 Let (Y, d) be a separable and complete metric space and $f : \mathbb{R} \longrightarrow Y$ be a continuous mapping. The function f is said to be almost periodic if, for every $\delta > 0$, there exists a constant $l(\delta) > 0$ such that any interval of length $l(\delta)$ contains at least a number τ for which

$$\sup_{t\in\mathbb{R}} d(f(t + \tau), f(t)) < \delta.$$

Now, we are in a position to prove the last result of the work.

Proof of Theorem 1.5 Consider y, a, b and f satisfying (2.4), (2.5) and (2.6). The straightforward calculation, thanks to the following change of variable $\tau = s - \sigma$, shows that

$$\begin{aligned}\|y(t+\sigma) - y(t)\|_{H^2(I)} = \Bigg\| &\int_{-\infty}^{t+\sigma} S(t+\sigma-s)(-yy_x - ay_x - by + f)(s)ds \\ &- \int_{-\infty}^{t} S(t-s)(-yy_x - ay_x - by + f)(s)ds \Bigg\|_{H^2(I)}\end{aligned}$$

$$\leq \left\| \int_{-\infty}^{t} S(t-s)\left(y(s+\sigma)y_x(s+\sigma)-y(s)y_x(s)\right)ds \right\|_{H^2(I)} \quad (3.3)$$
$$+ \left\| \int_{-\infty}^{t} S(t-s)\left(a(s+\sigma)y_x(s+\sigma)-a(s)y_x(s)\right)ds \right\|_{H^2(I)}$$
$$+ \left\| \int_{-\infty}^{t} S(t-s)\left(b(s+\sigma)y(s+\sigma)-b(s)y(s)\right)ds \right\|_{H^2(I)}$$
$$+ \left\| \int_{-\infty}^{t} S(t-s)\left(f(s+\sigma)-f(f)\right)ds \right\|_{H^2(I)}.$$

Set $z(\cdot,s)=y(\cdot,s+\sigma)$ in the expression (2.17), we get

$$\begin{aligned} &\left\| \int_{-\infty}^{t} S(t-s)\left(y(s+\sigma)y_x(s+\sigma)-y(s)y_x(s)\right)ds \right\|_{H^2(I)} \\ &\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X^2. \end{aligned} \quad (3.4)$$

Therefore, follows by (2.14) and (2.12) with $a(\cdot+\sigma)-a(\cdot)$ instead of a that

$$\begin{aligned} &\left\| \int_{-\infty}^{t} S(t-s)\left(a(s+\sigma)y_x(s+\sigma)-a(s)y_x(s)\right)ds \right\|_{H^2(I)} \\ &\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a\|_X\|y(\cdot+\sigma)-y(\cdot)\|_X \\ &\quad + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|a(\cdot+\sigma)-a(\cdot)\|_X\|y\|_X. \end{aligned} \quad (3.5)$$

Analogously, we get

$$\begin{aligned} &\left\| \int_{-\infty}^{t} S(t-s)\left(b(s+\sigma)y(s+\sigma)-b(s)y(s)\right)ds \right\|_{H^2(I)} \\ &\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b\|_X\|y(\cdot+\sigma)-y(\cdot)\|_X \\ &\quad + C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|b(\cdot+\sigma)-b(\cdot)\|_X\|y\|_X, \end{aligned} \quad (3.6)$$

where we have used (2.15) and (2.13) with $b(\cdot+\sigma)-b(\cdot)$ instead of b. Due to the Theorem 1.1 and using the change of variables $\tau=t-s$, yields that

$$\left\| \int_{-\infty}^{t} S(t-s)\left(f(\cdot,s+\sigma)-f(\cdot,s)\right)ds \right\|_{H^2(I)} \leq \frac{C}{\omega}\|f(\cdot+\sigma)-f(\cdot)\|_X. \quad (3.7)$$

Now, replacing (3.4), (3.5), (3.6) and (3.7) into (3.3), we ensures that

$$\begin{aligned}
&\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\quad\leq C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X^2 \\
&\qquad+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\qquad+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X)\,\|y\|_X \\
&\qquad+\frac{C}{\omega}\,\|f(\cdot+\sigma)-f(\cdot)\|_X\,.
\end{aligned} \tag{3.8}$$

Thus, taking $y\in X_\rho$ and $0<\epsilon\ll 1$ in the proof of Theorem 1.2 such that

$$2C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\rho<\frac{1}{3}$$

and

$$C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)<\frac{1}{3},$$

we have that Theorem 1.2 is still valid and also is verified that

$$\begin{aligned}
C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X^2 &\leq 2\rho C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&<\frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X.
\end{aligned}$$

Finally, applying it in (3.8) we have

$$\begin{aligned}
&\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\quad\leq 2\rho C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\qquad+C\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a\|_X+\|b\|_X)\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\qquad+C\rho\left(\frac{1}{\sqrt{\omega}}+\frac{1}{\omega}\right)(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X) \\
&\qquad+\frac{C}{\omega}\,\|f(\cdot+\sigma)-f(\cdot)\|_X \\
&\quad\leq\frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X+\frac{1}{3}\|y(\cdot+\sigma)-y(\cdot)\|_X \\
&\qquad+\frac{1}{3}(\|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X) \\
&\qquad+\frac{C}{\omega}\,\|f(\cdot+\sigma)-f(\cdot)\|_X
\end{aligned} \tag{3.9}$$

and follows that

$$\begin{aligned}\|y(\cdot+\sigma)-y(\cdot)\|_X \leq & \|a(\cdot+\sigma)-a(\cdot)\|_X+\|b(\cdot+\sigma)-b(\cdot)\|_X \\ & +\frac{3C}{\omega}\|f(\cdot+\sigma)-f(\cdot)\|_X .\end{aligned} \tag{3.10}$$

Since a, b, and f are almost periodic functions, y is also an almost periodic function. According to the fact that

$$y(x,t):=u(x,t)-A(x)\varphi(t),$$

we obtain that u is also an almost periodic function. Thus, u is almost a periodic solution of (3.1), and the Theorem is achieved. □

4 Further Comments

In this work were able to present properties for a higher-order dispersive system, namely, the Kawahara equation, posed on a bounded domain. Many results in the literature, as we saw in the introduction, treated this equation from a control point of view. Here, we provide periodic properties for the following initial boundary value problem

$$\begin{cases} u_t+u_{xxx}-u_{xxxxx}+uu_x=0, & (x,t)\in I\times\mathbb{R}, \\ u(0,t)=\varphi(t),\ u(1,t)=u_x(1,t)=u_x(0,t)=0, & t\in\mathbb{R}, \\ u_{xx}(1,t)=\alpha u_{xx}(0,t), & t\in\mathbb{R}, \\ u(x,0)=u_0(x), & x\in I, \end{cases} \tag{4.1}$$

with a forcing boundary term $\varphi(t)$ and a term $\alpha u_{xx}(0,t)$ acting as a damping mechanism. Thus, we have succeeded to prove Massera-type theorems for the solution of (4.1). Concerning the generality of the work, let us make some additional comments.

- Theorems 1.3, 1.4 and 1.5 can be obtained for more general nonlinearities. Indeed, if we consider $u\in\mathcal{B}:=C(0,T;L^2(0,1))\cap\ L^2(0,T;H^2(0,1))$ and the nonlinearity u^pu_x, $p\in(2,4]$, we have that

  $$\int_0^T\int_0^1|u^{p+2}|dxdt\leqslant C\|u\|^p_{C([0,T];L^2(0,1))}\int_0^T\|u_x\|^2dt\leqslant C\|u\|^{p+2}_{\mathcal{B}},$$

  by the Gagliardo–Nirenberg inequality. Moreover, recently, Zhou [21] showed the well-posedness of the following initial boundary value problem

  $$\begin{cases} u_t-u_{xxxxx}=c_1uu_x+c_2u^2u_x+b_1u_xu_{xx}+b_2uu_{xxx}, & x\in(0,L),\ t\in\mathbb{R}^+, \\ u(t,0)=h_1(t),\quad u(t,L)=h_2(t),\quad u_x(t,0)=h_3(t), & t\in\mathbb{R}^+, \\ u_x(t,L)=h_4(t),\quad u_{xx}(t,L)=h(t), & t\in\mathbb{R}^+, \\ u(0,x)=u_0(x), & x\in(0,L). \end{cases} \tag{4.2}$$

  Thus, due to the previous inequality and the results proved in [21], when we consider $b_1=b_2=0$ and the combination $c_1uu_x+c_2u^2u_x$ instead of uu_x on (4.1), the main results of this work remains valid.

- As in the classical framework of Massera's theorem, a principal point is to prove that the initial boundary value problem (4.1) admits a bounded solution, to do that, an important step is the study of the energy associated with the linear system under consideration, this analysis was made in the Appendix.
- An important point of the previous remark is to deal with the energy of (4.1) we analyze the Kawahara operator removing the drift term u_x. This term presents an extra problem because a critical set appears, see [2] for details. In this way, to overcome this difficulty it was necessary to remove the drift term. Thus, an interesting open problem is to extend the result presented in this paper for the Kawahara equation (4.1) with the drift term taking into account that this equation, with $\varphi(t) = 0$, has the critical set phenomena, as conjectured in [2].
- It is important to point out that the Massera-type theorem has been extended to many differential equations as we can see in [9, 11, 13, 16, 22] and the references therein. The method employed in these works is to prove the existence of periodic solutions if the solution of the equation under consideration is bounded.
- Finally, there are two important points related to the Massera-type theorems for the Kawahara equation. The first one is that we can work with more general nonlinearities, as mentioned before. The second one is the strong relationship between the damping mechanism (stabilization problem) and the Massera-type theorems in our case.

Appendix: Additional Properties

In this appendix, we present some additional properties of the linear Kawahara system. For the sake of simplicity, we present the results for the linear system, however, the results obtained here can be also extended for the nonlinear system. Precisely, let us study the energy properties for the following linear system

$$\begin{cases} u_t + u_{xxx} - u_{xxxxx} = 0, & (x,t) \in I \times \mathbb{R}, \\ u(0,t) = u(1,t) = u_x(1,t) = u_x(0,t) = 0, & t \in \mathbb{R}, \\ u_{xx}(1,t) = \alpha u_{xx}(0,t), & t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{A.1}$$

where $|\alpha| < 1$. Note that multiplying (A.1) by u and integrating over $(0, L)$ yields

$$\frac{d}{dt}\int_0^L |u(x,t)|^2\,dx = \frac{1}{2}(\alpha^2 - 1)(u_{xx}(0))^2 \le 0,\ \forall t \ge 0. \tag{A.2}$$

This indicates that the energy $E(t) = \frac{1}{2}\|u\|^2(t)$ associated with (A.1) is not increasing, and the term $\alpha u_{xx}(0,t)$ designs a damping mechanism. To ensure that this energy decays exponentially is natural to show an *observability inequality* associated with the solutions of (A.1). Before presenting it, let us first prove *a weak observability inequality*.

Proposition A.1 *Consider u solution of* (A.1) *belonging in $C(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$. Thus, we have*

$$\frac{1}{2}\|u_0\|^2 \le \frac{1}{2T}\int_0^T\int_0^1 |u(x,t)|^2 dxdt + \frac{1-\alpha^2}{2}\int_0^T |u_{xx}(0,t)|^2 dt, \tag{A.3}$$

for all $T > 0$.

Proof We prove the result for the initial data $u_0 \in D(A)$, the result in $L^2(0,1)$ follows by density. First, multiplying the system (A.1) by $(T-t)u$, integrating by parts in $(0,T)\times(0,1)$ and using the boundary conditions we have

$$-\frac{T}{2}\|u_0\|^2+\frac{1}{2}\int_0^T\int_0^1 u^2dxdt+\frac{1}{2}(\alpha^2-1)\int_0^T(T-t)(u_{xx}(0,t))^2dt=0,$$

or equivalently,

$$\frac{1}{2}\int_0^T\int_0^1 u^2dxdt+\frac{1}{2}(\alpha^2-1)\int_0^T(T-t)(u_{xx}(0,t))^2dt=\frac{T}{2}\|u_0\|^2.$$

Thus, we get

$$\frac{1}{2}\|u_0\|^2\le\frac{1}{2T}\int_0^T\int_0^1 u^2dxdt+\frac{(\alpha^2-1)}{2}\int_0^T|u_{xx}(0,t)|^2dt,$$

showing the proposition. □

Now, we are in a position to prove that the energy associated with (A.1) decays exponentially.

Theorem A.2 *There exists $C>0$ and $\mu>0$ such that*

$$E(t)\le C\|u_0\|^2e^{-\mu t}, \tag{A.4}$$

for all $t\ge0$ and u solution of (A.1) with $u_0\in L^2(0,1)$.

Proof This result is a consequence of the following observability inequality

$$\frac{1}{2}\int_0^T\int_0^1 u^2dxdt\le c_1\int_0^T|u_{xx}(0,t)|^2dt, \tag{A.5}$$

for some constant $c_1>0$ independent of the solution u.

In fact, replacing (A.5) in (A.3), we get

$$\begin{aligned}\frac{1}{2}\|u_0\|^2\le&\frac{1}{2T}\int_0^T\int_0^1|u(x,t)|^2dxdt+\left(\frac{1-\alpha^2}{2}\right)\int_0^T|u_{xx}(0,t)|^2dt\\ \le&\frac{1}{2T}\int_0^T|u_{xx}(0,t)|^2dt+\left(\frac{1-\alpha^2}{2}\right)\int_0^T|u_{xx}(0,t)|^2dt\\ =&C\int_0^T|u_{xx}(0,t)|^2dt,\end{aligned} \tag{A.6}$$

where $C=C(T,\alpha)>0$. As we have that

$$E'(t)=\frac{1}{2}(\alpha^2-1)(u_{xx}(0,t))^2\le0,$$

integrating in $(0,t)$ the previous equation and multiplying by $(1+C)$, where C is the same constant obtained previously, we have that

$$(1+C)E(T)\le CE(0). \tag{A.7}$$

Thus,

$$E(T) \leq \gamma E(0), \text{ where } 0 < \gamma = \frac{C}{1+C} < 1,$$

with $\gamma := \gamma(T, \alpha)$. Now, the same argument used on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots,$ yields that

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0).$$

Thus, we have

$$E(mT) \leq e^{-\nu m T} E(0)$$

with

$$\nu = \frac{1}{T} \ln\left(1 + \frac{1}{C}\right) > 0.$$

For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0),$$

showing the result. □

Let us now prove the observability inequality.

Proof of (A.5) We argue by contradiction. Suppose that (A.5) does not hold. Thus, there exist a sequence $\{u_n\}_{n\in\mathbb{N}}$ of (A.1) such that

$$\lim_{n\longrightarrow+\infty} \frac{\displaystyle\int_0^T \int_0^1 u^2 dx dt}{\displaystyle\int_0^T |u_{xx}(0,t)|^2 dt} = +\infty. \tag{A.8}$$

Now on, taking $\lambda_n = \sqrt{\displaystyle\int_0^T \int_0^1 |u_n(x,t)|^2 dx dt}$ and $v_n(x,t) = \dfrac{u_n(x,t)}{\lambda_n}$, we have that $\{v_n\}_{n\in\mathbb{N}}$ is a sequence satisfying (A.1) with initial data $v_n(x,0) = \dfrac{u_n(x,0)}{\lambda_n}$ and

$$\int_0^T \int_0^1 |v_n(x,t)|^2 dx dt = \frac{1}{\lambda_n^2} \int_0^T \int_0^1 |u_n(x,t)|^2 dx dt = 1. \tag{A.9}$$

Thanks to the equation (A.8), we have

$$\lim_{n\to+\infty}\int_0^T |(v_n)_{xx}(0,t)|^2dt = \lim_{n\to+\infty}\frac{\displaystyle\int_0^T |(u_n)_{xx}(0,t)|^2dt}{\lambda_n^2} = \lim_{n\to+\infty}\frac{\displaystyle\int_0^T |(u_n)_{xx}(0,t)|^2dt}{\displaystyle\int_0^T\int_0^1 |u_n(x,t)|^2dxdt}=0. \tag{A.10}$$

Due the relation (A.3), since (A.9) and (A.10) are verified, we have that $\{v_n(x,0)\}_{n\in\mathbb{N}}$ is a sequence bounded in $L^2(0,1)$. Therefore, Propositions 2.1 and 2.2 gives the existence of a constant $M>0$ such that

$$\|v_n\|^2_{L^2(0,T;H_0^2(0,1))}\le M, \tag{A.11}$$

for all $n\in\mathbb{N}$. Since $H_0^2(0,1)\hookrightarrow L^2(0,1)$ compactly, we have $\{v_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2(0,1))$. Thus, there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, such that

$$v_n \rightharpoonup v \text{ weakly in } L^2(0,T;H_0^2(0,1)).$$

Moreover, since $v_{n,t}$ is bounded in $L^2(0,T;H^{-3}(0,1))$, so thanks to the Aubin-Lions's theorem we have

$$v_n\to v \text{ strongly in } L^2(0,T;L^2(0,1)). \tag{A.12}$$

By (A.9), we get

$$\|v\|_{L^2(0,T;L^2(0,1))}=1, \tag{A.13}$$

and so using (A.10) and (A.12), verifies that

$$\int_0^T |v_{xx}(0,t)|^2dt\le \liminf_{n\to+\infty}\int_0^T |(v_n)_{xx}(0,t)|^2dt=0, \tag{A.14}$$

which ensures $v_{xx}(0,t)=0$, for all $t\in(0,T)$. Therefore, the function v satisfies

$$\begin{cases} v_t+v_{xxx}-v_{xxxxx}=0, & (x,t)\in I\times\mathbb{R},\\ v(0,t)=v(1,t)=v_x(1,t)=v_x(0,t) & \\ \qquad =v_{xx}(1,t)=v_{xx}(0,t)=0, & t\in\mathbb{R},\\ v(x,0)=v_0, & x\in I. \end{cases} \tag{A.15}$$

The result follows by using [20, Lemma 1.1] that gives us $v=0$, contradicting the hypotheses (A.13). Thus, the observability inequality holds. □

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Declarations

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References

1. Alvarez-Samaniego, B., Lannes, D.: Large time existence for 3D water-waves and asymptotics. Invent. Math. **171**(3), 485–541 (2008)
2. Araruna, F.D., Capistrano-Filho, R.A., Doronin, G.G.: Energy decay for the modified Kawahara equation posed in a bounded domain. J. Math. Anal. Appl. **385**(2), 743–756 (2012)
3. Bona, J.L., Lannes, D., Saut, J.-C.: Asymptotic models for internal waves. J. Math. Pures Appl. (9) **89**(6), 538–566 (2008)
4. Capistrano-Filho, R.A., de S. Gomes, M.M.: Well-posedness and controllability of Kawahara equation in weighted Sobolev spaces. Nonlinear Anal. **207**, 1–24 (2021)
5. Capistrano-Filho, R.A., de Sousa, L.S.: Control results with overdetermination condition for higher order dispersive system. J. Math. Anal. Appl. **506**(1), 1–22 (2022)
6. Capistrano-Filho, R.A., Gonzalez Martinez, V.H.: Stabilization results for delayed fifth-order KdV-type equation in a bounded domain. Math. Control Relat. Fields (2023). https://doi.org/10.3934/mcrf.2023004
7. Capistrano-Filho, R.A., de Sousa, L.S., Gallego, F.A.: Control of Kawahara equation with overdetermination condition: the unbounded cases. Math. Methods Appl. Sci. (2023). https://doi.org/10.1002/MMA.9368
8. Chen, M.: Internal controllability of the Kawahara equation on a bounded domain. Nonlinear Anal. **185**, 356–373 (2019)
9. Chen, M.: Recurrent solutions of the Korteweg–de Vries equation with boundary force. Indian J. Pure Appl. Math. **53**, 112–126 (2022)
10. Chentouf, B.: Well-posedness and exponential stability of the Kawahara equation with a time-delayed localized damping. Math. Methods Appl. Sci. **45**, 10312–10330 (2022)
11. Fleury, M., Mesquita, J.G., Slavík, A.: Massera's theorems for various types of equations with discontinuous solutions. J. Differ. Equ. **269**(12), 11667–11693 (2020)
12. Hasimoto, H.: Water waves. Kagaku **40**, 401–408 (1970) [Japanese]
13. Hernández, E.M.: A Massera type criterion for a partial neutral functional differential equation. Electron. J. Differ. Equ. **2002**, No. 40, 17 pp. (2002)
14. Kawahara, T.: Oscillatory solitary waves in dispersive media. J. Phys. Soc. Jpn. **33**, 260–264 (1972)
15. Lannes, D.: The Water Waves Problem. Mathematical Analysis and Asymptotics. Mathematical Surveys and Monographs, vol. 188, xx+321 pp. Am. Math. Soc., Providence (2013)
16. Liu, Q., Minh, N.V., Nguerekata, G., Yuan, R.: Massera type theorems for abstract functional differential equations. Funkc. Ekvacioj **51**(3), 329–350 (2008)
17. Massera, J.L.: The existence of periodic solutions of systems of differential equations. Duke Math. J. **17**, 457–475 (1950)
18. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Appl. Math. Sci., vol. 44. Springer, New York (1983)
19. Silva, P.N., Vasconcellos, C.F.: Stabilization of the linear Kawahara equation with localized damping. Asymptot. Anal. **58**, 229–252 (2008)
20. Silva, P.N., Vasconcellos, C.F.: Erratum: Stabilization of the linear Kawahara equation with localized damping. Asymptot. Anal. **66**, 119–124 (2010)
21. Zhou, D.: Non-homogeneous initial-boundary-value problem of the fifth-order Korteweg-de Vries equation with a nonlinear dispersive term. J. Math. Anal. Appl. **497**(1), 124848 (2021)
22. Zubelevich, O.E.: A note on theorem of Massera. Regul. Chaotic Dyn. **11**(4), 475–481 (2006)