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Control of Kawahara equation with overdetermination condition: The unbounded cases

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In this manuscript, we consider the internal control problem for the fifth-order KdV type equation, commonly called the Kawahara equation, on unbounded domains. Precisely, under certain hypotheses over the initial and boundary data, we can prove that there exists an internal control input such that solutions of the Kawahara equation satisfy an *integral overdetermination condition*. This condition is satisfied when the domain of the Kawahara equation is posed in the real line, left half-line, and right half-line. Moreover, we are also able to prove that there exists a minimal time in which the integral overdetermination condition is satisfied. Finally, we show a type of exact controllability associated with the "mass" of the Kawahara equation posed in the half-line.

KEYWORDS

higher order KdV type, integral overdetermination condition, internal controllability, unbounded domains

MSC CLASSIFICATION

Primary: 35G31, 35Q53, 93B05, Secondary: 37K10, 49N45

1 | INTRODUCTION

1.1 | Model under consideration

Water wave systems are too complex to easily derive and rigorously from it relevant qualitative information on the dynamics of the waves. Alternatively, under suitable assumptions on amplitude, wavelength, wave steepness, and so on, the study on asymptotic models for water waves has been extensively investigated to understand the full water wave system, see, for instance, [1–6] and references therein for a rigorous justification of various asymptotic models for surface and internal waves.

Formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two nondimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength, and the amplitude of the free surface are parameterized as h, λ, and a, respectively. Moreover, another nondimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces. The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long-wave approximations according to relations between ε and δ.

In this spirit, when we consider $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$, we are dealing with the Korteweg-de Vries (KdV) equation. Under this regime, Korteweg and de Vries [7][1] derived the following equation well known as a central equation among other dispersive or shallow water wave models called the KdV equation:

$$\pm 2u_t + 3uu_x + \left(\frac{1}{3} - \mu\right) u_{xxx} = 0.$$

Another alternative is to treat a new formulation, that is, when $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, to generate the so-called equation Kawahara equation. That equation was derived by Hasimoto and Kawahara [10, 11] as a fifth-order KdV equation and takes the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0.$$

Our main focus is to investigate a type of controllability for the higher-order KdV type equation. We will continue working with an *integral overdetermination condition* started in [12] however in another framework, to be precise, on an *unbounded domain*. To do that, consider the initial boundary value problem (IBVP)

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} + \xi u_{xxxxx} + uu_x = f_0(t) g(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = h_1(t),\ u_x(t,0) = h_2(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{1.1}$$

where α, β, and ξ are real numbers, $u = u(t,x)$, $g = g(x,t)$, and $h_i = h_i(t)$, for $i = 1, 2$, are well-known functions, and $f_0 = f_0(t)$ is a control input. It is important to mention that (1.1) is called KdV and Kawahara equation when $\xi = 0$ and $\xi = -1$, respectively.

1.2 | Framework of the problems

In this work, we will be interested in a kind of internal control property to the Kawahara equation when an *integral overdetermination condition*, on an unbounded domain, is required, namely,

$$\int_{\mathbb{R}^+} u(t,x)\omega(x)dx = \varphi(t),\ t \in [0,T], \tag{1.2}$$

where ω and φ are some known functions. To present the problems under consideration, take the following unbounded domain $Q_T^+ = (0,T) \times \mathbb{R}^+$, where T is a positive number, consider the boundary functions μ and ν, and a source term $f = f(t,x)$ with a special form, to be specified later. Thus, let us deal with the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t),\ u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{1.3}$$

Therefore, the goal of the article is concentrated on proving an *overdetermination control problem*. Precisely, we want to prove that if f take the following special form

$$f(t,x) = f_0(t) g(t,x),\ \ (t,x) \in Q_T^+, \tag{1.4}$$

the solution of (1.3) satisfies the integral overdetermination condition (1.2). In other words, we have the following issue.

Problem $\mathcal{A}$: For given functions u_0, μ, ν, and g in some appropriate spaces, can we find an internal control f_0 such that the solution associated with Equation (1.3) satisfies the integral condition (1.2)?

Naturally, another point to be considered is the following one.

[1]This equation was first introduced by Boussinesq [8], and Korteweg and de Vries rediscovered it 20 years later. Details can be found in [9] and the reference therein.

Problem $\mathcal{B}$: What assumptions are needed to ensure that the solution u of (1.3) is unique and verifies (1.2) for a unique f_0?

Finally, with these results in hand, the last problem of this article is related to the existence of a minimal time for which the integral overdetermination condition (1.2) be satisfied. Precisely, the problem can be seen as follows.

Problem C: Can we find a time $T_0 > 0$, depending on the boundary and initial data, such that if $T \leq T_0$, there exists a function f_0, in appropriated space, in that way that the solution u of (1.3) verifies (1.2)?

In summary, the main goal of this manuscript is to prove that these problems are indeed true. There are some features to be emphasized.

- The integral overdetermination condition is effective and gives good control properties. This kind of condition was first applied in the inverse problem (see, e.g., [13]) and, more recently, in control theory [12, 14, 15].
- One should be able of controlling the system, when the control acts in $[0, T]$, on an unbounded domain, which is new for the Kawahara equation.
- We are also able to prove the existence of a minimal $T > 0$ such that the overdetermination condition is still verified; however, we believe that this time is not optimal.

1.3 | Main results

In this paper, we can present answers to the problems $\mathcal{A}$ and $\mathcal{B}$ that were first proposed in [16]. Additionally, the results of this work extend the results presented in [16] for a new framework for the Kawahara equation, that is, the real line, right half-line, and left half-line. For the sake of simplicity, we will present here the *overdetermination control problem* in the right half-line; for details of the results for the real line and left half-line, we invite the reader to read Section 5 at the end of this article.

In this way, the first result ensures that the *overdetermination control problem*, that is, the internal control problem with an integral condition like (1.2) on an unbounded domain, follows for small data, giving answers for Problems $\mathcal{A}$ and $\mathcal{B}$.

Theorem 1.1. *Let $T > 0$ and $p \in [2, \infty]$. Consider $\mu \in H^{\frac{2}{5}}(0,T) \cap L^p(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T) \cap L^p(0,T)$, $u_0 \in L^2\left(\mathbb{R}^+\right)$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C\left(0,T; L^2\left(0,T; L^2\left(\mathbb{R}^+\right)\right)\right.$ and ω be a fixed function that belongs to the following set*

$$\mathcal{J} = \left\{\omega \in H^5\left(\mathbb{R}^+\right) : \omega(0) = \omega'(0) = \omega''(0) = 0\right\}, \tag{1.5}$$

satisfying

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx$$

and

$$\left|\int_{\mathbb{R}^+} g(t,x)\omega(x)dx\right| \geq g_0 > 0, \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$c_1 = \|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (1.3) satisfying (1.2).

Our second result gives us a small time interval for which the integral overdetermination condition (1.2) holds to the solutions of (1.3). Precisely, the answer to the Problem C can be read as follows.

Theorem 1.2. *Suppose the hypothesis of Theorem 1.1 be satisfied and consider $\delta := T^{\frac{1}{5}} \in (0,1)$, for $T > 0$. Then there exists $T_0 := \delta_0^{\frac{1}{5}} > 0$, depending on $c_1 = c_1(\delta)$ given by*

$$c_1(\delta) := \|u_{0\delta}\|_{L^2(\mathbb{R}^+)} + \|\varphi'_\delta\|_{L^2(0,T)} + \|\mu_\delta\|_{H^{\frac{2}{5}}(0,T)} + \|\nu_\delta\|_{H^{\frac{1}{5}}(0,T)},$$

such that if $T \leq T_0$, there exists a control function $f_0 \in L^p(0,T)$ and a solution u of (1.3) verifying (1.2).

As a consequence of the previous results, we can give a controllability result for the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = u_x(t,0) = 0 & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{1.6}$$

posed in the right half-line. Precisely, we present a control property involving the overdetermination condition (1.2) and the initial state u_0 and final state u_T. To do that, consider the following notation

$$[u(x,t)] = \int_{\mathbb{R}^+} u(x,t) d\eta(x), \tag{1.7}$$

which one will be called of mass, for some σ-finite measure η in $\mathbb{R}^+$. With this in hand, as a consequence of Theorem 1.1, the following exact controllability in the right half-line holds.

Corollary 1.3. *Let $T > 0$ and $p \in [2, \infty]$. Consider u_0, $u_T \in L^2\left(\mathbb{R}^+\right)$ and $g \in C\left(0, T; L^2\left(0, T; L^2\left(\mathbb{R}^+\right)\right)\right)$, satisfying*

$$\left| \int_{\mathbb{R}^+} g(t,x) dx \right| \geq g_0 > 0, \ \forall t \in [0,T], \tag{1.8}$$

where g_0 is a constant. Additionally, consider ω be a fixed function that belongs to the set $\mathcal{J}$ defined in (1.5) and $\varphi \in W^{1,p}(0,T)$ satisfying

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx \ \text{and} \ \varphi(T) = \int_{\mathbb{R}^+} u_T(x)\omega(x)dx. \tag{1.9}$$

Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$\|u_0\|_{L^2(\mathbb{R}^+)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$, a unique solution u of (1.6) and a σ-finite measure η in $\mathbb{R}^+$ such that

$$[u(T,x)] = [u_T(x)]. \tag{1.10}$$

1.4 | Historical background

It is well known that the Cauchy problem and control theory for the Kawahara equation

$$u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = 0 \tag{1.11}$$

has been studied by several mathematicians in recent years in different frameworks: bounded domain of $\mathbb{R}$, on the real line $\mathbb{R}$, on the torus $\mathbb{T}$, right half-line $\mathbb{R}^+$, and left half-line $\mathbb{R}^-$.

Concerning the well-posedness of the Kawahara equation, the first local result is due to Cui and Tao [17]. The authors proved a Strichartz estimate for the fifth-order operator and obtained the local well-posedness in $H^s(\mathbb{R})$, for $s > 1/4$. After that, Cui et al. [18] improved the previous result to the negative regularity of Sobolev space $H^s(\mathbb{R}), s > -1$. It is important to point out that Wang et al. [19] improved to a lower regularity, in this case, $s \geq -7/5$. These papers treated the problem using the Fourier restriction norm method. In [20] and [21], authors showed the local well-posedness in $H^s(\mathbb{R}), s > -7/4$, while their methods are the same, particularly, the Fourier restriction norm method in addition to Tao's $[K; Z]$-multiplier norm method. At the critical regularity Sobolev space $H^{-7/4}(\mathbb{R})$, Chen and Guo [22] proved local and global well-posedness by using Besov-type critical space and I-method. Kato [23] studied local well-posedness for $s \geq -2$ by modifying $X^{s,b}$ space and the ill-posedness for $s < -2$ in the sense that the flow map is discontinuous.

Finally, still regarding the well-posedness results, we refer to two recent works that treat the Kawahara equation. Recently, Cavalcante and Kwak [24] studied the IBVP of the Kawahara equation posed on the right and left half-lines with the nonlinearity as in (1.11). To be precise, they proved the local well-posedness in the low regularity Sobolev space, that is, $s \in \left(-\frac{7}{4}, \frac{5}{2}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}$. Additionally, the authors in [25] extended the argument of [24] to fifth-order KdV-type equations with different nonlinearities, in specific, where the scaling argument does not hold. They are established in some range of s where the local well-posedness of the IBVP fifth-order KdV-type equations on the right half-line and the left half-line holds.

Stabilization and control problems (see [26, 27] for details of these kinds of issues) have been studied in recent years for the Kawahara Equation, however with few results in the literature. A first work concerning the stabilization property for the Kawahara equation in a bounded domain $Q_T = (0,T) \times (0,L)$,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t),\, u(t,L) = h_2(t),\, u_x(t,0) = h_3(t) & \text{on } [0,T], \\ u_x(t,L) = h_4(t),\, u_{xx}(t,L) = h(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L], \end{cases} \tag{1.12}$$

is due to Capistrano-Filho et al. in [26]. In this paper, the authors were able to introduce an internal feedback law in (1.12), considering general nonlinearity $u^p u_x$, $p \in [1,4)$, instead of uu_x, and $h(t) = h_i(t) = 0$, for $i = 1,2,3,4$. To be precise, they proved that under the effect of the damping mechanism the energy associated with the solutions of the system decays exponentially.

Now, some references to internal control problems are presented. This problem was first addressed in [28] and after that in [27]. In both cases, the authors considered the Kawahara equation in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(t,x) = (Gh)(t,x) := g(x)(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)dy),$$

where $g \in C^\infty(\mathbb{T})$ supported in $\omega \subset \mathbb{T}$ and h is a control input. Here, it is important to observe that the control in consideration has a different form as presented in (1.4), and the result is proven in a different direction from what we will present in this manuscript.

Still related to internal control issues, Chen [29] presented results considering the Kawahara equation (1.12) posed on a bounded interval with distributed control $f(t,x)$ and homogeneous boundary conditions. She showed the result by taking advantage of a Carleman estimate associated with the linear operator of the Kawahara equation with an internal observation. With this in hand, she was able to get a null controllable result when f is effective in a $\omega \subset (0,L)$. As the results obtained by her do not answer all the issues of internal controllability, in a recent article [16], the authors closed some gaps left in [29]. Precisely, considering the system (1.12) with an internal control $f(t,x)$ and homogeneous boundary conditions, the authors can show that the equation in consideration is exactly controllable in L^2-weighted Sobolev spaces, and additionally, the Kawahara equation is controllable by regions on L^2-Sobolev space; for details, see [16].

Finally, concerning a new tool to find control properties for dispersive systems, we can cite a recent work of the first two authors [12]. In this work, the authors showed a new type of controllability for a dispersive fifth-order equation that models water waves, what they called *overdetermination control problem*. Precisely, they can find a control acting at the boundary that guarantees that the solution of the problem under consideration satisfies an integral overdetermination condition. In addition, when the control acts internally in the system, instead of the boundary, the authors proved that this condition is satisfied. These problems give answers that were left open in [16] and present a new way to prove boundary and internal controllability results for a fifth-order KdV-type equation.

1.5 | Heuristic and outline of the article

The goal of this manuscript is to investigate and discuss control problems with an integral condition on an unbounded domain. Precisely, we study the internal control problem when the solution of the system satisfies (1.2), so we intend to extend—for unbounded domains—a new way to prove internal control results for the system (1.12), initially proposed in [14, 15], for KdV equation, and more recently in [12], for Kawahara equation in a bounded domain. Thus, for this type of *integral overdetermination condition*, the first results on the solvability of control problems for the IBVP of the Kawahara equation on unbounded domains are obtained in the present paper.

The first result, Theorem 1.1, is concerning the *internal overdetermination control problem*. Roughly speaking, we can find an appropriate control f_0, acting on $[0,T]$ such that integral condition (1.2) it turns out. First, we borrowed the existence of solutions for the IBVP (1.3) of [24]. With these results in hand, for the special case when $s = 0$, Theorem

1.1 is first proved for the linear system associated with (1.3) and after that, using a fixed point argument, extended to the nonlinear system. The main ingredients are auxiliary lemmas presented in Section 3. In one of these lemmas (see Lemma 3.3), we can find two appropriate applications that link the internal control term $f_0(t)$ with the overdetermination condition (1.2), namely,

$$\begin{aligned}\Lambda : L^p(0,T) &\to \tilde{W}^{1,p}(0,T)\\ f_0 &\mapsto (\Lambda f_0)(\cdot) = \int_{\mathbb{R}^+} u(\cdot,x)\omega(x)dx\end{aligned}$$

and

$$\begin{aligned}A : L^p(0,T) &\to L^p(0,T)\\ f_0 &\mapsto (Af_0)(\cdot) = \frac{\varphi'(\cdot)}{g_1(\cdot)} - \frac{1}{g_1(\cdot)}\int_{\mathbb{R}^+} u(t,x)\left(\alpha\omega' + \beta\omega''' - \omega'''''\right) dx,\end{aligned}$$

where

$$g_1(\cdot) = \int_0^L g(\cdot,x)\omega(x)dx.$$

So, we prove that such application has an inverse that is continuous, by Banach's theorem, showing the lemma in question, and so, reaching our goal, to prove Theorem 1.1.

With the previous result in hand, the answer to the Problem C is given by Theorem 1.2. This result gives us a minimal time in which the integral condition (1.2) is satisfied. To be more precise, Theorem 1.2 is proved in three parts. In the first part, we give a refinement of Lemma 3.3, namely, Lemma 3.4. With this in hand, we need, in a second moment, to use the scaling of our Equation (1.3) to produce a "new" Kawahara equation on Q_T^+. This gives us the possibility to use the Theorem 1.1 and, with the help of Lemma 3.4, reach the proof of Theorem 1.2.

Finally, as a consequence of Theorem 1.1, we produce a type of exact controllability result (Corollary 1.3). More precisely, we show that the mass of the system (1.7) is reached on the final time T; that is, (1.10) holds.

Thus, we finish our introduction by showing the structure of the manuscript. Section 2 is devoted to presenting some preliminaries, which are used throughout the article. Precisely, we present the Fourier restriction spaces related to the operator of the Kawahara and reviewed the main results of the well-posedness for the fifth-order KdV equation in these spaces. In Section 3, we present some auxiliary lemmas that help us to prove the internal controllability results. The overdetermination control results, when the control is acting internally, are presented in Section 4; that is, we present the proof of the main results of the manuscript, Theorems 1.1 and 1.2 and Corollary 1.3. Finally, in Section 5, we present some further comments and some conclusions about the generality of the work.

2 | PRELIMINARIES

2.1 | Fourier restriction spaces

Let f be a Schwartz function, that is, $f \in \mathcal{S}_{t,x}(\mathbb{R}\times\mathbb{T})$, $\widetilde{f}$ or $\mathcal{F}(f)$ denotes the space-time Fourier transform of f defined by

$$\widetilde{f}(\tau,\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\xi}e^{-it\tau} f(t,x)\, dxdt.$$

Moreover, we use $\mathcal{F}_x$ (or $\widehat{\ }$) and $\mathcal{F}_t$ to denote the spatial and temporal Fourier transform, respectively.

For given $s, b \in \mathbb{R}$, we define the space $X^{s,b}$ associated to (1.3) as the closure of $\mathcal{S}_{t,x}(\mathbb{R}\times\mathbb{T})$ under the norm

$$\|f\|^2_{X^{s,b}} = \int_{\mathbb{R}^2} \langle\xi\rangle^{2s}\left\langle\tau - \xi^5\right\rangle^{2b}|\tilde{f}(\tau,\xi)|^2 d\xi d\tau$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

As well known, the $X^{s,b}$ space with $b > \frac{1}{2}$ is well adapted to study the IVP of dispersive equations. The function space equipped with the Fourier restriction norm, which is the so-called $X^{s,b}$ spaces, has been proposed by Bourgain [30, 31] to solve the periodic NLS and generalized KdV. Since then, it has played a crucial role in the theory of dispersive equations and has been further developed by many researchers, in particular, Kenig, Ponce, and Vega [32] and Tao [33].

In our case, to study the IBVP (1.3) is requested to introduce modified $X^{s,b}$-type spaces. So, we define the (time-adapted) Bourgain space $Y^{s,b}$ associated with (1.3) as the completion of $\mathcal{S}\left(\mathbb{R}^2\right)$ under the norm

$$\|f\|_{Y^{s,b}}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^{\frac{2s}{5}} \left\langle \tau - \xi^5 \right\rangle^{2b} |\tilde{f}(\tau,\xi)|^2 d\xi d\tau.$$

Additionally, due to the study of the IBVP introduced in [24], they used the low frequency localized $X^{0,b}$-type space with $b > \frac{1}{2}$ in the nonlinear estimates. Hence, we need also define D^α space as the completion of $\mathcal{S}\left(\mathbb{R}^2\right)$ under the norm

$$\|f\|_{D^\alpha}^2 = \int_{\mathbb{R}^2} \langle \tau \rangle^{2\alpha} 1_{\{\xi : |\xi| \le 1\}}(\xi) |\tilde{f}(\tau,\xi)|^2 d\xi d\tau$$

where 1_A is the characteristic functions on a set A. With this in hand, now we set the solution space denoted by $Z_1^{s,b,\alpha}$ with the following norm

$$\|f\|_{Z_1^{s,b,\alpha}(\mathbb{R}^2)} = \sup_{t\in\mathbb{R}} \|f(t,\cdot)\|_{H^s} + \sum_{j=0}^{1} \sup_{x\in\mathbb{R}} \left\|\partial_x^j f(\cdot,x)\right\|_{H^{\frac{s+2-j}{5}}} + \|f\|_{X^{s,b}\cap D^\alpha}.$$

The spatial and time restricted space of $Z_1^{s,b,\alpha}\left(\mathbb{R}^2\right)$ is defined by the standard way:

$$Z_1^{s,b,\alpha}\left((0,T)\times\mathbb{R}^+\right) = Z_1^{s,b,\alpha}\Big|_{(0,T)\times\mathbb{R}^+}$$

equipped with the norm

$$\|f\|_{Z_1^{g,b,\alpha}((0,T)\times\mathbb{R}^+)} = \inf_{g\in Z_1^{s,b,\alpha}} \left\{ \|g\|_{Z_1^{s,b,\alpha}} : g(t,x) = f(t,x) \text{ on } (0,T)\times\mathbb{R}^+ \right\}.$$

2.2 | Overview of the well-posedness results

In this section, we are interested to present the well-posedness results for the Kawahara system, namely,

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t),\ u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{2.1}$$

The results presented here are borrowed from [24] and give us good properties of the IBVP (2.1). The first one gives a relation of the nonlinearity involved in our problem with the Fourier restriction spaces introduce in the previous subsection. Precisely, we have the nonlinear term $f = uu_x$ that can be controlled in the $X^{s,-b}$ norm.

Proposition 2.1. *For $-7/4 < s$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have*

$$\|\partial_x(uv)\|_{X^{s,-b}} \le c\|u\|_{X^{s,b}\cap D^\alpha} \|v\|_{X^{s,b}\cap D^\alpha}. \tag{2.2}$$

Proof. See [24, Proposition 5.1]. □

Now on, we will consider the following: $s = 0$, $b(s) = b_0$, $\alpha(s) = \alpha_0$ and $Z_1^{0,b_0,\alpha_0}\left(Q_T^+\right) = Z\left(Q_T^+\right)$. As a consequence of the previous proposition, we have the following.

Corollary 2.2. *There exists $b_0 \in \left(0, \frac{1}{2}\right)$ such that for all $\alpha_0 > \frac{1}{2}$, it follows that*

$$\|\partial_x(uv)\|_{X^{0,-b_0}(Q_T^+)} \le C\|u\|_{Z(Q_T^+)}\|v\|_{Z(Q_T^+)}, \tag{2.3}$$

for any $u, v \in Z\left(Q_T^+\right)$.

Now, we are interested in a special case of the well-posedness result presented in [24]. To be precise, considering $s = 0$, [24, Theorem 1.1] gives us the following result.

Theorem 2.3. *Let $T > 0$ and $u_0 \in L^2\left(\mathbb{R}^+\right)$, $\mu \in H^{\frac{2}{5}}(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T)$ and $f \in X^{0,-b_0}\left(Q_T^+\right)$, for $b_0 \in \left(0, \frac{1}{2}\right)$. Then there exists a unique solution $u := S(u_0, \mu, \nu, f) \in Z\left(Q_T^+\right)$ of (2.1) such that*

$$\|u\|_{Z(Q_T^+)} \le C_0\left(\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|f\|_{X^{0,-b_0}(Q_T^+)}\right) \tag{2.4}$$

where $C_0 > 0$ is a positive constant depending only of b_0, α_0, and T.

3 | KEY LEMMAS

In this section, we are interested to prove some auxiliary lemmas for the solutions of the system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t),\ u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{3.1}$$

These lemmas will be the key to proofing the main results of this work.
To do this, consider $\omega \in \mathcal{J}$ defined by (1.5) and define $q : [0,T] \to \mathbb{R}$ as follows

$$q(t) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx, \tag{3.2}$$

where $u := S(u_0, \mu, \nu, f_1 + f_{2x})$ is the solution of (3.1) guaranteed by Theorem 2.3. The next two auxiliary lemmas are the key point to show the main results of this work. The first one gives that $q \in W^{1,p}(0,T)$ and can be read as follows.

Lemma 3.1. *Let $T > 0$, $p \in [2,\infty]$ and the assumptions of Theorem 2.3 are satisfied, with $f = f_1 + f_{2x}$, where $f_1 \in L^p(0,T;L^2\left(\mathbb{R}^+\right))$, $f_2 \in L^p(0,T;L^1\left(\mathbb{R}^+\right))$ and $\mu, \nu \in L^p(0,T)$. If $u \in Z\left(Q_T^+\right)$ is a solution of (2.1) and $\omega \in \mathcal{J}$, defined in (1.5), then the function $q \in W^{1,p}(0,T)$ and the relation*

$$\begin{aligned} q'(t) = {} & \omega'''(0)\nu(t) - \omega''''(0)\mu(t) + \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\ & + \int_{\mathbb{R}^+} u(t,x)[\alpha\omega'(x) + \beta\omega'''(x) - \omega'''''(x)]dx \end{aligned} \tag{3.3}$$

holds for almost all $t \in [0,T]$. In addition, the function $q' \in L^p(0,T)$ can be estimated in the following way:

$$\begin{aligned} \|q'\|_{L^p(0,T)} \le C\Bigg(& (\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{\left(L^p\cap H^{\frac{2}{5}}\right)(0,T)} + \|\nu\|_{\left(L^p\cap H^{\frac{1}{5}}\right)(0,T)} \\ & + \|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))} + \|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))} + \|f_{2x}\|_{X^{0,-b_0}(Q_T^+)}\Bigg) \end{aligned} \tag{3.4}$$

with $C = C(|\alpha|, |\beta|, T, \|\omega\|_{\mathbb{R}^+}) > 0$ a constant that is non-decreasing with increasing T.

Proof. Considering $\psi \in C_0^\infty(0,T)$, multiplying (3.1) by $\psi\omega$ and integrating by parts in $[0,T]\times[0,R]$, for some $R>0$, we get, using the boundary condition of (3.1) and the hypothesis that $\omega \in \mathcal{J}$, that

$$\begin{aligned}
-\int_0^T \psi'(t)q(t)dt &= \int_0^T \int_{\mathbb{R}^+} u_t(t,x)\psi(t)\omega(x)dxdt \\
&= \int_0^T \psi(t)\left(\int_{\mathbb{R}^+} u(t,x)\left(\alpha\omega'(x)+\beta\omega'''(x)-\omega'''''(x)\right)dx\right. \\
&\quad + \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\
&\quad \left.-\omega''''(0)\mu(t)+\omega'''(0)\nu(t)\right)dt \\
&= \int_0^T \psi(t)r(t)dt,
\end{aligned}$$

with $r:[0,T]\mapsto\mathbb{R}$ defined by

$$\begin{aligned}
r(t) &= \int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x)+\beta\omega'''(x)-\omega'''''(x))dx-\omega''''(0)\mu(t)+\omega'''(0)\nu(t) \\
&\quad + \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\
&:= I_1+I_2+I_3,
\end{aligned}$$

which gives us $q'(t)=r(t)$, where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x)+\beta\omega'''(x)-\omega'''''(x))dx-\omega''''(0)\mu(t)+\omega'''(0)\nu(t), \\
I_2 &= -\int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx, \\
I_3 &= \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx.
\end{aligned}$$

It remains for us to prove that $q'\in L^p(0,T)$, for $p\in[2,\infty]$. To do it, we need to bound each term of (3.3). We will split this analysis into two steps.

Step 1. $2\le p<\infty$

Let us first bound I_1. To do this, note that, for $t\in[0,T]$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x)+\beta\omega'''(x)-\omega'''''(x))dx\right| \\
&\le \left(|\alpha|\|\omega'\|_{L^2(\mathbb{R}^+)}+|\beta|\|\omega'''\|_{L^2(\mathbb{R}^+)}+\|\omega'''''\|_{L^2(\mathbb{R}^+)}\right)\|u(t,\cdot)\|_{L^2(\mathbb{R}^+)}.
\end{aligned}$$

Moreover, the trace terms are bounded thanks to the fact that $\omega\in\mathcal{J}$. Thus, this yields that

$$\left\|\int_{\mathbb{R}^+} u(t,x)\left(\alpha\omega(x)+\beta\omega'''(x)-\omega'''''(x)\right)dx\right\|_{L^p(0,T)} \le C\left(|\alpha|,|\beta|,\|\omega\|_{H^5(\mathbb{R}^+)}\right)\|u\|_{L^p(0,T;L^2(\mathbb{R}^+))}.$$

Since

$$\|u\|_{L^p(0,T;L^2(\mathbb{R}^+))}\le T^{\frac{1}{p}}\|u\|_{C(0,T;L^2(\mathbb{R}^+))},$$

we have that

$$\left\| \int_{\mathbb{R}^+} u(t,x) \left(\alpha\omega(x) + \beta\omega'''(x) - \omega'''''(x) \right) dx \right\|_{L^p(0,T)} \leq C \left(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)} \right) T^{\frac{1}{p}} \|u\|_{C(0,T;L^2(\mathbb{R}^+))}.$$

Now, let us estimate I_2. In this case, we start observing that

$$\begin{aligned} \left| \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \right| &\leq \int_{\mathbb{R}^+} |f_2(t,x)\omega'(x)|dx \\ &\leq \|\omega'\|_{C(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ &\leq C\|\omega'\|_{H^1(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ &\leq C\|\omega\|_{H^5(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)}, \end{aligned}$$

where we have used the following continuous embedding

$$H^1\left(\mathbb{R}^+\right) \hookrightarrow L^\infty\left(\mathbb{R}^+\right) \cap C\left(\mathbb{R}^+\right).$$

Therefore, we get that

$$\left\| \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \right\|_{L^p(0,T)} \leq C \left(\|\omega\|_{H^5(\mathbb{R}^+)} \right) \|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))}.$$

Similarly, we can bound I_3 as

$$\left\| \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx \right\|_{L^p(0,T)} \leq \|\omega\|_{L^2(\mathbb{R}^+)} \|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))}.$$

With these estimates in hand and using the hypothesis over μ and ν, that is, μ and ν belonging to $L^p(0,T)$, we have $r \in L^p(0,T)$, which implies that $q \in W^{1,p}(0,T)$ and

$$\begin{aligned} \|q'\|_{L^p(0,T)} \leq \tilde{C}(|\alpha|, |\beta|, T, \|\omega\|_{H^5(\mathbb{R}^+)}) &\Big(\|\mu\|_{L^p(0,T)} + \|\nu\|_{L^p(0,T)} + \|u\|_{Z(Q_T^+)} \\ &+ \|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))} + \|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))} \Big). \end{aligned}$$

Finally, using (2.4) in the previous inequality, (3.4) holds.

Step 2. $p = \infty$

Observe that thanks to the relation (3.3) and the fact that

$$H^1\left(\mathbb{R}^+\right) \hookrightarrow (L^\infty\left(\mathbb{R}^+\right) \cap C\left(\mathbb{R}^+\right),$$

we get that

$$\begin{aligned} |q'(t)| \leq &\left(|\alpha|\|\omega'\|_{L^2(\mathbb{R}^+)} + |\beta|\|\omega'''\|_{L^2(\mathbb{R}^+)} + \|\omega'''''\|_{L^2(\mathbb{R}^+)} \right) \|u(t,\cdot)\|_{L^2(\mathbb{R}^+)} \\ &+ \|\omega\|_{L^2(\mathbb{R}^+)} \|f_1(t,\cdot)\|_{L^2(\mathbb{R}^+)} + \|\omega'\|_{H^1(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ &+ \left|\omega''''(0)\right| |\mu(t)| + \left|\omega'''(0)\right| |\nu(t)|. \end{aligned}$$

Thus,

$$\|q'\|_{C(0,T)} \leq C \left(\|u\|_{Z_1(Q_+^T))} + \|f_2\|_{C(0,T;L^1(\mathbb{R}^+))} + \|f_1\|_{C(0,T;L^2(\mathbb{R}^+))} + \|\mu\|_{C(0,T)} + \|\nu\|_{C(0,T)} \right),$$

with $C = C(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)}, |\omega''''(0)|, |\omega'''(0)|) > 0$. Thus, Step 2 is achieved using (2.4), and the proof of the lemma is complete.

□

Remarks. We will give some remarks in order related to the previous lemma.

i. We are implicitly assuming that $f_{2x} \in L^1\left(0,T;L^2\left(\mathbb{R}^+\right)\right)$, but it is not a problem, since the function that we will take for f_2, in our purposes, satisfies that condition.
ii. When $p = \infty$, the spaces $L^p(0,T)$, $L^p\left(0,T;L^2\left(\mathbb{R}^+\right)\right)$ and $L^p\left(0,T;L^2\left(0,T;L^1\left(\mathbb{R}^+\right)\right)\right)$ are replaced by the spaces $C([0,T])$, $C([0,T];L^2\left(\mathbb{R}^+\right))$ and $C\left([0,T];L^1\left(\mathbb{R}^+\right)\right)$, respectively. So, we can obtain $q \in C^1([0,T])$.

Now, consider a special case of the system (3.1), precisely, the following

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = u_x(t,0) = 0 & \text{on } [0,T], \\ u(0,x) = 0 & \text{in } \mathbb{R}^+. \end{cases} \tag{3.5}$$

For the solutions of this system, the next lemma holds.

Lemma 3.2. *Suppose that $f \in L^2(0,T;L^2\left(\mathbb{R}^+\right))$ and $u := S(0,0,0,f)$ is solution of (3.5), then*

$$\int_{\mathbb{R}^+} |u(t,x)|^2 dx \le 2\int_0^t \int_{\mathbb{R}^+} f(\tau,x)u(\tau,x)dxdt, \ \forall t \in [0,T]. \tag{3.6}$$

Proof. Consider $f \in C_0^\infty\left(Q_T^+\right)$ and $u = S(0,0,0,f_1)$ a smooth solution of (3.5). Multiplying (3.5) by $2u$, integrating by parts on $[0,R]$, for $R > 0$, yields that

$$\begin{aligned} \frac{d}{dt}\int_0^R |u(t,x)|^2 dx = {}& 2\int_0^R f(t,x)u(t,x)dx - \alpha\left(|u(t,R)|^2 - |u(t,0)|^2\right) \\ &+ \beta\left(|u_x(t,R)|^2 - |u_x(t,0)|^2\right) + \left(|u_{xx}(t,R)|^2 - |u_{xx}(t,0)|^2\right) \\ &- 2\beta(u_{xx}(t,R)u(t,R) - u_{xx}(t,0)u(t,0)) \\ &+ 2(u_{xxxx}(t,R)u(t,R) - u_{xxxx}(t,0)u(t,0)) \\ &- 2(u_{xxx}(t,R)u_x(t,R) - u_{xxx}(t,0)u_x(t,0)). \end{aligned}$$

So, taking $R \to \infty$, integrating in $[0,t]$ and using the boundary condition of (3.5), we get

$$\int_{\mathbb{R}^+} |u(t,x)|^2 dx \le 2\int_0^t \int_{\mathbb{R}^+} f(\tau,x)u(\tau,x)dxd\tau,$$

showing (3.6) for smooth solutions. The result for the general case follows by density argument. □

Consider the space

$$\tilde{W}^{1,p}(0,T) = \left\{\varphi \in W^{1,p}(0,T); \varphi(0) = 0\right\}, p \in [2,\infty]$$

and define the following linear operator Q

$$Q(u)(t) := q(t),$$

where $q(t)$ is defined by (3.2). Here, we consider the following norm associated to $\tilde{W}^{1,p}(0,T)$

$$\|Q(u)\|_{\tilde{W}^{1,p}(0,T)} = \|q\|_{\tilde{W}^{1,p}(0,T)} = \|q'\|_{L^p(0,T)}.$$

With this in hand, we have the following result.

Lemma 3.3. *Consider $\omega \in \mathcal{J}$, defined by (1.5), and $\varphi \in \tilde{W}^{1,p}(0,T)$, for some $p \in [2,\infty]$, $g \in C\left(0,T;L^2\left(\mathbb{R}^+\right)\right)$. If the following assumption holds*

$$\left|\int_{\mathbb{R}^+} g(t,x)\omega(x)dx\right| \geq g_0 > 0, \forall\, t \in [0,T], \tag{3.7}$$

then there exist a unique function $f_0 = \Gamma(\varphi) \in L^p(0,T)$, such that for $f(t,x) := f_0(t)g(t,x)$ the function $u := S(0,0,0,f)$ solution of (3.5) satisfies (1.2). Additionally, the linear operator

$$\begin{array}{lcl} \Gamma : \tilde{W}^{1,p}(0,T) & \to & L^p(0,T) \\ \varphi & \mapsto & \Gamma(\varphi) = f_0 \end{array} \tag{3.8}$$

is bounded.

Proof. Consider the function

$$G : L^p(0,T) \to L^2\left(0,T;L^2\left(\mathbb{R}^+\right)\right)$$

defined by

$$f_0 \mapsto G(f_0) = f_0 g.$$

By the definition, G is linear. Moreover, we have

$$\begin{aligned} \|G(f_0)\|^2_{L^2(0,T;L^2(\mathbb{R}^+))} &\leq \|g\|^2_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|^2_{L^2(0,T)} \\ &\leq T^{\frac{p-2}{p}}\|g\|^2_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|^2_{L^p(0,T)}. \end{aligned}$$

Thus,

$$\|G(f_0)\|_{L^2(0,T;L^2(\mathbb{R}^+))} \leq T^{\frac{p-2}{2p}}\|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|_{L^p(0,T)}. \tag{3.9}$$

Consider the application

$$\Lambda = Q \circ S \circ G : L^p(0,T) \to \tilde{W}^{1,p}(0,T)$$

which one will be defined by

$$f_0 \mapsto \Lambda(f_0) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx,$$

where $u := S(0,0,0,f)$. Therefore, since Q, S, and G are linear and bounded, we have that Λ is linear and bounded and have the following property

$$(\Lambda f_0)(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx = 0$$

that is, Λ is well-defined.

Introduce the operator

$$\Lambda = A : L^p(0,T) \to L^p(0,T)$$

by

$$f_0 \mapsto A(f_0) \in L^p(0,T),$$

where

$$(Af_0)(t) = \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)}\int_{\mathbb{R}^+} u(t,x)\left(\alpha\omega' + \beta\omega''' - \omega'''''\right)dx.$$

Here, $u = S(0,0,0,f)$ and

$$g_1(t) = \int_{\mathbb{R}^+} g(t,x)\omega(x)dx,$$

for all $t \in [0,T]$. Observe that, using (3.3), $\Lambda(f_0) = \varphi$ if and only if $f_0 = A(f_0)$.

Now we show that the operator A is a contraction on $L^p(0,T)$ if we choose an appropriate norm in this space. To show it, let us split our proof into two cases.

Case one: $2 \le p < \infty$.

Let $f_{01}, f_{02} \in L^p(0,T)$, $u_1 = (S\circ G)f_{01}$ and $u_2 = (S\circ G)f_{02}$, so thanks to (3.6) we get

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)} \le 2\|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|f_{01} - f_{02}\|_{L^1(0,t)},\ \forall t \in [0,T]. \tag{3.10}$$

Consider $\gamma > 0$ and $t \in [0,T]$, using Hölder inequality, we have

$$\begin{aligned}\left|e^{-\gamma t}\left((Af_{01})(t) - (Af_{02})(t)\right)\right| &\le \frac{e^{-\gamma t}}{|g_1(t)|}\int_{\mathbb{R}^+}\left|(u_1(t,x) - u_2(t,x))\left(\alpha\omega' + \beta\omega''' - \omega'''''\right)\right|dx\\ &\le \frac{e^{-\gamma t}}{g_0}\|\alpha\omega' + \beta\omega''' - \omega'''''\|_{L^2(\mathbb{R}^+)}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)}\\ &\le \frac{1}{g_0}\|\omega\|_{H^5(\mathbb{R}^+)}e^{-\gamma t}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)}.\end{aligned}$$

Therefore, now, using (3.10), yields that

$$\begin{aligned}\|e^{-\gamma t}(Af_{01} - Af_{02})\|_{L^p(0,T)} &\le \frac{2\|\omega\|_{H^5(\mathbb{R}^+)}\|g\|_{C(0,T;L^2(\mathbb{R}^+))}}{g_0}\left(\int_0^T e^{-\gamma pt}\left(\int_0^t |f_{01}(\tau) - f_{02}(\tau)|d\tau\right)^p dt\right)^{\frac{1}{p}}\\ &\le C\left(\int_0^T e^{-\gamma pt}\left(\int_0^T |f_{01}(\tau) - f_{02}(\tau)|d\tau\right)^p dt\right)^{\frac{1}{p}}.\end{aligned}$$

Finally, using the last inequality for $p \in [2,\infty)$, such that $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned}\|e^{-\gamma t}(Af_{01} - Af_{02})\|_{L^p(0,T)} &\le c_0\left(\int_0^T e^{-\gamma pt}\left(\int_0^T |f_{01}(\tau) - f_{02}(\tau)|d\tau\right)^p dt\right)^{\frac{1}{p}}\\ &\le c_0\|e^{-\gamma\tau}(f_{01} - f_{02})\|_{L^p(0,T)}\left[\int_0^T e^{-p\gamma t}\left(\int_0^t e^{p'\gamma\tau}\mathrm{d}\tau\right)^{p/p'}\mathrm{d}t\right]^{1/p}\\ &\le \frac{c_0T^{1/p}}{(p'\gamma)^{1/p'}}\|e^{-\gamma t}(f_{01} - f_{02})\|_{L^p(0,T)},\end{aligned} \tag{3.11}$$

where $c_0 = c_0(\|\omega\|_{H^5(\mathbb{R}^+)}, g_0, \|g\|_{C(0,T;L^2(\mathbb{R}^+))})$ is defined by

$$c_0 := \frac{2}{g_0}\|g\|_{C([0,T];L_2(\mathbb{R}^+))}\left(|\alpha|\|\omega'\|_{L_2(\mathbb{R}^+)} + |\beta|\|\omega'''\|_{L_2(\mathbb{R}^+)} + \|\omega'''''\|_{L_2(\mathbb{R}^+)}\right). \tag{3.12}$$

Therefore, it is enough to take $\gamma > \frac{\left(c_0T^{\frac{1}{p}}\right)^{p'}}{p'}$, and so A is contraction, showing the case one for $p \in [2,\infty)$.

Case two: $p = \infty$.

In this case, we have

$$\begin{aligned}\left\|e^{-\gamma t}(Af_{01} - Af_{02})\right\|_{L^\infty(0,T)} &\leq c_0 \sup_{t\in[0,T]} e^{-\gamma t}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)} \\ &\leq c_0 \sup_{t\in[0,T]} e^{-\gamma t}\|f_{01} - f_{02}\|_{L^1(0,t)} \\ &\leq \frac{c_0}{\gamma}\left\|e^{-\gamma t}(f_{01} - f_{02})\right\|_{L^\infty(0,T)},\end{aligned} \tag{3.13}$$

where $c_0 = c_0(T, p, \|\omega\|_{H^5(\mathbb{R}^+)}, g_0, \|g\|_{C(0,T;L^2(\mathbb{R}^+))})$ is defined by (3.12). Therefore, taking $\gamma > c_0$, we have that A is contraction, showing case two.

Thus, in both cases, the operator A is a contraction, and so, for any $\varphi \in \tilde{W}^{1,p}(0,T)$, there exists a unique $f_0 \in L^p(0,T)$ such that $f_0 = A(f_0)$, or equivalently, $\varphi = \Lambda(f_0)$. Thus, it follows that Λ is invertible. Due to the Banach theorem, its inverse

$$\Gamma : L^p(0,T) \mapsto \tilde{W}^{1,p}(0,T)$$

is bounded. Particularly,

$$\|\Gamma\varphi\|_{L^p(0,T)} \leq C(T)\|\varphi'\|_{L^p(0,T)}. \tag{3.14}$$

□

To prove our second main result of this work, we need one refinement of Lemma 3.3.

Lemma 3.4. *Under the hypothesis of Lemma 3.3, if $c_0T \leq p^{1/p}/2$, c_0 given by (3.12), and $p^{1/p} = 1$ for $p = +\infty$, we have the following estimate*

$$\|\Gamma\varphi\|_{L_p(0,T)} \leq \frac{2}{g_0}\|\varphi'\|_{L_p(0,T)}, \tag{3.15}$$

for the operator $\Gamma : \tilde{W}^{1,p}(0,T) \mapsto L^p(0,T)$.

Proof. Since $f_0 = Af_0 = \Gamma\varphi$, taking $\gamma = 0$, similar as we did in (3.11), we get that

$$\left\|f_0 - \frac{\varphi'}{g_1}\right\|_{L^p(0,T)} \leq c_0\left[\int_0^T\left(\int_0^t |f_0(\tau)|\,\mathrm{d}\tau\right)^p \mathrm{d}t\right]^{1/p} \leq \frac{c_0T}{p^{1/p}}\|f_0\|_{L^p(0,T)},$$

and in a way analogous to the one made in (3.13), we also have

$$\left\|f_0 - \frac{\varphi'}{g_1}\right\|_{L^\infty(0,T)} \leq c_0\int_0^T |f_0(\tau)|\,\mathrm{d}\tau \leq c_0T\|f_0\|_{L^\infty(0,T)}.$$

Thus, for $p \in [2, +\infty]$, we get

$$\|\Gamma\varphi\|_{L^p(0,T)} \leq \frac{1}{g_0}\|\varphi'\|_{L^p(0,T)} + \frac{c_0T}{p^{1/p}}\|\Gamma\varphi\|_{L^p(0,T)},$$

and the estimate (3.15) holds true. □

4 | CONTROL RESULTS

In this section, the *overdetermination control problem* is studied. Precisely, we will give answers to some questions left at the beginning of this work. Here, let us consider the full system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.1}$$

First, we prove that when we have the linear system associated with (4.1), the control problem with an integral overdetermination condition holds. After that, we can extend this result, by using the regularity in Bourgain spaces, to the nonlinear one. Finally, we give, under some hypothesis, a minimal time such that the solution of (4.1) satisfies (1.2).

4.1 | Linear case

In this section, let us present the following result.

Theorem 4.1. *Let $T > 0$, $p \in [2, \infty]$, $u_0 \in L^2(\mathbb{R}^+)$, $\mu \in \left(H^{\frac{2}{5}} \cap L^p\right)(0,T)$ and $\nu \in \left(H^{\frac{1}{5}} \cap L^p\right)(0,T)$. Consider $g \in C(0,T; L^2(\mathbb{R}^+))$, $\omega \in \mathcal{J}$, defined by (1.5), and $\varphi \in W^{1,p}(0,T)$ such that*

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx. \tag{4.2}$$

Additionally, if

$$\left|\int_{\mathbb{R}^+} g(t,x)\omega(x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T], \tag{4.3}$$

then there exists a unique $f_0 \in L^p(0,T)$ such that for $f(t,x) := f_0(t)g(t,x) + f_{2x}(t,x)$, with $f_2 \in L^p(0,T; L^1(\mathbb{R}^+))$ and $f_{2x} \in X^{0,-b_0}(Q_T^+)$, the solution $u := S(u_0, \mu, \nu, f_0 g + f_{2x})$ of

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & in \ \ [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & on \ \ [0,T], \\ u(0,x) = u_0(x) & in \ \ \mathbb{R}^+, \end{cases} \tag{4.4}$$

satisfies (1.2).

Proof. Pick $v_1 = S(u_0, \mu, \nu, -f_{2x})$ solution of

$$\begin{cases} v_{1t} + \alpha v_{1x} + \beta v_{1xxx} - v_{1xxxxx} = -f_{2x} & \text{in} \ \ Q_T^+, \\ v_1(t,0) = \mu(t), v_{1x}(t,0) = \nu(t) & \text{on} \ \ [0,T], \\ v_1(0,x) = u_0(x) & \text{in} \ \ \mathbb{R}^+. \end{cases}$$

Define the following function

$$\varphi_1 = \varphi - Q(v_1) : [0,T] \to \mathbb{R}$$

by

$$\varphi_1(t) = \varphi(t) - \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx.$$

Since $\varphi \in W^{1,p}(0,T)$, using Lemma 3.1 together with (4.2), it follows that $\varphi_1 \in \tilde{W}^{1,p}(0,T)$. Therefore, Lemma 3.3 ensures that there exists a unique $\Gamma\varphi_1 = f_0 \in L^p(0,T)$ such that the solution $v_2 := S(0,0,0,f_0 g)$ of

$$\begin{cases} v_{2t} + \alpha v_{2x} + \beta v_{2xxx} - v_{2xxxxx} = f_0 g & \text{in} \ \ Q_T^+, \\ v_2(t,0) = 0, v_{2x}(t,0) = 0 & \text{on} \ \ [0,T], \\ v_2(0,x) = 0 & \text{in} \ \ \mathbb{R}^+, \end{cases}$$

satisfies the following integral condition

$$\int_{\mathbb{R}^+} v_2(t,x)\omega(x)dx = \varphi_1(t), \ t \in [0,T].$$

Thus, taking $u = v_1 + v_2 := S(u_0, \mu, \nu, f_0 g - f_{2x})$, we have u solution of (4.10) satisfying

$$\begin{aligned}\int_{\mathbb{R}^+} u(t,x)\omega(x)dx &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \int_{\mathbb{R}^+} v_2(t,x)\omega(x)dx \\ &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \varphi_1(t) \\ &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \varphi(t) - \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx \\ &= \varphi(t),\end{aligned}$$

for all $t \in [0,T]$, that is, (1.2) holds, showing the result. □

4.2 | Nonlinear case

We are in a position to prove the first main result of this manuscript, that is, Theorem 1.1. Here, is essential the estimates in Bourgain space proved by [24] and presented in Section 2.

Proof of Theorem 1.1. Let $u, v \in Z\left(Q_T^+\right)$. The following estimate holds, using Hölder inequality,

$$\|u(t,\cdot)v(t,\cdot)\|_{L^1(\mathbb{R}^+)} \leq \|u(t,\cdot)\|_{L^2(\mathbb{R}^+)}\|v(t,\cdot)\|_{L^2(\mathbb{R}^+)}, \forall t \in [0,T].$$

So, we get

$$\|uv\|_{C(0,T;L^1(\mathbb{R}^+))} \leq \|u\|_{C(0,T;L^2(\mathbb{R}^+))}\|v\|_{C(0,T;L^2(\mathbb{R}^+))}.$$

Since we have the following embedding $C\left(0,T;L^1\left(\mathbb{R}^+\right)\right) \hookrightarrow L^p\left(0,T;L^1\left(\mathbb{R}^+\right)\right)$ for each $p \in [2,\infty]$, we can find

$$\|uv\|_{L^p(0,T;L^1(\mathbb{R}^+))} \leq T^{\frac{1}{p}}\|u\|_{C(0,T;L^2(\mathbb{R}^+))}\|v\|_{C(0,T;L^2(\mathbb{R}^+))},$$

or equivalently,

$$\|uv\|_{L^p(0,T;L^1(\mathbb{R}^+))} \leq T^{\frac{1}{p}}\|u\|_{Z(Q_T^+)}\|v\|_{Z(Q_T^+)}, \tag{4.5}$$

for any $u, v \in Z\left(Q_T^+\right)$.

Now, pick $f = f_1 - f_{2x}$ in the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.6}$$

Consider so $f_2 = \frac{v^2}{2}$, where $v \in Z\left(Q_T^+\right)$ and $f_1 \in L^2\left(0,T;L^2\left(\mathbb{R}^+\right)\right)$. The estimate (2.3) yields that $f_{2x} = vv_x \in X^{0,-b_0}\left(Q_T^+\right)$, for some $b_0 \in \left(0,\frac{1}{2}\right)$. Moreover, thanks to (4.5), we have that $f_2 \in L^p\left(0,T;L^1\left(\mathbb{R}^+\right)\right)$.

On the space $Z\left(Q_T^+\right)$, let us define the functional $\Theta : Z\left(Q_T^+\right) \to Z\left(Q_T^+\right)$ by

$$u := \Theta v = S\left(u_0, \mu, \nu, \Gamma\left(\varphi - Q(S(u_0,\mu,\nu,-vv_x))\right)g - vv_x\right). \tag{4.7}$$

Note that using Lemma 3.3 and Theorem 4.1, the operator Θ is well-defined.

Considering $p=2$, thanks to (2.4), the embedding $L^2\left(0,T;L^2\left(\mathbb{R}^+\right)\right)\hookrightarrow X^{0,-b_0}\left(Q_T^+\right)$, (3.9), (3.14), and (3.4), we get

$$\begin{aligned}
\|\Theta v\|_{Z(Q_T^+)} \leq & C\left(\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|v\|_{H^{\frac{1}{5}}(0,T)}\right.\\
&\left.+\|\Gamma\left(\varphi-Q(S(u_0,\mu,\nu,-vv_x))\right)g-vv_x\|_{X^{0,-b_0}(Q_T^+)}\right)\\
\leq & C\left(\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|v\|_{H^{\frac{1}{5}}(0,T)}+\|vv_x\|_{X^{0,-b_0}(Q_T^+)}\right.\\
&\left.+\|\Gamma\left(\varphi-Q(S(u_0,\mu,\nu,-vv_x))\right)g\|_{L^2(0,T;L^2(\mathbb{R}^+))}\right)\\
\leq & C\left(\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|v\|_{H^{\frac{1}{5}}(0,T)}+\|vv_x\|_{X^{0,-b_0}(Q_T^+)}\right.\\
&\left.+\|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|\left(\varphi-Q(S(u_0,\mu,\nu,-vv_x))\right)\|_{\tilde{W}^{1,2}(0,T)}\right)\\
\leq & C(\|g\|_{C(0,T;L^2(\mathbb{R}^+))},T)\left(\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|v\|_{H^{\frac{1}{5}}(0,T)}\right.\\
&\left.+\|vv_x\|_{X^{0,-b_0}(Q_T^+)}+\|\varphi'\|_{L^2(0,T)}+\|q'\|_{L^2(0,T)}\right)\\
\leq & 2C(|\alpha|,|\beta|,\|\omega\|_{H^5(\mathbb{R}^+)},\|g\|_{C(0,T;L^2(\mathbb{R}^+))},T)\left(c_1+\|vv_x\|_{X^{0,-b_0}(Q_T^+)}+\|v\|^2_{L^2(0,T;L^1(\mathbb{R}^+))}\right),
\end{aligned}$$

or equivalently,

$$\|\Theta v\|_{Z(Q_T^+)}\leq 2C(|\alpha|,|\beta|,\|\omega\|_{H^5(\mathbb{R}^+)},\|g\|_{C(0,T;L^2(\mathbb{R}^+))},T)\left(c_1+\|vv_x\|_{X^{0,-b_0}(Q_T^+)}+\left\|v^2\right\|_{L^2(0,T;L^1(\mathbb{R}^+))}\right).$$

Now, using the estimates (4.5) and (2.3), we have that

$$\|\Theta v\|_{Z(Q_T^+)}\leq C\left(c_1+\left(T^{\frac{1}{2}}+1\right)\|v\|^2_{Z(Q_T^+)}\right). \tag{4.8}$$

Here, $c_1>0$ is a constant depending such that

$$c_1:=\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|v\|_{H^{\frac{1}{5}}(0,T)}+\|\varphi'\|_{L^2(0,T)}$$

and $C>0$ is a constant depending of $C:=C(|\alpha|,|\beta|,\|\omega\|_{H^5(\mathbb{R}^+)},\|g\|_{C(0,T;L^2(\mathbb{R}^+))},T)$.

Similarly, using the linearity of the operator S, Q, and Γ, once again thanks to (4.5) and (2.3), we have

$$\|\Theta v_1-\Theta v_2\|_{Z(Q_T^+)}\leq C\left(T^{\frac{1}{2}}+1\right)(\|v_1\|_{Z(Q_T^+)}+\|v_2\|_{Z(Q_T^+)})\|v_1-v_2\|_{Z(Q_T^+)} \tag{4.9}$$

Finally, for fixed $c_1>0$, take $T_0>0$ such that

$$8C^2_{T_0}\left(T_0^{\frac{1}{2}}+1\right)c_1\leq 1$$

then, for any $T\in(0,T_0]$, choose

$$r\in\left[2C_Tc_1,\frac{1}{\left(4C_T\left(T^{\frac{1}{2}}+1\right)\right)}\right].$$

On the other hand, for fixed $T>0$, pick

$$r=\frac{1}{\left(4C_T\left(T^{\frac{1}{2}}+1\right)\right)}$$

and

$$c_1 \leq \gamma = \frac{1}{\left(8C_T^2\left(T^{\frac{1}{2}}+1\right)\right)}.$$

Therefore,

$$C_T c_1 \leq \frac{r}{2},\ C_T\left(T^{\frac{1}{2}}+1\right) r \leq \frac{1}{4}.$$

So, Θ is a contraction on the ball $B(0,r) \subset Z\left(Q_T^+\right)$. Theorem (4.1) ensures that the unique fixed point $u = \Theta u \in Z\left(Q_T^+\right)$ is a desired solution for $f_0 := \Gamma(\varphi - Q(S(u_0, \mu, \nu, -uu_x))) \in L^p(0,T)$. Thus, the result is achieved. □

4.3 | Minimal time for the integral condition

We are able now to prove that the integral overdetermination condition (5.2) holds for a minimal time T_0. To do that, let us prove the second main result of this work, namely, Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, let us assume $T \leq 1$. It is well known that the Kawahara equation

$$\begin{cases} u_t - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.10}$$

enjoys the scaling symmetry: If u is a solution to (4.10), $u_\delta(t,x)$ defined by

$$u_\delta(t,x) := \delta^4 u(\delta^5 t, \delta x),\ \delta > 0$$

is solution of (4.10) as well as. Indeed, let $\delta = T^{\frac{1}{5}} \in (0,1)$; thus,

$$\begin{aligned} u_{0\delta}(x) &:= \delta^4 u_0(\delta x),\ \mu_\delta(t) := \delta^4\mu(\delta^5 t),\ \nu_\delta(t) := \delta^4\nu(\delta^5 t) \\ g_\delta(t,x) &:= \delta g(\delta^5 t, \delta x),\ \omega_\delta(x) := \omega(\delta x),\ \varphi_\delta(t) := \delta^4\varphi(\delta^5 t). \end{aligned}$$

Therefore, if the par (f_0, u) is solution of (4.10), a straightforward calculation gives that

$$\{f_{0\delta}(t) := \delta^8 f_0(\delta^5 t),\ u_\delta(t,x) := \delta^4 u(\delta^5 t, \delta x)\}$$

is solution of

$$\begin{cases} u_{\delta t} - u_{\delta xxxxx} + u_\delta u_{\delta x} = f_{0\delta}(t) g_\delta(t,x) & \text{in } [0,1]\times\mathbb{R}^+, \\ u_\delta(t,0) = \mu_\delta(t),\ u_{\delta x}(t,0) = \nu_\delta(t) & \text{on } [0,1], \\ u_\delta(0,x) = u_{0\delta}(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.11}$$

Additionally, we have that (f_0, u) satisfies (1.2) if and only if $(f_{0\delta}(t), u_\delta(t,x))$ satisfies the following integral condition

$$\int_{\mathbb{R}^+} u_\delta(t,x)\omega_\delta(x)dx = \varphi_\delta(t),\ t \in [0,1]. \tag{4.12}$$

Now, using the change of variables theorem and the definition of δ, we verify that

$$\|u_{0\delta}\|_{L^2(\mathbb{R}^+)} = \delta^{\frac{1}{2}}\delta^4\|u_0\|_{L^2(\mathbb{R}^+)} \leq \delta^{\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^+)}$$

and

$$\|\varphi_\delta'\|_{L^2(0,1)} = \delta^{\frac{1}{2}}\delta^{11}\|\varphi_\delta\|_{L^2(0,T)} \leq \delta^{\frac{1}{2}}\|\varphi_\delta\|_{L^2(0,T)}.$$

Thus, we have that

$$c_1(\delta) := \|u_{0\delta}\|_{L^2(\mathbb{R}^+)} + \|\varphi'_\delta\|_{L^2(0,1)} + \|\mu_\delta\|_{H^{\frac{2}{5}}(0,1)} + \|\nu_\delta\|_{H^{\frac{1}{5}}(0,1)} \leq \delta^{\frac{1}{2}} c_1.$$

Moreover,

$$\|g_\delta\|_{C([0,1];L^2(\mathbb{R}^+))} \leq \delta^{\frac{1}{2}} \|g\|_{C([0,T];L^2(\mathbb{R}^+))},$$
$$\left| \int_{\mathbb{R}^+} g_\delta(t,x)\omega_\delta dx \right| \geq g_0, \ \forall t \in [0,1]$$

and

$$\|\omega_\delta'''''\|_{L^2(\mathbb{R}^+)} \leq \delta^{\frac{9}{2}} \|\omega'''''\|_{L^2(\mathbb{R}^+)}.$$

So, as we want that $c_{0\delta}$ be one corresponding to c_0, which was defined by (3.12), therefore,

$$c_{0\delta} \leq \delta^5 c_0.$$

Pick $\delta_0 = (2c_0)^{-1/5}$, so for $0 < \delta \leq \delta_0$ we can apply Lemma 3.4, and according to (3.15), the corresponding operator to Γ, which one will be called of γ_δ, satisfies

$$\|\Gamma_\delta \varphi_\delta\|_{L^p(0,1)} \leq \frac{2}{g_0} \|\varphi'_\delta\|_{L^p(0,1)}. \tag{4.13}$$

Therefore, for Θ_δ defined in the same way as in (4.7) and using, similarly as in (4.8) and (4.9), the relation (4.13) we have that

$$\|\Theta_\delta v_\delta\|_{Z(Q_1^+)} \leq C\left(\delta^{\frac{1}{2}} c_1 + \left(T^{\frac{1}{2}} + 1\right) \|v_\delta\|^2_{Z(Q_1^+)}\right)$$

and

$$\|\Theta_\delta v_{1_\delta} - \Theta_\delta v_{2_\delta}\|_{Z(Q_1^+)} \leq C\left(T^{\frac{1}{2}} + 1\right)\left(\|v_{1_\delta}\|_{Z(Q_1^+))} + \|v_{2_\delta}\|_{Z(Q_1^+)}\right) \|v_{1_\delta} - v_{2_\delta}\|_{Z(Q_1^+)},$$

where the constant C is uniform with respect to $0 < \delta \leq \delta_0$. Taking δ_0 small enough, if necessary, to satisfy the following inequality

$$\delta_0^{\frac{1}{2}} c_1 \leq \frac{1}{8c^2\left(T^{\frac{1}{2}} + 1\right)},$$

so using the same arguments as done in Theorem 1.1, the operator Θ_δ becomes, at least, locally, a contraction on a certain ball. Lastly, taking the time T_0 defined by $T_0 := \delta_0^5$, and if $T \leq T_0$, we have that (1.2) holds, showing the result. □

Remark 4.2. Note that the system (4.1) does not admit the scaling symmetry due to the presence of the terms $\alpha u_x + \beta u_{xxx}$. So, in this case, we analyzed Equation (4.10), since in the analysis of Theorem 1.2, the most important term is of order 5, so we can neglect the terms of order 1 (αu_x) and 3 (βu_{xxx}).

4.4 | An exact controllability result

The goal of this subsection is to prove the Corollary 1.3, showing that if the overdetermination condition is verified, for given any initial data u_0 and final data u_T, the mass (1.7) of the system (1.6) is reached on the time T.

Proof of Corollary 1.3. Thanks to the Theorem 1.1 with $\mu = \nu = 0$, there exist $f_0 \in L^p(0,T)$ and a unique solution u of (1.6) such that

$$\varphi(t) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx, \ t \in [0,T]. \tag{4.14}$$

On the other hand, we know that ω defined a measure in $\mathbb{R}^+$ given by

$$\eta(E) = \int_E w(x)dx,$$

for all Lebesgue measure set E of $\mathbb{R}^+$ and

$$\int_{\mathbb{R}^+} f d\eta = \int_{\mathbb{R}^+} f(x)\omega(x)dx,$$

for all measurable function f in $\mathbb{R}^+$. Hence, from (1.9) and (4.14), we conclude that

$$[u(T)] = \int_{\mathbb{R}^+} u(T)d\eta = \int_{\mathbb{R}^+} u(T,x)\omega(x)dx = \int_{\mathbb{R}^+} u_T(x)\omega(x)dx = \int_{\mathbb{R}^+} u_T d\eta = [u_T],$$

and the corollary is achieved. □

5 | FURTHER COMMENTS

This work deals with the internal controllability problem with an integral overdetermination condition on unbounded domains. Precisely, we consider the higher order KdV type equation, the so-called, Kawahara equation on the right half-line

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t),\ u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{5.1}$$

where $f := f_0(t)g(x,t)$, with f_0 as a control input. In this case, we prove that given functions u_0, μ, ν, and g, the following integral overdetermination condition

$$\int_{\mathbb{R}^+} u(t,x)\omega(x)dx = \varphi(t),\ t \in [0,T], \tag{5.2}$$

holds. Additionally, that condition can be verified for a small time T_0. These points answer the previous questions introduced in [16] and extend to other domains the results of [12].

5.1 | Comments about the main results

Let us give some remarks order concerning the generality of this manuscript.

1. Theorems 1.1 and 1.2 can be obtained for more general nonlinearity u^2u_x. This is possible due to the result of Cavalcante and Kwak [25] that showed the following:

   Theorem 5.1. *The following estimates hold.*

   - *For* $-1/4 \leq s$*, there exists* $b = b(s) < 1/2$ *such that for all* $\alpha > 1/2$*, we have*

   $$\|\partial_x(uvw)\|_{X^{s,-b}} \lesssim \|u\|_{X^{s,b}\cap D^\alpha} \|v\|_{X^{s,b}\cap D^\alpha} \|w\|_{X^{s,b}\cap D^\alpha}.$$

   - *For* $-1/4 \leq s \leq 0$*, there exists* $b = b(s) < 1/2$ *such that for all* $\alpha > 1/2$*, we have*

   $$\|\partial_x(uvw)\|_{Y^{s,-b}} \lesssim \|u\|_{X^{s,b}\cap D^\alpha} \|v\|_{X^{s,b}\cap D^\alpha} \|w\|_{X^{s,b}\cap D^\alpha}.$$

   Thus, Theorems 1.1 and 1.2 remain valid for u^2u_x; however, for the sake of simplicity, we consider only the nonlinearity as uu_x.
2. Due to the boundary traces defined in [24, Theorems 1.1 and 1.2], the regularities of the functions involved in this manuscript are sharp.

3. The results presented in this manuscript are still valid when we consider the following domains: the real line ($\mathbb{R}$) or the left half-line ($\mathbb{R}^-$). Precisely, let us consider the following systems

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{5.3}$$

and

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T]\times\mathbb{R}^-, \\ u(t,0) = \mu(t),\ u_x(t,0) = \nu(t),\ u_{xx}(t,0) = h(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^-. \end{cases} \tag{5.4}$$

For given $T > 0$, φ, ω, and ω^-, consider the following integral conditions

$$\int_{\mathbb{R}} u(t,x)\omega(x)dx = \varphi(t),\, t \in [0,T] \tag{5.5}$$

and

$$\int_{\mathbb{R}^-} u(t,x)\omega^-(x)dx = \varphi(t),\, t \in [0,T]. \tag{5.6}$$

Thus, the next two theorems give us answers for the Problems $\mathcal{A}$ and $\mathcal{B}$, presented at the beginning of the manuscript, for real line and left half-line, respectively.

Theorem 5.2. *Let $T > 0$ and $p \in [2,\infty]$. Consider $u_0 \in L^2(\mathbb{R})$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T;L^2(\mathbb{R}))$ and $\omega \in H^5(\mathbb{R})$ be a fixed function satisfying*

$$\varphi(0) = \int_{\mathbb{R}} u_0(x)\omega(x)dx$$

and

$$\left|\int_{\mathbb{R}} g(t,x)\omega(x)dx\right| \geq g_0 > 0,\ \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if $c_1 = \|u_0\|_{L^2(\mathbb{R}}+\|\varphi'\|_{L^2(0,T)} \leq \gamma$, we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (5.3) satisfying (5.5).

Theorem 5.3. *Let $T > 0$ and $p \in [2,\infty]$. Consider $\mu \in H^{\frac{2}{5}}(0,T) \cap L^p(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T) \cap L^p(0,T)$, $h \in L^p(0,T)$, $u_0 \in L^2(\mathbb{R}^-)$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T;L^2(\mathbb{R}^-))$ and ω^- be a fixed function which belongs to the following set*

$$\mathcal{J} = \{\omega \in H^5(\mathbb{R}^-) : \omega(0) = \omega'(0) = 0\} \tag{5.7}$$

satisfying

$$\varphi(0) = \int_{\mathbb{R}^-} u_0(x)\omega^-(x)dx$$

and

$$\left|\int_{\mathbb{R}^-} g(t,x)\omega^-(x)dx\right| \geq g_0 > 0,\ \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$c_1 = \|u_0\|_{L^2(\mathbb{R}^-)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|h\|_{L^2(0,T)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (5.4) satisfying (5.6).

4. The difference between the numbers of boundary conditions in (5.1) and (5.4) is motivated by integral identities on smooth solutions to the linear Kawahara equation

$$u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = 0.$$

5. Theorem 1.2 is also true for the systems (5.3) and (5.4). Additionally, due to the results presented in [24, 25], the functions involved in Theorems 5.2 and 5.3 are also sharp, and we can introduce a more general nonlinearity like $u^2 u_x$ in these systems.
6. Corollary 1.3 may be extended for the system (5.3) taking into account the integral condition (5.5). Also for the system (5.4), with $u(t,0) = u_x(t,0) = u_{xx}(t,0) = 0$ and the integral condition (5.6), this corollary is verified.

5.2 | General control result

Finally, we would like to comment on a more general control result. Thanks to the Corollary 1.3 it is possible to obtain an exact controllability property related to the mass of the system. However, we would like to show the following exact controllability result:

Exact control problem: *Given* $u_0, u_T \in L^2(\mathbb{R}^+)$ *and* $g \in C(0,T; L^2(\mathbb{R}^+))$ *satisfying* (1.8), *can we find a control* $f_0 \in L^p(0,T)$ *such that the solution* u of (1.6) *satisfies* $u(T,x) = u_T(x)$?
A possibility to answer this question is to modify the overdetermination condition (1.2). For example, if Theorem 1.1 is verified for the following integral condition

$$\widetilde{\varphi}(t) = \int_{\mathbb{R}^+} u^2(t,x) w(x) dx, \tag{5.8}$$

we are able to get the exact controllability in $L^2(\mathbb{R}^+)$ with internal control $f_0 \in L^2(0,T)$ by using the same argument as in Corollary 1.3. However, with the approach used in this manuscript, it is not clear that the Lemma 3.3 can be replied to for the condition (5.8).
Indeed, if we consider

$$q(t) = \int_{\mathbb{R}^+} u^2(t,x) w(x) dx,$$

analyzing $q'(t)$ for $u = S(0,0,0, f_0(t)g(t,x))$ (see Lemma 3.2) we obtain

$$\begin{aligned} q'(t) = & \int_{\mathbb{R}^+} u^2(t,x) \left[\alpha w'(x) + \beta w''(x) - 2w'''''(x)\right] dx \\ & + \int_{\mathbb{R}^+} u_x^2(t,x) \left[5w'''(x) - 3\beta w'(x) - 2w'''''(x)\right] dx \\ & - 5\int_{\mathbb{R}^+} u_{xx}^2(t,x) w'(x) dx + f_0(t) \int_{\mathbb{R}^+} g(t,x) u(t,x) w(x) dx. \end{aligned}$$

Now, introduce the operator

$$\widetilde{A} : L^p(0,T) \to L^p(0,T)$$

defined by

$$f_0 \mapsto \widetilde{A}(f_0) \in L^p(0,T),$$

where

$$\begin{aligned} \left(\widetilde{A} f_0\right)(t) = & \varphi'(t) - \int_{\mathbb{R}^+} u^2(t,x) \left[\alpha w'(x) + \beta w''(x) - 2w'''''(x)\right] dx \\ & - \int_{\mathbb{R}^+} u_x^2(t,x) \left[5w'''(x) - 3\beta w'(x) - 2w'''''(x)\right] dx + 5\int_{\mathbb{R}^+} u_{xx}^2(t,x) w'(x) dx. \end{aligned}$$

If we assume that $\Lambda(f_0) = \widetilde{\varphi}$, we deduce that

$$\left(\widetilde{A} f_0\right)(t) = f_0(t) \int_{\mathbb{R}^+} g(t,x) u(t,x) w(x) dx.$$

Note that this expression depends on the solution of the system (1.6), then we do not be able to obtain the overdetermination control condition for $S(0,0,0,f_0(t)g(t,x))$ by using a fixed point argument for the operator

$$\left[\int_{\mathbb{R}^+} g(t,x) u(t,x) w(x) dx\right]^{-1} \left(\widetilde{A} f_0\right)(t),$$

as in the proof of Lemma 3.3. Therefore, the exact controllability with internal control does not hold. Hence, the following open question arises:
Question: *Is it possible to prove Theorem* 1.1 *for the overdetermination condition* (5.8)?

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Roberto Capistrano Filho: Conceptualization; formal analysis; investigation; writing—original draft; writing—review and editing. **Luan Soares de Sousa**: Conceptualization; formal analysis; investigation; writing—original draft; writing—review and editing. **Fernando Gallego**: Conceptualization; formal analysis; investigation; writing—original draft; writing—review and editing.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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