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SOME CONTROL RESULTS FOR THE KdV-TYPE EQUATION

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SOME CONTROL RESULTS FOR THE KDV-TYPE EQUATION

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To God, my family, and my friends.

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RESUMO

Este trabalho trata da controlabilidade e estabilização de equações dispersivas de quinta ordem em domínio limitado e ilimitado. No primeiro resultado, provamos um novo tipo de controlabilidade para uma equação dispersiva de quinta ordem que modela ondas de água, o qual chamamos de *problema de controle sobredeterminado*. Precisamente, somos capazes de encontrar um controle agindo na fronteira que nos fornece que as soluções do problema considerado satisfazem uma condição integral sobredeterminada. Adicionalmente, quando o controle age internamente no sistema, em vez de na fronteira, também somos capazes de provar um resultado de controlabilidade. No segundo resultado, estendemos a propriedade de controle sobredeterminado para domínios ilimitados. Essa condição é satisfeita quando o domínio da equação Kawahara é a reta real, a semi-reta positiva e a semi-reta negativa. Além disso, mostramos um tipo de controle exato associado com a ”massa” da equação Kawahara sobre a semi-reta positiva. O terceiro, e último, trabalho trata do decaimento exponencial da energia associada às soluções da equação de Kawahara. Precisamente, provamos que o sistema dispersivo de quinta ordem, com termos de amortecimento e delay na fronteira, é exponencialmente estável. Fazemos isto usando dois procedimentos distintos: O primeiro resultado é obtido utilizando o método de Lyapunov, que assegura o decaimento exponencialmente. O segundo resultado, é obtido por meio do argumento de *compacidade-unicidade*, o qual reduz nosso estudo a provar uma desigualdade de observabilidade.

Palavras-chave: Equação de Kawahara, Controlabilidade, Condição sobredeterminada, Estabilidade, Funcional de Lyapunov.

ABSTRACT

This work deals with the controllability and stabilization of fifth-order dispersive equations in bounded and unbounded domains. In the first result, we prove a new type of controllability for a fifth-order dispersive equation that models water waves, which we call *overdetermination control problem*. Precisely, we can find a control acting on the boundary that provides us that the solutions of the considered problem satisfy an overdetermined integral condition. Additionally, when the control acts internally in the system rather than at the boundary, we are also able to prove a controllability result. In the second result, we extend the overdetermined control property to unbounded domains. This condition is satisfied when the domain of the Kawahara equation is the real line, left half-line, and right half-line. Furthermore, we show a type of exact control associated with the "mass" of the Kawahara equation over the right half-line. The third, and last, work deals with the exponential decay of the energy associated with the solutions of the Kawahara equation. Precisely, we prove that the fifth-order dispersive system, with a damping mechanism and delay terms on the boundary, is exponentially stable. We do this using two different procedures: The first result is obtained using the Lyapunov method, which ensures exponential decay. The second result is obtained through the *compactness-uniqueness* argument, which reduces our study to proving an observability inequality.

Keywords: Kawahara equation, Controllability, Overdetermination condition, Stability, Lyapunov function.

Contents

List of symbols

$\|\cdot\|$	when it was not specified, denotes a norm on a vector space X;
$\langle\cdot,\cdot\rangle$	when it was not specified, denotes different types of dualities;
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$	denotes the gradient of u;
$\Delta u = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$	denotes the laplacian of u;
C	when it was not specified, denotes an arbitrary positive constant;
$\mathcal{L}(X,Y)$	denotes the space of continuous linear operators from X onto Y;
$D(f)$	denotes the domain of f;
X^*	denotes the topologic dual of the vector space X;
$\mathcal{S}'(\mathbb{R})$	denotes the space of tempered distributions;
δ_ζ	denotes the Dirac measure at $x = \zeta$.

1 Introduction

The ideas of control and stabilization are present in all fields of the known universe, from our brain, when we try to balance a stick at the tip of the finger, to the entire solar system, which moves slowly in space-time. In this work, we will deal with some results of controllability and stabilization of the fifth-order dispersive equation, commonly called the Kawahara equation, which, among other phenomena, models water waves in shallow channels. The results obtained here were motivated by studies on the standard Korteweg–de Vries (KdV) equation, due to this let us start by giving some historical review of the water waves.

1.1 Historical review

1.1.1 The water waves

In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon which he called a wave of translation [85]. He saw a particular wave traveling through this channel without losing its shape or velocity and was so captivated by this event that he focused his attention on these waves for several years and asked the mathematical community to find a specific mathematical model describing them. More precisely, his words were:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of a vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation …."

Russell was fascinated with his discovery to the point that he not only built water wave tanks at his home but also did practical and theoretical research into these types of waves. His experiments, well-known as The wave line system of hull construction, consisted of raising an area of fluid behind an obstacle, then removing the obstacle so that a long, heap-shaped wave propagated down the channel. His developments revolutionized naval architecture in the nineteenth century, and he was awarded the gold medal of the Royal Society of Edinburgh for his work in 1837. Russell's experiments contradicted physical conjectures such as G.B. Airy's water wave theory [1], in which the traveling wave could not exist because it eventually changed its speed or its shape, or G.G. Stokes' theory [91], where waves of finite amplitude and fixed form were possible, but only in deep water and only in periodic form. However, Stokes was aware of the unfinished state of Russell's theory:

"It is the opinion of Mr. Russell that the solitary wave is a phenomenon sui generis, in nowise deriving its character from the circumstances of the generation of the wave. His experiments seem to render this conclusion probable. Should it be correct, the analytical character of the solitary wave remains to be discovered."

Consequently, to convince the physics community, Scott Russell challenged the mathematical community to prove theoretically the existence of the phenomenon that he witnessed:

"Having ascertained that no one had succeeded in predicting the phenomenon which I have ventured to call the wave of translation,... it was not to be supposed that after its existence had been discovered and its phenomena determined, endeavors would not be made... to show how it ought to have been predicted from the known general equations of fluid motion. In other words, it now remained to the mathematician to predict the discovery after it had happened, i.e. to give a priori demonstration a posteriori."

Several researchers took up Russell's challenge. The first mathematician to respond was Joseph Boussinesq, a French mathematician, and physicist who got important results [18] in 1871. In 1876, the English physicist Lord Rayleigh obtained a different result [81], and in 1895 the Dutch mathematicians D.J. Korteweg and his student G. de Vries gave the last significant result of the 19th-century [69]. Boussinesq considered a model of long, incompressible, and rotation-free waves in a shallow channel with a rectangular cross-section neglecting the friction along the boundaries, and he obtained the equation

$$\frac{\partial^2 h}{\partial t^2} = gH\frac{\partial^2 h}{\partial x^2} + gH\frac{\partial^2}{\partial x^2}\Big[\frac{3h^2}{2H} + \frac{H^2}{3}\frac{\partial^2 h}{\partial x^2}\Big],$$

where (t, x) are the coordinates of a fluid particle at time t, h is the amplitude of the wave, H is the height of the water in equilibrium and g is the gravitational constant. Rayleigh independently considered the same phenomenon and added the assumption of the existence of a stationary wave vanishing at infinity. He considered only spatial dependence and captured the desired behavior in the equation

$$\left(\frac{dh}{dx}\right)^2 + \frac{3}{H^3}h^2(h - h_0) = 0,$$

with h_0 being the crest of the wave and the other parameters defined as before. This equation has an explicit solution given by

$$h(x) = h_0 Sech^2\left(\sqrt{\frac{3h_0}{4H^3}}x\right).$$

In 1876, Rayleigh wrote in his article [81]:

"I have lately seen a memoir by M. Boussinesq, Comptes Rendus, Vol. LXXII, which is contained a theory of the solitary wave very similar to that of this paper. So as far as our results are common, the credit of priority belongs of course to Boussinesq J."

The last proof of the existence of "translation waves" was given by Diederik Johannes Korteweg and Gustav de Vries. They constructed a nonlinear partial differential equation that has a solution describing the phenomenon discovered by Russell, thus giving the Korteweg–de Vries equation its name, often abbreviated as the KdV equation. In 1895, they published an article deriving the equation

$$\frac{\partial \eta}{\partial l} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right),$$

in which η is the surface elevation above the equilibrium level l, α is a small arbitrary constant related to the motion of the liquid, g is the gravitational constant, and $\sigma = \frac{l^3}{3} - \frac{Tl}{\rho g}$, with surface capillary tension T and density ρ. Eliminating the physical constants by the change of variables

$$t \to \frac{1}{2}\sqrt{\frac{g}{l\sigma}}t, \quad x \to -\frac{x}{\sqrt{\sigma}} \quad \text{e} \quad u \to -\left(\frac{1}{2}\eta + \frac{1}{3}\alpha\right)$$

one obtains the standard Korteweg- de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

which is a model describing the propagation of small amplitude, long wavelength waves on an air-sea interface in a canal of the rectangular cross-section. The steady periodic wave-train solution is called the cnoidal wave. C.S. Gardner and G.K. Morikawa [55] found a new application of this model in the study of collision-free hydro-magnetic waves in hopes of describing the unidirectional propagation of small but finite amplitude waves in a nonlinear dispersive medium. Also, M. Kruskal and N. Zabusky [100] showed that the KdV equation models the Fermi-Pasta-Ulam problem, as it describes longitudinal waves propagating in a one-dimensional lattice of equal masses coupled by nonlinear springs. Other applications have been found and are studied until the present day.

Still talking about water wave equations, under certain circumstances, the coefficient of the third-order derivative in the KdV equation may become very small or even zero. So it is necessary to take into account the higher-order effect of dispersion to balance the nonlinear effects. Hence, in 1972, Takuji Kawahara, in an article titled "Oscillatory Solitary Waves in Dispersive Media" [65], generalized the KdV equation, whose most common denominations are Kawahara equation or generalized KdV equation, which is

$$u_t + \frac{3}{2}uu_x + \alpha u_{xxx} - \beta u_{xxxxx} = 0,$$

where $\alpha > 0$ and $\beta > 0$. This equation differs from KdV by the fifth-order derivative. Fifth-order dispersive differential equations, the main type of equation analyzed in this thesis, describe the propagation of waves of small amplitudes in one dimension. In addition, fluid-related problems and plasma physics are generally physical formulations represented by the Kawahara equation.

1.2 Derivation of the equations

In this section, we presented a deduction of two dispersive models of shallow water waves, a Boussinesq system, and the Korteweg-de Vries equation which is based on [6,82]. These equations are consequences of the Euler equation under certain conditions. Our interest is to motivate the study of the water waves equation.

1.2.1 The material derivative

Fluid flow may be represented mathematically as a continuous transformation of three-dimensional Euclidean space into itself. The transformation is parametrized by a real parameter t representing time.

Let us introduce a fixed rectangular coordinate system (x_1, x_2, x_3). We refer to the coordinate triple (x_1, x_2, x_3) as the position and denote by x. Now consider a particle P moving with the fluid, and suppose that at time $t = 0$ it occupies a position $X = (X_1, X_2, X_3)$ and that at some other time $-\infty < t < +\infty$, it has moved to a position $x = (x_1, x_2, x_3)$. Then x is determined as a function of X and t:

$$x = x(t, X), \;\; (or \;\, x_i = x_i(t, X), i = 1, 2, 3.). \tag{1.1}$$

The initial coordinates X of the particle will be referred to as the material coordinates of the particle. The spatial coordinates x may be referred to as its position, or place. If x is fixed and t varies, the equation (1.1) specifies the path of the particle initially at X. On the other hand, for fixed t, (1.1) determines a transformation of a region initially occupied by the fluid into its position at time t.

We assume that the transformation (1.1) is continuous and invertible, that is, there exists its inverse

$$X = X(t, x), \;\; (or \;\, X_i = X_i(t, x), i = 1, 2, 3.).$$

Besides that, to be able to differentiate, we assume that the functions x_i and X_i are sufficiently smooth. From the condition that the transformation (1.1) possesses an inverse and it is differentiable, it follows that its Jacobian

$$J(t, X) = \det \left(\frac{\partial x_j}{\partial X_j} \right)$$

is such that $0 < J(t, X) < +\infty$.

The representation of fluid motion as a point transformation violates the concept of the kinetic theory of fluids, because in this theory the particles are molecules, and they are in random motion. In the theory of continuum mechanics the state of motion at a given point x and at a given time t is described by several variables such as density $\rho = \rho(t, x)$, velocity $\vec{V} = \vec{V}(t, x)$, pressure $P = P(t, x)$ and other hydrodynamics variables. Within these hydrodynamics variables, the velocity is a very important one, and the reason for this is as follows. The velocity $\vec{V}$ at a time t of a particle X is given, by definition, as

$$\frac{dx}{dt}(t, X) = \vec{V}(t, x(t, X)). \tag{1.2}$$

Above, X is treated as a parameter representing a given fixed particle. Hence, that is the reason why we use the ordinary derivative in (1.2). Also, supposing the velocity field $\vec{V}(t, x)$ known, we can (in principle) determine the transformation 1.1, solving the Cauchy problem

$$\begin{cases} \frac{dx}{dt}(t, X) = \vec{V}(t, x(t, X)), \\ x(0, X) = X. \end{cases}$$

So, due to the transformation (1.1), each hydrodynamic variable f can also be expressed in terms of material coordinates:

$$f(t, x) = f(t, x(t, X)) = F(t, X). \tag{1.3}$$

Suppose f is differentiable. Applying the Chain's Rule to derivatives, we have

$$\frac{dF}{dt}(t, X) = \frac{df}{dt}(t, x(t, X)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_j}(t, x(t, X)) \frac{dx_j}{dt} + \frac{\partial f}{\partial t}(t, x(t, X)).$$

So, rewriting the last equation in terms of the gradient operator we have

$$\frac{dF}{dt}(t, X) = \left(\vec{V}.\nabla + \frac{\partial}{\partial t} \right) f(t, x(t, X)) \equiv \frac{Df}{Dt}(t, x(t, X)).$$

Hence, the differential operator $\frac{D}{Dt}$ given by

$$f \longmapsto \frac{Df}{Dt} = (\vec{V}.\nabla)f + \frac{\partial f}{\partial t} \tag{1.4}$$

is known as *material derivative.*

Remark 1.1. The trajectories of the velocity field are called streamlines. Thus, when the velocity field depends on t (we call such flow unsteady), streamlines may change with time. At a given moment t the streamline $s \longrightarrow x(s, X)$, which for $s = 0$ passes through a point X, is described by the equation

$$\frac{dx}{dt}(t, X) = \vec{V}(t, x(t, X)), \; x(0, X) = X.$$

then, at $s = 0$ the streamline is at the initial point X. From the above equations for streamlines and path lines, we conclude that when the velocity field $\vec{V}$ does not depend on t, that is $\vec{V} = \vec{V}(x)$ (we call flow steady), we may identify s with time, so the streamlines and path lines coincide.

1.2.2 The Boussinesq system

Let us consider an incompressible homogeneous inviscid fluid, with constant density ρ, on a horizontal waterproof flat bottom, with air above and in a constant gravitational field $\vec{g}$. Define $(t, X) = (x(t), y(t), z(t))$ the position at an instant t of a fluid particle $X = (x, y, z)$ and $\vec{v}(t, X) = (u(t, X), v(t, X), w(t, X))$ the fluid velocity field, such that

$$rot\vec{v} = 0. \tag{1.5}$$

Also, $P(t, X)$ is the fluid pressure, and $b(t, X)$ is the external forces acting on the fluid. Taking $\mathcal{O}$ a body fluid, the Newton Second Law says that the balance of the forces acting on $\mathcal{O}$ is given by

$$\rho \frac{d(\vec{v})}{dt} = \sum \vec{F}. \tag{1.6}$$

As the fluid is supposed to inviscid, just two forces are acting on $\mathcal{O}$: the pressure P, whose acceleration vector field is given by $-\nabla P$ and the gravity (external force), given by $\rho\vec{g}$. Replacing this on (1.6) and remembering that $\frac{d(\vec{v})}{dt} = \frac{D\vec{v}}{Dt}$ we get

$$\rho \frac{D\vec{v}}{Dt} = -\nabla P + \rho\vec{g}. \tag{1.7}$$

Besides, the principle of conservation of mass says that the mass of $\mathcal{O}$ does not change as $\mathcal{O}$ moves with the fluid, that is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{v}) = 0, \tag{1.8}$$

on the other hand, $\nabla.(\rho\vec{v}) = (\vec{v}.\nabla)\rho + \rho\nabla\vec{v}$, then

$$\frac{\partial \rho}{\partial t} + (\vec{v}.\nabla)\rho + \rho\nabla\vec{v} = \frac{D\rho}{Dt} + \rho\nabla\vec{v} = 0,$$

which means

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla\vec{v}.$$

By the hypothesis of incompressibility, follows that $\frac{D\rho}{Dt} = 0$, hence

$$\nabla\vec{v} = 0. \tag{1.9}$$

Combining (1.7) and (1.9), we get the equations that describe the balance of forces of the system, the well-known Euler equations:

$$\begin{cases} \frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla P + \vec{g}, \\ \nabla\vec{v} = 0. \end{cases} \tag{1.10}$$

As we know, the gravitational field acts in negative z direction, then $\vec{g} = -g\hat{k}$, with g a constant. So, replacing this information in (1.10) and remember that $(\vec{v}.\nabla)\vec{v} = \nabla(\frac{1}{2}\|\vec{v}\|^2)$ we get

$$\begin{cases} \frac{\partial\vec{v}}{\partial t} + \nabla(\frac{1}{2}\|\vec{v}\|^2) = -\frac{1}{\rho}\nabla P - g\hat{k}, \\ \nabla\vec{v} = 0. \end{cases} \tag{1.11}$$

Besides, we need equations that describe the boundary conditions on the bottom and the free surface. Remember that the bottom is waterproof. A surface defined implicitly by the equation $\Sigma(t, x, y, z) = 0$ will satisfy the condition of no transport fluid particles throughout it if and only if

$$\frac{D\Sigma}{Dt} = 0.$$

To the flat bottom, we have $\Sigma(t, x, y, z) = z + \mathbf{h}$, hence

$$\frac{D}{Dt}(z + \mathbf{h}) = \vec{v}.(0, 0, 1) = w$$

which means

$$w = 0, \text{ on } z = -\mathbf{h}. \tag{1.12}$$

On the other hand, to the free surface, we have $\Sigma(t, x, y, z) = \eta(t, x, y) - z$, then

$$\frac{D}{Dt}(\eta(t, x, y) - z) = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} + v\frac{\partial\eta}{\partial y} - w$$

so we get the boundary condition to free surface

$$\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} + v\frac{\partial\eta}{\partial y} = w, \text{ on } z = \eta(t, x, y). \tag{1.13}$$

In addition, we need to establish a final boundary condition for the pressure in the free surface. To do this, note that the tension is zero on the free surface, which means the pressure in the fluid and the pressure in the air should be equal there. Hence, if we suppose that p is the pressure in the fluid and p_0 is the pressure in the air, we have $\frac{p-p_0}{\rho}$ on interior of Ω and

$$p = p_0 \text{ on } z = \eta(t, x, y). \tag{1.14}$$

The equations (1.10)-(1.14) are called **Euler equations with free boundary**.

Now, by (1.5), there exists a potential function φ such that $\nabla\varphi = \vec{v}$ and satisfies the Laplace equation

$$\Delta\varphi = 0, \text{ on } -\mathbf{h} < z < \eta(t, x, y). \tag{1.15}$$

Hence, rewriting (1.11) in terms of potential function

$$\frac{\partial}{\partial t}(\nabla\varphi)+\nabla(\frac{1}{2}\|\nabla\varphi\|^2)=-\frac{1}{\rho}\nabla P-g\hat{k},$$

which means

$$\nabla\left(\frac{\partial\varphi}{\partial t}+\frac{1}{2}\|\nabla\varphi\|^2+\frac{1}{\rho}(p-p_0)+gz\right)=0.$$

Integrating this equation on spatial variables we find

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\|\nabla\varphi\|^2+\frac{1}{\rho}(p-p_0)=-gz+f(t),\ \text{to}-\mathbf{h}\leq z\leq\eta(t,x,y)$$

where f is a constant of integration that only depends on t. If we suppose that f is absorbed into φ, this is $f=0$, and combining the above equation with the pressure condition (1.14) we get

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}(\nabla\varphi)^2+g\eta=0,\ \text{on } z=\eta(t,x,y). \tag{1.16}$$

Besides, we can rewrite the equations (1.12) and (1.13) in terms of the potential function, then

$$\frac{\partial\eta}{\partial t}+\nabla\eta\nabla\varphi=\frac{\partial\varphi}{\partial z},\ \text{on } z=\eta(t,x,y) \tag{1.17}$$

and

$$\frac{\partial\varphi}{\partial z}=0,\ \text{on } z=-\mathbf{h}. \tag{1.18}$$

The Laplace equation (1.15) and the equations (1.16) - (1.18) are called **Bernoulli equations with free boundary**.

In order to facilitate the mathematical analysis of the equations (1.16) - (1.18), let us consider

$$\psi(t,\mathbf{x})=\varphi(t,\mathbf{x},\eta(t,\mathbf{x})) \tag{1.19}$$

as the potential function on the surface $\eta(t,\mathbf{x})$. Here, $\mathbf{x}=(x,y)\in\mathbb{R}^2$. Note that if ψ and η are known, then the potential function is completely determined on the fluid by the Laplace equation (1.15) with Dirichlet condition $\varphi=\psi$ on the free surface and homogeneous Neumann condition on the bottom. In particular, with ψ and η on hands, we can establish $\frac{\partial\varphi}{\partial z}|_{z=\eta}$. Now, let us introduce the linear operator

$$Y(\eta):\psi\longmapsto Y(\eta)\psi=\frac{\partial\varphi}{\partial z}|_{z=\eta}.$$

then applying the chain's rule on (1.19) we have

$$\begin{aligned}\frac{\partial\psi}{\partial t}&=\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial z}|_{z=\eta}\frac{\partial\eta}{\partial t}\\&=\frac{\partial\varphi}{\partial t}+(Y(\eta)\psi)\frac{\partial\eta}{\partial t}\end{aligned}$$

and

$$\begin{aligned}\nabla\psi =& \nabla\varphi + \frac{\partial\varphi}{\partial z}|_{z=\eta}\nabla\eta \\ =& \nabla\varphi + (Y(\eta)\psi)\nabla\eta\end{aligned}$$

which is equivalent to

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\psi}{\partial t} - (Y(\eta)\psi)\frac{\partial\eta}{\partial t} \tag{1.20}$$

and

$$\nabla\varphi = \nabla\psi - (Y(\eta)\psi)\nabla\eta. \tag{1.21}$$

Multiplying (1.17) by $\frac{\partial\varphi}{\partial z}|_{z=\eta} = (Y(\eta)\psi)$ we get

$$(Y(\eta)\psi)\frac{\partial\eta}{\partial t} + \nabla\eta\nabla\varphi(Y(\eta)\psi) = (Y(\eta)\psi)^2. \tag{1.22}$$

Now replacing (1.20) and (1.21) in (1.16) , yields

$$\frac{\partial\psi}{\partial t} - (Y(\eta)\psi)\frac{\partial\eta}{\partial t} + \frac{1}{2}\|\nabla\psi - (Y(\eta)\psi)\nabla\eta\|^2 + g\eta = 0.$$

Replacing (1.22) in the previous equation we get

$$\frac{\partial\psi}{\partial t} + \nabla\eta\nabla\varphi(Y(\eta)\psi) - (Y(\eta)\psi)^2 + \frac{1}{2}\|\nabla\psi - (Y(\eta)\psi)\nabla\eta\|^2 + g\eta = 0.$$

Now, replacing (1.21) in the last equation gives

$$\frac{\partial\psi}{\partial t} + \nabla\eta\Big(\nabla\psi - (Y(\eta)\psi)\nabla\eta\Big)(Y(\eta)\psi) - (Y(\eta)\psi)^2 + \frac{1}{2}\|\nabla\psi - (Y(\eta)\psi)\nabla\eta\|^2 + g\eta = 0.$$

Rewriting this last equation we have

$$\frac{\partial\psi}{\partial t} - (Y(\eta)\psi)^2\left(1 + \frac{\|\nabla\eta\|^2}{2}\right) + \frac{1}{2}\|\nabla\psi\|^2 + g\eta = 0. \tag{1.23}$$

On the other hand, replacing (1.21) in (1.17) yields that

$$\frac{\partial\eta}{\partial t} + \nabla\eta\Big(\nabla\psi - (Y(\eta)\psi)\nabla\eta\Big) = \frac{\partial\varphi}{\partial z}$$

which means

$$\frac{\partial\eta}{\partial t} + \nabla\eta\nabla\psi - (1 + \|\nabla\eta\|^2)(Y(\eta)\psi) = 0. \tag{1.24}$$

Remember that the bottom is flat by hypothesis. On top of that, Y is linear on the variable ψ and $\eta = 0$ is the water undisturbed position. So the linear models are obtained simply neglecting the nonlinear effects on the equations (1.23)-(1.24). So, let us consider the linearity around $(\psi, \eta) = (0, 0)$. Hence, from (1.23) and (1.24) we get

$$\begin{cases} \frac{\partial\psi}{\partial t} + g\eta = 0, \\ \frac{\partial\eta}{\partial t} - Y(0)\psi = 0. \end{cases} \tag{1.25}$$

Then, eliminating η we find

$$\frac{\partial^2\psi}{\partial t^2} + gY(0)\psi = 0. \tag{1.26}$$

Remark 1.2. Let us consider as notation the following multiplier Fourier: Let $f \in L^\infty(\mathbb{R}^d)$ and $u \in L^2(\mathbb{R}^d),\ d \in \{1,2\}$. Then $f(D)u$ is defined by

$$\mathcal{F}(f(D)u)(\xi) = f(\xi)\hat{u}(\xi)$$

with

$$\mathcal{F}(u)(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} u(x)e^{-ix.\xi}dx = \hat{u}(\xi)$$

is the Fourier transform.

To calculate $Y(0)\psi$, note that by (1.15), (1.18) and (1.19) we have

$$\begin{cases} \Delta\varphi = 0,\ on\ \mathbb{R}^d \times (-\mathbf{h}, 0) \\ \psi = \varphi,\ on\ z = 0, \\ \frac{\partial\varphi}{\partial z} = 0\ on\ z = -\mathbf{h}. \end{cases} \tag{1.27}$$

So using Fourier transform, the system (1.27) and making some computations we find

$$\varphi(t, \mathbf{x}, z) = \frac{\cosh((z + \mathbf{h})|D|)}{\cosh(|D|\mathbf{h})}\psi. \tag{1.28}$$

Hence, we conclude

$$Y(0)\psi = |D| \tanh(\mathbf{h}|D|)\psi. \tag{1.29}$$

Replacing (1.29) in (1.26) we have

$$\frac{\partial^2\psi}{\partial t^2} + g|D| \tanh(\mathbf{h}|D|)\psi = 0. \tag{1.30}$$

The equation (1.30) has an explicit solution $\eta(t, \mathbf{x})$ given by

$$\eta(t, \mathbf{x}) = \mathbf{a} \cos(\mathbf{k}.\mathbf{x} - \omega t), \tag{1.31}$$

where $\mathbf{k} = (k_1, k_2) \in \mathbb{R}^2$ is called **wave number** and $\mathbf{a} \in \mathbb{R}$ is constant with physical means, which we will see soon. The solution (1.31) leads to the relation

$$\omega^2 = g|\mathbf{k}| \tanh(|\mathbf{k}|\mathbf{h}), \tag{1.32}$$

where $|\mathbf{k}| = \sqrt{k_1^2 + k_2^2}$ and $\omega \in \mathbb{R}$ is the **pulsation**. This equation is called **dispersion relation**.

To make a qualitative analysis of the surface waves, let us consider the equations in unidimensional form. To do this, let us consider the following physics magnitudes:

- The letter $\mathbf{a}$ denotes the amplitude of the observed waves;
- The letter $\mathbf{h}$ denotes the depth of the channel;
- The letter λ denotes the wavelength and is defined by $\lambda = \frac{1}{|\mathbf{k}|}$.

Now, let us establish some unidimensional variables:

$$\tilde{\eta} = \frac{\eta}{\mathbf{a}},\ \tilde{z} = \frac{z}{\mathbf{h}},\ \tilde{\mathbf{x}} = \frac{\mathbf{x}}{\lambda}.$$

Moreover, rewrite the time and the velocity potential as

$$\tilde{t} = \frac{\sqrt{g\mathbf{h}}}{\lambda}t,\ \tilde{\varphi} = \frac{\varphi}{\mathbf{a}\lambda\sqrt{\frac{g}{\mathbf{h}}}}.$$

Finally, we introduce two unidimensional parameters μ and ε, given by

$$\mu = \left(\frac{\mathbf{h}}{\lambda}\right)^2\ \varepsilon = \frac{\mathbf{a}}{\mathbf{h}},$$

Remark 1.3. Note that as $\lambda = \frac{1}{|\mathbf{k}|}$, we can rewrite $\mu = (|\mathbf{k}|\mathbf{h})^2$. So, the shallow water wave condition means that $\mu \ll 1$. On the other hand, the parameter ε measures the flow amplitude. We are talking about low amplitude waves when $\varepsilon \ll 1$.

Now, we rewrite the Bernoulli equations with free boundary under the unidimensional form:

$$\mu\Delta_{\mathbf{x}}\tilde{\varphi} + \frac{\partial^2\tilde{\varphi}}{\partial\tilde{z}^2} = 0,\ \text{on}\ -1 < \tilde{z} < \varepsilon\tilde{\eta}, \tag{1.33}$$

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{t}} + \frac{\varepsilon}{2}\|\nabla\tilde{\varphi}\|^2 + \tilde{\eta} + \frac{1}{2}\frac{\varepsilon}{\mu}\left|\frac{\partial\tilde{\varphi}}{\partial\tilde{z}}\right|^2 = 0,\ \text{on}\ \tilde{z} = \varepsilon\tilde{\eta}, \tag{1.34}$$

$$\frac{\partial\tilde{\eta}}{\partial\tilde{t}} + \varepsilon\nabla\tilde{\eta}\nabla\tilde{\varphi} - \frac{1}{\mu}\frac{\partial\varphi}{\partial\tilde{z}} = 0,\ \text{on}\ \tilde{z} = \varepsilon\tilde{\eta} \tag{1.35}$$

and

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{z}} = 0,\ \text{on}\ \tilde{z} = -1, \tag{1.36}$$

here, $\Delta_{\mathbf{x}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In the same way, we rewrite

$$\tilde{\psi}(t, \tilde{\mathbf{x}}) = \tilde{\varphi}(t, \tilde{\mathbf{x}}, \eta(t, \tilde{\mathbf{x}})) \tag{1.37}$$

and

$$Y_\mu(\varepsilon\tilde{\eta}) : \tilde{\psi} \longmapsto Y_\mu(\varepsilon\tilde{\eta})\tilde{\psi} = \frac{\partial\tilde{\varphi}}{\partial\tilde{z}}|_{\tilde{z}=\varepsilon\tilde{\eta}}.$$

We can rewrite the equations (1.20) and (1.21) on unidimensional form

$$\frac{\partial\tilde{\psi}}{\partial\tilde{t}} - \frac{\varepsilon}{2}(Y_\mu\,(\tilde{\eta})\,\tilde{\psi})^2\left(1 + \frac{\|\nabla\tilde{\eta}\|^2}{2}\right) + \frac{1}{2}\|\nabla\tilde{\psi}\|^2 + \tilde{\eta} = 0 \tag{1.38}$$

and

$$\frac{\partial\tilde{\eta}}{\partial\tilde{t}} + \varepsilon\nabla\tilde{\eta}\nabla\tilde{\psi} - \left(\frac{1}{\mu} + \varepsilon^2\|\nabla\tilde{\eta}\|^2\right)(Y_\mu(\varepsilon\tilde{\eta})\tilde{\psi}) = 0. \tag{1.39}$$

Remark 1.4. From equations (1.33)-(1.36) we can see that ε measures the nonlinear effects and μ measures the dispersive effects. In addition, the number $S = \frac{\varepsilon}{\mu}$ is called **Stokes number** and measures the relation between ε and μ.

From now on, our purpose is to get the asymptotic equations involving the nonlinear and dispersive effects. With this aim in mind, we can consider $\mu, \varepsilon \ll 1$, which are shallow water waves with low amplitude. Also, we are supposing $S \simeq 1$, which means $\mu \simeq \varepsilon$. Let us admit that

$$Y_\varepsilon(\varepsilon\eta)\psi = -\varepsilon\Delta\psi - \varepsilon^2\left(\frac{1}{3}\Delta^2\psi + \eta\Delta\psi\right) + O(\varepsilon^2). \tag{1.40}$$

where $O(\varepsilon^2)$ indicates the approximation error[1]. Replacing $Y_\varepsilon(\varepsilon\eta)\psi$ on (1.38)-(1.39) by the expression (1.40) and drop the terms with order $O(\varepsilon^2)$ we find

$$\begin{cases} \frac{\partial\psi}{\partial t} + \eta + \frac{\varepsilon}{2}|\nabla\psi|^2 = 0, \\ \frac{\partial\eta}{\partial t} + \varepsilon\nabla\psi.\nabla\eta + \Delta\psi + \varepsilon(\frac{1}{3}\Delta^2\psi + \eta\Delta\psi) = 0. \end{cases}$$

Taking the gradient operator on the first equation and making $U = \nabla\psi$ we find **the Boussinesq system**

$$\begin{cases} \frac{\partial U}{\partial t} + \nabla\eta + \frac{\varepsilon}{2}\nabla|U|^2 = 0, \\ \frac{\partial\eta}{\partial t} + \nabla.U + \varepsilon(\nabla.(\eta U) + \frac{1}{3}\Delta(\nabla.U)) = 0. \end{cases} \tag{1.41}$$

Remark 1.5. We have by definition $U = \nabla\psi = \nabla\varphi + \varepsilon Y_\varepsilon(\varepsilon\eta)\psi\nabla\eta$. So by (1.40) it follows that $U = \nabla\varphi + O(\varepsilon^2)$, which means U is the horizontal component of the velocity field on the surface, with approximation error $O(\varepsilon^2)$.

1.2.3 The Korteweg-de Vries (KdV) equation

The Boussinesq system (1.41) presented in the last subsection had an interesting property: It can be converted into a wave equation with a constant velocity equal 1 when $\varepsilon = 1$. On the other hand, the horizontal component $U = (u, v)$ is bidimensional but, if we consider dimension $d = 1$, we have two components that propagate in opposite directions. Besides, when $\varepsilon \neq 0$, the nonlinear and dispersive terms modify the behavior of the waves and, in particular, the wave components are coupled. The equation that describes the evolution of such waves is called the Korteweg–de Vries (KdV) equation.

From now on, let us consider only one direction, which is $d = 1$. Then, from (1.41), with $U = u$, we find

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial\eta}{\partial x} + \varepsilon u\frac{\partial u}{\partial x} = 0, \\ \frac{\partial\eta}{\partial t} + \frac{\partial u}{\partial x} + \varepsilon\left(\frac{\partial(u\eta)}{\partial x} + \frac{1}{3}\frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial x})\right) = 0. \end{cases} \tag{1.42}$$

[1] An explanation about the asymptotic development of the operator $Y_\varepsilon(\varepsilon\eta)\psi$ given by (1.40) can be found in [2].

So, we are interested in describing the behavior of variables u and η that appear in the equation (1.42) for a wave propagating to the right-hand side. To do this, consider the following change of variable

$$u(t,x) = v(\varepsilon t, x-t) \text{ and } \eta(t,x) = \zeta(\varepsilon t, x-t).$$

So, by the chain's rule and (1.42), the functions $v(\tau,\xi)$ and $\zeta(\tau,\xi)$ must satisfy

$$\begin{cases} \varepsilon\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial \xi} + \frac{\partial \zeta}{\partial \xi} + \varepsilon v\frac{\partial v}{\partial \xi} = 0, \\ \varepsilon\frac{\partial \zeta}{\partial \tau} - \frac{\partial \zeta}{\partial \xi} + \frac{\partial v}{\partial \xi} + \varepsilon\left(\frac{\partial(v\zeta)}{\partial \xi} + \frac{1}{3}\frac{\partial^2}{\partial \xi^2}\left(\frac{\partial v}{\partial \xi}\right)\right) = 0. \end{cases} \tag{1.43}$$

It follows from the second equation of (1.43) that

$$\frac{\partial v}{\partial \xi} = \frac{\partial \zeta}{\partial \xi} + O(\varepsilon).$$

So, we can replace v by ζ on the dispersive and nonlinear terms, once the error approximation is small. Hence, doing this and adding the two result equations, we find the well-known **Korteweg- de Vries equation**

$$\zeta_\tau + \frac{3}{2\varepsilon}\zeta\zeta_\xi + \frac{1}{6\varepsilon}\zeta_{\xi\xi\xi} = 0,$$

where

$$\frac{3}{2\varepsilon} = \frac{3\mathbf{h}}{2\mathbf{a}}$$

and

$$\frac{1}{6\varepsilon} = \frac{\mathbf{h}}{6\mathbf{a}}.$$

1.3 Semi-group theory

In this section, we present some definitions and results of the semi-group theory that will be used in the text. This section is based on [31, 76]. Denotes by $(X, \|\cdot\|)$ a Banach space, $D(A) \subset X$ it is a nonempty subset, and $A : D(A) \subset X \longrightarrow X$ is a linear operator.

1.3.0.1 Preliminaries

Definition 1.3.1. *Let $(X, \|\cdot\|)$ a Banach space and $\mathcal{L}(X)$ the algebra of linear and bounded operators of X. We said that $S : \mathbb{R}^+ \longrightarrow \mathcal{L}(X)$ is a C_0-semigroup of bounded operators on X if:*

(i) $S(0) = I$, *where I is the identity operator on X;*

(ii) $S(t+s) = S(t)S(s)$, *for all $t, s \in \mathbb{R}^+$;*

(iii) $\lim_{t\to 0^+} \|(S(t)-I)x\| = 0$, *for all $x \in X$.*

Proposition 1.3.1. *Let $S : \mathbb{R}^+ \longrightarrow \mathcal{L}(X)$ be a C_0-semigroup, then*

$$\lim_{t\to\infty} \frac{ln\|S(t)\|}{t} = \inf_{t>0} \frac{ln\|S(t)\|}{t} = \omega_0.$$

Besides, for every $\omega > \omega_0$, there exists a constant $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0. \tag{1.44}$$

Remark 1.6. When $\omega_0 < 0$, we can take $\omega_0 < \omega < 0$ and by (1.44), there exists $M \geq 1$ such that

$$\|S(t)\| \leq M, \text{ for all } t \geq 0.$$

Besides, when $M \leq 1$, we said that $S : \mathbb{R}^+ \longrightarrow \mathcal{L}(X)$ is a C_0-semigroup of contractions.

Definition 1.3.2. *Let Let $S : \mathbb{R}^+ \longrightarrow \mathcal{L}(X)$ be a C_0-semigroup. The operator $A : D(A) \subset X \longmapsto X$ defined by*

$$D(A) = \left\{ x \in X; \textit{there exists} \lim_{h\to 0^+} \left(\frac{S(h) - I}{h} \right) x \right\}$$

and

$$Ax = \lim_{h\to 0} \left(\frac{S(h) - I}{h} \right) x$$

is the infinitesimal generator of the C_0-semigroup.

Remark 1.7. It is easy to see that $D(A) \subset X$ is a subspace of X and A is a linear operator.

Proposition 1.8. *Let $S : \mathbb{R}^+ \longrightarrow \mathcal{L}(X)$ be a C_0-semigroup and $A : D(A) \subset X \longmapsto X$ the infinitesimal generator of S, then:*

(i) If $x \in D(A)$, then $S(t)x \in D(A)$, for all $t \geq 0$ and also

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \forall t \geq 0.$$

(ii) If $x \in D(A)$, then

$$S(t)x - S(s)x = \int_s^t AS(\xi)xd\xi = \int_s^t S(\xi)Axd\xi, \ 0 \leq s \leq t.$$

(iii) If $x \in X$, then $\int_0^t S(\xi)xd\xi \in D(A)$ and

$$A\int_0^t S(\xi)xd\xi = S(t)x - x.$$

Definition 1.3.3. *Let $A : D(A) \subset X \longmapsto X$ be a linear operator. The resolvent set $\rho(A)$ of A is the set of all the complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda, A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operators is called resolvent of A.*

1.3.0.2 The Hille-Yosida and Lumer-Phillips theorems

In this subsection, we present two theorems that establish necessary and sufficient conditions for a linear operator $A : D(A) \subset X \longmapsto X$ be a C_0-semigroup generator.

Theorem 1.9. *(Hille-Yosida) A linear operator $A : D(A) \subset X \longmapsto X$ is a C_0-semigroup generator satisfying $\|S(t)\| \leq Me^{\omega t},\ t \geq 0$ if and only if:*

(i) A is closed and $\overline{D(A)} = X$;

(ii) The resolvent set $\rho(A)$ of A is such that $\{\lambda;\ im(\lambda) = 0,\ \lambda > \omega\} \subset \rho(A)$ and for such λ we have

$$\|R(\lambda : A)\| \leq \frac{M}{\lambda - \omega}.$$

In order to A be a C_0-semigroup generator of contractions, we must replace (ii) by the following condition:

$(\tilde{ii})$ For all $\lambda > 0,\ \lambda \in \rho(A)$ we have

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

Before we presented the next result, we need another concept. Let X be a Banach space and let X^* its dual. We denote the value of $x^* \in X^*$ at X by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{x^*;\ x^* \in X^* \text{ and } \langle x^*, x \rangle = \|x\|^* = \|x^*\|^2\right\}.$$

Remark 1.10. From the Hahn-Banach theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$.

Definition 1.3.4. *A linear operator $A : D(A) \subset X \longmapsto X$ is dissipative if for every $x \in D(A)$ there is a x^* such that $Re\langle Ax, x^* \rangle \leq 0$.*

Theorem 1.11. *(Lumer-Phillips) Let $A : D(A) \subset X \longmapsto X$ be a linear operator, with $\overline{D(A)} = X$.*

(a) If A is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(b) If A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $Re\langle Ax, x^* \rangle \leq 0$.

Corollary 1.12. *Let $A : D(A) \subset X \longmapsto X$ be a linear closed operator, with $\overline{D(A)} = X$. If A and A^* (adjoint of A) are dissipative, then A is a generator of a C_0-semigroup of contractions on X.*

1.3.0.3 The abstract Cauchy problem

Let $A : D(A) \subset X \longmapsto X$ be a linear operator. Given $u_0 \in X$, the abstract Cauchy problem for A with initial data u_0 consists of finding a solution $u(t)$ to the homogeneous Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \ t > 0, \\ u(0) = u_0, \end{cases} \tag{1.45}$$

Now, let us introduce a notion of a solution to the problem (1.45):

Definition 1.3.5. *(Classical solution) A function $u : \mathbb{R}^+ \longmapsto X$ is a classical solution of (1.45) for all $t \geq 0$ if u is continuous for all $t \geq 0$, continuously differentiable on $\mathbb{R}^+$, $u(t) \in D(A)$ for all $t \in \mathbb{R}^+$ and (1.45) is satisfied for all $t \geq 0$.*

Remark 1.13. By proposition (1.8), if $u_0 \in D(A)$ and A is a infinitesimal generator of C_0-semigroup $S : \mathbb{R}^+ \longmapsto X$, then $u(\cdot) = S(\cdot)u_0 : \mathbb{R}^+ \longmapsto D(A)$ is a classical solution of (1.45). Actually $S(\cdot)u_0$ is the only solution of (1.45).

From now on, we will assume that A is an infinitesimal generator of a C_0- semigroup S and $u(\cdot) = S(\cdot)u_0$, with $u_0 \in D(A)$ is a classical solution of (1.45). Next, let $T > 0$ be a fixed constant and $f : [0, T) \longmapsto X$. Consider the inhomogeneous Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.46}$$

Now, let $f \in L^1(0, T; X)$. So, $v(s) = S(t-s)u(s)$ is differentiable for $0 < s < t$, then

$$\begin{aligned} \frac{dv}{ds} &= -AS(t-s)u(s) + S(t-s)\frac{du}{ds} \\ &= -AS(t-s)u(s) + S(t-s)Au(s) + -AS(t-s)f(s) \\ &= S(t-s)f(s). \end{aligned}$$

Hence, as $v(s)$ is integrable on $[0, t]$, integrating from 0 to t yields

$$v(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds. \tag{1.47}$$

As consequence, (1.47) has at most one solution u. Also, if u exists, then $u \in C([0, T]; X)$. So, it is natural to consider as a generalized solution of (1.46) as follows:

Definition 1.14. Let $u_0 \in X$ and $f \in L^1([0, T]; X)$. The function $u \in C([0, T]; X)$ given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds, \ 0 \leq t \leq T.$$

is the mild solution of the inhomogeneous Cauchy problem (1.46) on $[0, T]$.

Remark 1.15. In general, the homogeneous Cauchy problem (1.45) does not have a classical solution, once $u_0 \notin D(A)$. So making $f \equiv 0$ on definition (1.14), $u(\cdot) = S(\cdot)u_0$ is the mild solution of (1.45), since that $u_0 \in X$. It is therefore clear that not every mild solution of (1.46) is indeed a classical solution even in the case $f \equiv 0$.

Now, let us present another notion of solution to the Cauchy problem (1.46):

Definition 1.3.6. *(Strong solution): A function u which is differentiable almost everywhere on $[0,T]$ such that $\frac{du}{dt} \in L^1([0,T] : X)$ is called a strong solution of the Cauchy problem (1.46) if $u(0) = u_0$ and*

$$\frac{du}{dt} = Au + f, \text{ almost everywhere on } [0,T].$$

Remark 1.16. We note that if $A = 0$ and $f \in L^1([0,T] : X)$, the Cauchy problem (1.46) has usually no solution unless $f \in C([0,T] : X)$. However, (1.46) has always a strong solution given by $u(t) = u_0 + \int_0^t f(s)ds$. Besides, it is easy to show that if u is a strong solution of (1.46) and $f \in L^1([0,T] : X)$, then such u is a mild solution as well.

1.3.0.4 The nonlinear problem

Let $(X, \|.\|)$ be a reflexive Banach space. Consider the initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + F(u(t)), \ t > 0, \\ u(0) = u_0. \end{cases} \tag{1.48}$$

where $F : X \longmapsto X$ is a continuous function and $A : D(A) \subset X \longmapsto X$ is an infinitesimal generator of a C_0-semigroup $S : \mathbb{R}^+ \longmapsto \mathcal{L}(X)$ such that $\|S(t)\| \leq M, \forall t \geq 0$. if u is a classical solution or a strong solution of (1.48), it is not difficult to see that u satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds,$$

which means that u is a mild solution.

Theorem 1.17. *Let $F : X \longmapsto X$ be a Lipschitz function, i.e. there exists $L > 0$ such that*

$$\|F(u) - F(v)\| \leq L\|u - v\|, \ \forall u, v \in X.$$

So, for every $u_0 \in X$, there exists an unique mild solution $u \in C(\mathbb{R}^+ : X)$. Besides:

(i) If $u_0, v_0 \in X$ are initial data and u, v are its respective mild solutions of (1.48), then

$$\|u(t) - v(t)\| \leq Me^{LMt}\|u_0 - v_0\|.$$

(ii) If $u_0 \in D(A)$, then u is a strong solution of (1.48) on $[0,T], T > 0$.

1.4 Control Theory

Throughout history, mankind has developed significantly their skills to manipulate nature to improve their lifestyle. Simple examples of these modifications are in animal husbandry and agriculture, where man realized that could interact with nature for it to work on his behalf. Then the idea of control is born, as an action or actions of man as a means to obtain a predetermined goal. This idea developed over the years and began to make part of the daily life of humanity. In the beginning, the notion of control was closely linked to engineering, the construction of dams, and irrigation systems, in the creation of the steam machine, a crucial point of the industrial revolution, among other examples. However, with the development of calculus and differential equations, control theory was separated from engineering, from the point of view of scientific studies, and went on to take its first steps as a branch of mathematics.

After the evolution of differential calculus, in the last century, due to the emergence of important tools such as Functional Analysis, the ideas, and concepts of finite dimension systems theory has been extended to the theory of systems with infinite dimension and in particular to differential equations. Consequently, great advances were obtained in the study of partial differential equations (PDEs) that describe the most varied physical-mathematical phenomena. Currently, any physical phenomena that can be modeled by a differential equation is a potential object of study of control theory[2]. In this spirit, we are interested in obtaining controllability results for PDEs-governed systems. For this, we will formally introduce the concepts of control: Internal controllability, boundary controllability, and exponential stability.

1.4.1 Internal and boundary controllability

Let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$, be a bounded subset with smooth boundary Γ, $\mathcal{P} : W \longrightarrow K$ a differential operator that associates each function $y \in W$ to its derivative $\mathcal{P}(y) \in K$ and $\mathcal{B}$ a boundary condition, this is, the behavior of y on the boundary Γ. For example, $\mathcal{B}(y) = 0$ means that y vanishes on Γ. Consider also, X and Y suitable function spaces.

Given $\omega \subset \Gamma$ and a function $y_0 \in X$, a **boundary control system** is an evolution equation given by the expression:

$$\begin{cases} y'(t,x) - \mathcal{P}(y)(t,x) = 0, \quad t > 0, \ x \in \Omega, \\ \mathcal{B}_1(y)(t,x) = 0, \quad t > 0, \ x \in \Gamma, \\ \mathcal{B}_2(y)(t,x) = u(t,x), \quad t > 0, \ x \in \omega, \\ y(0,x) = y_0, \ x \in \Omega. \end{cases} \tag{1.49}$$

Here, $t > 0$ is the temporal variable, $y = y(t,x)$ is the state (in principle unknown), and $u = u(t,x) \in Y$ is the control. In this model, y' represents $\frac{dy}{dt}$.

[2] To read more about control theory and others result in this direction, we suggest [21, 90].

Now let $\omega \subset \Omega$. An **internal control system** is an evolution equation given by the expression:

$$\begin{cases} y'(t,x) - \mathcal{P}(y)(t,x) = \mathcal{X}_\omega(x)u(t,x), \quad t > 0,\ x \in \Omega, \\ \mathcal{B}_1(y)(t,x) = 0, \quad t > 0,\ x \in \Gamma, \\ y(0,x) = y_0,\ x \in \Omega. \end{cases} \tag{1.50}$$

Here, $\mathcal{X}_\omega$ denotes the characteristic function of ω.

Now, we present the main definitions of controllability, including a kind of control property called integral overdetermination condition, which is explored in the chapters 3 and 4.

Definition 1.4.1. ***(Exact controllability)*** *Let $T > 0$ and $y_0, y_1 \in X$ be two states of the system (1.49) (respec. (1.50)). Such system is said exactly controllable if there exists $u \in Y$ such that the solution $y = y(y_0, u)$ of the system (1.49) (respec. (1.50)) fulfill $y(T,.) = y_1$.*

Definition 1.4.2. ***(Null controllability)*** *Let $T > 0$ and $y_0 \in X$ be a state of the system (1.49) (respec. (1.50)). The system is said to be null controllable if there exists $u \in Y$ such that the solution $y = y(y_0, u)$ of the system (1.49) (respec. (1.50)) fulfill $y(T,.) = 0$.*

Definition 1.4.3. ***(Integral overdetermination condition)*** *Let $y_0 \in L^1(\Omega)$ be an initial state of the system (1.49) (respec. (1.50)). Given $T > 0$, $\gamma \in L^\infty(\Omega)$ and $\varphi \in L^1(0,T)$ such that*

$$\int_\Omega y_0(x)\gamma(x)dx = \varphi(0).$$

We say that the system (1.49) (respec. (1.50)) satisfies the integral overdetermination condition if there exists $u \in Y$ such that the solution $y = y(y_0, u)$ of the system (1.49) (respec. (1.50)) satisfies

$$\int_\Omega y(t,x)\gamma(x)dx = \varphi(t),\ t \in [0,T].$$

Now, let us present some mathematical concepts and techniques very useful to find the control u. Let us consider that $y_0 \in H$, and $u \in L^2(0,T;U)$. Besides, the function $y : [0,T] \longrightarrow H$ denotes the solution of the Cauchy problem

$$\begin{cases} y' = Ay + Bu, \\ y(0) = y_0. \end{cases} \tag{1.51}$$

Recall that for any $y_0 \in \mathcal{D}(A)$ and $u \in W^{1,1}(0,T;U)$, the Cauchy problem (1.51) admits a unique classical solution $y \in C([0,T];\mathcal{D}(A)) \cap C^1(0,T;H)$ given by Duhamel formula

$$y(t) = S(t)y_0 + \int_0^t S(T-t)Bu(s)ds,\ \forall t \in [0,T],$$

where $B : L^1(0,T;H) \longmapsto L^1(0,T;H)$ it is a linear operator and $\{S(t)\}_{t\geq 0}$ is a semigroup generated by the operator A. For $y_0 \in H$ and $u \in L^1(0,T;U)$, the above formula is still meaningful and defines the mild solution of (1.51).

Let us introduce the operator $\mathcal{L}_T : L^2(0,T;U) \longrightarrow H$ defined by

$$\mathcal{L}_T u = \int_0^T S(T-t)Bu(s)ds.$$

Hence,

$$\text{Exact controllability in time } T \Leftrightarrow Im\mathcal{L}_T = H;$$

$$\text{Null controllability in time } T \Leftrightarrow S(T)H \subset Im\mathcal{L}_T.$$

Remark 1.18. In finite dimension, which is when $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, the three previous definitions are equivalent and equals to a purely algebraic condition, the famous Kalman rank condition: $rank(B, AB, ..., A^{n-1}) = n$. As a consequence, the time T plays no role (for more details, see [41] and [99]). Conversely, treating PDEs on the infinite dimension, the situation is more complicated, once that:

- There is no algebraic test for the controllability;
- The control time plays a role in hyperbolic PDE;
- The converses
$$\text{Exact controllability } \Longrightarrow \text{ Null controllability}$$
is not true in general.

1.4.1.1 Adjoint operators

In general, controllability problems require proving an observability inequality for the solution of the adjoint system. To make it more precise, let us introduce the formal definition of the adjoint operator.

The *adjoint* of the bounded operator $B \in \mathcal{L}(U,H)$ is the operator $B^* \in \mathcal{L}(H,U)$ defined by $(Bz,u)_H = (z, B^*u)_U$ for all $z \in U$ and $u \in H$. Thus, the adjoint of the (unbounded) operator A is the unbounded operator A^* with domain

$$\mathcal{D}(A^*) = \{z \in H : \exists C \in \mathbb{R}^+; |(Ay,z)_H| \leq C\|y\|_H, \forall y \in \mathcal{D}(A)\}$$

and defined by

$$(Ay,z)_H = (y, A^*z)_H, \forall y \in \mathcal{D}(A), \forall z \in \mathcal{D}(A^*).$$

Therefore, A^* also generates a continuous semigroup $(e^{tA^*})_{t\geq 0}$ fulfilling $e^{tA^*} = S^*(t), \forall t \geq 0$. If $A^* = A$ (resp. $A^* = -A$) the operator A is said self-adjoint (resp. skew-adjoint)[3].

[3] A skew-adjoint operator generates a continuous group of isometries (e.g. [76]).

1.4.1.2 Hilbert Uniqueness Method (H.U.M)

With the previous notations in hand, we can introduce the Hilbert Uniqueness Method (H.U.M), developed by J. L. Lions. This method is a tool of great importance for the study of the controllability of EDPs. If we consider an initial boundary value problem

$$\Sigma : \qquad \begin{cases} y' = Ay + Bu, \\ y(0) = 0, \end{cases}$$

its adjoint problem, obtained by taking the distributional adjoint of the operator $\partial_t - A$, this is, $-\partial_t - A^*$ is given by

$$\Sigma^* : \qquad \begin{cases} z' = -A^* z, \\ z(T) = z_T. \end{cases}$$

Note that Σ^* is without control and backward in time. For any $z_T \in H$, the solution z of Σ^* is given by $z(t) = S^*(T-t)\, z_T$.

We can assume the following *key identity*:

$$(y(t), z_T)_H = \int_0^T (u, B^* z)_U \, dt,$$

to ensure the equivalence between observability inequality and controllability of the system Σ. In addition, we can conclude that:

1. The evolution equation in the *adjoint problem* $y' = -A^* y$ differs from the one for the *adjoint operator* $z' = A^* z$ by sign minus. Solutions of the second one give solutions of the first one just by changing t by $T - t$;

2. H.U.M provides a bounded operator $\Lambda : z_T \longmapsto u$ that give us the control;

3. In general, we do not need explicitly of B and B^*. The important ingredients in H.U.M are the *key identity* and the *observability inequality*. In summary, the following two results give a relation between controllability and observability.

Theorem 1.19. *The system (1.51) is exactly controllable in time $T > 0$ if and only if there exists a constant $c > 0$ such that*

$$\int_0^T \|B^* S^*(t) z_0\|_U dt \geq c\|z_0\|_H^2, \ \forall z_0 \in H. \tag{1.52}$$

Theorem 1.20. *The system (1.51) is null controllable in time $T > 0$ if and only if there exists a constant $c > 0$ such that*

$$\int_0^T \|B^* S^*(t) z_0\|_U dt \geq c\|S^*(T) z_0\|_H^2, \ \forall z_0 \in H. \tag{1.53}$$

The inequality (1.52) is called the observability inequality. Such inequality means that the map

$$\Psi : z_0 \longmapsto B^*S^*(.)z_0, \tag{1.54}$$

is boundedly invertible; i.e., it is possible to recover the complete information about the initial state z_0 from a measure on $[0, T]$ of the output $B^*[S^*(t)z_0]t$ (observability property). Additionally, the inequality (1.53) is the weak observability inequality, i.e., only $S^*(T)z_0$ may be recovered, but not z_0. These tests are based on H.U.M and allow us to solve a problem by proving a mathematical inequality[4].

1.4.1.3 Inverse problems and integral overdetermination condition

In this subsection, we talk about the concepts of inverse problems and the integral overdetermination condition. This last one will be studied in the context of the Kawahara equation, in chapters 3 and 4. This subsection is based on [80].

The source of the theory of inverse problems may be found late in the 19th century or early 20th century. They include the problem of equilibrium figures for the rotating fluid, the kinematic problems in seismology, the inverse Sturm-Liuville problem, and more. Newton's problem of discovering forces making planets move by Kepler's laws was one of the first inverse problems in the dynamics of mechanical systems solved in the past.

In the study of the so-called direct problems, the solution of a given differential equation or system of equations is realized by employing supplementary conditions. However, in inverse problems, the equation itself is also unknown. The determination of both the governing equation and its solution necessitates imposing more additional conditions than in related direct problems.

The influence of inverse problems of recovering mathematical physics equations, in which supplementary conditions help assign either the values of some or other arguments or the values of certain functional of a solution, began to spread to more and more branches as they gradually took on an important place in applied problems arising in "real life" situations.

It is worth noting here that for the first time problems with integral overdetermination conditions for the Kawahara equation are posed and analyzed in this thesis. To better understand the meaning of the integral overdetermination condition, we present the problems studied in Chapter 3 from a point of view of inverse problems. The same problem is analyzed in Chapter 4, but this time, on unbounded domains.

Let us consider an inverse problem in which, to solve the differential equation for u, it is necessary to know the value of some operator or functional $Bu(t) = \varphi(t)$ as a function of the time t which represent a physical value measurement carried out by a

[4] The proof of these results can be found in [41], [73], [99] and [105].

perfect sensor of finite size. In our case, the operator B is an integral operator and the partial differential equation is the Kawahara equation

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^p u_x = F(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t), \ u(t,L) = h_2(t), \ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t), \ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{1.55}$$

where $Q_T = (0,T) \times (0,L)$, with T and L positive numbers, $p \in \{1,2\}$. The initial state u_0, the functions h, h_1, h_2, h_3, h_4 and the source term F are known functions. Hence, if we take $F(t,x)$ in (1.55) on an specific form

$$F(t,x) = f(t)g(t,x), \ t \in [0,T].$$

with g a known function and f is unknown, our problem is to find a pair (u,f) such that

- u is a solution of (1.55);
- u depends on f;
- u must fulfill
$$\int_0^L u(t,x)\omega(x)dx = \varphi(t), \ t \in [0,T], \tag{1.56}$$
with ω and φ two known functions.

The condition (1.56) is called **integral overdetermination**. The second problem is very similar to the last one, the big difference is that we try to recover a boundary term of (1.55), namely

$$u_{xx}(t,L) = h(t), \ t \in [0,T].$$

Hence, the problem consists to determine a pair (u,h) such that

- u is a solution of (1.55);
- u depends on h;
- u must fulfill
$$\int_0^L u(t,x)\omega(x)dx = \varphi(t), \ t \in [0,T], \tag{1.57}$$
with ω and φ two known functions.

Remark 1.21. The integral overdetermination condition is an extra equation on variable u which help us solve the problem. The function ω (sensor) is the link between us and the physical phenomena. For this reason, a such function must be chosen carefully. We can not take, for instance, $\omega \equiv 1$, once we lost all the derivative properties.

Remark 1.22. It is well known that the equation (1.55) describes the water waves on a channel and other physical phenomena. So, roughly speaking, if we suppose that u describes a water wave on a channel, then the integral overdetermination condition (1.56) may be understood as a function φ which provides the mass at a part of the channel with length $L > 0$ on an instant $t \geq 0$. Such measure is carried out by the function ω.

1.4.2 Stabilizability

Stabilization of differential equations has been shown in recent years as one of the most important applications of control theory in engineering and technology problems. In this section, we present some results about the stabilization of systems of differential equations, including the Lyapunov method, which is used in Chapter 5.

1.4.2.1 Concepts of stabilization

In order to address the stabilization of the control system (1.51), let us start consider $K \in \mathcal{L}(H, U)$, A_K the operator $A_K z = Az + BKz$ with domain $\mathcal{D}(A_K) = \mathcal{D}(A)$ and by $(S_K(t))_{t\geq 0}$ the semigroup generated by A_K.

The system (1.51) is said to be exponentially stabilizable if there exists a feedback $K \in \mathcal{L}(H, U)$ such that the operator A_K is exponentially stable; i.e., for some constants $C > 0$ and $\mu > 0$,

$$\|S_K(t)\| \leq Ce^{-\mu t}, \ \forall t \geq 0.$$

On the other hand, the system (1.51) is said to be completely stabilizable if it is exponentially stabilizable with an arbitrary exponential decay rate; i.e., for arbitrary $\mu \in \mathbb{R}$, there exists a feedback $K \in \mathcal{L}(H, U)$ and a constant $C > 0$ such that

$$\|S_K(t)\| \leq Ce^{-\mu t}, \ \forall t \geq 0.$$

Stabilization of the system (1.51) is strongly related to controllability[5].

Theorem 1.23. *If the system (1.51) is null controllable, then it is exponentially stabilizable.*

Next, we have an equivalence between controllability and stability:

Theorem 1.24. *Assume that A generates a group $(S(t))_{t\geq 0}$ of operators. Then the following properties are equivalent.*

(i). The system (1.51) is exactly controllable in some time $T > 0$;

(ii). The system (1.51) is null controllable in some time $T > 0$;

[5] The first result in this direction was given by Datko in [45].

(iii). The system (1.51) *is completely stabilizable.*

Remark 1.25. Note that $(i) \implies (iii)$ is due to Slemrod [88]. Additionally, $(iii) \implies (i)$ is proved by Megan [74] (see also Zabczyk [99, Theorem 3.4 p. 229]). Finally, $(i) \implies (ii)$ is obvious.

1.4.2.2 The Lyapunov approach

A basic tool to study the asymptotic stability of an equilibrium point is the Lyapunov function (see, for example, [5]). In the case of a control system, the control is at our disposal, there are more chances that a given function could be a Lyapunov function for a suitable choice of feedback laws. Hence Lyapunov functions are even more useful for the stabilization of control systems than for dynamical systems without control. This subsection is based on [4, 8, 41].

Let us start considering an open subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$, with $(0,0) \in U, f \in C^\infty(U; \mathbb{R}^n)$ a function such that

$$f(0,0) = 0$$

and the following nonlinear control system

$$\frac{dz}{dt} = f(z,u), \tag{1.58}$$

where $z \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control. Now, suppose equation (1.58) has an equilibrium position, and choose a coordinate system such that the equilibrium position is at the origin. Besides, let us assume that the only solution of (1.58) with initial condition $\varphi(t_0) = 0$ is $\varphi = 0$. So, we are interested in the behavior of solutions with neighboring initial conditions to 0. So, we have the following notion of stability.

Definition 1.4.4. *An equilibrium position $\tilde{\varphi} = 0$ of the equation* (1.58) *is said to be stable (in Lyapunov's sense) if given any $\epsilon > 0$, there exists a $\delta > 0$ (depending only on ϵ and not on t) such that for every x_0 which $|x_0| < \delta$ the solution φ of* (1.58) *with initial condition $\varphi(0) = \varphi_0$ can be extended onto the whole half-line $t > 0$ and satisfies the inequality $|\varphi(t)| < \epsilon$ for all $t > 0$.*

With the previous notation in hand, let us present the Lyapunov method.

Definition 1.4.5. *A function $V : U \longmapsto \mathbb{R}$, where U is an neighborhood of $\tilde{\varphi} = 0$ is called Lyapunov control function to the system* (1.58) *when fulfill*

(i) V is continuous on U and differentiable on $U \backslash \{0\}$;

(ii) $V(0) = 0$ and $V > 0$ for all $x \in U \backslash \{0\}$;

(iii) For all $x \in U \backslash \{0\}$, there exists $u \in \mathbb{R}^m$ such that $\nabla V(x).f(x,u) < 0$.

Besides, we say that V satisfies the ***small control property*** *if, for every $\epsilon > 0$, there exists $\eta > 0$ such that for every $x \in U$, with $0 < \|x\| < \eta$, there exists $u \in \mathbb{R}^m$ satisfying $\|u\| < \epsilon$ and $\nabla V(x).f(x,u) < 0$.*

Now, we present some theorems for local stabilization of the control system (1.58) based on Lyapunov's approach. The first one uses continuous periodic time-varying feedback laws and it is proved in [42].

Theorem 1.26. *The control system (1.58) can be local asymptotically stabilized using continuous periodic time-varying feedback laws if it admits a control Lyapunov function satisfying the small control property.*

Before presenting the next theorem, we need another definition:

Definition 1.4.6. *The control system (1.58) is said to be control affine if there are $m+1$ maps $f_i, i \in \{0, ...m\}$ such that*

$$f(z,u) = f_0(z) + \sum_{i=1}^{m} u_i f_i(z), \forall (z,u) \in U.$$

The second theorem, proved by Eduardo Sontag in [89] gives us explicit and simple feedback laws.

Theorem 1.27. *Suppose that V is a control Lyapunov function satisfying the small control property for the control system (1.58) and also, assume that (1.58) is the control affine. Then $u = (u_1, ..., u_m)^{tr} : \mathbb{R}^n \longmapsto \mathbb{R}^m$ defined by*

$$u_i(x) := -\phi\Big(f_0(z).\nabla V(z), \sum_{j=1}^{m} (f_j(x).\nabla V(z))^2\Big) f_i(z).\nabla V(z),$$

with

$$\phi(a,b) = \begin{cases} \frac{a+\sqrt{a^2+b^2}}{b}, & \text{if } b \neq 0 \\ 0, & \text{if } b = 0, \end{cases}$$

is continuous, vanishes at $0 \in \mathbb{R}^n$ and globally asymptotically stabilizes the control system (1.58).

As we see above, control Lyapunov functions are a very powerful tool used to get stability on control systems. So, to achieve stability, the big challenge is to define good functions of this type. At least, for mechanical systems, a natural candidate for a control Lyapunov function is given by the total energy of the system, i.e., the sum of potential and kinetic energies; however, in general, it does not work.

2 Problems and main Results

With the introduction presented before, in this chapter, we present the main problems and results of this thesis. The first part is related to the results contained in [24] and treats the controllability of the Kawahara equation on a bounded domain, with a kind of control condition called integral overdetermination. The second part, based on [25], deals with the controllability of the Kawahara equation with the overdetermination condition in the unbounded domain. Finally, the third part of this section presents the result proved in [22], and deals with the stability of the Kawahara equation on a bounded domain, under the action of time-delayed boundary control.

2.0.1 Controllability of Kawahara equation with overdetermination condition

The first result of this thesis [24], in collaboration with Roberto de A. Capistrano Filho, investigated a kind of control property to the Kawahara equation when an *integral overdetermination condition* is required, namely

$$\int_0^L u(t,x)\omega(x)dx = \varphi(t), \ t \in [0,T], \tag{2.1}$$

with some known functions ω and φ. To present the problem, let us consider the Kawahara equation in the bounded rectangle $Q_T = (0,T) \times (0,L)$, where T and L are positive numbers with boundary function h_i, for $i = 1,2,3,4$ and h or the right-hand side f of a special form to specify latter and $p \in \{1,2\}$, namely,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^p u_x = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t), \ u(t,L) = h_2(t), \ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t), \ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{2.2}$$

Thus, we are interested in studying two control problems, which we will call them from now on by *overdetermination control problem.* The first one can be read as follows:

Problem $\mathcal{A}$: For given functions u_0, h_i, for $i = 1,2,3,4$ and f in some appropriated spaces, can we find a boundary control h such that the solution associated with the equation (2.2) satisfies the integral overdetermination (2.1)?

The second problem of this work is concentrated to prove that for a special form of the function

$$f(t,x) = f_0(t)g(t,x), \quad (t,x) \in Q_T, \tag{2.3}$$

the integral overdetermination (2.1) is verified.

Problem $\mathcal{B}$: For given functions u_0, h_i, for $i = 1, 2, 3, 4$, h and g in some appropriated spaces, can we find an internal control f_0 such that the solution associated with the equation (2.2) satisfies the integral overdetermination (2.1)?

Before trying to answer the questions right above, let us introduce some notations.

i. Denote by
$$X(Q_T) = C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2(0,L)),$$
the space equipped with the following norm
$$\|v\|_{X(Q_T)} = \max_{t\in[0,T]} \|v(t,\cdot)\|_{L^2(0,L)} + \|v_{xx}\|_{L^2(Q_T)} = \|v\|_{C([0,T];L^2(0,L))} + \|v_{xx}\|_{L^2(Q_T)}.$$

ii. Consider
$$\mathcal{H} = H^{\frac{2}{5}}(0,T) \times H^{\frac{2}{5}}(0,T) \times H^{\frac{1}{5}}(0,T) \times H^{\frac{1}{5}}(0,T),$$
with the norm
$$\|\widetilde{h}\|_{\mathcal{H}} = \|h_1\|_{H^{\frac{2}{5}}(0,T)} + \|h_2\|_{H^{\frac{2}{5}}(0,T)} + \|h_3\|_{H^{\frac{1}{5}}(0,T)} + \|h_4\|_{H^{\frac{1}{5}}(0,T)},$$
where $\widetilde{h} = (h_1, h_2, h_3, h_4)$.

iii. The intersection $(L^p \cap L^q)(0,T)$ will be considered with the following norm
$$\|\cdot\|_{(L^p\cap L^q)(0,T)} = \|\cdot\|_{L^p(0,T)} + \|\cdot\|_{L^q(0,T)}.$$

iv. Finally, for any $p \in [1,\infty]$, we denote by
$$\widetilde{W}^{1,p}(0,T) = \{\varphi \in W^{1,p}(0,T); \varphi(0) = 0\},$$
with the norm defined by
$$\|\varphi\|_{\widetilde{W}^{1,p}(0,T)} = \|\varphi'\|_{L^p(0,T)}.$$

vi. Consider ω be a fixed function that belongs to the following set
$$\mathcal{J} = \{\omega \in H^5(0,L) \cap H_0^2(0,L);\ \omega''(0) = 0\}. \tag{2.4}$$

The first result of the chapter gives us an answer for Problem $\mathcal{A}$. The answer for the boundary overdetermination control problem for the system (2.2) can be read as follows.

Theorem 2.1. *Let $p \in [2,\infty]$. Suppose that $u_0 \in L^2(0,L)$, $f \in L^p(0,T; L^2(0,L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0,T)$, for $i = 1,2,3,4$. If $\varphi \in L^p(0,T)$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$ and*
$$\int_0^L u_0(x)\omega(x)dx = \varphi(0), \tag{2.5}$$
considering $c_0 = \|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^2(0,T)}$, the following assertions hold true.

1. *For a fixed c_0, there exists $T_0 > 0$ such that for $T \in (0, T_0]$, then we can find a unique function $h \in L^p(0,T)$ in such a way that the solution $u \in X(Q_T)$ of (2.2) satisfies (2.1).*

2. *For each $T > 0$ fixed, exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, then we can find a unique boundary control $h \in L^p(0,T)$ with the solution $u \in X(Q_T)$ of (2.2) satisfying (2.1).*

The next theorem ensures for the first time that we can control the Kawahara equation with a function f_0 supported in $[0,T]$. Precisely, we will respond to the Problem $\mathcal{B}$.

Theorem 2.2. *Let $p \in [1, \infty]$, $u_0 \in L^2(0,L)$, $h \in L^{max\{2,p\}}(0,T; L^2(0,L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0,T)$, for $i = 1,2,3,4$. If $\varphi \in L^p(0,T)$, $g \in C([0,T]; L^2(0,L))$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$, and there exists a positive constant g_0 such that (2.5) is satisfied and*

$$\left| \int_0^L g(t,x)\omega(x)dx \right| \geq g_0 > 0,$$

considering $c_0 = \|u_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^1(0,T)}$, we have that:

1. *For a fixed c_0, so there exists $T_0 > 0$ such that for $T \in (0, T_0]$, exists a unique $f_0 \in L^p(0,T)$ and a solution $u \in X(Q_T)$ of (2.2), with f defined by (2.3), satisfying (2.1).*

2. *For a fixed $T > 0$, there exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, we have the existence of a control input $f_0 \in L^p(0,T)$ which the solution $u \in X(Q_T)$ of (2.2), with f as in (2.3), verifies (2.1).*

2.0.2 Controllability of Kawahara equation with overdetermination condition: The unbounded cases

In the second work of this thesis [25], in collaboration with Roberto de A. Capistrano Filho and Fernando Andrés Gallego Restrepo, we will continue working with an *integral overdetermination condition* started in [24], however in another framework, on an *unbounded domain*. We begin considering the initial boundary value problem (IBVP)

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} + \xi u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = h_1(t), \ u_x(t,0) = h_2(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{2.6}$$

where α, β and ξ are real number, $u = u(t,x)$, $g = g(t,x)$ and $h_i = h_i(t)$, for $i = 1,2$, are known function and $f_0 = f_0(t)$ is a control input. The equation (2.6) is called KdV equation when $\xi = 0$ and Kawahara equation when $\xi = -1$.

Let us now consider the *integral overdetermination condition*, but this time on an unbounded domain, namely

$$\int_{\mathbb{R}^+} u(t,x)\omega(x)dx = \varphi(t), \;\; t \in [0,T], \tag{2.7}$$

where ω and φ are some known functions. To present the problems under consideration, take the following unbounded domain $Q_T^+ = (0,T) \times \mathbb{R}^+$, where T is a positive number, consider the boundary functions μ and ν, and a source term $f = f(t,x)$ with a special form, which is

$$f(t,x) = f_0(t)g(t,x), \quad (t,x) \in Q_T^+. \tag{2.8}$$

Thus, let us deal with the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), \;\; u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{2.9}$$

Hence, the goal of the work is to answer the following questions:

Problem $\mathcal{A}$: For given functions u_0, μ, ν and g in some appropriated spaces, can we find an internal control f_0 such that the solution associated with the equation (2.9) satisfies the integral condition (2.7)?

Problem $\mathcal{B}$: What assumptions are needed to ensure that the solution u of (2.9) is unique and verifies (2.7) for a unique f_0?

Problem $\mathcal{C}$: Can we find a time $T_0 > 0$, depending on the boundary and initial data, such that if $T \leq T_0$, there exists a function f_0, in appropriated space, in that way that the solution u of (2.9) verifies (2.7)?

In this way, the first result that we provided gives answers for the Problems $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.3. *Let $T > 0$ and $p \in [2,\infty]$. Consider $\mu \in H^{\frac{2}{5}}(0,T) \cap L^p(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T) \cap L^p(0,T)$, $u_0 \in L^2(\mathbb{R}^+)$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T;L^2(\mathbb{R}^+))$ and ω be a fixed function which belongs to the following set*

$$\mathcal{J} = \{\omega \in H^5(\mathbb{R}^+) : \omega(0) = \omega'(0) = \omega''(0) = 0\}, \tag{2.10}$$

satisfying

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx$$

and

$$\left| \int_{\mathbb{R}^+} g(t,x)\omega(x)dx \right| \geq g_0 > 0, \;\; \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$c_1 = \|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (2.9) satisfying (2.7).

Our second result gives us a small time interval for which the integral overdetermination condition (2.7) follows for solutions of (2.9). Precisely, the answer for the Problem $\mathcal{C}$ can be read as follows.

Theorem 2.4. *Suppose the hypothesis of Theorem 2.3 be satisfied and consider $\delta := T^{\frac{1}{5}} \in (0,1)$, for $T > 0$. Then there exists $T_0 := \delta_0^{\frac{1}{5}} > 0$, depending on $c_1 = c_1(\delta)$ given by*

$$c_1(\delta) := \|u_{0\delta}\|_{L^2(\mathbb{R}^+)} + \|\varphi'_\delta\|_{L^2(0,T)} + \|\mu_\delta\|_{H^{\frac{2}{5}}(0,T)} + \|\nu_\delta\|_{H^{\frac{1}{5}}(0,T)},$$

such that if $T \leq T_0$, there exist a control function $f_0 \in L^p(0,T)$ and a solution u of (2.9) verifying (2.7).

From the previous results, we can give a consequence related to the controllability of the following system (2.6) with $\xi = -1$ and $h_1 = h_2 = 0$ posed in the right half-line. Precisely, we present a control property involving the overdetermination condition (2.7) and the initial state u_0 and final state u_T. To do that, consider the following notation

$$[u(t,x)] = \int_{\mathbb{R}^+} u(t,x)d\eta, \tag{2.11}$$

which one will be called of mass, for some σ-finite measure η in $\mathbb{R}^+$. With this in hand, as a consequence of Theorem 2.3, the following exact controllability in the right half-line holds.

Corollary 2.5. *Let $T > 0$ and $p \in [2,\infty]$. Consider u_0, $u_T \in L^2(\mathbb{R}^+)$ and $g \in C(0,T;L^2(\mathbb{R}^+))$, satisfying*

$$\left|\int_{\mathbb{R}^+} g(t,x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T], \tag{2.12}$$

where g_0 is a constant. Additionally, consider ω be a fixed function that belongs to the set defined by (2.10) and $\varphi \in W^{1,p}(0,T)$ satisfying

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx \quad \text{and} \quad \varphi(T) = \int_{\mathbb{R}^+} u_T(x)\omega(x)dx. \tag{2.13}$$

Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$\|u_0\|_{L^2(\mathbb{R}^+)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$, a unique solution u of (2.6) and a σ-finite measure η in $\mathbb{R}^+$ such that

$$[u(T)] = [u_T]. \tag{2.14}$$

2.0.3 Two stability results for the Kawahara equation with a time-delayed

In the third work of this thesis [22], in collaboration with Roberto de Almeida Capistrano Filho, Boumendiène Chentouf, and Victor H. Gonzalez Martinez, we deal with the stability of the Kawahara equation in a bounded domain under the action of time-delayed boundary control, namely

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) + u^p(t,x)\partial_x u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t > 0, \\ \partial_x^2 u(t,L) = \mathcal{F}(t,h), & t > 0, \\ \partial_x^2 u(t,0) = z_0(t), & t \in \mathcal{T}, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2.15}$$

In (2.15), $\Omega = (0,L)$, where $L > 0$, while $a > 0$ and $b > 0$ are physical parameters. Moreover, $p \in \{1,2\}$ and $\mathcal{F}(t,h)$ is the delayed control given by

$$\mathcal{F}(t) = \alpha\partial_x^2 u(t,0) + \beta\partial_x^2 u(t-h,0), \tag{2.16}$$

in which $h > 0$ is the time-delay, α and β are two feedback gains satisfying the restriction

$$|\alpha| + |\beta| < 1. \tag{2.17}$$

Finally, $\mathcal{T} = (-h,0)$, while u_0 and z_0 are initial conditions.

Thereafter, the functional energy associated to the system (2.15)-(2.16) is

$$E(t) = \int_0^L u^2(t,x)dx + h|\beta| \int_0^1 (\partial_x^2 u(t-h\rho,0))^2 d\rho, \ t \geq 0. \tag{2.18}$$

Now, recall that if $\alpha = \beta = 0$, then the term $\partial_x^2 u(t,0)$ represents a feedback damping mechanism (see for instance [3], where $a = 1$ and [94], where $a = 0$) but extra internal damping $a(x)u(t,x)$ is required to achieve the stability of the solutions. Note that $a(x)$ is a nonnegative function and positive only on an open subset of $(0,L)$. Therefore, taking into account the action of the time-delayed boundary control (2.16) in (2.15), the following issue will be addressed:

Does $E(t) \longrightarrow 0$, as $t \to \infty$? If it is the case, can we provide a decay rate?

It is also noteworthy that the answer to the above question is crucial in the understanding of the behavior of the solutions to the Kawahara equation when it is subject to a delayed boundary control $\mathcal{F}(t)$. In other words, are the solutions to our problem stable despite the action of the delay? If yes, then how robust is the stability property of the solutions?

First of all, let us introduce the following notations that we will use throughout this work:

(i) We consider the space of solutions

$$X(Q_T) = C(0,T;L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$$

equipped with the norm

$$\|v\|_{X(Q_T)} = \max_{t\in(0,T)} \|v(t,\cdot)\|_{L^2(0,L)} + \left(\int_0^T \|v(t,\cdot)\|^2_{H_0^2(0,L)} dt\right)^{\frac{1}{2}}.$$

(ii) Denote by

$$\tilde{H} = L^2(0,L) \times L^2(-h,0)$$

the Hilbert space equipped with the inner product

$$\langle (u_1,z_1),(u_2,z_2)\rangle_{\tilde{H}} = \int_0^L u_1u_2dx + |\beta| \int_{-h}^0 z_1(s)z_2(s)\ ds,$$

which yields the following norm

$$\|(u,z)\|^2_{\tilde{H}} = \int_0^L u^2(x)dx + |\beta| \int_{-h}^0 z^2(\rho)d\rho.$$

(iii) Throughout all the work, $(\cdot,\cdot)_{\mathbb{R}^2}$ denotes the canonical inner product of $\mathbb{R}^2$.

With the above notations in hand, let us state our first main result:

Theorem 2.6. *Let $\alpha \neq 0$ and $\beta \neq 0$ be two real constants satisfying (2.17) and suppose that the spatial length L fulfills*

$$0 < L < \sqrt{\frac{3b}{a}}\pi. \tag{2.19}$$

Then, there exists $r > 0$ sufficiently small, such that for every $(u_0,z_0) \in H$ with $\|(u_0,z_0)\|_H < r$, the energy of the system (2.15)-(2.16), denoted by E and defined by (2.18) exponentially decays, that is, there exist two positive constants κ and λ such that

$$E(t) \leq \kappa E(0)e^{-2\lambda t},\ t > 0. \tag{2.20}$$

Here,

$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2+|\beta|)}, \frac{3b\pi^2 - r^2L - L^2a}{2L^2(1+L\mu_1)}\mu_1\right\} \tag{2.21}$$

and

$$\kappa \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{|\beta|}\right\}\right),$$

for $\mu_1,\mu_2 \in (0,1)$ sufficiently small.

The second main result gives another answer to the question presented in this summary. Indeed, using a different approach based on an observability inequality, we can highlight the critical lengths phenomenon observed in [3] for the Kawahara equation:

Theorem 2.7. *Assume that α and β satisfy (2.17), whereas $L > 0$ is taken so that the problem $(\mathcal{N})$ (see Lemma 5.12) has only the trivial solution. Then, there exists $r > 0$ such that for every $(u_0, z_0) \in H$ satisfying*

$$\|(u_0, z_0)\|_H \leq r,$$

the energy of system (2.15)-(2.16), denoted by E and defined by (2.18), decays exponentially. More precisely, there exist two positive constants ν and κ such that

$$E(t) \leq \kappa E(0) e^{-\nu t}, \quad t > 0.$$

With the presentation of this summary, we can give details of these results in the next three chapters.

3 Control results with overdetermination condition to higher order dispersive system

3.1 Introduction

3.1.1 Setting of the problem

The Kawahara equation proposed in 1972 by T. Kawahara [65] is a fifth-order Korteweg-de Vries equation (KdV) that can be viewed as a generalization of the KdV equation, which occurs in the theory of shallow water waves and take the form

$$u_t + u_x + u_{xxx} + \alpha u_{xxxxx} + uu_x = 0, \tag{3.1}$$

when $\alpha = -1$ and $u = u(t,x)$ is a real-valued function of two real variables (t,x). It is important to point out that there are other physical backgrounds of the Kawahara equation or viewed as a perturbed equation of KdV[1]. To see more about such matters, the reader can see [19, 59, 78], among others.

In this chapter, we will be interested in a kind of control property to the Kawahara equation when an *integral overdetermination condition* is required, namely

$$\int_0^L u(t,x)\omega(x)dx = \varphi(t), \ \ t \in [0,T], \tag{3.2}$$

with some known functions ω and φ. To present the problem, let us consider the Kawahara equation in the bounded rectangle $Q_T = (0,T) \times (0,L)$, where T and L are positive numbers with boundary function h_i, for $i = 1,2,3,4$ and h or the right-hand side f of a special form to specify latter, namely,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t), \ u(t,L) = h_2(t), \ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t), \ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{3.3}$$

Thus, we are interested in studying two control problems, which we will call them from now on by *overdetermination control problem*. The first one can be read as follows:

Problem $\mathcal{A}$: For given functions u_0, h_i, for $i = 1,2,3,4$ and f in some appropriated spaces, can we find a boundary control h such that the solution associated to the equation (3.3) satisfies the integral overdetermination (3.2)?

The second problem of this work is concentrated to prove that for a special form of the function

$$f(t,x) = f_0(t)g(t,x), \quad (t,x) \in Q_T, \tag{3.4}$$

[1] Considering $\alpha = 0$ in (3.1) we have the so-called KdV equation, for a historic review of this equation we can cite [11] and the reference therein.

the integral overdetermination (3.2) is verified, in other words.

Problem $\mathcal{B}$: For given functions u_0, h_i, for $i = 1, 2, 3, 4$, h and g in some appropriated spaces, can we find an internal control f_0 such that the solution associated to the equation (3.3) satisfies the integral overdetermination (3.2)?

3.1.2 Bibliographical comments

We comment briefly on the bibliography emphasizing the works related with the *well-posedness* and *controllability*. Before presenting it, we caution that this is only a small sample of the extant works existent for the Kawahara equation since there are other subjects of interest from a mathematical point of view.

3.1.2.1 Well-posedness results

Regarding the Cauchy problem some authors showed the local and global well-posedness results. For example, Kenig *et al.* [67] proved the well-posedness result for a general nonlinear dispersive equation, which one with some restrictions, can be viewed as (3.1). In this celebrated work, the authors are able to prove that the associated initial value problem (IVP) is locally well-posed in weighted Sobolev spaces. We would like to mention that in [27,68] the authors also treated the theory of well-posedness in weighted Sobolev spaces for the Kawahara equation. Recently, Cui *et al.* [43] studied the Cauchy problem of the Kawahara equation in L^2-space, precisely, they proved the global well-posedness for (3.1). Considering the initial boundary value problem (IBVP) we can see relevant advances in [47], for homogeneous boundary conditions, and in [52], for the half-line. In addition to these works, some other works treat the well-posedness theory, we can cite, for example, [51,79].

3.1.2.2 Controllability results

As is well known the control theory can be studied in two ways: Stabilization problems and internal or boundary control problems (see [3,103] for details of these kinds of issues).

In this spirit, let us start to mention a pioneer work concerning the stabilization property for the Kawahara equation. In [3], the first author with some collaborators was able to introduce an internal feedback law in (3.3), considering the nonlinearity u^2u_x instead of uu_x and $h(t) = h_i(t) = 0$, for $i = 1, 2, 3, 4$. To be precise, they proved that under the effect of the damping mechanism the energy associated with the solutions of the system decays exponentially. Additionally, they conjecture the existence of important phenomena, the so-called *critical set phenomenon* as occurs with the single KdV equation [20,83] and the Boussinesq KdV-KdV system [29][2]. We also would like to suggest to the reader the reference [48] to stabilization problems related to the Kawahara equation in the real line.

Now, some references to internal control problems are presented. This problem was first addressed in [102] and after that in [103]. In both cases, the authors considered the Kawahara equation in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(t,x) = (Gh)(t,x) := g(x)(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)dy),$$

where $g \in C^\infty(\mathbb{T})$ supported in $\omega \subset \mathbb{T}$ and h is a control input. Here, it is important to observe that the control in consideration has a different form as presented in (3.4), and the result is proven in a different direction from what we will present in this chapter.

Still related to internal control issues, Chen [34] presented results considering the Kawahara equation (3.3) posed in a bounded interval with a distributed control $f(t,x)$ and homogeneous boundary conditions. She showed the result by taking advantage of a Carleman estimate associated with the linear operator of the Kawahara equation with an internal observation. With this in hand, she was able to get a null controllability result when f is effective in a $\omega \subset (0,L)$. As the results obtained by her do not answer all the issues of internal controllability, in a recent article [27] the authors closed some gaps left in [34]. Precisely, considering the system (3.3) with an internal control $f(t,x)$ and homogeneous boundary conditions, the authors can show that the equation in consideration is exactly controllable in L^2-weighted Sobolev spaces and, additionally, the Kawahara equation is controllable by regions on L^2-Sobolev space, for details see [27].

Finally, related to the boundary control problem, there is a unique result that was proved in [56]. The authors consider the boundary conditions as in (3.3) and show that exact controllability holds when two or up to five controls are inputted in these boundary conditions.

3.1.3 Notations and Main results

With these previous results in hand, we can present our main results that try to answer questions left open in the manuscript [27] and present an alternative way for the boundary and internal control problems of the Kawahara equation. First of all, let us introduce the following notation that we will use from now on.

i. Denote by

$$X(Q_T) = C([0,T];L^2(0,L)) \cap L^2(0,T;H^2(0,L)),$$

the space equipped with the following norm

$$\|v\|_{X(Q_T)} = \max_{t\in[0,T]} \|v(t,\cdot)\|_{L^2(0,L)} + \|v_{xx}\|_{L^2(Q_T)} = \|v\|_{C([0,T];L^2(0,L))} + \|v_{xx}\|_{L^2(Q_T)}.$$

[2] Differently what happens with KdV and Boussinesq KdV-KdV the characterization of the critical set for the Kawahara equation is an open issue, we cite [46] for details of this subject.

ii. Consider

$$\mathcal{H} = H^{\frac{2}{5}}(0,T) \times H^{\frac{2}{5}}(0,T) \times H^{\frac{1}{5}}(0,T) \times H^{\frac{1}{5}}(0,T),$$

with the norm

$$\|\widetilde{h}\|_{\mathcal{H}} = \|h_1\|_{H^{\frac{2}{5}}(0,T)} + \|h_2\|_{H^{\frac{2}{5}}(0,T)} + \|h_3\|_{H^{\frac{1}{5}}(0,T)} + \|h_4\|_{H^{\frac{1}{5}}(0,T)},$$

where $\widetilde{h} = (h_1, h_2, h_3, h_4)$.

iii. The intersection $(L^p \cap L^q)(0,T)$ will be considered with the following norm

$$\|\cdot\|_{(L^p\cap L^q)(0,T)} = \|\cdot\|_{L^p(0,T)} + \|\cdot\|_{L^q(0,T)}.$$

iv. Finally, for any $p \in [1,\infty]$, we denote by

$$\widetilde{W}^{1,p}(0,T) = \{\varphi \in W^{1,p}(0,T); \varphi(0) = 0\},$$

with the norm defined by

$$\|\varphi\|_{\widetilde{W}^{1,p}(0,T)} = \|\varphi'\|_{L^p(0,T)}.$$

vi. Consider ω be a fixed function that belongs to the following set

$$\mathcal{J} = \{\omega \in H^5(0,L) \cap H_0^2(0,L);\ \omega''(0) = 0\}. \tag{3.5}$$

The first result of this chapter gives us an answer for Problem $\mathcal{A}$, presented at the beginning of the introduction. The answer for the boundary overdetermination control problem for the system (3.3) can be read as follows.

Theorem 3.1. *Let $p \in [2,\infty]$. Suppose that $u_0 \in L^2(0,L)$, $f \in L^p(0,T;L^2(0,L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0,T)$, for $i = 1,2,3,4$. If $\varphi \in W^{1,p}(0,T)$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$ and*

$$\int_0^L u_0(x)\omega(x)dx = \varphi(0), \tag{3.6}$$

considering $c_0 = \|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^2(0,T)}$, the following assertions hold true.

1. *For a fixed c_0, there exists $T_0 > 0$ such that for $T \in (0,T_0]$, then we can find a unique function $h \in L^p(0,T)$ in such a way that the solution $u \in X(Q_T)$ of (3.3) satisfies (3.2).*

2. *For each $T > 0$ fixed, exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, then we can find a unique boundary control $h \in L^p(0,T)$ with the solution $u \in X(Q_T)$ of (3.3) satisfying (3.2).*

The next result ensures for the first time that we can control the Kawahara equation with a function f_0 supported in $[0,T]$. Precisely, we will give an affirmative answer to the Problem $\mathcal{B}$ mentioned in this introduction.

Theorem 3.2. *Let $p \in [1,\infty]$, $u_0 \in L^2(0,L)$, $h \in L^{max\{2,p\}}(0,T;L^2(0,L))$, $\widetilde{h} \in \mathcal{H}$ and $h_i \in L^p(0,T)$, for $i=1,2,3,4$. If $\varphi \in W^{1,p}(0,T)$, $g \in C([0,T];L^2(0,L))$ and $\omega \in \mathcal{J}$ are such that $\omega''(L) \neq 0$, and there exists a positive constant g_0 such that (3.6) is satisfied and*

$$\left| \int_0^L g(t,x)\omega(x)dx \right| \geq g_0 > 0,$$

considering $c_0 = \|u_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^1(0,T)}$, we have that:

1. *For a fixed c_0, so there exists $T_0 > 0$ such that for $T \in (0,T_0]$, exists a unique $f_0 \in L^p(0,T)$ and a solution $u \in X(Q_T)$ of (3.3), with f defined by (3.4), satisfying (3.2).*

2. *For a fixed $T > 0$, there exists a constant $\gamma > 0$ such that for $c_0 \leq \gamma$, we have the existence of a control input $f_0 \in L^p(0,T)$ which the solution $u \in X(Q_T)$ of (3.3), with f as in (3.4), verifies (3.2).*

3.1.4 Heuristic of the chapter and further comments

In this chapter, we investigate and discuss overdetermination control problems for the boundary and internal variations. As can be seen in this introduction, the agenda of the research of control theory for the fifth-order KdV equation is quite new, and there are not many results in the literature. With this proposal to fill this gap, we intend to present a new way to prove internal and boundary control results for this system. Thus, for this type of *integral overdetermination condition* the first results on the solvability of control problems for the Kawahara equation are obtained in the present work.

3.1.4.1 Heuristic of the chapter

The first result is concerning the *boundary overdetermination control problem*, roughly speaking, we are able to find an appropriate control h, acting on the boundary term $u_{xx}(t,L)$, such that integral condition (3.2) holds. Theorem 3.1 is first proved for the linear system associated with (3.3) and after that, using a fixed point argument, extended to the nonlinear system. The main ingredients are the Lemmas 3.7 and 3.10. In the Lemma 3.10 we can find two appropriate applications that link the boundary control term $h(t)$ with the overdetermination condition (3.2), namely

$$\begin{aligned} \Lambda : L^p(0,T) &\longrightarrow \widetilde{W}^{1,p}(0,T) \\ h \quad &\longmapsto (\Lambda h)(\cdot) = \int_0^L u(\cdot,x)\omega(x)dx \end{aligned}$$

and

$$\begin{aligned} A : L^p(0,T) &\longrightarrow L^p(0,T) \\ h &\longmapsto (Ah)(t) = \varphi'(t) - \int_0^L u(t,x)(\omega'(x) + \omega'''(x) - \omega'''''(x))dx, \quad \forall t \in [0,T]. \end{aligned}$$

So, we prove that such application Λ has an inverse $\Gamma = \Lambda^{-1}$ which is continuous, by Banach's theorem, showing the lemma in question, and so, reaching our goal, to prove Theorem 3.1.

Theorem 3.2 follows the same idea, the strictly different point is related to the appropriated applications which in this case links the internal control f_0 with the overdetermination condition (3.2) (see Lemma 3.13), defined as follows

$$(\Lambda f_0)(\cdot) = \int_0^L u(\cdot, x)\omega(x)dx$$

and

$$(Af_0)(t) = \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)} \int_0^L u(t,x)(\omega' + \omega''' - \omega''''')dx,$$

where,

$$g_1(t) = \int_0^L g(t,x)\omega(x)dx.$$

3.1.4.2 Further comments

To conclude this introduction, we outline additional comments. It is important to point out that the method used here is commonly applied to inverse problems in optimal control. For the readers we cite this excellent book [80] for details of the integral conditions applied in inverse problems.

Concerning the generality of the work, we have the following points:

- Theorems 3.1 and 3.2 can be obtained for more general nonlinearities. Indeed, if we consider $v \in X(Q_T)$ and $p \in (2,4]$, we have that

  $$\int_0^T \int_0^L |v^{p+2}| dxdt \leqslant C \left\| v \right\|^p_{C([0,T];L^2(0,L))} \int_0^T \left\| v_x \right\|^2 dt \leqslant C \left\| v \right\|^{p+2}_{X(Q_T)},$$

  by the Gagliardo–Nirenberg inequality. Moreover, recently, Zhou [104] showed the well-posedness of the following initial boundary value problem

  $$\begin{cases} u_t - u_{xxxxx} = c_1 uu_x + c_2 u^2 u_x + b_1 u_x u_{xx} + b_2 uu_{xxx}, & x \in (0,L),\ t \in \mathbb{R}^+, \\ u(t,0) = h_1(t), \quad u(t,L) = h_2(t), \quad u_x(t,0) = h_3(t), & t \in \mathbb{R}^+, \\ u_x(t,L) = h_4(t), \quad u_{xx}(t,L) = h(t), & t \in \mathbb{R}^+, \\ u(0,x) = u_0(x), & x \in (0,L), \end{cases} \tag{3.7}$$

  Thus, due to the previous inequality and the results proved in [104], when we consider $b_1 = b_2 = 0$ and the combination $c_1 uu_x + c_2 u^2 u_x$ instead of uu_x on (3.3), Theorems 3.1 and 3.2 remains valid, however, for sake of simplicity, we consider only the nonlinearity as uu_x.

- Note that the regularity of the boundary terms is sharp in $H^s(0, L)$, for $s \geq 0$. In fact, due to the method introduced by Bona *et al.* [11] for the KdV equation the authors in [101] and [104] can provide sharp regularity for the traces function in both IBVP (3.3) and (3.7). So, in this sense, Theorems 3.1 and 3.2 give a sharp regularity of the functions involved.

- Unlike what happens in the case of the control problem considered in [83] for the KdV equation and on [29] for Boussinesq KdV–KdV equation, here, due to the method used, we can take $h_i = 0$, for $i = 1, 2, 3, 4$, and only consider a control acting in the trace $u_{xx}(t, L)$, without concern with the critical set phenomenon.

- The arguments presented in this work have prospects to be applied to other non-linear dispersive equations in the context of the bounded domains. Our motivation was because Faminskii [53] proved a result for the KdV equation, that is when considering the system (3.1) with $\alpha = 0$. However, note that in [53] the author decides to use the solution in a weak sense, ensuring that the results are verified for the function $\frac{u^2}{2}$, but, in our case, we can deal with more general the terms like $\frac{u^2}{2}$, uu_x and u^2u_x.

- Finally, this work presents another way to prove control results for the higher order dispersive system which are completely different from what was presented in [27, 56, 103].

3.1.5 Outline of the work

Section 3.2 is devoted to reviewing the main results of the well-posedness for the fifth-order KdV equation in Sobolev spaces. In Section 3.3 we present two auxiliary lemmas which help us to prove the controllability results. The overdetermination control results, when the control is acting in the boundary and internally, are presented in Sections 3.4 and 3.5, respectively, that is, we will present the proof of the main results of the chapter, Theorems 3.1 and 3.2.

3.2 A fifth-order KdV equation: A review of well-posedness results

In this section let us treat the well-posedness of the fifth-order KdV equation, that is, we are interested in the well-posedness of the following system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t),\ u(t,L) = h_2(t),\ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t),\ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L], \end{cases} \tag{3.8}$$

where $L, T > 0$ are fixed real numbers, $Q_T = [0,T] \times [0,L]$ and $u_0, h_1, h_2, h_3, h_4, h$ and f are well-known functions. Precisely, we will put together the mains results of well-posedness to (3.8).

3.2.1 Homogeneous case

The first result is due to the first author [3, Lemma 2.1] and provided the well-posedness results for the linear problem

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0 & \text{in } Q_T, \\ u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = u_{xx}(t,L) = 0 & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{3.9}$$

Lemma 3.3. *Let $u_0 \in L^2(0,L)$. Then (3.9) possesses a unique (mild) solution $u \in X(Q_T)$ with*

$$u_{xx}(0,t) \in L^2(0,T).$$

Moreover, there exists a constant $C = C(T,L) > 0$ such that

$$\|u\|_{C^0([0,T];L^2(0,L))} + \|u\|_{L^2(0,T;H^2(0,L))} \leq C \|u_0\|_{L^2(0,L)}$$

and

$$\|u_{xx}(0,t)\|_{L^2(0,T)} \leq \|u_0\|_{L^2(0,L)}.$$

The proof of this lemma is a direct consequence of the semigroup theory and multipliers method. In the way to prove global well-posedness results for the nonlinear system, in [3, Lemma 2.2 and 2.3], the authors can prove some results for the following system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^p u_x = 0 & \text{in } Q_T, \\ u(t,0) = u(t,L) = u_x(t,0) = u_x(t,L) = u_{xx}(t,L) = 0 & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L], \end{cases} \tag{3.10}$$

with $p \in (2,4]$. The global well-posedness for this system can be read as follows (we infer the read see [3, Lemmas 2.2 and 2.3 and Remark 2.1] for details).

Lemma 3.4. *Let $T_0 > 0$ and $u_0 \in L^2(0,L)$ be given. Then there exists $T \in (0, T_0]$ such that (3.10) possesses a unique solution $u(t,x) \in Q_T$. Moreover, if $\|u_0\| \ll 1$, then*

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leq c_1 \|u_0\|^2_{L^2(0,L)} \left(1 + \|u_0\|^4_{L^2(0,L)}\right),$$

where $c_1 = c_1(T,L)$ is a positive constant. Moreover,

$$u_t \in L^{4/3}\left(0,T; H^{-3}(0,L)\right).$$

3.2.2 Non-homogeneous case

For the non-homogeneous initial-boundary value problem (IBVP)

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t), \ u(t,L) = h_2(t), \ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t), \ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L], \end{cases} \tag{3.11}$$

Zhao and Zhang [101, Lemma 3.1] showed the following result:

Lemma 3.5. *Let $T > 0$ be given, there is a $C > 0$ such that for any $f \in L^1(0,T;L^2(0,L))$, $u_0 \in L^2(0,L)$, $h \in L^2(0,T)$ and $\tilde{h} \in \mathcal{H}$, IBVP (3.11) admits a unique solution (mild) $u := S(u_0, h, f, \tilde{h}) \in X(Q_T)$ satisfying*

$$\|u\|_{X(Q_T)} \leq C\left(\|u_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|\tilde{h}\|_{\mathcal{H}} + \|f\|_{L^1(0,T;L^2(0,L))}\right).$$

Considering the full system (3.8), in this same work, Zhao and Zhang [101, Lemma 3.2], showed the following result.

Lemma 3.6. *There exists a constant $C > 0$ such that for any $T > 0$ and $u, v \in X(Q_T)$ satisfying the following inequalities:*

$$\int_0^T \|uv_x\|_{L^2(0,L)} dt \leq C(T^{\frac{1}{2}} + T^{\frac{1}{4}}) \|u\|_{X(Q_T)} \|v\|_{X(Q_T)}$$

and

$$\|uv_x\|_{L^1(0,T;L^2(0,L))} \leq C(T^{\frac{1}{2}} + T^{\frac{1}{4}}) \|u\|_{X(Q_T)} \|v\|_{X(Q_T)}.$$

3.3 Auxiliary results

In this section, we are interested to prove some auxiliary lemmas for the solutions of the system

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } Q_T, \\ u(t,0) = h_1(t), \ u(t,L) = h_2(t), \ u_x(t,0) = h_3(t), & \text{in } [0,T], \\ u_x(t,L) = h_4(t), \ u_{xx}(t,L) = h(t) & \text{in } [0,T], \\ u(0,x) = u_0(x) & \text{in } [0,L]. \end{cases} \tag{3.12}$$

To do this, consider $\omega \in \mathcal{J}$ defined by (3.5) and define $q : [0,T] \longrightarrow \mathbb{R}$ as follows

$$q(t) = \int_0^L u(t,x)\omega(x)dx, \tag{3.13}$$

where $u = S(u_0, h, f, \widetilde{h})$ is solution of (3.12) guaranteed by Lemma 3.5. The next two auxiliary lemmas are the key point to show the main results of this work. The first one gives that $q \in W^{1,p}(0,L)$ and can be read as follows.

Lemma 3.7. *Let $p \in [1,\infty]$ and the assumptions of Lemma 3.5 be satisfied. Suppose that h_i, for $i = 1,2,3,4$, and h belonging in $L^p(0,T)$, $f = f_1 + \frac{\partial f_2}{\partial x}$, where $f_1 \in L^p(0,T;L^2(0,L))$ and $f_2 \in L^p(0,T;H^1(0,L))$. If $u = S(u_0, h, f_1 + \frac{\partial f_2}{\partial x}, \widetilde{h})$ is a mild solution of (3.12) and $\omega \in \mathcal{J}$, then the function q given by (3.13) belongs to $W^{1,p}(0,T)$ and the relation*

$$\begin{aligned} q'(t) = {} & \omega''(L)h(t) - \omega'''(L)h_4(t) + \omega'''(0)h_3(t) + \omega''''(L)h_2(t) - \omega''''(0)h_1(t) \\ & + \int_0^L f_1(t,x)\omega(x)dx - \int_0^L f_2(t,x)\omega'(x)dx + \int_0^L u(t,x)[\omega'(x) + \omega'''(x) - \omega'''''(x)]dx \end{aligned} \tag{3.14}$$

holds for almost all $t \in [0,T]$. In addition, the function $q' \in L^p(0,T)$ can be estimate in the following way

$$\begin{aligned} \|q'\|_{L^p(0,T)} \leq C \Big(& \|u_0\|_{L^2(0,L)} + \|h\|_{(L^p\cap L^2)(0,T)} + \|h_1\|_{(L^p\cap H^{\frac{2}{5}})(0,T)} \\ & + \|h_2\|_{(L^p\cap H^{\frac{2}{5}})(0,T)} + \|h_3\|_{(L^p\cap H^{\frac{1}{5}})(0,T)} + \|h_4\|_{(L^p\cap H^{\frac{1}{5}})(0,T)} \\ & + \|f_1\|_{L^p(0,T;L^2(0,L))} + \|f_2\|_{L^p(0,T;H^1(0,L))} \Big) \end{aligned} \tag{3.15}$$

with $C > 0$ a constant that is nondecreasing with increasing T.

Proof. Considering $\psi \in C_0^\infty(0,T)$, multiplying (3.12) by $\psi\omega$ and integrating by parts in Q_T we have that

$$\begin{aligned} \int_0^T \psi'(t)q(t)dt = & \int_0^T \psi(t)\Big(\omega''(L)h(t) - \omega'''(L)h_4(t) + \omega'''(0)h_3(t) \\ & + \omega''''(L)h_2(t) - \omega''''(0)h_1(t) \\ & + \int_0^L f_1(t,x)\omega(x)dx - \int_0^L f_2(t,x)\omega'(x)dx \\ & + \int_0^L u(t,x)(\omega'(x) + \omega'''(x) - \omega'''''(x))dx \Big) dt \\ = & -\int_0^T \psi(t)r(t)dt \end{aligned}$$

with $r : [0,T] \longmapsto \mathbb{R}$ defined by

$$\begin{aligned} r(t) = {} & \omega''(L)h(t) - \omega'''(L)h_4(t) + \omega'''(0)h_3(t) + \omega''''(L)h_2(t) - \omega''''(0)h_1(t) \\ & + \int_0^L f_1(t,x)\omega(x)dx - \int_0^L f_2(t,x)\omega'(x)dx + \int_0^L u(t,x)[\omega'(x) + \omega'''(x) - \omega'''''(x)]dx, \end{aligned}$$

which gives us $q'(t) = r(t)$.

It remains for us to prove that $q' \in L^p(0,T)$, for $p \in [1,\infty]$. To do it, we need to estimate each term of (3.14). We will split the proof in two cases, namely, $p \in [1,\infty)$ and $p = +\infty$.

Case 1. $1 \leq p < \infty$.

First, note that

$$\begin{aligned}\left|\int_0^L u(t,x)(\omega'(x)+\omega'''(x)-\omega'''''(x))dx\right| &\leq \|\omega\|_{H^5(0,L)}\|u(t,\cdot)\|_{L^2(0,L)}\\ &\leq T^{\frac{1}{p}}\|\omega\|_{H^5(0,L)}\|u\|_{C([0,T];L^2(0,L))}\\ &\leq C(T,\|\omega\|_{H^5(0,L)})\|u\|_{X(Q_T)}.\end{aligned}$$

To estimate the last term of (3.14), note that

$$\begin{aligned}\left|\int_0^L f_2(t,x)\omega'(x)dx\right| &\leq C(L)\|\omega'\|_{H_0^1(0,L)}\|f_2(t,\cdot)\|_{L^1(0,L)}\\ &\leq C(L)\|\omega\|_{H^5(0,L)}\|f_2(t,\cdot)\|_{L^1(0,L)},\end{aligned}$$

since $H^1(0,L) \hookrightarrow L^\infty(0,L) \cap C[0,L]$. So, the last inequality yields that

$$\left\|\int_0^L f_2(t,x)\omega'(x)dx\right\|_{L^p(0,T)} \leq C(L,\|\omega\|_{H^5(0,L)})\|f_2\|_{L^p(0,T;H^1(0,L))}.$$

Also, we have

$$\left\|\int_0^L f_1(t,x)\omega(x)dx\right\|_{L^p(0,T)} \leq \|\omega\|_{L^2(0,L)}\|f_1\|_{L^p(0,T;L^2(0,L))}.$$

To finish this case note that h_i, for $i=1,2,3,4$ and h belong to $L^p(0,T)$. Thus, we have that $q' \in L^p(0,T)$, which ensures that $q \in W^{1,p}(0,T)$. Moreover, follows that

$$\begin{aligned}\|q'\|_{L^p(0,T)} \leq &\widetilde{C}(T,L,\|\omega\|_{H^5(0,L)})\Big(\|h\|_{L^p(0,T)}+\|h_1\|_{L^p(0,T)}+\|h_2\|_{L^p(0,T)}+\|h_3\|_{L^p(0,T)}\\ &+\|h_4\|_{L^p(0,T)}+\|u\|_{X(Q_T)}+\|f_1\|_{L^p(0,T;L^2(0,L))}+\|f_2\|_{L^p(0,T;H^1(0,L))}\Big).\end{aligned}$$

Thus, estimate (3.15) holds, showing Case 1.

Case 2. $p = \infty$.

This case follows noting that

$$\begin{aligned}\|q'\|_{C([0,T])} \leq C\Big(&\|u\|_{X(Q_T)}+\|f_2\|_{C(0,T;H^1(0,L))}+\|f_1\|_{C([0,T];L^2(0,L))}\\ &+\|h\|_{C([0,T])}+\|h_1\|_{C([0,T])}+\|h_2\|_{C([0,T])}+\|h_3\|_{C([0,T])}+\|h_4\|_{C([0,T])}\Big).\end{aligned}$$

Thus, Case 2 is achieved and the proof of the lemma is complete. □

The next proposition gives us a relation between u, h and f_1 and will be the key point to prove the control problems, presented in the next sections.

Lemma 3.8. *Suppose that $h \in L^2(0,L)$, $f_1 \in L^1(0,T;L^2(0,L))$ and $u = S(0,h,f_1,0)$ mild solution of (3.12), then*

$$\int_0^L |u(t,x)|^2 dx \leq \int_0^t |h(t)|^2 d\tau + 2\int_0^t\int_0^L f_1(\tau,x)u(\tau,x)dxdt \tag{3.16}$$

for all $t \in [0,T]$.

Proof. Pick any function $h \in C_0^\infty(0,T)$ and consider $f_1 \in C_0^\infty(Q_T)$. Therefore, there exists a smooth solution $u = S(0,h,f_1,0)$ of (3.12). Thus, multiplying (3.12) by $2u$, integrating in $[0,L]$ and using the boundary conditions (remembering that $h_1 = h_2 = h_3 = h_4 = 0$), we get that

$$\begin{aligned}\frac{d}{dt}\int_0^L |u(t,x)|^2 dx &= \int_0^L f_1(t,x)u(t,x)dx + |u_{xx}(t,L)|^2 - |u_{xx}(t,0)|^2 \\ &\leq 2\int_0^L f_1(t,x)u(t,x)dx + |u_{xx}(t,L)|^2.\end{aligned}$$

So, using the fact that $u_{xx}(t,L) = h(t)$, integrating in $[0,t]$ and taking account that $u(0,\cdot) = 0$ yields

$$\int_0^L |u(t,x)|^2 dx \leq 2\int_0^t \int_0^L f_1(\tau,x)u(\tau,x)dxd\tau + \int_0^t |h(\tau)|^2 d\tau$$

which implies inequality (3.16). By density argument and the continuity of the operator S, the result is proved. □

Remark 3.9. We are now giving some remarks.

i. We are implicitly assuming that $\frac{\partial f_2}{\partial x} \in L^1(0,T;L^2(0,L))$ in the Lemma 3.6, but it is not a problem, since the function that we will take for f_2, in our purposes, satisfies that condition.

ii. When $p = \infty$, in Lemma 3.8, the spaces $L^p(0,T)$, $L^p(0,T;L^2(0,L))$ and $L^p(0,T;L^1(0,L))$ are replaced by the spaces $C([0,T])$, $C([0,T];L^2(0,L))$ and $C([0,T];L^1(0,L))$, respectively. So, we can obtain $q \in C^1([0,T])$.

3.4 Boundary control

In this section, we are interested in providing answers to overdetermination controllability results for the system (3.3) when the control is acting at the boundary. Precisely, we want to find a control function $h(t)$ acting in the boundary such that the solution of the system in consideration satisfies an overdetermination condition which will take an integral form.

3.4.1 Linear result

In this spirit presented above, the first lemma helps to prove a controllability result for the linear case.

Lemma 3.10. *Suppose that $f = \widetilde{h} = u_0 = 0$ and $\omega \in \mathcal{J}$, with $\omega''(L) \neq 0$ and $\varphi \in \widetilde{W}^{1,p}(0,T)$, for some $p \in [2,\infty]$. Then there exists a unique function $h = \Gamma\varphi \in L^p(0,T)$ whose corresponding generalized solution (mild) $u = S(0,h,0,0)$ of (3.12) satisfies the condition (3.2). Moreover, the linear operator*

$$\Gamma : \widetilde{W}^{1,p}(0,T) \longmapsto L^p(0,T)$$

is bounded and its norm is nondecreasing with increasing T.

Proof. Without loss of generality we consider here $\omega''(L) = 1$, in order to simplify the computations. First, define the application $\Lambda : L^p(0,T) \longrightarrow \widetilde{W}^{1,p}(0,T)$ as

$$(\Lambda h)(\cdot) = \int_0^L u(\cdot,x)\omega(x)dx,$$

with $u = S(0,h,0,0)$, assured by Lemma 3.3. Observe that $(\Lambda h)(0) = 0$ and $\Lambda = (Q \circ S)$, with the functions $Q : X(Q_T) \longrightarrow W^{1,p}(0,T)$ defined as

$$(Qv)(t) = \int_0^L v(t,x)\omega(x)dx, \ t \in [0,T]$$

and, in this case, S can be viewed as

$$S : L^2(0,L) \longrightarrow X(Q_T)$$

defined by $u = S(0,h,0,0)$, respectively. Since S and Q are linear, we have that Λ is also linear, and thanks to (3.15), we get that

$$\|\Lambda(h)\|_{\widetilde{W}^{1,p}(0,T)} = \|(Q\circ S)(h)\|_{\widetilde{W}^{1,p}(0,T)} \leq C(T)\|q'\|_{L^p(0,T)} \leq C(T)\|h\|_{L^2(0,T)} \quad \forall h \in L^p(0,T).$$

Therefore, Λ is continuous.

Observing that the relation $\varphi = \Lambda h$, for $h \in L^p(0,T)$, clearly means that the function h gives the desired solution of the control problem under consideration. So, our objective is to apply Banach's theorem to prove that the inverse of the operator Λ is continuous.

To do it, for a fixed function $\varphi \in \widetilde{W}^{1,p}(0,T)$, consider the mapping $A : L^p(0,T) \longrightarrow L^p(0,T)$ defined by

$$(Ah)(t) = \varphi'(t) - \int_0^L u(t,x)(\omega'(x) + \omega'''(x) - \omega'''''(x))dx, \ \forall t \in [0,T].$$

Firstly, the following claim holds true.

Claim 1. *$\varphi = \Lambda h$ if and only if $h = Ah$.*

Indeed, if $\varphi = \Lambda h$, then $q(t) = (\Lambda h)(t) = \varphi(t)$, that is, $q'(t) = \varphi'(t),\ t \in [0,T]$. Therefore,

$$(Ah)(t) = q'(t) - \int_0^L u(t,x)(\omega'(x) + \omega'''(x) - \omega'''''(x))dx = h(t),$$

thanks to (3.14). Conversely, if $Ah = h$, we have

$$h(t) = \varphi'(t) - \int_0^L u(t,x)(\omega'(x) + \omega'''(x) - \omega'''''(x))dx.$$

Here, the identity $q' = \varphi'$ holds for the function $q(t) = \Lambda h$ due to the identity (3.14). Since, $\varphi(0) = q(0) = 0$, follows that $\varphi = q$ in $\widetilde{W}^{1,p}(0,T)$, and the Claim 1 is proved.

The second claim ensures that:

Claim 2. *A is a contraction.*

In fact, let $2 \leq p < \infty$, $\mu_1, \mu_2 \in L^p(0,T)$, $u_1 = S(0,\mu_1,0,0)$ and $u_2 = S(0,\mu_2,0,0)$ in $X(Q_T)$. Therefore,

$$A\mu_1 - A\mu_2 = -\int_0^L (u_1 - u_2)(\omega' + \omega''' - \omega''''')dx.$$

Moreover, making $u = u_1 - u_2,\ h = \mu_1 - \mu_2$ we have, using (3.16), that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,L)} \leq \|\mu_1 - \mu_2\|_{L^2(0,t)},\ \forall t \in [0,T]. \tag{3.17}$$

Consider $\gamma > 0$. For $t \in [0,T]$, by Hölder inequality, follows that

$$\left| e^{-\gamma t}(A\mu_1 - A\mu_2)(t) \right| \leq C(\|\omega\|_{H^5(0,L)}) e^{-\gamma t} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,L)}. \tag{3.18}$$

The relation (3.17) gives,

$$\begin{aligned}
\|e^{-\gamma t}(A\mu_1 - A\mu_2)\|_{L^p(0,T)} &\leq C(\|\omega\|_{H^5(0,L)}) \left(\int_0^T e^{-\gamma p t} \|u_1(t,\cdot) - u_2(t,\cdot)\|^p_{L^2(0,L)} dt \right)^{\frac{1}{p}} \\
&\leq C(\|\omega\|_{H^5(0,L)}) \left(\int_0^T e^{-\gamma p t} \left(\int_0^t (\mu_1(\tau) - \mu_2(\tau))^2 d\tau \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}} \\
&\leq C(p, \|\omega\|_{H^5(0,L)}) \left(\int_0^T e^{-\gamma p t} \int_0^t |\mu_1(\tau) - \mu_2(\tau)|^p d\tau dt \right)^{\frac{1}{p}} \\
&\leq C(p, \|\omega\|_{H^5(0,L)}) \left(\int_0^T e^{-\gamma p t} |\mu_1(\tau) - \mu_2(\tau)|^p \int_t^T e^{p\gamma(\tau - t)} d\tau dt \right)^{\frac{1}{p}} \\
&\leq C(p, \|\omega\|_{H^5(0,L)}) \|e^{-\gamma t}(\mu_1 - \mu_2)\|_{L^p(0,T)} \left(\int_0^T e^{-\gamma p t} dt \right)^{\frac{1}{p}} \\
&\leq \frac{1}{(p\gamma)^{\frac{1}{p}}} C(p, T, \|\omega\|_{H^5(0,L)}) \|e^{-\gamma t}(\mu_1 - \mu_2)\|_{L^p(0,T)} \left(1 - e^{-\gamma p T} \right)^{\frac{1}{p}} \\
&= C_1 \|e^{-\gamma t}(\mu_1 - \mu_2)\|_{L^p(0,T)},
\end{aligned} \tag{3.19}$$

with $C_1 = C(p, T, \|\omega\|_{H^5(0,L)})$. Therefore, is enough to take $\gamma = \frac{(2C_1)^p}{p}$, and so A is contraction, showing the Claim 2 for the case $p \in [2, \infty)$.

Now, let us analyze the case $p = \infty$. Using (3.18), yields that

$$\begin{aligned}
\sup_{t\in[0,T]} e^{-\gamma t}\,|(A\mu_1 - A\mu_2)(t)| &= C(\|\omega\|_{H^5(0,L)}) \sup_{t\in[0,T]} e^{-\gamma t}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,L)} \\
&\leq C(\|\omega\|_{H^5(0,L)}) \sup_{t\in[0,T]} e^{-\gamma t}\|\mu_1 - \mu_2\|_{L^2(0,t)} \\
&\leq C(\|\omega\|_{H^5(0,L)}) \sup_{t\in[0,T]} \left(\int_0^t e^{2\gamma(\tau-t)}|\mu_1(\tau) - \mu_2(\tau)|^2 d\tau\right)^{\frac{1}{2}} \\
&\leq C(\|\omega\|_{H^5(0,L)})\|e^{-\gamma t}(\mu_1 - \mu_2)\|_{L^\infty(0,T)} \sup_{t\in[0,T]} \left(\frac{1}{2\gamma}[1 - e^{-2\gamma t}]\right)^{\frac{1}{2}}
\end{aligned} \tag{3.20}$$

so taking $\gamma = 2C_1^2$ yields that A is a contraction, showing the claim for $p = \infty$. This analysis ensures that the mapping A is a contraction, and Claim 2 is achieved.

Therefore, for each function $\varphi \in \widetilde{W}^{1,p}(0,T)$, there exists a unique function $h \in L^p(0,T)$ such that $h = A(h)$, that is, $\varphi = \Lambda(h)$. It follows that operator Λ is invertible, and so, its inverse $\Gamma := \Lambda^{-1} : L^p(0,T) \longmapsto \widetilde{W}^{1,p}(0,T)$ is continuous thanks to the Banach theorem. In particular,

$$\|\Gamma(\varphi)\|_{L^p(0,T)} \leq C(T)\|\varphi'\|_{L^p(0,T)}.$$

By a standard argument, we can extend continuously the function φ by the constant $\varphi(T)$ in (T, T_1) with the previous inequality still valid in $(0, T_1)$ with $C(T) \leq C(T_1)$, therefore the operator Γ in non-decreasing with increasing T, proving the result. □

With the previous result in hand, now let us show a controllability result for the linear case.

Theorem 3.11. *Consider $p \in [2, \infty]$, $\varphi \in W^{1,p}(0,T)$, $u_0 \in L^2(0,L)$, $\widetilde{h} \in \mathcal{H}$, with $h_i \in L^p(0,T)$, for $i = 1,2,3,4$ and $f = f_1 + \frac{\partial f_2}{\partial x}$, where $f_1 \in L^p(0,T;L^2(0,L))$. Moreover, if $f_2 \in L^p(0,T;L^1(0,L))$ such that $\frac{\partial f_2}{\partial x} \in L^1(0,T;L^2(0,L))$ and $\omega \in \mathcal{J}$, with $\omega''(L) \neq 0$, satisfies (3.6), then there exists a unique function $h \in L^p(0,T)$ such that the mild solution $u = S(u_0, h, f_1 + \frac{\partial f_2}{\partial x}, \widetilde{h})$ of (3.12) verifies the overdetermination condition (3.2).*

Proof. Here, consider

$$S : L^2(0,L) \times L^2(0,L) \times L^1(0,T;L^2(0,L)) \times \mathcal{H} \longrightarrow X(Q_T),$$

with $\widehat{u} = S(u_0, 0, f_1 + \frac{\partial f_2}{\partial x}, \widetilde{h})$ mild solution of the system (3.12). Now, consider the applications

$$(Q\widehat{u})(.) = \int_0^L \widehat{u}(., x)\omega(x)dx \text{ and } \widehat{\varphi} = \varphi - (Q\widehat{u}),$$

where $\varphi \in W^{1,p}(0,T)$. Lemma 3.7 together with (3.6), ensures that $\widehat{\varphi} \in \widetilde{W}^{1,p}(0,T)$. Thus, Lemma 3.16 guarantees the existence of a unique $\Gamma\widehat{\varphi} = h \in L^p(0,T)$ such that the solution $v = S(0,h,0,0)$ of (3.12) satisfies

$$\int_0^L v(t,x)\omega(x)dx = \widehat{\varphi}(t) = \varphi(t) - (Q\widehat{u})(t), \; t \in [0,T].$$

Thus, if $u = \widehat{u} + v = S(u_0, h, f_1 + \frac{\partial f_2}{\partial x}, \widetilde{h})$, we have that u is solution of (3.12) satisfying

$$\int_0^L u(t,x)\omega(x)dx = (Q\widehat{u})(t) + \varphi(t) - (Q\widehat{u})(t) = \varphi(t), \quad t \in [0,T].$$

So, the proof of the theorem is complete. □

Remark 3.12. Note that the proof of theorem (3.11) suggests to us a candidate of control h to solve the nonlinear problem, namely

$$h = \Gamma(\varphi - Q(\widehat{u})), \tag{3.21}$$

and such h is unique, once the application Γ is invertible, by the the lemma (3.10). In fact, if we fixed $f_2 = \frac{u^2}{2}$, then $\frac{\partial f_2}{dx} = uu_x$. So, the solution of the nonlinear problem is a fixed point of the application $u \longmapsto S(u_0, h, f_1 + uu_x, \widetilde{h})$, with h given by (3.21).

3.4.2 Nonlinear result

In this section we are able to prove the first main result of this work.

Proof of Theorem 3.1. In the assumptions of Theorem 3.11 consider $f_1 = f$ and $f_2 = \frac{v^2}{2}$, with $v \in X(Q_T)$ and $f \in L^1(0,T;L^2(0,L))$. Note that $v^2 \in C(0,T;L^1(0,L)) \hookrightarrow L^p(0,T;L^1(0,L))$, for $p \in [1,\infty]$. So, using the following inequality

$$\sup_{x\in[0,L]} |g(x)|^2 \leq C(L)\Big(\|g'\|_{L^2(0,L)}\|g\|_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}\Big).$$

we have $v^2 \in L^2(Q_T)$ and

$$
\begin{aligned}
\|v^2\|_{L^2(Q_T)} &\leq \int_0^T \left(\sup_{x\in[0,L]} |v(t,\cdot)|^2 \int_0^L |v(t,x)|^2 dxdt \right)^{\frac{1}{2}} \\
&\leq C(L) \left(\int_0^T \|v(t,\cdot)\|^3_{L^2(0,L)} . \|v_{xx}(t,\cdot)\|_{L^2(0,L)} dt + \int_0^T \|v(t,\cdot)\|^4_{L^2(0,L)} dt \right)^{\frac{1}{2}} \\
&\leq C(L) \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|^3_{L^2(0,L)} \int_0^T \|v_{xx}(t,\cdot)\|_{L^2(0,L)} dt + T \sup_{t\in[0,T]} \|v(t,\cdot)\|^4_{L^2(0,L)} \right)^{\frac{1}{2}} \\
&\leq C(L) \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|^{\frac{3}{2}}_{L^2(0,L)} \left(\int_0^T \|v_{xx}(t,\cdot)\|_{L^2(0,L)} dt \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \sup_{t\in[0,T]} \|v(t,\cdot)\|^2_{L^2(0,L)} \right) \\
&\leq C(L) \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|^{\frac{3}{2}}_{L^2(0,L)} \left(\int_0^T \|v_{xx}(t,\cdot)\|_{L^2(0,L)} dt \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \|v\|^2_{X(Q_T)} \right) \\
&\leq C(L) \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|^{\frac{3}{2}}_{L^2(0,L)} \left(T^{\frac{1}{2}} \|v_{xx}\|_{L^2(0,T;L^2(0,L))} \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \|v\|^2_{X(Q_T)} \right) \\
&\leq C(L) \left(T^{\frac{1}{4}} \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^2(0,L)} \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^2(0,L)} \|v_{xx}\|_{L^2(Q_T)} \right)^{\frac{1}{2}} + T^{\frac{1}{2}} \|v\|^2_{X(Q_T)} \right) \\
&\leq C(L) \left(T^{\frac{1}{4}} \|v\|_{X(Q_T)} \left(\sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^2(0,L)} + \|v_{xx}\|_{L^2(Q_T)} \right) + T^{\frac{1}{2}} \|v\|^2_{X(Q_T)} \right) \\
&= C(L) \left(T^{\frac{1}{4}} + T^{\frac{1}{2}} \right) \|v\|^2_{X(Q_T)},
\end{aligned}
$$

showing that

$$
\|v^2\|_{L^2(0,T;L^2(0,L))} = \|v^2\|_{L^2(Q_T)} \leq C(L)(T^{\frac{1}{2}} + T^{\frac{1}{4}}) \|v\|^2_{X(Q_T)}. \tag{3.22}
$$

On the space $X(Q_T)$ define the application $\Theta : X(Q_T) \longrightarrow X(Q_T)$ by

$$
\Theta v = S\left(u_0, \Gamma\left(\varphi - Q(S(u_0, 0, f - vv_x, \widetilde{h}))\right), f - vv_x, \widetilde{h}\right).
$$

Let $\varphi \in W^{1,p}(0,T)$, $u_0 \in L^2(0,L)$, $\widetilde{h} \in \mathcal{H}$ be given such that $h_i \in L^p(0,T)$, for $i = 1, 2, 3, 4$ and $f \in L^1(0,T;L^2(0,L))$. Applying the results of Lemma 3.6, for $s = 0$, Theorem 3.11 and inequality (3.22) we have, for $p = 2$, that

$$
\begin{aligned}
\|\Theta v\|_{X(Q_T)} \leq C(T) &\Big(\|u_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} \\
&\quad + \|vv_x\|_{L^1(0,T;L^2(0,L))} + \left\| \varphi - Q(S(u_0, 0, f - vv_x, \widetilde{h})) \right\|_{\widetilde{W}^{1,p}(0,T)} \Big) \\
\leq C(T) &\Big(\|u_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} \\
&\quad + \|vv_x\|_{L^1(0,T;L^2(0,L))} + \|\varphi'\|_{L^2(0,T)} + \|q'\|_{L^2(0,T)} \Big) \\
\leq C(T) &\Big(\|u_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^2(0,T)} \\
&\quad + \|f\|_{L^2(0,T;L^2(0,L))} + \|vv_x\|_{L^1(0,T;L^2(0,L))} + \left\| \frac{v^2}{2} \right\|_{L^2(0,T;L^1(0,L))} \Big) \\
\leq C(T) &\left(c_0 + T^{\frac{1}{4}} \right) \|v\|^2_{X(Q_T)},
\end{aligned}
$$

and in a similar way

$$\|\Theta v_1 - \Theta v_2\|_{X(Q_T)} \leq C(T)T^{\frac{1}{4}}(\|v_1\|_{X(Q_T)} + \|v_1\|_{X(Q_T)})\|v_1 - v_2\|_{X(Q_T)},$$

where

$$c_0 = \|u_0\|_{L^2(0,L)} + \|f\|_{L^2(0,T;L^2(0,L))} + \|\widetilde{h}\|_{\mathcal{H}} + \|\varphi'\|_{L^2(0,T)}$$

and the constant $C(T)$ is nondecreasing with increasing T.

Fix c_0 and consider $T_0 > 0$ such that $8C(T_0)^2 T_0^{\frac{1}{4}} c_0 \leq 1$ and then, for any $T \in (0, T_0]$, we can choose

$$r \in \left[2C(T)c_0, \frac{1}{\left(4C(T)T^{\frac{1}{4}}\right)}\right].$$

By the other hand, for a fixed $T > 0$, pick

$$r = \frac{1}{\left(4C(T,L)T^{\frac{1}{4}}\right)}$$

and

$$c_0 \leq \gamma = \frac{1}{\left(8C(T)^2 T^{\frac{1}{4}}\right)},$$

so in both cases

$$C(T)c_0 \leq \frac{r}{2} \quad \text{and} \quad C(T)T^{\frac{1}{4}} r \leq \frac{1}{4},$$

thus Θ is a contraction on $B(0,r) \subset X(Q_T)$. In this way, there exists a unique fixed point

$$u = \Theta u \in X(Q_T)$$

satisfying (3.3) and the integral condition (3.16) when

$$h = \Gamma\left(\varphi - Q(S(u_0, 0, f - uu_x, \widetilde{h}))\right).$$

The uniqueness of u is a direct consequence of the fixed point theorem and the uniqueness of h follow by (3.12). Hence, Theorem 3.1 is proved. □

3.5 Internal control

This section is dedicated to proving the internal controllability result for system (3.3) when f assumes a special form, namely $f(t,x) = f_0(t)g(t,x)$. First, we prove that the linear system associated with (3.3) is controllable in the sense proposed in the introduction, finally, we extend this result for the full system using a fixed point theorem, as made in the previous section.

3.5.1 Linear result

The next lemma is a key point to prove one of the main results of this chapter and can be read as follows.

Lemma 3.13. *Assuming that $h = \widetilde{h} = u_0 = 0$ in the system (3.12), for $g \in C(0,T;L^2(0,L))$ and $\omega \in \mathcal{J}$ be given such that*

$$\left|\int_0^L g(t,x)\omega(x)dx\right| \geq g_0 > 0, \ \forall\, t \in [0,T],$$

and $\varphi \in \widetilde{W}^{1,p}(0,T)$, for some $p \in [1,\infty]$, there exists a unique function $f_0 = \Gamma(\varphi) \in L^p(0,T)$ such that the solution $u = S(0,0,f_0g,0)$ of (3.12) satisfies the overdeternination condition (3.2). Moreover,

$$\Gamma : \widetilde{W}^{1,p}(0,T) \longmapsto L^p(0,T)$$

is a linear bounded operator and its norm is non-decreasing with increasing T.

Proof. With this hypothesis in hand, define the following linear application

$$G : L^p(0,T) \longrightarrow L^1(0,T;L^2(0,L))$$

by $G(f_0) = f_0g$, which satisfies

$$\|G(f_0)\|_{L^1(0,T;L^2(0,L))} \leq T^{\frac{p-1}{p}}\|g\|_{C([0,T];L^2(0,L))}\|f_0\|_{L^p(0,T)}$$

Now, considering the mapping $\Lambda = Q \circ S \circ G : L^p(0,T) \longrightarrow \widetilde{W}^{1,p}(0,T)$ as

$$(\Lambda f_0)(t) = \int_0^L u(t,x)\omega(x)dx,$$

where $u = S(0,0,f_0g,0)$, since Q, S and G are linear and bounded operators, we have that Λ is a bounded linear operator. Additionally, using Lemma 3.7 and the continuity of the operator S, Λ acts boundedly from the spaces $L^p(0,T)$ to the space $\widetilde{W}^{1,p}(0,T)$.

Note that $\varphi = \Lambda f_0$, for $f_0 \in L^p(0,T)$, means that the function f_0 gives the desired solution to our control problem. So, with this in hand define the operator $A : L^p(0,T) \longrightarrow L^p(0,T)$ by

$$(Af_0)(t) = \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)}\int_0^L u(t,x)(\omega' + \omega''' - \omega''''')dx,$$

where $u = S(0,0,f_0g,0)$ and

$$g_1(t) = \int_0^L g(t,x)\omega(x)dx,$$

for all $t \in [0,T]$. Thus, as done in Lemma 3.10 we have that $\Lambda f_0 = \varphi$ if and only if $f_0 = Af_0$.

Now, we are concentrating on proving the following.

Claim 3. *A is a contraction.*

For the case when $1 \leq p < \infty$, consider f_{01} and f_{02} in $L^p(0,T)$, $u_1 = S(0,0,f_{01}g,0)$ and $u_2 = S(0,0,f_{02}g,0)$. Thus,

$$Af_{01} - Af_{02} = -\frac{1}{g_1}\int_0^L (u_1 - u_2)(\omega' + \omega''' - \omega''''')dx.$$

Moreover, rewrite the following functions as $u = u_1 - u_2$ and $f_1 = f_{01} - f_{02}$, thanks to the inequality (3.16), it holds that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,L)} \leq 2\|g\|_{C(0,T;L^2(0,L))}\|f_{01} - f_{02}\|_{L^1(0,t)}, \ \forall t \in [0,T].$$

Let $\gamma > 0$, for $p < +\infty$, in an analogously way as we did in (3.19), we have

$$\begin{aligned}
\|e^{-\gamma t}(Af_{01} - Af_{02})\|_{L^p(0,T)} \leq & \frac{1}{g_0}\|\omega\|_{H^5(0,L)}\left(\int_0^T e^{-\gamma pt}\|u_1(t,\cdot) - u_2(t,\cdot)\|^p_{L^2(0,L)}dt\right)^{\frac{1}{p}} \\
\leq & \ C(p)\left(\int_0^T e^{-\gamma pt}\int_0^T |f_{01}(\tau) - f_{02}(\tau)|^p d\tau dt\right)^{\frac{1}{p}} \\
\leq & \frac{2^{\frac{1}{p}}C(p)}{(\gamma p)^{\frac{1}{p}}}\|f_{01} - f_{02}\|_{L^p(0,T)}\left(\frac{1 - e^{-\gamma pT}}{2}\right)^{\frac{1}{p}} \\
\leq & \ \frac{C_1}{(\gamma p)^{\frac{1}{p}}}\|e^{-\gamma t}(f_{01} - f_{02})\|_{L^p(0,T)}
\end{aligned}$$

where $C_1 = C_1(T, p, \|\omega\|_{H^5(0,L)}, g_0, \|g\|_{C(0,T;L^2(0,L))})$. So, just take $\gamma = \frac{(2C_1)^p}{p}$ and A is a contraction in this case.

Now, consider $p = \infty$. For $\gamma > 0$, we get similarly to what was done in (3.20) that

$$\begin{aligned}
\sup_{t\in[0,T]} e^{-\gamma t}|((Ah_1)(t) - (Ah_2)(t)))| \leq & \ 2C(g_0, \|g\|, \|\omega\|_{H^5(0,L)}) \sup_{t\in[0,T]} e^{-\gamma t}\|f_{01} - f_{02}\|_{L^1(0,t)} \\
\leq 2C(g_0, \|g\|, \|\omega\|_{H^5(0,L)}) & \sup_{t\in[0,T]}\int_0^t e^{\gamma(\tau - t)}|f_{01}(\tau) - f_{02}(\tau)|d\tau \\
\leq 2C(g_0, \|g\|, \|\omega\|_{H^5(0,L)}) & \|f_{01} - f_{02}\|_{L^\infty(0,T)} \sup_{t\in[0,T]}\frac{1}{\gamma}[1 - e^{-\gamma t}] \\
\leq \frac{1}{\gamma}2C(g_0, \|g\|, & \|\omega\|_{H^5(0,L)})\|f_{01} - f_{02}\|_{L^\infty(0,T)} \\
\leq \frac{C_1}{\gamma}\|f_{01} - f_{02}\|_{L^\infty(0,T)}, &
\end{aligned}$$

where $C_1 = C_1(T, \|\omega\|_{H^5(0,L)}, g_0, \|g\|_{C(0,T;L^2(0,L))})$. Thus, if $\gamma = 2C_1$ we have A is contractions, finishing the case $p = +\infty$, proving Claim 3.

Therefore, for each $\varphi \in \widetilde{W}^{1,p}(0,T)$, there exists a unique $f_0 \in L^p(0,T)$ such that $f_0 = A(f_0)$, i.e., $\varphi = \Lambda(f_0)$. It follows that Λ is invertible, and its inverse $\Gamma : L^p(0,T) \longmapsto \widetilde{W}^{1,p}(0,T)$ is a continuous operator thanks to the Banach's theorem. Additionally, we have

$$\|\Gamma(\varphi)\|_{L^p(0,T)} \leq C(T)\|\varphi'\|_{L^p(0,T)}.$$

The end of the proof follows in the same way as in Lemma 3.10, and so, the proof is complete. □

Let us now enunciate a result concerning the internal controllability of the linear system. The result is the following one.

Theorem 3.14. *Assume that $p \in [1,\infty]$, $u_0 \in L^2(0,L)$, $h \in L^{max\{2,p\}}(0,T)$, $\widetilde{h} \in \mathcal{H}$, with $h_i \in L^p(0,T)$, for $i = 1,2,3,4$ and $f_2 \in L^p(0,T;L^1(0,L))$ such that $\frac{\partial f_2}{\partial x} \in L^1(0,T;L^2(0,L))$. If $g \in C([0,T];L^2(0,L))$, $\omega \in \mathcal{J}$, $\omega''(L) \neq 0$ and $\varphi \in W^{1,p}(0,T)$ satisfies (3.6) and*

$$\left|\int_0^L g(t,x)\omega(x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T]$$

then there exists a unique function $f_0 \in L^p(0,T)$ such that the solution $u = S(u_0, h, f_0 g + \frac{\partial f_2}{\partial x}, \widetilde{h})$ of (3.12) satisfies

$$\int_0^L u(t,x)\omega(x)dx = \varphi(t), \ t \in [0,T].$$

Proof. Pick $\widehat{u} = S(u_0, h, -\frac{\partial f_2}{\partial x}, \widetilde{h})$ solution of (3.12) with $f = -\frac{\partial f_2}{\partial x}$. Now, consider $\widehat{\varphi} = \varphi - Q(\widehat{u})$ with $\varphi \in W^{1,p}(0,T)$. By Lemma 3.7, together with (3.6), follows that $\widehat{\varphi} \in \widetilde{W}^{1,p}(0,T)$. Therefore, due to the Lemma 3.13, there exists a unique $\Gamma\widehat{\varphi} = f_0 \in L^p(0,T)$ such that the solution $v = S(0,0,f_0 g, 0)$ of (3.12) with $f = f_0 g$ satisfies

$$\int_0^L v(t,x)\omega(x)dx = \widehat{\varphi}(t), \ t \in [0,T].$$

Thus, taking $u = \widehat{u} + v = S(u_0, h, f_0 g - \frac{\partial f_2}{\partial x}, \widetilde{h})$, we have that u solution of (3.12) have the following property

$$\int_0^L u(t,x)\omega(x)dx = \varphi(t),$$

for $t \in [0,T]$, showing the result. □

3.5.2 Nonlinear result

In this section we can prove the second main result of this chapter.

Proof of Theorem 3.2. In the assumption of Theorem 3.14, pick $f_2 = -\frac{v^2}{2}$ for an arbitrary $v \in X(Q_T)$. Now, define the mapping $\Theta : X(Q_T) \longrightarrow X(Q_T)$ as follows

$$\Theta v = S\left(u_0, h, \Gamma\left(\varphi - Q(S(u_0, h, -vv_x, \widetilde{h}))\right) g - vv_x, \widetilde{h}\right).$$

In the same way, as done in the proof of Theorem 3.1, we have

$$\|\Theta v\|_{X(Q_T)} \leq C(T)\left(c_0 + T^{\frac{1}{4}}\|v\|^2_{X(Q_T)}\right)$$

and

$$\|\Theta v_1 - \Theta v_2\|_{X(Q_T)} \leq C(T)T^{\frac{1}{4}}(\|v_1\|_{X(Q_T)} + \|v_1\|_{X(Q_T)})\|v_1 - v_2\|_{X(Q_T)}.$$

With this in hand we can proceed as the Theorem 3.1 to conclude that Θ is a contraction and there exists a unique fixed point $u \in X(Q_T)$ such that $f_0 = \Gamma\Big(\varphi - Q(S(u_0, h, -uu_x, \widetilde{h}))\Big)$.

$\square$

4 Control of Kawahara equation with overdetermination condition: The unbounded cases

4.1 Introduction

4.1.1 Model under consideration

Water wave systems are too complex to easily derive and rigorously from them relevant qualitative information on the dynamics of the waves. Alternatively, under suitable assumption on amplitude, wavelength, wave steepness, and so on, the study on asymptotic models for water waves has been extensively investigated to understand the full water wave system, see, for instance, [2, 13–15, 70, 86] and references therein for a rigorous justification of various asymptotic models for surface and internal waves.

Formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces. The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long-wave approximations according to relations between ε and δ.

In this spirit, when we consider $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$, we are dealing with the Korteweg-de Vries (KdV) equation. Under this regime, Korteweg and de Vries [69][1] derived the following equation well-known as a central equation among other dispersive or shallow water wave models called the KdV equation

$$\pm 2u_t + 3uu_x + \left(\frac{1}{3} - \mu\right) u_{xxx} = 0.$$

Another alternative is to treat a new formulation, that is when $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, to generate the so-called equation Kawahara equation. That equation was derived by Hasimoto and Kawahara [57, 65] as a fifth-order KdV equation and take the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0.$$

Our main focus is to investigate a type of controllability for the higher-order KdV type equation. We will continue working with an *integral overdetermination condition*

[1] This equation was first introduced by Boussinesq [18], and Korteweg and de Vries rediscovered it twenty years later. Details can be found in [30] and the reference therein.

started in [24] however in another framework, to be precise, on an *unbounded domain*. To do that, consider the initial boundary value problem (IBVP)

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} + \xi u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = h_1(t), \; u_x(t,0) = h_2(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{4.1}$$

where α, β and ξ are real number, $u = u(t,x)$, $g = g(t,x)$ and $h_i = h_i(t)$, for $i = 1,2$, are well-known function and $f_0 = f_0(t)$ is a control input. It is important to mention that (4.1) is called KdV and Kawahara equation when $\xi = 0$ and $\xi = -1$, respectively.

4.1.2 Framework of the problems

In this work, we will be interested in a kind of internal control property to the Kawahara equation when an *integral overdetermination condition*, on an unbounded domain, is required, namely

$$\int_{\mathbb{R}^+} u(t,x)\omega(x)dx = \varphi(t), \; t \in [0,T], \tag{4.2}$$

where ω and φ are some known functions. To present the problems under consideration, take the following unbounded domain $Q_T^+ = (0,T) \times \mathbb{R}^+$, where T is a positive number, consider the boundary functions μ and ν, and a source term $f = f(t,x)$ with a special form, to be specified later. Thus, let us deal with the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), \; u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{4.3}$$

Therefore, the goal is concentrated on proving an *overdetermination control problem*. Precisely, we want to prove that if f takes the following special form

$$f(t,x) = f_0(t)g(t,x), \quad (t,x) \in Q_T^+, \tag{4.4}$$

the solution of (4.3) satisfies the integral overdetermination condition (4.2). In other words, we have the following issue.

Problem $\mathcal{A}$: For given functions u_0, μ, ν and g in some appropriated spaces, can we find an internal control f_0 such that the solution associated with the equation (4.3) satisfies the integral condition (4.2)?

Naturally, another point to be considered is the following one.

Problem $\mathcal{B}$: What assumptions are needed to ensure that the solution u of (4.3) is unique and verifies (4.2) for a unique f_0?

Finally, with these results in hand, the last problem of this chapter is related to the existence of a minimal time for which the integral overdetermination condition (4.2) be satisfied. Precisely, the problem can be seen as follows.

Problem $\mathcal{C}$: Can one find a time $T_0 > 0$, depending on the boundary and initial data, such that if $T \leq T_0$, there exists a function f_0, in appropriated space, in that way that the solution u of (4.3) verifies (4.2)?

4.1.3 Main results

In this work, we can present answers to the problems $\mathcal{A}$ and $\mathcal{B}$ that were first proposed in [27]. Additionally, the results of this work extend the results presented in [27] for a new framework for the Kawahara equation, that is: The real line, right half-line, and left half-line. For sake of simplicity, we will present here the *overdetermination control problem* in the right half-line, for details of the results for the real line and left half-line we invite the reader to read Section 4.5.

In this way, the first result ensures that the *overdetermination control problem*, that is, the internal control problem with an integral condition like (4.2) on unbounded domain follows for small data, giving answers for the Problem $\mathcal{A}$ and $\mathcal{B}$.

Theorem 4.1. *Let $T > 0$ and $p \in [2, \infty]$. Consider $\mu \in H^{\frac{2}{5}}(0,T) \cap L^p(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T) \cap L^p(0,T)$, $u_0 \in L^2(\mathbb{R}^+)$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T; L^2(\mathbb{R}^+))$ and ω be a fixed function which belongs to the following set*

$$\mathcal{J} = \{\omega \in H^5(\mathbb{R}^+) : \omega(0) = \omega'(0) = \omega''(0) = 0\}, \tag{4.5}$$

satisfying

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx$$

and

$$\left| \int_{\mathbb{R}^+} g(t,x)\omega(x)dx \right| \geq g_0 > 0, \ \ \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$c_1 = \|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (4.3) satisfying (4.2).

Our second result gives us a small time interval for which the integral overdetermination condition (4.2) holds for solutions of (4.3). Precisely, the answer for the Problem $\mathcal{C}$ can be read as follows.

Theorem 4.2. *Suppose the hypothesis of Theorem 4.1 be satisfied and consider* $\delta := T^{\frac{1}{5}} \in (0,1)$*, for* $T > 0$*. Then there exists* $T_0 := \delta_0^{\frac{1}{5}} > 0$*, depending on* $c_1 = c_1(\delta)$ *given by*

$$c_1(\delta) := \|u_{0\delta}\|_{L^2(\mathbb{R}^+)} + \|\varphi'_\delta\|_{L^2(0,T)} + \|\mu_\delta\|_{H^{\frac{2}{5}}(0,T)} + \|\nu_\delta\|_{H^{\frac{1}{5}}(0,T)},$$

such that if $T \leq T_0$*, there exist a control function* $f_0 \in L^p(0,T)$ *and a solution* u *of* (4.3) *verifying* (4.2).

As a consequence of the previous results, we can give a controllability result for the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = u_x(t,0) = 0 & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{4.6}$$

posed in the right half-line. Precisely, we present a control property involving the overdetermination condition (4.2) and the initial state u_0 and final state u_T. To do that, consider the following notation

$$[u(t)] = \int_{\mathbb{R}^+} u(t,x)d\eta(x), \tag{4.7}$$

which one will be called of mass, for some σ-finite measure η in $\mathbb{R}^+$. With this in hand, as a consequence of Theorem 4.1, the following exact controllability in the right half-line holds.

Corollary 4.3. *Let* $T > 0$ *and* $p \in [2, \infty]$*. Consider* u_0, $u_T \in L^2(\mathbb{R}^+)$ *and* $g \in C(0,T; L^2(\mathbb{R}^+))$*, satisfying*

$$\left| \int_{\mathbb{R}^+} g(t,x)\omega(x)dx \right| \geq g_0 > 0, \ \forall t \in [0,T], \tag{4.8}$$

where g_0 *is a constant. Additionally, consider* ω *be a fixed function which belongs to the* $\mathcal{J}$ *defined in* (4.5) *and* $\varphi \in W^{1,p}(0,T)$ *satisfying*

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx \quad \textit{and} \quad \varphi(T) = \int_{\mathbb{R}^+} u_T(x)\omega(x)dx. \tag{4.9}$$

Then, for each $T > 0$ *fixed, there exists a constant* $\gamma > 0$ *such that if*

$$\|u_0\|_{L^2(\mathbb{R}^+)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$*, a unique solution* u *of* (4.6) *and a* σ*-finite measure* η *in* $\mathbb{R}^+$ *such that*

$$[u(T)] = [u_T]. \tag{4.10}$$

4.1.4 Historical background

Concerning the well-posedness of the Kawahara equation, the first local result is due to Cui and Tao [44]. The authors proved a Strichartz estimate for the fifth-order operator and obtained the local well-posedness in $H^s(\mathbb{R})$, for $s > 1/4$. After that, Cui *et al.* [43] improved the previous result to the negative regularity Sobolev space $H^s(\mathbb{R}), s > -1$. Is important to point out that Wang *et al.* [96] improved to a lower regularity, in this case, $s \geq -7/5$. These papers treated the problem using the Fourier restriction norm method. In [35] and [61], authors showed the local well-posedness in $H^s(\mathbb{R}), s > -7/4$, while their methods are the same, particularly, the Fourier restriction norm method in addition to Tao's $[K; Z]$-multiplier norm method. At the critical regularity Sobolev space, $H^{-7/4}(\mathbb{R})$, Chen and Guo [36] proved local and global well-posedness by using Besov-type critical space and I-method. Kato [63] studied local wellposedness for $s \geq -2$ by modifying $X^{s,b}$ space and the ill-posedness for $s < -2$ in the sense that the flow map is discontinuous.

Finally, still regarding the well-posedness results, we refer to two recent works that treat the Kawahara equation. Recently, Cavalcante and Kwak [33] studied the IBVP of the Kawahara equation posed on the right and left half-lines with the nonlinearity uu_x. Being precise, they proved the local well-posedness in the low regularity Sobolev space, that is, $s \in \left(-\frac{7}{4}, \frac{5}{2}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}$. Additionally, the authors in [32] extended the argument of [33] to fifth-order KdV-type equations with different nonlinearities, in specific, where the scaling argument does not hold. They are established in some range of s where the local well-posedness of the IBVP fifth-order KdV-type equations on the right half-line and the left half-line holds.

Stabilization and control problems (see [3, 103] for details of these kinds of issues) have been studied in recent years for the Kawahara Equation, however with few results in the literature. A first work concerning the stabilization property for the Kawahara equation in a bounded domain $Q_T = (0, T) \times (0, L)$,

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = f(t, x) & \text{in } Q_T, \\ u(t, 0) = h_1(t), \; u(t, L) = h_2(t), \; u_x(t, 0) = h_3(t) & \text{on } [0, T], \\ u_x(t, L) = h_4(t), \; u_{xx}(t, L) = h(t) & \text{on } [0, T], \\ u(0, x) = u_0(x) & \text{in } [0, L], \end{cases} \tag{4.11}$$

is due to Capistrano-Filho *et al.* in [3]. In this paper the authors were able to introduce an internal feedback law in (4.11), considering general nonlinearity $u^p u_x$, $p \in [1, 4)$, instead of uu_x, and $h(t) = h_i(t) = 0$, for $i = 1, 2, 3, 4$. To be precise, they proved that under the effect of the damping mechanism the energy associated with the solutions of the system decays exponentially.

Now, some references to internal control problems are presented. This problem was first addressed in [102] and after that in [103]. In both cases, the authors considered the Kawahara equation in a periodic domain $\mathbb{T}$ with a distributed control of the form

$$f(t,x) = (Gh)(t,x) := g(x)(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)dy),$$

where $g \in C^{\infty}(\mathbb{T})$ supported in $\omega \subset \mathbb{T}$ and h is a control input. Here, it is important to observe that the control in consideration has a different form as presented in (4.4), and the result is proven in a different direction from what we will present in this chapter.

Still related to internal control issues, Chen [34] presented results considering the Kawahara equation (4.11) posed on a bounded interval with a distributed control $f(t,x)$ and homogeneous boundary conditions. She showed the result by taking advantage of a Carleman estimate associated with the linear operator of the Kawahara equation with an internal observation. With this in hand, she was able to get a null controllability result when f is effective in a $\omega \subset (0,L)$. As the results obtained by her do not answer all the issues of internal controllability, in a recent article [27] the authors closed some gaps left in [34]. Precisely, considering the system (4.11) with an internal control $f(t,x)$ and homogeneous boundary conditions, the authors can show that the equation in consideration is exactly controllable in L^2-weighted Sobolev spaces and, additionally, the Kawahara equation is controllable by regions on L^2-Sobolev space, for details see [27].

Finally, concerning a new tool to find control properties for dispersive systems, we can cite a recent work of the first two authors [24]. In this work, the authors showed a new type of controllability for a dispersive fifth-order equation that models water waves, what they called *overdetermination control problem*. Precisely, they can find a control acting at the boundary that guarantees that the solution of the problem under consideration satisfies an integral overdetermination condition. In addition, when the control acts internally in the system, instead of the boundary, the authors proved that this condition is satisfied. These problems give answers that were left open in [27] and present a new way to prove boundary and internal controllability results for a fifth-order KdV-type equation.

4.1.5 Heuristic and outline of the chapter

The goal of this chapter is to investigate and discuss control problems with an integral condition on an unbounded domain. Precisely, we study the internal control problem when the solution of the system satisfies (4.2), so we intend to extend - for unbounded domains - a new way to prove internal control results for the system (4.11), initially proposed in [53, 54], for KdV equation, and more recently in [24], for Kawahara equation in a bounded domain. Thus, for this type of *integral overdetermination condition*, the first results on the solvability of control problems for the IBVP of Kawahara equation on unbounded domains are obtained in the present chapter.

The first result, Theorem 4.1, is concerning the *internal overdetermination control problem*. Roughly speaking, we are able to find an appropriate control f_0, acting on $[0,T]$

such that integral condition (4.2) it turns out. First, we borrowed the existence of solutions for the IBVP (4.3) of [33]. With these results in hand, for the special case when $s = 0$, Theorem 4.1 is first proved for the linear system associated to (4.3) and after that, using a fixed point argument, extended to the nonlinear system. The main ingredients are auxiliary lemmas presented in Section 4.3. In one of these lemmas (see Lemma 4.10 below) we can find two appropriate applications that link the internal control term $f_0(t)$ with the overdetermination condition (4.2), namely

$$\begin{aligned} \Lambda : L^p(0,T) &\longrightarrow \widetilde{W}^{1,p}(0,T) \\ f_0 &\longmapsto (\Lambda f_0)(\cdot) = \int_{\mathbb{R}^+} u(\cdot,x)\omega(x)dx \end{aligned}$$

and

$$\begin{aligned} A : L^p(0,T) &\longrightarrow L^p(0,T) \\ f_0 &\longmapsto (Af_0)(\cdot) = \frac{\varphi'(\cdot)}{g_1(\cdot)} - \frac{1}{g_1(\cdot)}\int_{\mathbb{R}^+} u(t,x)(\alpha\omega' + \beta\omega''' - \omega''''')dx, \end{aligned}$$

where,

$$g_1(\cdot) = \int_{\mathbb{R}^+} g(\cdot,x)\omega(x)dx.$$

So, we prove that such application has a continuous inverse, by Banach's theorem, showing the lemma in question, and so, reaching our goal, to prove Theorem 4.1.

With the previous result in hand, the answer for the Problem $\mathcal{C}$ is given by Theorem 4.2. This result gives us a minimal time in which the integral condition (4.2) is satisfied. More precisely, Theorem 4.2 is proved in three parts. In the first part, we give a refinement of Lemma 4.10, namely, Lemma 4.11. With this in hand, we need, in a second moment, to use the scaling of our equation (4.3) to produce a "new" Kawahara equation on Q_T^+. This gives us the possibility to use the Theorem 4.1 and, with help of Lemma 4.11, reach the proof of Theorem 4.2.

Finally, as a consequence of Theorem 4.1, we produce a type of exact controllability result (Corollary 4.3). More precisely, we show that the mass of the system (4.7) is reached on the final time T, that is, (4.10) holds.

Thus, we finish our introduction by showing the structure of the chapter. Section 4.2 is devoted to presenting some preliminaries, which are used throughout. Precisely, we present the Fourier restriction spaces related to the operator of the Kawahara, moreover, reviewed the main results of the well-posedness for the fifth-order KdV equation in these spaces. In Section 4.3 we present some auxiliary lemmas which help us to prove the internal controllability results. The overdetermination control results, when the control is acting internally, are presented in Section 4.4, that is, we present the proof of the main results, Theorems 4.1, 4.2 and Corollary 4.3. Finally, in Section 5.4, we present some further comments and some conclusions about the generality of the work.

4.2 Preliminaries

4.2.1 Fourier restriction spaces

Let f be a Schwartz function, i.e., $f \in \mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$, $\widetilde{f}$ or $\mathcal{F}(f)$ denotes the space-time Fourier transform of f defined by

$$\widetilde{f}(\tau, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(t,x)\, dxdt.$$

Moreover, we use $\mathcal{F}_x$ (or $\widehat{\ }$) and $\mathcal{F}_t$ to denote the spatial and temporal Fourier transform, respectively.

For given $s, b \in \mathbb{R}$, we define the space $X^{s,b}$ associated to (4.3) as the closure of $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$ under the norm

$$\|f\|^2_{X^{s,b}} = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \left\langle \tau - \xi^5 \right\rangle^{2b} |\widetilde{f}(\tau,\xi)|^2 d\xi d\tau$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

As well-known, the $X^{s,b}$ space with $b > \frac{1}{2}$ is well-adapted to study the IVP of dispersive equations. The function space equipped with the Fourier restriction norm, which is the so-called $X^{s,b}$ spaces, has been proposed by Bourgain [16, 17] to solve the periodic NLS and generalized KdV. Since then, it has played a crucial role in the theory of dispersive equations and has been further developed by many researchers, in particular, Kenig, Ponce, and Vega [66] and Tao [92].

In our case, to study the IBVP (4.3) is requested to introduce modified $X^{s,b}$-type spaces. So, we define the (time-adapted) Bourgain space $Y^{s,b}$ associated to (4.3) as the completion of $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|f\|^2_{Y^{s,b}} = \int_{\mathbb{R}^2} \langle \tau \rangle^{\frac{2s}{5}} \left\langle \tau - \xi^5 \right\rangle^{2b} |\widetilde{f}(\tau,\xi)|^2 d\xi d\tau.$$

Additionally, due to the study of the IBVP introduced in [33], they used the low frequency localized $X^{0,b}$-type space with $b > \frac{1}{2}$ in the nonlinear estimates. Hence, we need also define D^α space as the completion of $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|f\|^2_{D^\alpha} = \int_{\mathbb{R}^2} \langle \tau \rangle^{2\alpha} 1_{\{\xi : |\xi| \le 1\}}(\xi) |\widetilde{f}(\tau,\xi)|^2 d\xi d\tau$$

where 1_A is the characteristic function on a set A. With this in hand, now we set the solution space denoted by $Z_1^{s,b,\alpha}$ given by

$$Z_1^{s,b,\alpha}(\mathbb{R}^2) = \{f \in C(\mathbb{R}; H^s(\mathbb{R})) \cap (X^{s,b} \cap D^\alpha)(\mathbb{R}^2); \partial_x^j f \in C(\mathbb{R}_x; H^{\frac{s+2-j}{5}}(\mathbb{R}_t)), j = 0, 1.\}$$

with the following norm

$$\|f\|_{Z_1^{s,b,\alpha}(\mathbb{R}^2)} = \sup_{t\in\mathbb{R}} \|f(t,\cdot)\|_{H^s} + \sum_{j=0}^{1} \sup_{x\in\mathbb{R}} \left\|\partial_x^j f(\cdot,x)\right\|_{H^{\frac{s+2-j}{5}}} + \|f\|_{X^{s,b}\cap D^\alpha}.$$

The spatial and time restricted space of $Z_1^{s,b,\alpha}(\mathbb{R}^2)$ is defined by the standard way:

$$Z_1^{s,b,\alpha}\left((0,T)\times\mathbb{R}^+\right) = Z_1^{s,b,\alpha}\Big|_{(0,T)\times\mathbb{R}^+}$$

equipped with the norm

$$\|f\|_{Z_1^{g,b,\alpha}((0,T)\times\mathbb{R}^+)} = \inf_{g\in Z_1^{s,b,\alpha}} \left\{\|g\|_{Z_1^{s,b,\alpha}} : g(t,x)=f(t,x) \text{ on } (0,T)\times\mathbb{R}^+\right\}.$$

4.2.2 Overview of the well-posedness results

In this section we are interested to present the well-posedness results for the Kawahara system, namely,

$$\begin{cases} u_t+\alpha u_x+\beta u_{xxx}-u_{xxxxx}=f(t,x) & \text{em } [0,T]\times\mathbb{R}^+, \\ u(t,0)=\mu(t), \ u_x(t,0)=\nu(t) & \text{em } [0,T], \\ u(0,x)=u_0(x) & \text{em } \mathbb{R}^+. \end{cases} \tag{4.12}$$

The results presented here are borrowed from [33] and give us good properties of the IBVP (4.12). The first one gives a relation of the nonlinearity involved in our problem with the Fourier restriction spaces introduced in the previous subsection. Precisely, we have the nonlinear term $f=uu_x$ that can be controlled in the $X^{s,-b}$ norm.

Proposition 4.4. *For $-7/4<s$, there exists $b=b(s)<1/2$ such that for all $\alpha>1/2$, we have*

$$\|\partial_x(uv)\|_{X^{s,-b}} \leq c\|u\|_{X^{s,b}\cap D^\alpha}\|v\|_{X^{s,b}\cap D^\alpha}. \tag{4.13}$$

Proof. See [33, Proposition 5.1]. □

Now on, we will consider the following: $s=0$, $b(s)=b_0$, $\alpha(s)=\alpha_0$ and $Z_1^{0,b_0,\alpha_0}(Q_T^+)=Z(Q_T^+)$. As a consequence of the previous proposition, we have the following.

Corollary 4.5. *There exists $b_0\in(0,\frac{1}{2})$ such that for all $\alpha_0>\frac{1}{2}$, follows that*

$$\|\partial_x(uv)\|_{X^{0,-b_0}(Q_T^+)} \leq C\|u\|_{Z(Q_T^+)}\|v\|_{Z(Q_T^+)}, \tag{4.14}$$

for any $u,v\in Z(Q_T^+)$.

Now, we are interested in a special case of the well-posedness result presented in [33]. Precisely, considering $s=0$, [33, Theorem 1.1] gives us the following result.

Theorem 4.6. *Let $T>0$ and $u_0\in L^2(\mathbb{R}^+)$, $\mu\in H^{\frac{2}{5}}(0,T)$, $\nu\in H^{\frac{1}{5}}(0,T)$ and $f\in X^{0,-b_0}(Q_T^+)$, for $b_0\in(0,\frac{1}{2})$. Then there exists a unique solution $u:=S(u_0,\mu,\nu,f)\in Z(Q_T^+)$ of (4.12) such that*

$$\|u\|_{Z(Q_T^+)} \leq C_0\left(\|u_0\|_{L^2(\mathbb{R}^+)}+\|\mu\|_{H^{\frac{2}{5}}(0,T)}+\|\nu\|_{H^{\frac{1}{5}}(0,T)}+\|f\|_{X^{0,-b_0}(Q_T^+)}\right) \tag{4.15}$$

where $C_0>0$ is a positive constant depending only of b_0,α_0 and T.

4.3 Key lemmas

In this section, we are interested to prove some auxiliary lemmas for the solutions of the system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0)=\mu(t),\ u_x(t,0)=\nu(t) & \text{on } [0,T], \\ u(0,x)=u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.16}$$

These lemmas will be the key to proving the main results of this work.

To do this, consider $\omega \in \mathcal{J}$ defined by (4.5) and define $q:[0,T]\longrightarrow\mathbb{R}$ as follows

$$q(t) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx, \tag{4.17}$$

where $u := S(u_0,\mu,\nu,f_1+\frac{\partial f_2}{\partial x})$ is solution of (4.16) guaranteed by Theorem 4.6. The next two auxiliary lemmas are the key point to show the main results of this work. The first one gives that $q\in W^{1,p}(0,T)$ and can be read as follows.

Lemma 4.7. *Let $T>0$, $p\in[2,\infty]$ and the assumptions of Theorem 4.6 are satisfied, with $f=f_1+\frac{\partial f_2}{\partial x}$, where $f_1\in L^p(0,T;L^2(\mathbb{R}^+))$, $f_2\in L^p(0,T;L^1(\mathbb{R}^+))$ and $\mu,\nu\in L^p(0,T)$. If $u\in Z(Q_T^+)$ is a solution of (4.12) and $\omega\in\mathcal{J}$, defined in (4.5), then the function $q\in W^{1,p}(0,T)$ and the relation*

$$\begin{aligned} q'(t) = {} & \omega'''(0)\nu(t) - \omega''''(0)\mu(t) + \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\ & + \int_{\mathbb{R}^+} u(t,x)[\alpha\omega'(x)+\beta\omega'''(x)-\omega'''''(x)]dx \end{aligned} \tag{4.18}$$

holds for almost all $t\in[0,T]$. In addition, the function $q'\in L^p(0,T)$ can be estimate in the following way

$$\begin{aligned} \|q'\|_{L^p(0,T)} \le C \Big(& \|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{(L^p\cap H^{\frac{2}{5}})(0,T)} + \|\nu\|_{(L^p\cap H^{\frac{1}{5}})(0,T)} \\ & + \|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))} + \|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))} + \|\frac{\partial f_2}{\partial x}\|_{X^{0,-b_0}(Q_T^+)} \Big) \end{aligned} \tag{4.19}$$

with $C=C(|\alpha|,|\beta|,T,\|\omega\|_{\mathbb{R}^+})>0$ a constant that is nondecreasing with increasing T.

Proof. Considering $\psi\in C_0^\infty(0,T)$, multiplying (4.16) by $\psi\omega$ and integrating by parts in $[0,T]\times[0,R]$, for some $R>0$, we get, using the boundary condition of (4.16) and the hypothesis that $\omega \in \mathcal{J}$, that

$$\begin{aligned} -\int_0^T \psi'(t)q(t)dt &= \int_0^T \int_{\mathbb{R}^+} u_t(t,x)\psi(t)\omega(x)dxdt \\ &= \int_0^T \psi(t) \bigg(\int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x) + \beta\omega'''(x) - \omega'''''(x))dx \\ &\quad + \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\ &\quad - \omega''''(0)\mu(t) + \omega'''(0)\nu(t) \bigg) dt \\ &= \int_0^T \psi(t)r(t)dt, \end{aligned}$$

with $r : [0,T] \longmapsto \mathbb{R}$ defined by

$$\begin{aligned} r(t) =& \int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x) + \beta\omega'''(x) - \omega'''''(x))dx - \omega''''(0)\mu(t) + \omega'''(0)\nu(t) \\ &+ \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx - \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \\ :=& I_1 + I_2 + I_3, \end{aligned}$$

which gives us $q'(t) = r(t)$, where

$$\begin{aligned} &I_1 = \int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x) + \beta\omega'''(x) - \omega'''''(x))dx - \omega''''(0)\mu(t) + \omega'''(0)\nu(t), \\ &I_2 = -\int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx, \\ &I_3 = \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx. \end{aligned}$$

It remains for us to prove that $q' \in L^p(0,T)$, for $p \in [2,\infty]$. To do it, we need to estimate each term of (4.18). We will split this analysis into two steps.

Step 1. $2 \leq p < \infty$

Let us first estimate I_1. To do this, note that, for $t \in [0,T]$, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^+} u(t,x)(\alpha\omega'(x) + \beta\omega'''(x) - \omega'''''(x))dx \right| \\ &\qquad\qquad \leq (|\alpha|\|\omega'\|_{L^2(\mathbb{R}^+)} + |\beta|\|\omega'''\|_{L^2(\mathbb{R}^+)} + \|\omega'''''\|_{L^2(\mathbb{R}^+)})\|u(t,\cdot)\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

Moreover, the trace terms are bounded thanks to the fact that $\omega \in \mathcal{J}$. Thus, this yields that

$$\left\| \int_{\mathbb{R}^+} u(t,x)(\alpha\omega(x) + \beta\omega'''(x) - \omega'''''(x))dx \right\|_{L^p(0,T)} \leq C(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)})\|u\|_{L^p(0,T;L^2(\mathbb{R}^+))}.$$

Since

$$\|u\|_{L^p(0,T;L^2(\mathbb{R}^+))} \leq T^{\frac{1}{p}}\|u\|_{C(0,T;L^2(\mathbb{R}^+))},$$

we have that

$$\left\| \int_{\mathbb{R}^+} u(t,x)(\alpha\omega(x)+\beta\omega'''(x)-\omega'''''(x))dx \right\|_{L^p(0,T)} \leq C(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)})T^{\frac{1}{p}}\|u\|_{C(0,T;L^2(\mathbb{R}^+))}.$$

Now, let us estimate I_2. In this case, we start observing that

$$\begin{aligned} \left| \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \right| \leq & \int_{\mathbb{R}^+} |f_2(t,x)\omega'(x)|dx \\ \leq & \|\omega'\|_{C(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ \leq & C\|\omega'\|_{H^1(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ \leq & C\|\omega\|_{H^5(\mathbb{R}^+)} \|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)}, \end{aligned}$$

where we have used the following continuous embedding

$$H^1(\mathbb{R}^+) \hookrightarrow L^\infty(\mathbb{R}^+) \cap C(\mathbb{R}^+).$$

Therefore, we get that

$$\left\| \int_{\mathbb{R}^+} f_2(t,x)\omega'(x)dx \right\|_{L^p(0,T)} \leq C(\|\omega\|_{H^5(\mathbb{R}^+)})\|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))}.$$

Similarly, we can bound I_3 as

$$\left\| \int_{\mathbb{R}^+} f_1(t,x)\omega(x)dx \right\|_{L^p(0,T)} \leq \|\omega\|_{L^2(\mathbb{R}^+)}\|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))}.$$

With these estimates in hand and using the hypothesis over μ and ν, that is, μ and ν belonging to $L^p(0,T)$, we have $r \in L^p(0,T)$, which implies that $q \in W^{1,p}(0,T)$ and

$$\begin{aligned} \|q'\|_{L^p(0,T)} \leq & \widetilde{C}(|\alpha|, |\beta|, T, \|\omega\|_{H^5(\mathbb{R}^+)}) \Big(\|\mu\|_{L^p(0,T)} + \|\nu\|_{L^p(0,T)} + \|u\|_{Z(Q_T^+)} \\ & + \|f_1\|_{L^p(0,T;L^2(\mathbb{R}^+))} + \|f_2\|_{L^p(0,T;L^1(\mathbb{R}^+))} \Big). \end{aligned}$$

Finally, using (4.15) in the previous inequality, (4.19) holds.

Step 2. $p = \infty$

Observe that thanks to the relation (4.18) and the fact that

$$H^1(\mathbb{R}^+) \hookrightarrow L^\infty(\mathbb{R}^+) \cap C(\mathbb{R}^+),$$

we get that

$$\begin{aligned} |q'(t)| \leq & (|\alpha|\|\omega'\|_{L^2(\mathbb{R}^+)} + |\beta|\|\omega'''\|_{L^2(\mathbb{R}^+)} + \|\omega'''''\|_{L^2(\mathbb{R}^+)})\|u(t,\cdot)\|_{L^2(\mathbb{R}^+)} \\ & + \|\omega\|_{L^2(\mathbb{R}^+)}\|f_1(t,\cdot)\|_{L^2(\mathbb{R}^+)} + \|\omega'\|_{H^1(\mathbb{R}^+)}\|f_2(t,\cdot)\|_{L^1(\mathbb{R}^+)} \\ & + |\omega''''(0)||\mu(t)| + |\omega'''(0)||\nu(t)|. \end{aligned}$$

Thus,

$$\|q'\|_{C(0,T)} \leq C\left(\|u\|_{Z(Q_+^T)} + \|f_2\|_{C(0,T;L^1(\mathbb{R}^+))} + \|f_1\|_{C(0,T;L^2(\mathbb{R}^+))} + \|\mu\|_{C(0,T)} + \|\nu\|_{C(0,T)}\right),$$

with $C = C(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)}, |\omega''''(0)|, |\omega'''(0)|) > 0$. Thus, Step 2 is achieved using (4.15) and the proof of the lemma is complete. □

Remark 4.8. We will give some remarks in order related to the previous lemma.

i. We are implicitly assuming that $\frac{\partial f_2}{\partial x} \in L^1(0,T;L^2(\mathbb{R}^+))$ but it is not a problem, since the function that we will take for f_2, in our purposes, satisfies that condition.

ii. When $p = \infty$ the spaces $L^p(0,T)$, $L^p(0,T;L^2(\mathbb{R}^+))$ and $L^p(0,T;L^1(\mathbb{R}^+))$ are replaced by the spaces $C([0,T])$, $C([0,T];L^2(\mathbb{R}^+))$ and $C([0,T];L^1(\mathbb{R}^+))$, respectively. So, we can obtain $q \in C^1([0,T])$.

Now, consider a special case of the system (4.16), precisely, the following

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = u_x(t,0) = 0 & \text{on } [0,T], \\ u(0,x) = 0 & \text{in } \mathbb{R}^+. \end{cases} \tag{4.20}$$

For the solutions of the system (4.20) the next lemma holds.

Lemma 4.9. *Suppose that $f \in L^2(0,T;L^2(\mathbb{R}^+))$ and $u := S(0,0,0,f)$ is solution of (4.20), then*

$$\int_{\mathbb{R}^+} |u(t,x)|^2 dx \leq 2\int_0^t \int_{\mathbb{R}^+} f(\tau,x)u(\tau,x)dxdt, \quad \forall t \in [0,T]. \tag{4.21}$$

Proof. Consider $f \in C_0^\infty(Q_T^+)$ and $u = S(0,0,0,f_1)$ a smooth solution of (4.20). Multiplying (4.20) by $2u$, integrating by parts on $[0,R]$, for $R > 0$, yields that

$$\begin{aligned} \frac{d}{dt}\int_0^R |u(t,x)|^2 dx = {} & 2\int_0^R f(t,x)u(t,x)dx - \alpha(|u(t,R)|^2 - |u(t,0)|^2) \\ & + \beta(|u_x(t,R)|^2 - |u_x(t,0)|^2) + (|u_{xx}(t,R)|^2 - |u_{xx}(t,0)|^2) \\ & - 2\beta(u_{xx}(t,R)u(t,R) - u_{xx}(t,0)u(t,0)) \\ & + 2(u_{xxxx}(t,R)u(t,R) - u_{xxxx}(t,0)u(t,0)) \\ & - 2(u_{xxx}(t,R)u_x(t,R) - u_{xxx}(t,0)u_x(t,0)). \end{aligned}$$

So, taking $R \to \infty$, integrating in $[0,t]$ and using the boundary condition of (4.20), we get

$$\int_{\mathbb{R}^+} |u(t,x)|^2 dx \leq 2\int_0^t \int_{\mathbb{R}^+} f(\tau,x)u(\tau,x)dxd\tau,$$

showing (4.21) for smooth solutions. The result for the general case follows by density argument. □

Consider the space

$$\widetilde{W}^{1,p}(0,T) = \{\varphi \in W^{1,p}(0,T); \varphi(0) = 0\},\ p \in [2,\infty]$$

and define the following linear operator Q

$$Q(u)(t) := q(t),$$

where $q(t)$ is defined by (4.17). Here, we consider the following norm associated to $\widetilde{W}^{1,p}(0,T)$

$$\|Q(u)\|_{\widetilde{W}^{1,p}(0,T)} = \|q\|_{\widetilde{W}^{1,p}(0,T)} = \|q'\|_{L^p(0,T)}.$$

With this in hand, we have the following result.

Lemma 4.10. *Consider $\omega \in J$, defined by (4.5), and $\varphi \in \widetilde{W}^{1,p}(0,T)$, for some $p \in [2,\infty]$, $g \in C(0,T;L^2(\mathbb{R}^+))$. If the following assumption holds*

$$\left|\int_{\mathbb{R}^+} g(t,x)\omega(x)dx\right| \geq g_0 > 0,\ \forall\, t \in [0,T],$$

then there exist a unique function $f_0 = \Gamma(\varphi) \in L^p(0,T)$, such that for $f(t,x) := f_0(t)g(t,x)$ the function $u := S(0,0,0,f)$ solution of (4.20) satisfies (4.2). Additionally, the linear operator

$$\begin{array}{rccc} \Gamma: & \widetilde{W}^{1,p}(0,T) & \longrightarrow & L^p(0,T) \\ & \varphi & \longmapsto & \Gamma(\varphi) = f_0 \end{array} \tag{4.22}$$

is bounded.

Proof. Consider the function

$$G : L^p(0,T) \longrightarrow L^2(0,T;L^2(\mathbb{R}^+))$$

defined by

$$f_0 \longmapsto G(f_0) = f_0 g.$$

By the definition, G is linear. Moreover, we have

$$\begin{aligned} \|G(f_0)\|^2_{L^2(0,T;L^2(\mathbb{R}^+))} &\leq \|g\|^2_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|^2_{L^2(0,T)} \\ &\leq T^{\frac{p-2}{p}}\|g\|^2_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|^2_{L^p(0,T)}. \end{aligned}$$

Thus,

$$\|G(f_0)\|_{L^2(0,T;L^2(\mathbb{R}^+))} \leq T^{\frac{p-2}{2p}}\|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|f_0\|_{L^p(0,T)}. \tag{4.23}$$

Consider the application

$$\Lambda = Q \circ S \circ G : L^p(0,T) \longrightarrow \widetilde{W}^{1,p}(0,T)$$

which one will be defined by

$$f_0 \longmapsto \Lambda(f_0) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx,$$

where $u := S(0,0,0,f)$. Therefore, since Q, S and G are linear and bounded, we have that Λ is linear and bounded and have the following property

$$(\Lambda f_0)(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx = 0$$

that is, Λ is well-defined. Introduce the operator

$$A : L^p(0,T) \longrightarrow L^p(0,T)$$

by

$$f_0 \longmapsto A(f_0) \in L^p(0,T),$$

where

$$(Af_0)(t) = \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)}\int_{\mathbb{R}^+} u(t,x)(\alpha\omega' + \beta\omega''' - \omega''''')dx.$$

Here, $u = S(0,0,0,f)$ and

$$g_1(t) = \int_{\mathbb{R}^+} g(t,x)\omega(x)dx,$$

for all $t \in [0,T]$. Observe that, using (4.18) $\Lambda(f_0) = \varphi$ if and only if $f_0 = A(f_0)$.

Now we show that the operator A is a contraction on $L^p(0,T)$ if we choose an appropriate norm in this space. To show it let us split our proof into two cases.

Case one: $2 \leq p < \infty$.

Let $f_{01}, f_{02} \in L^p(0,T)$, $u_1 = (S\circ G)f_{01}$ and $u_2 = (S\circ G)f_{02}$, so thanks to (4.21) we get

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)} \leq 2\|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|f_{01} - f_{02}\|_{L^1(0,t)}, \ \forall t \in [0,T]. \tag{4.24}$$

Consider $\gamma > 0$ and $t \in [0,T]$, using Hölder inequality, we have

$$\begin{aligned}\left|e^{-\gamma t}\Big((Af_{01})(t) - (Af_{02})(t)\Big)\right| \leq & \frac{e^{-\gamma t}}{|g_1(t)|}\int_{\mathbb{R}^+} |(u_1(t,x) - u_2(t,x))(\alpha\omega' + \beta\omega''' - \omega''''')|dx \\ \leq & \frac{e^{-\gamma t}}{g_0}\|\alpha\omega' + \beta\omega''' - \omega'''''\|_{L^2(\mathbb{R}^+)}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)} \\ \leq & \frac{1}{g_0}\|\omega\|_{H^5(\mathbb{R}^+)}e^{-\gamma t}\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R}^+)}.\end{aligned}$$

Therefore, now, using (4.24), yields that

$$\begin{aligned}\|e^{-\gamma t}(Af_{01} - Af_{02})\|_{L^p(0,T)} \leq & C\left(\int_0^T e^{-\gamma p t}\left(\int_0^t |f_{01}(\tau) - f_{02}(\tau)|d\tau\right)^p dt\right)^{\frac{1}{p}} \\ \leq & C\left(\int_0^T e^{-\gamma p t}\left(\int_0^T |f_{01}(\tau) - f_{02}(\tau)|d\tau\right)^p dt\right)^{\frac{1}{p}},\end{aligned}$$

where $C = \frac{2\|\omega\|_{H^5(\mathbb{R}^+)}\|g\|_{C(0,T;L^2(\mathbb{R}^+))}}{g_0}$.

Finally, using the last inequality for $p \in [2, \infty)$, such that $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$
\begin{aligned}
\|e^{-\gamma t}(Af_{01} - Af_{02})\|_{L^p(0,T)} &\leq c_0 \left(\int_0^T e^{-\gamma p t} \left(\int_0^T |f_{01}(\tau) - f_{02}(\tau)| d\tau \right)^p dt \right)^{\frac{1}{p}} \\
&\leq c_0 \left\| e^{-\gamma \tau} \left(f_{01} - f_{02}\right) \right\|_{L^p(0,T)} \left[\int_0^T e^{-p\gamma t} \left(\int_0^t e^{p'\gamma\tau} \mathrm{d}\tau \right)^{p/p'} \mathrm{d}t \right]^{1/p} \\
&\leq \frac{c_0 T^{1/p}}{(p'\gamma)^{1/p'}} \left\| e^{-\gamma t} \left(f_{01} - f_{02}\right) \right\|_{L^p(0,T)},
\end{aligned}
\tag{4.25}
$$

where $c_0 = c_0(\|\omega\|_{H^5(\mathbb{R}^+)}, g_0, \|g\|_{C(0,T;L^2(\mathbb{R}^+))})$ is defined by

$$
c_0 := \frac{2}{g_0} \|g\|_{C([0,T];L_2(\mathbb{R}^+))} \left(|\alpha| \, \|\omega'\|_{L_2(\mathbb{R}^+)} + |\beta| \, \|\omega'''\|_{L_2(\mathbb{R}^+)} + \|\omega'''''\|_{L_2(\mathbb{R}^+)} \right). \tag{4.26}
$$

Therefore, is enough to take $\gamma > \frac{(c_0 T^{\frac{1}{p}})^{p'}}{p'}$, and so A is contraction, showing the case one for $p \in [2, \infty)$.

Case two: $p = \infty$.

In this case, we have

$$
\begin{aligned}
\left\| e^{-\gamma t} \left(Af_{01} - Af_{02}\right) \right\|_{L^\infty(0,T)} &\leq c_0 \sup_{t \in [0,T]} e^{-\gamma t} \left\|u_1(t, \cdot) - u_2(t, \cdot)\right\|_{L^2(\mathbb{R}^+)} \\
&\leq c_0 \sup_{t \in [0,T]} e^{-\gamma t} \left\|f_{01} - f_{02}\right\|_{L^1(0,t)} \\
&\leq \frac{c_0}{\gamma} \left\| e^{-\gamma t} \left(f_{01} - f_{02}\right) \right\|_{L^\infty(0,T)},
\end{aligned}
\tag{4.27}
$$

where $c_0 = c_0(T, p, \|\omega\|_{H^5(\mathbb{R}^+)}, g_0, \|g\|_{C(0,T;L^2(\mathbb{R}^+))})$ is defined by (4.26). Therefore, taking $\gamma > c_0$, we have that A is contraction, showing case two.

Thus, in both cases the operator A is a contraction and, so, for any $\varphi \in \widetilde{W}^{1,p}(0,T)$, there exists a unique $f_0 \in L^p(0,T)$ such that $f_0 = A(f_0)$, or equivalently, $\varphi = \Lambda(f_0)$. Thus, follows that Λ is invertible. Due to the Banach theorem its inverse

$$
\Gamma : L^p(0,T) \longmapsto \widetilde{W}^{1,p}(0,T)
$$

is bounded. Particularly,

$$
\|\Gamma\varphi\|_{L^p(0,T)} \leq C(T) \|\varphi'\|_{L^p(0,T)}. \tag{4.28}
$$

□

To prove our second main result of this work we need one refinement of Lemma 4.10.

Lemma 4.11. *Under the hypothesis of Lemma 4.10, if $c_0 T \le p^{1/p}/2$, c_0 given by (4.26), and $p^{1/p} = 1$ for $p = +\infty$, we have the following estimate*

$$\|\Gamma\varphi\|_{L_p(0,T)} \le \frac{2}{g_0} \|\varphi'\|_{L_p(0,T)}, \tag{4.29}$$

for the operator $\Gamma : \widetilde{W}^{1,p}(0,T) \longmapsto L^p(0,T)$.

Proof. Since $f_0 = Af_0 = \Gamma\varphi$, taking $\gamma = 0$, similar as we did in (4.25), we get that

$$\left\| f_0 - \frac{\varphi'}{g_1} \right\|_{L^p(0,T)} \le c_0 \left[\int_0^T \left(\int_0^t |f_0(\tau)| \, \mathrm{d}\tau \right)^p \, \mathrm{d}t \right]^{1/p} \le \frac{c_0 T}{p^{1/p}} \|f_0\|_{L^p(0,T)},$$

and in a way analogous to the one made in (4.27) we also have

$$\left\| f_0 - \frac{\varphi'}{g_1} \right\|_{L^\infty(0,T)} \le c_0 \int_0^T |f_0(\tau)| \, \mathrm{d}\tau \le c_0 T \|f_0\|_{L^\infty(0,T)}.$$

Thus, for $p \in [2, +\infty]$, we get

$$\|\Gamma\varphi\|_{L^p(0,T)} \le \frac{1}{g_0} \|\varphi'\|_{L^p(0,T)} + \frac{c_0 T}{p^{1/p}} \|\Gamma\varphi\|_{L^p(0,T)},$$

and the estimate (4.29) holds true. □

4.4 Control results

In this section, the *overdetermination control problem* is studied. Precisely we will give answers to some questions left at the beginning of this chapter. Here, let us consider the full system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.30}$$

First, we prove that when we have the linear system associated with (4.30) the control problem with an integral overdetermination condition holds. After that, we can extend this result, by using the regularity in Bourgain spaces, to the nonlinear one. Finally, we give, under some hypothesis, a minimal time such that the solution of (4.30) satisfies (4.2).

4.4.1 Linear case

In this section let us present the following result.

Theorem 4.12. *Let $T > 0$, $p \in [2, \infty]$, $u_0 \in L^2(\mathbb{R}^+)$, $\mu \in (H^{\frac{2}{5}} \cap L^p)(0,T)$ and $\nu \in (H^{\frac{1}{5}} \cap L^p)(0,T)$. Consider $g \in C(0,T; L^2(\mathbb{R}^+))$, $\omega \in \mathcal{J}$, defined by (4.5), and $\varphi \in W^{1,p}(0,T)$ such that*

$$\varphi(0) = \int_{\mathbb{R}^+} u_0(x)\omega(x)dx. \tag{4.31}$$

Additionally, if

$$\left|\int_{\mathbb{R}^+} g(t,x)\omega(x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T], \tag{4.32}$$

then there exists a unique $f_0 \in L^p(0,T)$ *such that for* $f(t,x) := f_0(t)g(t,x) + \frac{\partial f_2}{\partial x}(t,x)$, *with* $f_2 \in L^p(0,T;L^1(\mathbb{R}^+))$ *and* $\frac{\partial f_2}{\partial x} \in X^{0,-b_0}(Q_T^+)$, *the solution* $u := S(u_0,\mu,\nu,f_0g + \frac{\partial f_2}{\partial x})$ *of*

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{4.33}$$

satisfies (4.2).

Proof. Pick $v_1 = S(u_0,\mu,\nu,-\frac{\partial f_2}{\partial x})$ solution of

$$\begin{cases} v_{1t} + \alpha v_{1x} + \beta v_{1xxx} - v_{1xxxxx} = -\frac{\partial f_2}{\partial x} & \text{in } Q_T^+, \\ v_1(t,0) = \mu(t), v_{1x}(t,0) = \nu(t) & \text{on } [0,T], \\ v_1(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases}$$

Define the following function

$$\varphi_1 = \varphi - Q(v_1) : [0,T] \longrightarrow \mathbb{R}$$

by

$$\varphi_1(t) = \varphi(t) - \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx.$$

Since $\varphi \in W^{1,p}(0,T)$, using Lemma 4.7 together with (4.31), follows that $\varphi_1 \in \widetilde{W}^{1,p}(0,T)$. Therefore, Lemma 4.10, ensures that there exists a unique $\Gamma\varphi_1 = f_0 \in L^p(0,T)$ such that the solution $v_2 := S(0,0,0,f_0g)$ of

$$\begin{cases} v_{2t} + \alpha v_{2x} + \beta v_{2xxx} - v_{2xxxxx} = f_0 g & \text{em } Q_T^+, \\ v_2(t,0) = 0, v_{2x}(t,0) = 0 & \text{em } [0,T], \\ v_2(0,x) = 0 & \text{em } \mathbb{R}^+, \end{cases}$$

satisfying the following integral condition

$$\int_{\mathbb{R}^+} v_2(t,x)\omega(x)dx = \varphi_1(t), \ t \in [0,T].$$

Thus, taking $u = v_1 + v_2 := S(u_0,\mu,\nu,f_0g - \frac{\partial f_2}{\partial x})$, we have u solution of (4.33) satisfying

$$\begin{aligned} \int_{\mathbb{R}^+} u(t,x)\omega(x)dx &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \int_{\mathbb{R}^+} v_2(t,x)\omega(x)dx \\ &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \varphi_1(t) \\ &= \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx + \varphi(t) - \int_{\mathbb{R}^+} v_1(t,x)\omega(x)dx \\ &= \varphi(t), \end{aligned}$$

for all $t \in [0,T]$, that is, (4.2) holds, showing the result. $\square$

4.4.2 Nonlinear case

We are in a position to prove the first main result of this chapter, that is, Theorem 4.1. Here, the estimates proved in [33], presented in Section 4.2, is essential to obtain the results.

Proof of Theorem 4.1. Let $u, v \in Z(Q_T^+)$. The following estimate holds, using Hölder inequality,

$$\|u(t,\cdot)v(t,\cdot)\|_{L^1(\mathbb{R}^+)} \leq \|u(t,\cdot)\|_{L^2(\mathbb{R}^+)}\|v(t,\cdot)\|_{L^2(\mathbb{R}^+)}, \ \forall t \in [0,T].$$

So, we get

$$\|uv\|_{C(0,T;L^1(\mathbb{R}^+))} \leq \|u\|_{C(0,T;L^2(\mathbb{R}^+))}\|v\|_{C(0,T;L^2(\mathbb{R}^+))}.$$

Since we have the following embedding $C(0,T;L^1(\mathbb{R}^+)) \hookrightarrow L^p(0,T;L^1(\mathbb{R}^+))$ for each $p \in [2,\infty]$, we can find

$$\|uv\|_{L^p(0,T;L^1(\mathbb{R}^+))} \leq T^{\frac{1}{p}}\|u\|_{C(0,T;L^2(\mathbb{R}^+))}\|v\|_{C(0,T;L^2(\mathbb{R}^+))},$$

or

$$\|uv\|_{L^p(0,T;L^1(\mathbb{R}^+))} \leq T^{\frac{1}{p}}\|u\|_{Z(Q_T^+)}\|v\|_{Z_1(Q_T^+)}, \tag{4.34}$$

for any $u, v \in Z(Q_T^+)$.

Now, pick $f = f_1 - \frac{\partial f_2}{\partial x}$ in the following system

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = f(t,x) & \text{in } [0,T]\times\mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.35}$$

Consider so $f_2 = \frac{v^2}{2}$, where $v \in Z(Q_T^+)$ and $f_1 \in L^2(0,T;L^2(\mathbb{R}^+))$. The estimate (4.14) yields that $\frac{\partial f_2}{\partial x} = vv_x \in X^{0,-b_0}(Q_T^+)$, for some $b_0 \in (0,\frac{1}{2})$. Moreover, thanks to (4.34) we have that $f_2 \in L^p(0,T;L^1(\mathbb{R}^+))$.

On the space $Z(Q_T^+)$ let us define the functional $\Theta : Z(Q_T^+) \longrightarrow Z(Q_T^+)$ by

$$u := \Theta v = S\Big(u_0, \mu, \nu, \Gamma\big(\varphi - Q(S(u_0,\mu,\nu,-vv_x))\big)g - vv_x\Big). \tag{4.36}$$

Note that using Lemma 4.10 and Theorem 4.12 the operator Θ is well-defined.

Considering $p = 2$, thanks to (4.15), the embedding $L^2(0,T;L^2(\mathbb{R}^+)) \hookrightarrow X^{0,-b_0}(Q_T^+)$, (4.23), (4.28) and (4.19), we get

$$\begin{aligned}
\|\Theta v\|_{Z(Q_T^+)} \leq & C\Big(\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} \\
& + \|\Gamma\big(\varphi - Q(S(u_0,\mu,\nu,-vv_x))\big)g - vv_x\|_{X^{0,-b_0}(Q_T^+)}\Big) \\
\leq & C\Big(\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|vv_x\|_{X^{0,-b_0}(Q_T^+)} \\
& + \|\Gamma\big(\varphi - Q(S(u_0,\mu,\nu,-vv_x))\big)g\|_{L^2(0,T;L^2(\mathbb{R}^+))}\Big) \\
\leq & C\Big(\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|vv_x\|_{X^{0,-b_0}(Q_T^+)} \\
& + \|g\|_{C(0,T;L^2(\mathbb{R}^+))}\|\big(\varphi - Q(S(u_0,\mu,\nu,-vv_x))\big)\|_{\widetilde{W}^{1,2}(0,T)}\Big) \\
\leq & C(\|g\|_{C(0,T;L^2(\mathbb{R}^+))},T)\Big(\|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} \\
& + \Big\|vv_x\Big\|_{X^{0,-b_0}(Q_T^+)} + \|\varphi'\|_{L^2(0,T)} + \|q'\|_{L^2(0,T)}\Big) \\
\leq & 2C\Big(c_1 + \Big\|vv_x\Big\|_{X^{0,-b_0}(Q_T^+)} + \Big\|v\Big\|^2_{L^2(0,T;L^1(\mathbb{R}^+))}\Big),
\end{aligned}$$

or equivalently,

$$\|\Theta v\|_{Z(Q_T^+)} \leq 2C\Big(c_1 + \Big\|vv_x\Big\|_{X^{0,-b_0}(Q_T^+)} + \Big\|v^2\Big\|_{L^2(0,T;L^1(\mathbb{R}^+))}\Big).$$

Here, C is a constant depends on $|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)}, \|g\|_{C(0,T;L^2(\mathbb{R}^+))}$ and T. Now, using the estimates (4.34) and (4.14), we have that

$$\|\Theta v\|_{Z(Q_T^+)} \leq C\Big(c_1 + (T^{\frac{1}{2}}+1)\|v\|^2_{Z(Q_T^+)}\Big). \tag{4.37}$$

Here, $c_1 > 0$ is a constant depending such that

$$c_1 := \|u_0\|_{L^2(\mathbb{R}^+)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|\varphi'\|_{L^2(0,T)}$$

and $C > 0$ is a constant depending of $C := C(|\alpha|, |\beta|, \|\omega\|_{H^5(\mathbb{R}^+)}, \|g\|_{C(0,T;L^2(\mathbb{R}^+))}, T)$.

Similarly, using the linearity of the operator S, Q and Γ, once again thanks to (4.34) and (4.14), we have

$$\|\Theta v_1 - \Theta v_2\|_{Z(Q_T^+)} \leq C(T^{\frac{1}{2}}+1)(\|v_1\|_{Z(Q_T^+)} + \|v_2\|_{Z(Q_T^+)})\|v_1 - v_2\|_{Z(Q_T^+)} \tag{4.38}$$

Finally, for fixed $c_1 > 0$, take $T_0 > 0$ such that

$$8C^2_{T_0}(T_0^{\frac{1}{2}}+1)c_1 \leq 1$$

then, for any $T \in (0, T_0]$, choose

$$r \in \left[2C_T c_1, \frac{1}{\left(4C_T(T^{\frac{1}{2}}+1)\right)}\right].$$

On the other hand, for fixed $T > 0$ pick

$$r = \frac{1}{\left(4C_T(T^{\frac{1}{2}}+1)\right)}$$

and

$$c_1 \leq \gamma = \frac{1}{\left(8C_T^2(T^{\frac{1}{2}}+1)\right)}.$$

Therefore,

$$C_T c_1 \leq \frac{r}{2}, \quad C_T(T^{\frac{1}{2}}+1)r \leq \frac{1}{4}.$$

So, Θ is a contraction on the ball $B(0,r) \subset Z(Q_T^+)$. Theorem 4.12 ensures that the unique fixed point $u = \Theta u \in Z(Q_T^+)$ is a desired solution for $f_0 := \Gamma\left(\varphi - Q(S(u_0, \mu, \nu, -uu_x))\right) \in L^p(0,T)$. Thus, the result is achieved. □

4.4.3 Minimal time for the integral condition

We are now able to prove the integral overdetermination condition (4.45) holds for a minimal time T_0. To do that, let us prove the second main result of this work, namely, Theorem 4.2.

Proof of Theorem 4.2. Without loss of generality, let us assume $T \leq 1$. It is well known that the Kawahara equation

$$\begin{cases} u_t - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.39}$$

enjoys the scaling symmetry: If u is a solution to (4.39), $u_\delta(t,x)$ defined by

$$u_\delta(t,x) := \delta^4 u(\delta^5 t, \delta x), \quad \delta > 0$$

is solution of (4.39) as well. Indeed, let $\delta = T^{\frac{1}{5}} \in (0,1)$, then

$$u_{0\delta}(x) := \delta^4 u_0(\delta x), \ \mu_\delta(t) := \delta^4 \mu(\delta^5 t), \ \nu_\delta(t) := \delta^4 \nu(\delta^5 t)$$

$$g_\delta(t,x) := \delta g(\delta^5 t, \delta x), \ \omega_\delta(x) := \omega(\delta x), \ \varphi_\delta(t) := \delta^4 \varphi(\delta^5 t).$$

Therefore, if the par (f_0, u) is solution of (4.39), a straightforward calculation gives that

$$\{f_{0\delta}(t) := \delta^8 f_0(\delta^5 t), \ u_\delta(t,x) := \delta^4 u(\delta^5 t, \delta x)\}$$

is solution of

$$\begin{cases} u_{\delta t} - u_{\delta xxxxx} + u_\delta u_{\delta x} = f_{0\delta}(t) g_\delta(t,x) & \text{in } [0,1]\times\mathbb{R}^+, \\ u_\delta(t,0) = \mu_\delta(t), \ u_{\delta x}(t,0) = \nu_\delta(t) & \text{on } [0,1], \\ u_\delta(0,x) = u_{0\delta}(x) & \text{in } \mathbb{R}^+. \end{cases} \tag{4.40}$$

Additionally, we have that (f_0, u)satisfies (4.2) if and only if $(f_{0\delta}(t), u_\delta(t,x))$ satisfies the following integral condition

$$\int_{\mathbb{R}^+} u_\delta(t,x)\omega_\delta(x)dx = \varphi_\delta(t), \ t\in[0,1]. \tag{4.41}$$

Now, using the change of variables theorem and the definition of δ, we verify that

$$\|u_{0\delta}\|_{L^2(\mathbb{R}^+)} = \delta^{\frac{1}{2}}\delta^4\|u_0\|_{L^2(\mathbb{R}^+)} \leq \delta^{\frac{1}{2}}\|u_0\|_{L^2(\mathbb{R}^+)}$$

and

$$\|\varphi_\delta'\|_{L^2(0,1)} = \delta^{\frac{1}{2}}\delta^{11}\|\varphi_\delta\|_{L^2(0,T)} \leq \delta^{\frac{1}{2}}\|\varphi_\delta\|_{L^2(0,T)}.$$

Thus, we have that

$$c_1(\delta) := \|u_{0\delta}\|_{L^2(\mathbb{R}^+)} + \|\varphi_\delta'\|_{L^2(0,1)} + \|\mu_\delta\|_{H^{\frac{2}{5}}(0,1)} + \|\nu_\delta\|_{H^{\frac{1}{5}}(0,1)} \leq \delta^{\frac{1}{2}}c_1.$$

Moreover,

$$\|g_\delta\|_{C([0,1];L^2(\mathbb{R}^+))} \leq \delta^{\frac{1}{2}}\|g\|_{C([0,T];L^2(\mathbb{R}^+))},$$
$$\left|\int_{\mathbb{R}^+} g_\delta(t,x)\omega_\delta dx\right| \geq g_0, \ \forall t\in[0,1],$$

and

$$\|\omega_\delta'''''\|_{L^2(\mathbb{R}^+)} \leq \delta^{\frac{9}{2}}\|\omega'''''\|_{L^2(\mathbb{R}^+)}.$$

So, as we want that $c_{0\delta}$ be one corresponding to c_0, which was defined by (4.26), therefore,

$$c_{0\delta} \leq \delta^5 c_0.$$

Pick $\delta_0 = (2c_0)^{-1/5}$, so for $0<\delta\leq\delta_0$ we can apply Lemma 4.11 and according to (4.29), the corresponding operator to Γ, which one will be called of γ_δ satisfies

$$\|\Gamma_\delta\varphi_\delta\|_{L^p(0,1)} \leq \frac{2}{g_0}\|\varphi_\delta'\|_{L^p(0,1)}. \tag{4.42}$$

Therefore, for Θ_δ defined in the same way as in (4.36) and using the inequalities (4.37) and (4.38), however, now related to (4.42) instead of (4.28), we have

$$\|\Theta_\delta v_\delta\|_{Z(Q_1^+)} \leq C\left(\delta^{\frac{1}{2}}c_1 + (T^{\frac{1}{2}}+1)\|v_\delta\|^2_{Z(Q_1^+)}\right)$$

and

$$\|\Theta_\delta v_{1_\delta} - \Theta_\delta v_{2_\delta}\|_{Z(Q_1^+)} \leq C(T^{\frac{1}{2}}+1)\left(\|v_{1_\delta}\|_{Z(Q_1^+))} + \|v_{2_\delta}\|_{Z(Q_1^+)}\right)\|v_{1_\delta} - v_{2_\delta}\|_{Z(Q_1^+)},$$

where the constant C is uniform with respect to $0 < \delta \leq \delta_0$. Taking δ_0 small enough, if necessary, to satisfy the following inequality

$$\delta_0^{\frac{1}{2}} c_1 \leq \frac{1}{8c^2(T^{\frac{1}{2}}+1)},$$

so using the same arguments as done in Theorem 4.1, the operator Θ_δ becomes, at least, locally, a contraction on a certain ball. Finally, taking the time T_0 defined by $T_0 := \delta_0^5$, and if $T \leq T_0$ we have that (4.2) holds, showing the result. □

Remark 4.13. Note that the system (4.30) does not admit the scaling symmetry due to the presence of the terms $\alpha u_x + \beta u_{xxx}$. So, in this case, we analyzed the equation (4.39), since in the analysis of Theorem 5.2 the most important term is of order five, so we can neglect the terms of order one (αu_x) and three (βu_{xxx}).

4.4.4 An exact controllability result

The main objective of this subsection is to prove the Corollary 4.3, showing that if the overdetermination condition is verified, for given any initial data u_0 and final data u_T the mass (4.7) of the system (4.6) is reached on the time T.

Proof of Corollary 4.3. Thanks to the Theorem 4.1 with $\mu = v = 0$, there exist $f_0 \in L^p(0,T)$ and a unique solution u of (4.6) such that

$$\varphi(t) = \int_{\mathbb{R}^+} u(t,x)\omega(x)dx, \quad t \in [0,T]. \tag{4.43}$$

On the other hand, using [84, Sections 6.1-6.6], we know that ω defined a real measure in $\mathbb{R}^+$ given by

$$\eta(E) = \int_E w(x)dx,$$

for all Lebesgue measurable set E of $\mathbb{R}^+$ and

$$\int_{\mathbb{R}^+} f d\eta = \int_{\mathbb{R}^+} f(x)\omega(x)dx,$$

for all measurable function f in $\mathbb{R}^+$. Hence, from (4.9) and (4.43), we conclude that

$$[u(T)] = \int_{\mathbb{R}^+} u(T)d\eta = \int_{\mathbb{R}^+} u(T,x)\omega(x)dx = \int_{\mathbb{R}^+} u_T(x)\omega(x)dx = \int_{\mathbb{R}^+} u_T d\eta = [u_T],$$

and the corollary is achieved. □

4.5 Further comments

This chapter deals with the internal controllability problem with an integral overdetermination condition on unbounded domains. Precisely, we consider the higher order KdV type equation, the so-called, Kawahara equation on the right half-line

$$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f(t,x) & \text{in } [0,T] \times \mathbb{R}^+, \\ u(t,0) = \mu(t), \ u_x(t,0) = \nu(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^+, \end{cases} \tag{4.44}$$

where $f := f_0(t)g(t,x)$, with f_0 as a control input. In this case, we prove that given functions u_0, μ, ν and g, the following integral overdetermination condition

$$\int_{\mathbb{R}^+} u(t,x)\omega(x)dx = \varphi(t), \ \ t \in [0,T], \tag{4.45}$$

holds. Additionally, that condition can be verified for a small time T_0. These points answer the previous questions introduced in [27] and extend to other domains the results of [24].

4.5.1 Comments about the main results

Let us give some remarks in order concerning the generality of these results.

- Theorems 4.1 and 4.2 can be obtained for more general nonlinearity u^2u_x. This is possible due to the result of Cavalcante and Kwak [32] that showed the following:

  Theorem 4.14. *The following estimates hold.*

  a) For $-1/4 \leq s$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have

  $$\|\partial_x(uvw)\|_{X^{s,-b}} \lesssim \|u\|_{X^{s,b}\cap D^\alpha}\|v\|_{X^{s,b}\cap D^\alpha}\|w\|_{X^{s,b}\cap D^\alpha}.$$

  b) For $-1/4 \leq s \leq 0$, there exists $b = b(s) < 1/2$ such that for all $\alpha > 1/2$, we have

  $$\|\partial_x(uvw)\|_{Y^{s,-b}} \lesssim \|u\|_{X^{s,b}\cap D^\alpha}\|v\|_{X^{s,b}\cap D^\alpha}\|w\|_{X^{s,b}\cap D^\alpha}.$$

  Thus, Theorems 4.1 and 4.2 remain valid for u^2u_x, however, for sake of simplicity, we consider only the nonlinearity as uu_x.

- Due to the boundary traces defined in [33, Theorems 1.1 and 1.2], the regularities of the functions involved in this part of the work are sharp.

- The results presented here are still valid when we consider the following domains: the real line ($\mathbb{R}$) or the left half-line ($\mathbb{R}^-$). Precisely, let us consider the following systems

  $$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T]\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{4.46}$$

  and

  $$\begin{cases} u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} + uu_x = f_0(t)g(t,x) & \text{in } [0,T]\times\mathbb{R}^-, \\ u(t,0) = \mu(t), \ \ u_x(t,0) = \nu(t), \ \ u_{xx}(t,0) = h(t) & \text{on } [0,T], \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^-. \end{cases} \tag{4.47}$$

  For given $T > 0$, φ, ω and ω^-, consider the following integral conditions

  $$\int_{\mathbb{R}} u(t,x)\omega(x)dx = \varphi(t), t \in [0,T] \tag{4.48}$$

and

$$\int_{\mathbb{R}^-} u(t,x)\omega^-(x)dx = \varphi(t), t \in [0,T]. \tag{4.49}$$

Thus, the next two theorems give us answers for the Problems $\mathcal{A}$ and $\mathcal{B}$, presented at the beginning of the introduction, for the real line and left half-line, respectively.

Theorem 4.15. *Let $T > 0$ and $p \in [2,\infty]$. Consider $u_0 \in L^2(\mathbb{R})$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T;L^2(\mathbb{R}))$ and $\omega \in H^5(\mathbb{R})$ be a fixed function satisfying*

$$\varphi(0) = \int_{\mathbb{R}} u_0(x)\omega(x)dx$$

and

$$\left|\int_{\mathbb{R}} g(t,x)\omega(x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if $c_1 = \|u_0\|_{L^2(\mathbb{R}} + \|\varphi'\|_{L^2(0,T)} \leq \gamma$, we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (4.46) satisfying (4.48).

Theorem 4.16. *Let $T > 0$ and $p \in [2,\infty]$. Consider $\mu \in H^{\frac{2}{5}}(0,T) \cap L^p(0,T)$, $\nu \in H^{\frac{1}{5}}(0,T) \cap L^p(0,T)$, $h \in L^p(0,T)$, $u_0 \in L^2(\mathbb{R}^-)$ and $\varphi \in W^{1,p}(0,T)$. Additionally, let $g \in C(0,T;L^2(\mathbb{R}^-))$ and ω^- be a fixed function which belongs to the following set*

$$\mathcal{J} = \{\omega \in H^5(\mathbb{R}^-) : \omega(0) = \omega'(0) = 0\} \tag{4.50}$$

satisfying

$$\varphi(0) = \int_{\mathbb{R}^-} u_0(x)\omega^-(x)dx$$

and

$$\left|\int_{\mathbb{R}^-} g(t,x)\omega^-(x)dx\right| \geq g_0 > 0, \ \forall t \in [0,T],$$

where g_0 is a constant. Then, for each $T > 0$ fixed, there exists a constant $\gamma > 0$ such that if

$$c_1 = \|u_0\|_{L^2(\mathbb{R}^-)} + \|\mu\|_{H^{\frac{2}{5}}(0,T)} + \|\nu\|_{H^{\frac{1}{5}}(0,T)} + \|h\|_{L^2(0,T)} + \|\varphi'\|_{L^2(0,T)} \leq \gamma,$$

we can find a unique control input $f_0 \in L^p(0,T)$ and a unique solution u of (4.47) satisfying (4.49).

- The difference between the numbers of boundary conditions in (4.44) and (4.47) is motivated by integral identities on smooth solutions to the linear Kawahara equation

$$u_t + \alpha u_x + \beta u_{xxx} - u_{xxxxx} = 0.$$

- Theorem 5.2 is also true for the systems (4.46) and (4.47). Additionally, due to the results presented in [32,33] the functions involved in Theorems 4.15 and 4.16 are also sharp and we can introduce a more general nonlinearity like u^2u_x in these systems.

- Corollary 4.3 may be extended for the system (4.46) taking into account the integral condition (4.48). Also for the system (4.47), with $u(t,0) = u_x(t,0) = u_{xx}(t,0) = 0$ and the integral condition (4.49), this corollary is verified.

5 Two stability results for the Kawahara equation with a time-delayed boundary control

5.1 Introduction

5.1.1 Physical motivation and goal

It is well-known that the following fifth-order nonlinear dispersive equation

$$\pm 2\partial_t u + 3u\partial_x u - \nu\partial_x^3 u + \frac{1}{45}\partial_x^5 u = 0, \tag{5.1}$$

models numerous physical phenomena. Considering suitable assumptions on the amplitude, wavelength, wave steepness, and so on, the properties of the asymptotic models for water waves have been extensively studied in the last years, through (5.1), to understand the full water wave system[1].

In some situations, we can formulate the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form with at least two (non-dimensional) parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. In turn, if we introduce another non-dimensional parameter μ, so-called the Bond number, which measures the importance of gravitational forces compared to surface tension forces, then the physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves. On the other hand, there are several long-wave approximations depending on the relations between ε and δ. For instance, if we consider $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation, represented by (5.1), and derived by Hasimoto and Kawahara in [57, 65].

In the last years, there has been an extensive mathematical endeavor that focuses on the analytical and numerical methods for solving the Kawahara equation (5.1). These methods include the tanh-function method [9], extended tanh-function method [10], sine-cosine method [98], Jacobi elliptic functions method [59], a direct algebraic method [77] as well as the variational iterations and homotopy perturbations methods [62]. These approaches deal, as a rule, with soliton-like solutions obtained while one considers problems posed on a whole real line. For numerical simulations, however, there appears the question of cutting off the spatial domain [11, 12]. This motivates the detailed qualitative analysis of the problem (5.1) in bounded regions [51].

In this spirit, the main concern of this chapter is to deal with the Kawahara equation in a bounded domain under the action of time-delayed boundary control, namely

[1] See for instance [2, 15, 70] and references therein, for a rigorous justification of various asymptotic models for surface and internal waves.

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) + u^p(t,x)\partial_x u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t > 0, \\ \partial_x^2 u(t,L) = \mathcal{F}(t,h), & t > 0, \\ \partial_x^2 u(t,0) = z_0(t), & t \in \mathcal{T}, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{5.2}$$

In (5.2), $\Omega = (0,L)$, where $L > 0$, while $a > 0$ and $b > 0$ are physical parameters. Moreover, $p \in \{1,2\}$ and $\mathcal{F}(t,h)$ is the delayed control given by

$$\mathcal{F}(t) = \alpha\partial_x^2 u(t,0) + \beta\partial_x^2 u(t-h,0), \tag{5.3}$$

in which $h > 0$ is the time-delay, α and β are two feedback gains satisfying the restriction

$$|\alpha| + |\beta| < 1. \tag{5.4}$$

Finally, $\mathcal{T} = (-h,0)$, while u_0 and z_0 are initial conditions.

Thereafter, the functional energy associated to the system (5.2)-(5.3) is

$$E(t) = \int_0^L u^2(t,x)dx + h|\beta|\int_0^1 (\partial_x^2 u(t-h\rho,0))^2 d\rho, \ t \geq 0. \tag{5.5}$$

Now, recall that if $\alpha = \beta = 0$, then the term $\partial_x^2 u(t,0)$ represents a feedback damping mechanism (see for instance [3], where $a = 1$ and [94], where $a = 0$) but extra internal damping $a(x)u(t,x)$ is required to achieve the stability of the solutions. Note that $a(x)$ is a nonnegative function and positive only on an open subset of $(0,L)$. Therefore, taking into account the action of the time-delayed boundary control (5.3) in (5.2), the following issue will be addressed here:

Does $E(t) \longrightarrow 0$, as $t \to \infty$? If it is the case, can we provide a decay rate?

It is also worth noting that the answer to the above question is crucial in the understanding of the behavior of the solutions to the Kawahara equation when it is subject to a delayed boundary control $\mathcal{F}(t,h)$. In other words, are the solutions to our problem stable despite the action of the delay? If yes, then how robust is the stability property of the solutions?

5.1.2 Historical background

Let us first present a review of the main results available in the literature for the analysis of the Kawahara equation in a bounded interval. A pioneer work is due to Silva and Vasconcellos [93,94], where the authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second endeavor, in this line, is completed by the same authors in [95], where the problem (5.2)-(5.3) is considered with $a = \alpha = \beta = 0, b = p = 1$ and under the action of a localized interior control $a(x)u(t,x)$. Then, exponential stability results are obtained. Subsequently, on [26], the authors investigate two problems that appear in the mathematical theory when we consider the study of dispersive systems. The first one is the global well-posedness, in time, of a system so-called fifth-order KdV–type system or second-order Boussinesq system. The second one is concerned with boundary stabilization of the linearized system. Capistrano-Filho *et al.* [3] considered the generalized Kawahara equation in a bounded domain $Q_T = (0,T) \times (0,L)$:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + a(x)u = 0, & \text{in } Q_T, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = \partial_x^2 u(t,L) = 0, & \text{on } [0,T], \\ u(0,x) = u_0(x), & \text{in } [0,L], \end{cases} \tag{5.6}$$

with $p \in [1,4)$. It is proven that the solutions of the above system decay exponentially.

The internal controllability problem has been tackled by Chen [34] for the Kawahara equation with homogeneous boundary conditions. Using Carleman estimates associated with the linear operator of the Kawahara equation with an internal observation, a null controllability result is shown when the internal control is effective in a subdomain $\omega \subset (0,L)$. In [27], considering the system (5.6) with an internal control $f(t,x)$ and homogeneous boundary conditions, the equation is exactly shown to be controllable in L^2-weighted Sobolev spaces and, additionally, controllable by regions in L^2-Sobolev space.

We also note that the existence and uniqueness of solutions as well as their stability are investigated for the Kawahara type equation posed in the whole real line [38], [39], [40], [44], [60], the half-line [49], [72], a periodic domain [58, 64], and a non-periodic bounded domain [47], [50], [71], [72]. We conclude the literature review by mentioning the last works on the stabilization of the Kawahara equation with a localized time-delayed interior control. In [28, 37], under suitable assumptions on the time delay coefficients, the authors can prove that solutions of the Kawahara system are exponentially stable. The results are obtained using the Lyapunov approach and a compactness-uniqueness argument.

5.1.3 Novel contribution of this chapter

Now, after providing an overview of the results previously obtained in the literature, let us highlight the novelty and contribution of the present chapter.

We start providing a systematic study of the well-posedness and stability results for the Kawahara equation with delayed boundary control. As far as we know, no attempt has been made in this direction. To be more specific, the present work shows that the existence, uniqueness, and stability properties of the solutions of the Kawahara equation with a boundary remain “robust” concerning the presence of a time delay in the boundary control.

Not only that, we manage to show that the presence of a time-delayed term in the boundary control (5.3) may play a dissipation role in the system. This can be explained by the fact although it might be possible to take $\alpha = 0$ and $\beta > 0$ in (5.4), the solutions of the Kawahara problem (5.2)-(5.3) remain exponentially stable.

Concerning the main contribution, we have

(i) The results obtained in this chapter do not require the presence of localized interior damping control, which constitutes an improvement of the results in [93–95];

(ii) Contrary to the works [93–95], where the nonlinearity has the simple form $u\partial_x u$, we can extend the results of these works to the case where the more general nonlinearity term $u^p\partial_x u$, $p \in [1, 2]$. Additionally, unlike these works, where the boundary conditions are all homogeneous, a boundary delayed control is present in one of our boundary conditions;

(iii) The results of [3] are complemented by taking into consideration delayed boundary control. Specifically, the stability of the solutions to the Kawahara equations is conserved despite the presence of a time delay in one of the boundary conditions;

(iv) Our stability result is obtained via two different approaches, namely, the energy method and a compactness argument.

(v) Resembled it was done in ([26, Theorem 1.2], for the higher order Boussinesq KdV system, we give, for the Kawahara equation, a relation between the spatial length L and the Möbius transform (see Subsection 5.1.5 for more details about this point).

5.1.4 Notations and main results

First of all, let us introduce the following notations that we will use throughout here.

(i) We consider the space of solutions

$$X(Q_T) = C(0, T; L^2(0, L)) \cap L^2(0, T; H_0^2(0, L))$$

equipped with the norm

$$\|v\|_{X(Q_T)} = \max_{t\in(0,T)} \|v(t, \cdot)\|_{L^2(0,L)} + \left(\int_0^T \|v(t, \cdot)\|^2_{H_0^2(0,L)} dt\right)^{\frac{1}{2}}.$$

(ii) Denote by

$$\tilde{H} = L^2(0, L) \times L^2(-h, 0)$$

the Hilbert space equipped with the inner product

$$\langle (u_1, z_1), (u_2, z_2)\rangle_{\tilde{H}} = \int_0^L u_1u_2dx + |\beta| \int_{-h}^0 z_1(s)z_2(s)\, ds,$$

which yields the following norm

$$\|(u,z)\|^2_{\tilde{H}} = \int_0^L u^2(x)dx + |\beta| \int_{-h}^0 z^2(\rho)d\rho.$$

(iii) Throughout all this chapter, $(\cdot,\cdot)_{\mathbb{R}^2}$ denotes the canonical inner product of $\mathbb{R}^2$.

With the above notations in hand, let us state our first main result.

Theorem 5.1. *Let $\alpha \neq 0$ and $\beta \neq 0$ be two real constants satisfying (5.4) and suppose that the spatial length L fulfills*

$$0 < L < \sqrt{\frac{3b}{a}}\pi. \tag{5.7}$$

Then, there exists $r > 0$ sufficiently small, such that for every $(u_0, z_0) \in H$ with $\|(u_0,z_0)\|_H < r$, the energy of the system (5.2)-(5.3), denoted by E and defined by (5.5) exponentially decays, that is, there exist two positive constants κ and λ such that

$$E(t) \leq \kappa E(0)e^{-2\lambda t},\ t > 0. \tag{5.8}$$

Here,

$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2+|\beta|)}, \frac{3b\pi^2 - r^2L\pi^2 - L^2a}{2L^2(1+L\mu_1)}\mu_1\right\} \tag{5.9}$$

and

$$\kappa \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{|\beta|}\right\}\right),$$

for $\mu_1, \mu_2 \in (0,1)$ sufficiently small.

The second main result gives another answer to the question presented in this introduction. Indeed, using a different approach based on an observability inequality, we can highlight the critical lengths phenomenon observed in [3] for the Kawahara equation:

Theorem 5.2. *Assume that α and β satisfy (5.4), whereas $L > 0$ is taken so that the problem $(\mathcal{N})$ (see Lemma 5.12) has only the trivial solution. Then, there exists $r > 0$ such that for every $(u_0, z_0) \in H$ satisfying*

$$\|(u_0,z_0)\|_H \leq r,$$

the energy of system (5.2)-(5.3), denoted by E and defined by (5.5), decays exponentially. More precisely, there exist two positive constants ν and κ such that

$$E(t) \leq \kappa E(0)e^{-\nu t}, \quad t > 0.$$

5.1.5 Heuristic of the chapter and its structure

In this chapter, we can prove that the Kawahara system (5.2) is exponentially stable despite the presence of the boundary time-delayed control $\mathcal{F}(t)$ defined by (5.3).

To prove Theorem 5.1, we use the idea of the work that treated the delayed wave systems [97] (see also [75]). More precisely, choosing an appropriate Lyapunov functional associated with the solutions of (5.2)-(5.3) and with some restrictions on the spatial length L and an appropriate size of the initial data, that is, L bounded as in (5.7) and

$$\|(u_0, z_0)\|_H < \frac{\sqrt{2}}{\pi}\sqrt{\frac{3b\pi^2 - L^2 a}{L}},$$

the energy (5.5) decays exponentially. The key idea of this analysis is the relation between the linearized system associated with (5.2)-(5.3) and a transport equation (see Section 5.2 for more details). Let us mention that such an approach is also used for the Korteweg–de Vries (KdV) with a boundary delayed control in [7] and for the Kawahara equation with a localized time-delayed interior control [37]. However, the nonlinearity in [7, 37] is $u\partial_x u$, which becomes a special case in our study, that is, $p = 1$ in (5.2).

Note that the KdV equation studied in [7] is of order three, while the Kawahara equation is a fifth-order equation. Furthermore, in this work, the boundary conditions in [7] are two homogeneous Dirichlet boundary conditions and one Neumann right-end control $\partial_x u(t, L) =$ control. This means, unlike our case, that no second-order derivative is involved in the boundary conditions. Moreover, the reader can notice that the difference between the order of the derivative in the equation and the highest order of the derivatives in the boundary conditions is two, whereas it is three in our problem. These points are the main differences between our work and [7] although there are similarities in the proof of Theorem 5.1 and Theorem 1 in [7].

Concerning the proof of Theorem 5.2, we proceed as in [83], i.e, combining multipliers and compactness arguments which reduces the problem to show a unique continuation result for the state operator. To prove the latter, we extend the solution under consideration by zero in $\mathbb{R} \setminus [0, L]$ and take the Fourier transform. However, due to the complexity of the system, after taking the Fourier transform of the extended solution u, it is not possible to use the same techniques used in [83]. Thus, to prove our main result, we use a strategy inspired by [26] and invoke the result due Santos *et al.* [46]. Specifically, after taking the Fourier transform, the issue is to establish when a certain quotient of entire functions still turns out to be an entire function. We then pick a polynomial $q : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_\alpha : \mathbb{C} \times (0, \infty) \to \mathbb{C}, \tag{5.10}$$

with $\alpha \in \mathbb{C}^4 \setminus \{0\}$, whose restriction $N_\alpha(\cdot, L)$ is entire for each $L > 0$. Next, we consider a family of functions $f_\alpha(\cdot, L)$, defined by

$$f_\alpha(\mu, L) = \frac{N_\alpha(\mu, L)}{q(\mu)}, \tag{5.11}$$

in its maximal domain. The problem is then reduced to determine $L > 0$ for which there exists $\alpha \in \mathbb{C}^4 \setminus \{0\}$ such that $f_\alpha(\cdot, L)$ is entire. In contrast with the analysis developed in [83], this approach does not provide an explicit characterization of a critical set, if it exists, but only ensures that the roots of f have a relation with the Möbius transform (see the proof of Lemma 5.12 above). It is also worth recalling that the proof of Theorem 5.2 is inspired by [83], which in turn has been used in [7]. However, the result obtained in the last work directly relies on [83]. This is not the case for our Theorem 5.2 as we explained above. On the other hand, the so-called set of critical lengths of the KdV problem is explicitly known in [83], and this made the task easier for the exponential stability of the KdV case [7]. It is also important to point out that the derivation of the set of critical lengths for the Kawahara problem is more challenging, and we only manage to derive a relation between the length of L and the Mobius transformation, while an explicit deduction of the critical set phenomena remains an open problem. This happens because the roots of the function

$$N_\alpha(\xi, L) = \alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{i\xi L}$$

in (5.11) cannot be found explicitly as in the KdV case. So, due to these facts, before explained, our problems are more challenging than those of [7, 83].

Finally, let us present the outline of our work: First, in Section 5.2, we prove the regularity properties of the solutions to the linear system associated with (5.2)-(5.3) and then show that the well-posedness of the problem (5.2)-(5.3). Section 5.3 is devoted to the proof of the first main result of this chapter, Theorem 5.1. In Section 5.4, with the help of the result established in [46], we show Theorem 5.2. Finally, in Section 5.5, we present some additional comments and open questions.

5.2 Well-posedness results

The goal of this section is to prove that the full nonlinear Kawahara system (5.2)-(5.3) is well-posed. The proof is divided into four parts by using the strategy due to Rosier [83]:

1. Well-posedness to the linear system associated to (5.2)-(5.3);
2. Properties of regularity of the linear system associated to (5.2)-(5.3).
3. Well-posedness of the linear system associated to (5.2)-(5.3) with a source term.
4. Well-posedness of the system (5.2)-(5.3).

5.2.1 Well-posedness: Linear system

We begin by proving the well-posedness of the linearized system

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t > 0, \\ \partial_x^2 u(t,L) = \alpha \partial_x^2 u(t,0) + \beta \partial_x^2 u(t-h,0), & t > 0, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{5.12}$$

In order to investigate (5.12), let $z(t,\rho) = \partial_x^2 u(t-\rho h, 0)$, which satisfies the transport equation [97] (see also [75])

$$\begin{cases} h\partial_t z(t,\rho) + \partial_\rho z(t,\rho) = 0, & \rho \in (0,1), \ t \ > 0, \\ z(t,0) = \partial_x^2 u(t,0), & t > 0, \\ z(0,\rho) = z_0(-h\rho), & \rho \in (0,1). \end{cases} \tag{5.13}$$

Next, we consider the Hilbert space $H = L^2(0,L) \times L^2(0,1)$ equipped with the following inner product

$$\langle (u_1, z_1), (u_2, z_2) \rangle_H = \int_0^L u_1 u_2 dx + |\beta| h \int_0^1 z_1 z_2 d\rho.$$

Subsequently, one can rewrite (5.12)-(5.13) as follows

$$\begin{cases} U_t(t) = AU(t), & t > 0, \\ U(0) = U_0 \in H, \end{cases} \tag{5.14}$$

where

$$A = \begin{bmatrix} -a\partial_x - b\partial_x^3 + \partial_x^5 & 0 \\ 0 & -\frac{1}{h}\partial_\rho \end{bmatrix}, U(t) = \begin{bmatrix} u(t,\cdot) \\ z(t,\cdot) \end{bmatrix}, U_0 = \begin{bmatrix} u_0(\cdot) \\ z_0(-h(\cdot)) \end{bmatrix}$$

and

$$\begin{aligned} D(A) = \{ &(u,z) \in (H^5(0,L) \cap H_0^2(0,L)) \times H^1(0,1); \\ &\qquad \partial_x^2 u(0) = z(0), \partial_x^2 u(L) = \alpha \partial_x^2 u(0) + \beta z(1) \}. \end{aligned}$$

The next result ensures the well-posedness for the problem (5.12).

Proposition 5.3. *Assume that the constants α and β satisfy (5.4) and that $U_0 \in H$. Then, there exists a unique mild solution $U \in C([0,+\infty), H)$ for the system (5.12). Additionally, considering $U_0 \in D(A)$, we have a classical solution with the following regularity*

$$U \in C([0,+\infty), D(A)) \cap C^1([0,+\infty), H).$$

Proof. As the proof uses standard arguments, only a sketch of it will be provided. Let $U = (u,z) \in D(A)$. Then, integrating by parts and using the boundary conditions of

(5.12) and (5.13), we obtain

$$\begin{aligned}\langle AU(t),U(t)\rangle_H =&\frac{1}{2}\left(\alpha^2(\partial_x^2u(t,0))^2+2\alpha\beta\partial_x^2u(t,0)\partial_x^2u(t-h,0)\right)\\&+\frac{1}{2}\left(\beta^2(\partial_x^2u(t-h,0))^2-(\partial_x^2u(t,0))^2\right)\\&+\frac{1}{2}\left(-|\beta|(\partial_x^2u(t-h,0))^2+|\beta|(\partial_x^2u(t,0))^2\right)\\=&\frac{1}{2}(M\eta(t),\eta(t))_{\mathbb{R}^2},\end{aligned}\tag{5.15}$$

where

$$\eta=\begin{bmatrix}\partial_x^2u(t,0)\\ \partial_x^2u(t-h,0))\end{bmatrix}\text{ and } M=\begin{bmatrix}\alpha^2-1+|\beta| & \alpha\beta\\ \alpha\beta & \beta^2-|\beta|\end{bmatrix}.\tag{5.16}$$

Now, observe that for the Adjoint of A, denoted by A^*, is defined by

$$A^*=\begin{bmatrix}a\partial_x+b\partial_x^3-\partial_x^5 & 0\\ 0 & \frac{1}{h}\partial_\rho\end{bmatrix}$$

with

$$\begin{aligned}D(A^*)=\{(\varphi,\psi)\in(H^5(0,L)\cap H_0^2(0,L))\times H^1(0,1):\varphi(0)=\varphi(L)=\partial_x\varphi(0)=\partial_x\varphi(L)=0,\\ \psi(1)=\frac{\beta}{|\beta|}\partial_x^2\varphi(L),\ \partial_x^2\varphi(0)=\alpha\partial_x^2\varphi(L)+|\beta|\psi(0)\}.\end{aligned}$$

Similarly, we have, for $V=(\varphi,\psi)\in D(A^*)$, that

$$\begin{aligned}\langle A^*V,V\rangle_H=&\frac{1}{2}\left[(\alpha^2-1+|\beta|^2)\partial_x^2\varphi(L)+2\alpha|\beta|\partial_x^2\varphi(L)\psi(0)+(|\beta|^2-|\beta|)\psi(0)^2\right]\\=&\frac{1}{2}(M^*\eta^*,\eta^*)_{\mathbb{R}^2},\end{aligned}$$

where

$$\eta^*=\begin{bmatrix}\partial_x^2\varphi(L)\\ \psi(0)\end{bmatrix}\text{ and } M^*=\begin{bmatrix}\alpha^2-1+|\beta| & \alpha|\beta|\\ \alpha|\beta| & \beta^2-|\beta|\end{bmatrix}.\tag{5.17}$$

Now, let us check that M and M^* are negative definite. For this, we will use the following lemma:

Lemma 5.4. *Let $M=(m_{ij})_{i,j}\in\mathbb{M}_{2\times 2}(\mathbb{R})$ be a symmetric matrix. If $m_{11}<0$ and $\det(M)>0$, then M is negative definite.*

Proof. It is sufficient to note that for all $u=(x\ y)\neq(0\ 0)$ we have

$$uMu^\top=m_{11}x^2+2xym_{12}+m_{22}y^2=m_{11}\left(x+\frac{m_{12}}{m_{11}}y\right)^2+\left(\frac{m_{11}m_{22}-m_{12}^2}{m_{11}}\right)y^2<0,$$

which completes the proof. □

Now, we are in a position to finish the proof. From (5.16), (5.17) and the condition (5.4), we see that $m_{11} = m^*_{11} = \alpha^2 - 1 + |\beta| < 0$ and

$$\det M = \det M^* = |\beta|((|\beta| - 1)^2 - \alpha^2) > 0,$$

where $M = (m_{ij})_{i,j\in\{1,2\}}$ and $M^* = (m^*_{i,j})_{i,j\in\{1,2\}}$. Therefore, by virtue of Lemma 5.4, it follows that M and M^* are negative definite and hence both A and A^* are dissipative in view of (5.15) and (5.2.1).

Finally, since A and A^* are densely defined closed linear operators and both A and A^* are dissipative, one can use semigroups theory of linear operators [76] to claim that A is a generator of a C_0–semigroups of contractions on H, together with the statements of Proposition 5.3. □

Remark 5.5. It is important to point out that considering $\alpha = \beta = 0$ or $\alpha \neq 0$ and $\beta = 0$, the well-posedness of (5.12) is easily obtained. Indeed, if $\alpha = \beta = 0$, the result follows from (see [3, Lemma 2.1]). In the case when $\alpha \neq 0$ and $\beta = 0$, we have $Au = -a\partial_x - b\partial^3_x u + \partial^5_x u$ with domain

$$D(A) = \{u \in H^5(0, L) : u(0) = u(L) = \partial_x u(0) = \partial_x u(L) = 0, \partial^2_x u(L) = \alpha\partial^2_x u(0)\}.$$

It may be seen that $A^*v = a\partial_x v + b\partial^3_x v - \partial^5_x v$ with domain

$$D(A^*) = \{v \in H^5(0, L) : v(0) = v(L) = \partial_x v(0) = \partial_x v(L) = 0, \partial^2_x v(0) = \alpha\partial^2_x v(L)\}$$

and we easily verifies that

$$(Au, u)_{L^2(0,L)} = \frac{(\alpha^2 - 1)}{2}(\partial^2_x u(0))^2 \quad \text{and} \quad (A^*v, v)_{L^2(0,L)} = \frac{(\alpha^2 - 1)}{2}(\partial^2_x v(L))^2,$$

so in this case, it is necessary to take $|\alpha| < 1$ in order to obtain the well posedness result.

5.2.2 Regularity estimates: Linear system

In the sequel, let $\{S(t)\}_{t\geq 0}$ be the semigroup of contractions associated with the operator A. We have some *a priori* estimates and regularity estimates for the linear systems (5.12) and (5.13).

Proposition 5.6. *Suppose that* (5.4) *holds. Then, the application*

$$\begin{aligned} \mathcal{S} : H &\longrightarrow X(Q_T) \times C(0, T; L^2(0, 1)) \\ (u_0, z_0(-h(\cdot))) &\longmapsto S(\cdot)(u_0, z_0(-h(\cdot))) \end{aligned} \tag{5.18}$$

is well-defined and continuous. Moreover, for every $(u_0(\cdot), z_0(-h(\cdot))) \in H$*, we have*

$$(\partial^2_x u(\cdot, 0), z(\cdot, 1)) \in L^2(0, T) \times L^2(0, T)$$

and the following estimates hold

$$\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)} \leq C\Big(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big), \tag{5.19}$$

$$\|u_0\|^2_{L^2(0,L)} \leq \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + \|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)}, \tag{5.20}$$

and

$$\|z_0(-h(\cdot))\|^2_{L^2(0,1)} \leq \|z(T,\cdot)\|^2_{L^2(0,1)} + \frac{1}{h}\|z(\cdot,1)\|^2_{L^2(0,T)} \tag{5.21}$$

for some constant $C>0$ that may depend of $a, b, \alpha, \beta, L, T$ and h.

Proof. We split the proof into several steps.

Step 1. *Main identities.*

For every $(u_0, z_0(-h(\cdot))) \in H$, the semigroups theory gives that

$$S(\cdot)(u_0, z_0(-h(\cdot))) \in C(0,T;H)$$

and due to the fact that A generates a C_0-semigroup of contractions, we have that

$$\|u(t)\|^2_{L^2(0,L)} + h|\beta|\|z(t)\|^2_{L^2(0,1)} \leq \|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}, \forall t \in [0,T]. \tag{5.22}$$

Now, let $\phi \in C^\infty([0,1]\times[0,T])$, $\psi \in C^\infty([0,L]\times[0,T])$ and $(u,z) \in D(A)$. Then, multiplying (5.13) by ϕz and (5.12) by ψu, using integrations by parts and the initial conditions, we have

$$\begin{aligned}\int_0^1 [\phi(T,\rho)z(T,\rho)^2 - \phi(0,\rho)z_0(-h\rho)^2]d\rho - \frac{1}{h}\int_0^T\int_0^1 [h\partial_t\phi(t,\rho) + \partial_\rho\phi(t,\rho)]z(t,\rho)^2 d\rho dt \\ + \frac{1}{h}\int_0^T [\phi(t,1)z(t,1)^2 - \phi(t,0)(\partial_x^2 u(t,0))^2]dt = 0\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}&-\int_0^T\int_0^L [\partial_t\psi(t,x) + a\partial_x\psi(t,x) + b\partial_x^3\psi(t,x) - \partial_x^5\psi(t,x)]u^2(t,x)dxdt \\ &+ 3b\int_0^T\int_0^L \partial_x\psi(t,x)(\partial_x u(t,x))^2 dxdt + \int_0^L [\psi(T,x)u^2(t,x) - \psi(0,x)u_0(x)^2]dx \\ &+ 5\int_0^T\int_0^L [\partial_x\psi(t,x)(\partial_x^2 u(t,x))^2 - \partial_x^3\psi(t,x)(\partial_x u(t,x))^2]dxdt \\ &- \int_0^T \psi(t,L)[\alpha\partial_x^2 u(t,0) + \beta z(t,1)]^2 dt + \int_0^T \psi(t,0)(\partial_x^2 u(t,0))^2 dt = 0.\end{aligned} \tag{5.24}$$

Step 2. *Proof of* (5.19).

Let us pick $\phi(t,\rho) = \rho$ in (5.23) to get

$$\int_0^1 (z(T,\rho)^2 - z_0(-\rho h)^2)\rho d\rho - \frac{1}{h}\int_0^T\int_0^1 z(t,\rho)^2 d\rho dt + \frac{1}{h}\int_0^T z(t,1)^2 dt = 0.$$

Owing to (5.22), the latter gives

$$\|z(\cdot,1)\|^2_{L^2(0,T)} \le (T+1)\left(1+\frac{1}{h|\beta|}\right)\left(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\right). \tag{5.25}$$

Now, choosing $\psi(t,x)=1$ in (5.24) yields

$$\int_0^L [u^2(t,x)-u_0(x)^2]dx + \int_0^T (\partial_x^2 u(t,0))^2 dt - \int_0^T [\alpha\partial_x^2 u(t,0)^2 + \beta z(t,1)]^2 dx = 0,$$

which implies

$$\int_0^T (\partial_x^2 u(t,0))^2 dt \le \int_0^T (\alpha\partial_x^2 u(t,0)+\beta z(t,1))^2 dt + \|u_0\|^2_{L^2(0,L)}. \tag{5.26}$$

Since

$$(\alpha\partial_x^2 u(t,0)+\beta z(t,1))^2 \le (\alpha^2+\beta^2)((\partial_x^2 u(t,0))^2 + (z(t,1))^2), \tag{5.27}$$

it follows from (5.26) and (5.27) that

$$\int_0^T \Big(1-(\alpha^2+\beta^2)\Big)(\partial_x^2 u(t,0))^2 dt \le \int_0^T (\alpha^2+\beta^2) z(t,1)^2 dt + \|u_0\|^2_{L^2(0,L)}.$$

In view of (5.25) and (5.4), the last estimate yields

$$\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} \le (T+1)\frac{1}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\left(\|u_0\|^2_{L^2(0,L)} + \|z(-h(\cdot)))\|^2_{L^2(0,1)}\right). \tag{5.28}$$

Combining (5.28) and (5.25), the estimate (5.19) follows.

Step 3. *The map* (5.18) *is well-defined and continuous.*

Letting $\psi(t,x)=x$ in (5.24) gives

$$\begin{aligned}-a\int_0^T\int_0^L u^2(t,x)dxdt + 3b\int_0^T\int_0^L (\partial_x u(t,x))^2 dxdt + 5\int_0^T\int_0^L (\partial_x^2 u(t,x))^2 dxdt \\ + \int_0^L x[u^2(T,x)-u_0(x)^2]dx - L\int_0^T [\alpha\partial_x^2 u(t,0)+\beta z(t,1)]^2 dt = 0.\end{aligned}$$

which implies, using (5.22) and (5.27), that

$$\begin{aligned}3b\int_0^T\int_0^L (\partial_x u(t,x))^2 dxdt + 5\int_0^T\int_0^L (\partial_x^2 u(t,x))^2 dxdt \le a\Big(\|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big) \\ + L\|u_0\|^2_{L^2(0,L)} + L(\alpha^2+\beta^2)\Big(\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)}\Big).\end{aligned}$$

In light of (5.19), we deduce that

$$\begin{aligned}\|\partial_x u\|^2_{L^2(0,T;L^2(0,L))} + \|\partial_{xx} u\|^2_{L^2(0,T,L^2(0,L))} \le{}& \frac{a}{\min\{3b,5\}}\Big(\|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big) \\ &+ (T+1)\frac{2-(\alpha^2+\beta^2)}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\frac{L}{\min\{3b,5\}}(\alpha^2+\beta^2) \\ &\times\Big(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big) + \frac{L}{\min\{3b,5\}}\|u_0\|^2_{L^2(0,L)} \\ \le{}& C_0(T+1)\Big(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big),\end{aligned} \tag{5.29}$$

where

$$C_0 = \max\left\{\frac{a}{\min\{3b,5\}}, \frac{a}{\min\{3b,5\}}|\beta|h, \left(\frac{2-(\alpha^2+\beta^2)}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\frac{L}{\min\{3b,5\}}(\alpha^2+\beta^2)\right)\right\}.$$

Combining (5.29) and (5.22), we obtain the desired result.

Step 4. *Proof of* (5.20) *and* (5.21).

In order to show these inequalities, choose $\psi = T-t$ in (5.24) and $\phi(t,\rho)=1$ in (5.23), respectively. Performing similar computations as we did in step 2, the result follows. Moreover, owing to the density of $D(A)$ in H, the proof of Proposition 5.6 is achieved. □

5.2.3 Well-posedness: Linear system with a source term

Now we consider the linear system with a source term

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) = f(t,x), & (t,x)\in\mathbb{R}^+\times\Omega,\\ u(t,0)=u(t,L)=\partial_x u(t,0)=\partial_x u(t,L)=0, & t>0,\\ \partial_x^2 u(t,L) = \alpha\partial_x^2 u(t,0)+\beta\partial_x^2 u(t-h,0), & t>0,\\ \partial_x^2 u(t,0) = z_0(t), & t>0,\\ u(0,x)=u_0(x), & x\in\Omega. \end{cases} \tag{5.30}$$

Then, we have the following result.

Proposition 5.7. *Let $|\alpha|$ and $|\beta|$ satisfying* (5.4)*. For every $(u_0,z_0)\in H$ and $f\in L^1(0,T;L^2(0,L))$, there exists a unique mild solution $(u,\partial_x^2 u(t-h.,0))\in X(Q_T)\times C(0,T;L^2(0,1))$ to* (5.30)*. Moreover, there exists a constant $C>0$ such that*

$$\|(u,z)\|_{C(0,T;H)} \leq C\Big(\|(u_0,z_0(-h(\cdot)))\|_H + \|f\|_{L^1(0,T;L^2(0,L))}\Big) \tag{5.31}$$

and

$$\|\partial_x^2 u\|^2_{L^2(0,T;L^2(0,L))} \leq C\Big(\|(u_0,z_0(-h(\cdot)))\|_H^2 + \|f\|^2_{L^1(0,T;L^2(0,L))}\Big). \tag{5.32}$$

Proof. This proof is analogous to that of [7, Proposition 2] and hence we omit it. □

5.2.4 Well-posedness of the nonlinear system (5.2)-(5.3)

Let us now prove that the system (5.2)-(5.3) is well-posed. To do so, we first deal with the properties of the nonlinearities, through the following lemma.

Lemma 5.8. *Let $u\in L^2(0,T;H^2(0,L)) = L^2(H^2)$. Then, $u\partial_x u$ and $u^2\partial_x u$ belong to $L^1(0,T;L^2(0,L))$. Besides, there exist positives constants C_0 and C_1, depending of L, such that for every $u,v\in L^2(0,T;H^2(0,L))$, one has*

$$\int_0^T \|u\partial_x u - v\partial_x v\|_{L^2(0,L)}dt \leq C_0(\|u\|_{L^2(H^2)}+\|v\|_{L^2(H^2)})\|u-v\|_{L^2(H^2)} \tag{5.33}$$

and

$$\int_0^T \|u^2\partial_x u - v^2\partial_x v\|_{L^2(0,L)}dt \le C_0(1+T^{\frac{1}{2}})\left(\|u\|^2_{X(Q_T)} + \|v\|^2_{X(Q_T)}\right)\|u-v\|_{X(Q_T)}. \quad (5.34)$$

Proof. Observe that (5.33) follows from [95, Lemma 2.1, p. 106]. Concerning (5.34), note that

$$\sup_{x\in(0,L)} |u(x)^2| \le \|u\|^2_{L^2(0,L)} + \|u\|_{L^2(0,L)}\|\partial_x u\|_{L^2(0,L)},$$

for $u \in H^1(0,L)$. Let $u, z \in X(Q_T)$, then

$$\begin{aligned}\|u^2(\partial_x u - \partial_x v)\|_{L^1(0,T;L^2(0,L))} &= \int_0^T \|u(t,\cdot)\|^2_{L^\infty(0,L)}\|(\partial_x u - \partial_x v)(t,\cdot)\|_{L^2(0,L)}dt \\ &\le T^{\frac{1}{2}}\|u\|^2_{L^\infty(0,T;L^2(0,L))}\|u-v\|_{L^2(0,T;H^2(0,L))} \\ &\quad + \|u\|_{L^\infty(0,T;L^2(0,L))}\|u\|_{L^2(0,T;H^2(0,L))}\|u-v\|_{L^2(0,T;H^2(0,L))}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|(u^2-v^2)\partial_x v\|_{L^1(0,T;L^2(0,L))} &= \int_0^T \left(\int_0^L |u+v|^2|u-v|^2|\partial_x v|^2dx\right)^{\frac{1}{2}} dt \\ &\le \int_0^T \left(\|(u+v)(t,\cdot)\|^2_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|^2_{L^\infty(0,L)}\int_0^L |\partial_x v|^2dx\right)^{\frac{1}{2}} dt \\ &= \int_0^T \|(u+v)(t,\cdot)\|_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|_{L^\infty(0,L)}\|\partial_x v(t,\cdot)\|_{L^2(0,L)}dt.\end{aligned}$$

Now, observe that

$$\begin{aligned}&\|(u+v)(t,\cdot)\|_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|_{L^\infty(0,L)} \le \\ &\left(\|(u+v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\|(\partial_x u + \partial_x v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\right) \\ &\times\left(\|(u-v)(t,\cdot)\|_{L^2(0,L)} + \|(u-v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\|(\partial_x u - \partial_x v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\right) \\ \le&\|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)} \\ &+ \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u - \partial_x v)(t,\cdot)\|_{L^2(0,L)} + \|(u-v)(t,\cdot)\|_{L^2(0,L)}\|(u+v)(t,\cdot)\|_{L^2(0,L)} \\ &+ \|(u-v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u + \partial_x v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u - \partial_x v)(t,\cdot)\|_{L^2(0,L)} \\ &+ \|(\partial_x u + \partial_x v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)}.\end{aligned}$$

Hence,

$$\|u^2\partial_x u - v^2\partial_x v\|_{L^1(0,T;L^2(0,L))} \le (1+T^{\frac{1}{2}})\left(\|u\|^2_{X(Q_T)} + \|v\|^2_{X(Q_T)}\right)\|u-v\|_{X(Q_T)},$$

and thus (5.34) is proved. □

Note that the arguments used for the proof of theorem (5.3), one can show after a simple calculation that the energy E defined by (5.5) is decreasing, that is,

$$E'(t) = (M\eta(t), \eta(t))_{\mathbb{R}^2} \le 0, \quad t > 0, \quad (5.35)$$

where M and η are defined by (5.16). Combining the previous lemma with the Proposition 5.7, with a classical fixed-point argument (see, for instance, [3]), we can obtain the following well-posedness result.

Theorem 5.9. *Let $L > 0$, $a, b > 0$ and $\alpha, \beta \in \mathbb{R}$ satisfying (5.4). Assume $p \in [1, 2]$ and $h > 0$. If $u_0 \in L^2(0, L)$ and $z_0 \in L^2(0, 1)$ are sufficient small, then the system (5.2)-(5.3) admits a unique solution $u \in X(Q_T)$.*

5.3 A stabilization result *via* Lyapunov approach

This part of the work aims to prove our first main result presented in Theorem 5.1. Precisely, we will prove the case $p = 2$, that is, when the nonlinearity takes the form $u^2\partial_x u$. The case $u\partial_x u$ can be shown similarly, therefore, we will omit its proof.

Proof of Theorem 5.1. First, we choose the following Lyapunov functional

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t).$$

Here $\mu_1, \mu_2 \in (0, 1)$, V_1 is defined by

$$V_1(t) = \int_0^L xu^2(t, x)dx \tag{5.36}$$

and V_2 is defined by

$$V_2(t) = h\int_0^1 (1-\rho)(\partial_x^2 u(t-h\rho, 0))^2 d\rho,$$

for any regular solution of (5.2)-(5.3). We have the following

$$E(t) \leq V(t), \tag{5.37}$$

for all $t \geq 0$. On the other hand, we have

$$\begin{aligned}
\mu_1 V_1(t) + \mu_2 V_2(t) =& \mu_1 \int_0^L xu^2(t, x)dx + h\mu_2 \int_0^1 (1-\rho)(\partial_x^2 u(t-h\rho, 0))^2 d\rho \\
\leq& \mu_1 L \int_0^L u^2(t, x)dx + \mu_2 \frac{h}{|\beta|}|\beta| \int_0^1 (1-\rho)(\partial_x^2 u(t-h\rho, 0))^2 d\rho \\
\leq& \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\} E(t),
\end{aligned}$$

that is,

$$E(t) \leq V(t) \leq \left(1 + \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\}\right) E(t), \tag{5.38}$$

for all $t \geq 0$.

Now, consider a sufficiently regular solution u of (5.2)-(5.3). Differentiating $V_1(t)$, using integration by parts and the boundary condition of (5.2)-(5.3), it follows that

$$\begin{aligned}\frac{d}{dt}V_1(t) =& -2\int_0^L xu(t,x)\Big[a\partial_x u + b\partial_x^3 u - \partial_x^5 u + u^2\partial_x u\Big](t,x)dx \\ =& a\int_0^L u^2(t,x)dx - 3b\int_0^L(\partial_x u(t,x))^2dx - 5\int_0^L(\partial_x^2 u(t,x))^2dx + \frac{1}{2}\int_0^L u^4(t,x)dx \\ &+ L\Big[\alpha^2(\partial_x^2 u(t,0))^2 + 2\alpha\beta\partial_x^2 u(t,0)\partial_x^2 u(t-h,0) + \beta^2(\partial_x^2 u(t-h,0))^2\Big].\end{aligned} \tag{5.39}$$

Similarly, because of (5.13), we have

$$\begin{aligned}\frac{d}{dt}V_2(t) =& 2h\int_0^1(1-\rho)\partial_x^2 u(t-\rho h,0)\frac{d}{dt}\partial_x^2 u(t-\rho h,0)d\rho \\ =& \partial_x^2 u(t,0)^2 - \int_0^1(\partial_x^2 u(t-\rho h,0))^2 d\rho.\end{aligned} \tag{5.40}$$

Consequently, (5.39) and (5.40) imply that for any $\lambda > 0$

$$\begin{aligned}\frac{d}{dt}V(t) + 2\lambda V(t) =& \Big(\alpha^2 - 1 + |\beta| + L\mu_1\alpha^2 + \mu_2\Big)(\partial_x^2 u(t,0))^2 \\ &+ \Big(\beta^2 - |\beta| + L\mu_1\beta^2\Big)(\partial_x^2 u(t-h,0))^2 \\ &+ 2\alpha\beta\Big(1 + L\mu_1\Big)\partial_x^2 u(t,0)\partial_x^2 u(t-h,0) + (2\lambda h|\beta| - \mu_2)\int_0^1(\partial_x^2 u(t-\rho h,0))^2 d\rho \\ &+ 2\lambda\mu_2 h\int_0^1(1-\rho)(\partial_x^2 u(t-\rho h,0))^2 d\rho + 2\lambda\mu_1\int_0^L xu^2(t,x)dx + \frac{\mu_1}{2}\int_0^L u^4(t,x)dx \\ &+ (\mu_1 a + 2\lambda)\int_0^L u^2(t,x)dx - 3b\mu_1\int_0^L(\partial_x u(t,x))^2dx - 5\mu_1\int_0^L(\partial_x^2 u(t,x))^2dx,\end{aligned}$$

or equivalently, by reorganizing the terms

$$\begin{aligned}\frac{d}{dt}V(t) + 2\lambda V(t) \le& \Big(M_{\mu_1}^{\mu_2}\ \eta(t),\eta(t)\Big)_{\mathbb{R}^2} - 3b\mu_1\int_0^L(\partial_x u(t,x))^2dx \\ &- 5\mu_1\int_0^L(\partial_x^2 u(t,x))^2dx \\ &+ \Big(2\lambda h(\mu_2 + |\beta|) - \mu_2\Big)\int_0^1(\partial_x^2 u(t-\rho h,0))^2 d\rho \\ &+ \Big(\mu_1 a + 2\lambda(1 + L\mu_1)\Big)\int_0^L u^2(t,x)dx + \frac{\mu_1}{2}\int_0^L u^4(t,x)dx,\end{aligned} \tag{5.41}$$

where $\eta(t) = (\partial_x^2 u(t,0), \partial_x^2 u(t-h,0))$ and

$$M_{\mu_1}^{\mu_2} = \begin{bmatrix}(1+L\mu_1)\alpha^2 - 1 + |\beta| + \mu_2 & \alpha\beta(1+L\mu_1) \\ \alpha\beta(1+L\mu_1) & \beta^2 - |\beta| + L\mu_1\beta^2\end{bmatrix}.$$

Observe that

$$M_{\mu_1}^{\mu_2} = M + L\mu_1\begin{bmatrix}\alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2\end{bmatrix} + \mu_2\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix},$$

where M is defined by (5.16). Since M is negative definite (see the proof of Proposition 5.3 and by the continuity of the determinant and the trace, one can claim that for μ_1 and $\mu_2 > 0$ small enough, the matric $M_{\mu_1}^{\mu_2}$ can also be made negative definite.

Finally, taking into account μ_1 and $\mu_2 > 0$ are small enough and using Poincaré inequality[2], we find

$$\begin{aligned}\frac{d}{dt}V(t) + 2\lambda V(t) \leq& \Big(2\lambda h(\mu_2+|\beta|) - \mu_2\Big)\int_0^1 (\partial_x^2 u(t-\rho h,0))^2 d\rho \\ &- 5\mu_1 \int_0^L (\partial_x^2 u(t,x))^2 dx + \frac{\mu_1}{2}\int_0^L u^4(t,x)dx \\ &+ \left(\frac{L^2}{\pi^2}(\mu_1 a + 2\lambda(1+L\mu_1)) - 3b\mu_1\right)\int_0^L (\partial_x^2 u(t,x))^2 dx.\end{aligned} \tag{5.42}$$

Additionally, applying Cauchy-Schwarz inequality and using the facts that the energy E defined by (5.5) is nonincreasing, together with $H_0^1(0,L) \hookrightarrow L^\infty(0,L)$, we have

$$\begin{aligned}\frac{\mu_1}{2}\int_0^L u^4(t,x)dx \leq& \frac{\mu_1}{2}\|u(t,\cdot)\|^2_{L^\infty(0,L)}\int_0^L u^2(t,x)dx \\ \leq& \frac{\mu_1}{2}L\|\partial_x u(t,\cdot)\|^2_{L^2(0,L)}\|u(t,x)\|^2_{L^2(0,L)} \\ \leq& \frac{L\mu_1}{2}\Big(\|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\Big)\|\partial_x u(t,\cdot)\|^2_{L^2(0,L)} \\ \leq& \frac{L\mu_1}{2}\|(u_0,z_0)\|^2_H\|\partial_x u(t,\cdot)\|^2_{L^2(0,L)} \leq r^2\frac{L\mu_1}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(0,L)}.\end{aligned} \tag{5.43}$$

Combining (5.42) and (5.43) yields

$$\frac{d}{dt}V(t)+2\lambda V(t) \leq \Xi\|\partial_x^2 u(t,x)\|^2_{L^2(0,L)} + \Big(2\lambda h(\mu_2+|\beta|)-\mu_2\Big)\|\partial_x^2 u(t-\rho h,0)\|^2_{L^2(0,1)}, \tag{5.44}$$

where

$$\Xi = \frac{L\mu_1}{2}r^2 + \frac{L^2}{\pi^2}\Big(\mu_1 a + 2\lambda(1+L\mu_1)\Big) - 3b\mu_1.$$

Here, we see the importance of the weights μ_1, μ_2 again. In fact, in order to make $\Xi \leq 0$ and $2\lambda h(\mu_2+|\beta|) - \mu_2 \leq 0$, we need to take

$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2+|\beta|)}, \frac{3b\pi^2 - r^2L\pi^2 - L^2a}{2L^2(1+L\mu_1)}\mu_1\right\}.$$

So, from $\Xi \leq 0$, as L, λ, a and b are supposed fixed, a straight computation leads us to the following inequality

$$|r| \leq \frac{1}{\sqrt{\mu_1}}\frac{\sqrt{2}}{\pi}\sqrt{\frac{3b\pi^2 - L^2a}{L}}. \tag{5.45}$$

In other words, (5.45) represents the relation between r and μ_1. Hence, fixed $\mu_1 \in (0,1)$ sufficient small, in view of the constraint (5.7) on the length L, we must choose r such that

$$0 < r < \frac{\sqrt{2}}{\pi}\sqrt{\frac{3b\pi^2 - L^2a}{L}}.$$

[2] $\|u\|^2_{L^2(0,L)} \leq \frac{L^2}{\pi^2}\|\partial_x u\|_{L^2(0,L)}$ for $u \in H_0^2(0,L)$.

Then, we pick $\lambda > 0$ such that (5.9) holds to ensure that

$$\frac{d}{dt}V(t) + 2\lambda V(t) \leq 0, \tag{5.46}$$

for all $t > 0$. Therefore, integrating (5.46) over $(0, t)$, and thanks to (5.37) and (5.38), yields that

$$E(t) \leq \left(1 + \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\}\right) E(0)e^{-2\lambda t}, \tag{5.47}$$

for all $t > 0$, which completes the proof. □

5.4 Second stability result *via* compactness-uniqueness argument

The second part of this chapter is devoted to the proof of another stability result of (5.2)-(5.3) stated in Theorem 5.2. To be more precise, we shall show a generic exponential stability result of the solutions to (5.2)-(5.3) by attempting to study the phenomenon of critical lengths of the system.

5.4.1 Stability of the linear system

We first prove that the following observability inequality ensures that the linear system (5.12) is exponentially stable.

Proposition 5.10. *Assume that α and β satisfies (5.4) and $L > 0$. Thus, there exists a constant $C > 0$, such that for all $(u_0, z_0) \in H$*

$$\int_0^L u_0^2(x)dx + |\beta|h \int_0^1 z_0^2(-h\rho)d\rho \,|\leq C \int_0^T \left((\partial_x^2 u(0,t))^2 + z^2(1,t)\right) dt \tag{5.48}$$

where $(u, z) = S(.)\,(u_0, z_0(-h\cdot))$ is the solution of the system (5.12)-(5.13).

Indeed, if (5.48) is true, we get

$$E(T) - E(0) \leq -\frac{E(0)}{C} \Rightarrow E(T) \leq E(0) - \frac{E(0)}{C} \leq E(0) - \frac{E(T)}{C},$$

where $E(t)$ is defined by (5.5). Thus,

$$E(T) \leq \gamma E(0), \quad \text{where} \quad \gamma = \frac{C}{1+C} < 1. \tag{5.49}$$

Now, the same argument used on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots$, yields that

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0).$$

Thus, we have

$$E(mT) \leq e^{-\nu mT}E(0) \quad \text{with} \quad \nu = \frac{1}{T}\ln\left(1 + \frac{1}{C}\right) > 0.$$

For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0),$$

showing the exponential stability result for the linear system.

For sake of clarity, the proof of Proposition 5.10 will be achieved in steps. Moreover, for sake of simplicity, we will take $a = b = 1$.

Step 1: Compactness-uniqueness argument

We argue by contradiction. Suppose that (5.48) does not hold and hence there exists a sequence $((u_0^n, z_0^n(-h\cdot)))_n \subset H$ such that

$$\int_0^L (u_0^n)^2(x)dx + |\beta| h \int_0^1 (z_0^n)^2(-h\rho)d\rho = 1 \tag{5.50}$$

and

$$\left\|\partial_x^2 u^n(0,.)\right\|_{L^2(0,T)}^2 + \|z^n(1,.)\|_{L^2(0,T)}^2 \to 0 \text{ as } n \to +\infty, \tag{5.51}$$

where $(u^n, z^n) = S(u_0^n, z_0^n(-h\cdot))$.

Owing to Proposition 5.3, $(u^n)_n$ is a bounded sequence in $L^2(0,T,H^2(0,L))$, and consequently

$$\partial_t u^n = -\partial_x u^n - \partial_x^3 u^n + \partial_x^5 u \quad \text{is bounded in} \quad L^2\left(0,T,H^{-3}(0,L)\right).$$

Thanks to a result of [87], $(u^n)_n$ is relatively compact in $L^2(0,T,L^2(0,L))$ and we may assume that $(u^n)_n$ is convergent in $L^2(0,T,L^2(0,L))$. Moreover, using (5.20) and (5.51), we have that $(u_0^n)_n$ is a Cauchy sequence in $L^2(0,L)$.

Claim 1. *If $T > h$, then $(z_0^n(-h\cdot))_n$ is a Cauchy sequence in $L^2(0,1)$.*

In fact, since $z^n(\rho, T) = u_{xx}^n(0, T-\rho h)$, if $T > h$, we have

$$\int_0^1 (z^n(\rho,T))^2 d\rho = \int_0^1 \left(\partial_x^2 u^n(0,T-\rho h)\right)^2 d\rho \leq \frac{1}{h}\int_0^T \left(\partial_x^2 u^n(0,t)\right)^2 dt.$$

Using (5.21), for $T > h$ yields that

$$\|z_0^n(-h\cdot)\|_{L^2(0,1)}^2 \leq \frac{1}{h}\left\|\partial_x^2 u^n(0,\cdot)\right\|_{L^2(0,T)}^2 + \frac{1}{h}\|z^n(1,\cdot)\|_{L^2(0,T)}^2.$$

Thus, $(z_0^n(-h\cdot))_n$ is a Cauchy sequence in $L^2(0,1)$ by means of (5.51) and hence the Claim 1 is ascertained.

Now, let us pick $(u_0, z_0(-h\cdot)) = \lim_{n\to\infty}(u_0^n, z_0^n(-h\cdot))$ in H. This, together with (5.50), yields that

$$\int_0^L u_0^2(x)dx + |\beta| h \int_0^1 z_0^2(-h\rho)d\rho = 1.$$

Furthermore, let $(u, z) = S(\cdot)\,(u_0, z_0(-h\cdot))$, which implies, thanks to Proposition 5.3, that

$$\left(\partial_x^2 u(0,\cdot), z(1,\cdot)\right) = \lim_{n\to\infty} \left(\partial_x^2 u^n(0,\cdot), z^n(1,\cdot)\right)$$

in $L^2(0,T)$. Combining the latter with (5.51) gives $(\partial_x^2 u(0,\cdot), z(1,.)) = 0$. As we have $z(1,t) = \partial_x^2 u(0,t-h) = 0$, we deduce that $z_0 = 0$ and $z = 0$. Consequently, taking, for sake of simplicity, $a = b = 1, u$ is solution of

$$\begin{cases} \partial_t u - \partial_x u + \partial_x^3 u - \partial_x^5 u = 0, & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = \partial_x u(L,t) = \partial_x u(0,t) = \partial_x^2 u(L,t) = \partial_x^2 u(0,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & x \in (0,L) \end{cases} \tag{5.52}$$

with

$$\|u_0\|_{L^2(0,L)} = 1. \tag{5.53}$$

Step 2: Reduction to a spectral problem

Lemma 5.11. *For any $T > 0$, let N_T denote the space of the initial state $u_0 \in L^2(0,L)$, such that the solution of the Kawahara system $u(t) = S(t)u_0$ satisfies (5.52). Then, $N_T = \{0\}$.*

Proof. We argue as in [83, Theorem 3.7]. If $N_T \neq \{0\}$, then the map $u_0 \in \mathbb{C}N_T \to A\,(N_T) \subset \mathbb{C}N_T$ ($\mathbb{C}N_T$ denotes the complexification of N_T) has (at least) one eigenvalue. Hence, there exists a pair $(\lambda, u_0) \in \mathbb{C} \times H^5(0,L)\backslash\{0\}$ such that

$$\begin{cases} \lambda u_0 + u_0' + u_0''' - u_0''''' = 0, & \text{in } (0,L), \\ u_0(0) = u_0(L) = u_0'(0) = u_0'(L) = u_0''(0) = u_0''(L) = 0. \end{cases}$$

To obtain the contradiction, it remains to prove that such a pair (λ, u_0) does not exist. This will be done in the next step. □

Step 3: Möbius transformation

To simplify the notation, henceforth we denote $u_0 := u$. Moreover, the notation $\{0, L\}$ means that the function is applied to 0 and L, respectively. The argument developed here is based on [26].

Lemma 5.12. *Let $L > 0$ and consider the assertion*

$$(\mathcal{N}):\quad \exists\lambda \in \mathbb{C}, \exists u \in H_0^2(0,L)\cap H^5(0,L) \text{ such that } \begin{cases} \lambda u + u' + u''' - u''''' = 0, & \text{on } (0,L), \\ u(x) = u'(x) = u''(x) = 0, & \text{in } \{0,L\}. \end{cases}$$

If $(\lambda, u) \in \mathbb{C} \times H_0^2(0,L) \cap H^5(0,L)$ is solution of $(\mathcal{N})$, then $u = 0$.

Proof. Consider the following system

$$\begin{cases} \lambda u + u' + u''' - u''''' = 0, & \text{on } (0, L), \\ u(x) = u'(x) = u''(x) = 0, & \text{in } \{0, L\}. \end{cases} \tag{5.54}$$

Multiplying the equation (5.54) by $\overline{u}$ and integrating in $[0, L]$, we have that λ is purely imaginary, i.e., $\lambda = ir$, for $r \in \mathbb{R}$. Now, extending the function u to $\mathbb{R}$ by setting $u = 0$ for $x \notin [0, L]$, we have that the extended function satisfies

$$\lambda u + u' + u''' - u''''' = -u''''(0)\delta_0' + u''''(L)\delta_L' - u'''(0)\delta_0 + u'''(L)\delta_L,$$

in $\mathcal{S}'(\mathbb{R})$, where δ_ζ denotes the Dirac measure at $x = \zeta$ and the derivatives $u''''(0)$, $u''''(L)$, $u'''(0)$ and $u'''(L)$ are those of the function u when restricted to $[0, L]$. Taking the Fourier transform of each term in the above system and integrating by parts, we obtain

$$\lambda\hat{u}(\xi) + i\xi\hat{u}(\xi) + (i\xi)^3\hat{u}(\xi) - (i\xi)^5\hat{u}(\xi) = -(i\xi)u'''(0) + (i\xi)u'''(L)e^{-iL\xi} - u''''(0) + u''''(L)e^{-iL\xi}.$$

Take $\lambda = -ir$ and let $f_\alpha(\xi, L) = i\hat{u}(\xi)$. The latter gives

$$f_\alpha(\xi, L) = \frac{N_\alpha(\xi, L)}{q(\xi)},$$

where $N_\alpha(\cdot, L)$ is defined by

$$N_\alpha(\xi, L) = \alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L} \tag{5.55}$$

and

$$q(\xi) = \xi^5 + \xi^3 - \xi + r,$$

where α_j, for $j = 1, 2, 3, 4$, are the traces of u''' and u''''.

For each $r \in \mathbb{R}$ and $\alpha \in \mathbb{C}^4 \setminus \{0\}$, let $\mathcal{F}_{\alpha r}$ be the set of $L > 0$ values, for which the function $f_\alpha(\cdot, L)$ is entire. Now, let us recall the equivalent following statements:

- A1. $f_\alpha(\cdot, L)$ is entire;
- A2. all zeros, taking the respective multiplicities into account, of the polynomial q are zeros of $N_\alpha(\cdot, L)$;
- A3. the maximal domain of $f_\alpha(\cdot, L)$ is $\mathbb{C}$.

Whereupon, the function $f_\alpha(\cdot, L)$ is entire, due to the equivalence between statement A1 and A2, if the following holds

$$\frac{\alpha_1 i\xi_j + \alpha_3}{\alpha_2 i\xi_j + \alpha_4} = e^{-iL\xi_j},$$

where ξ_i denotes the zeros of $q(\xi)$, for $j = 1, 2, 3, 4, 5$. Thereafter, let us define, for $\alpha \in \mathbb{C}^4 \setminus \{0\}$, the following discriminant

$$d(\alpha) = \alpha_1\alpha_3 - \alpha_2\alpha_4. \tag{5.56}$$

Then, for $\alpha \in \mathbb{C}^4 \setminus \{0\}$, such that $d(\alpha) \neq 0$ the Möbius transformations can be introduced by

$$M(\xi_i) = e^{-iL\xi_i}, \tag{5.57}$$

for each zero ξ_j of the polynomial $q(\xi)$.

The next claim describes the behavior of the roots of polynomial $q(\cdot)$:

Claim 2. *The polynomial $q(\cdot)$ has exactly one real root with multiplicity 1 and two pairs of complex conjugate roots.*

Indeed, we suppose that $r \neq 0$ (the case $r = 0$ will be discussed later). Note that the derivative of q is given by

$$q'(\xi) = 5\xi^4 + 3\xi^2 - 1,$$

and its zeros are $\pm z_1$ and $\pm z_2$, where

$$z_1 = \sqrt{\frac{-3 - \sqrt{29}}{10}} \quad \text{and} \quad z_2 = \sqrt{\frac{-3 + \sqrt{29}}{10}}.$$

It is easy to see that z_1 belongs to $\mathbb{C} \setminus \mathbb{R}$ and z_2 belongs to $\mathbb{R}$. Hence, the polynomial $q(\cdot)$ does not have critical points, which means that $q(\cdot)$ has exactly one real root. Suppose that $\xi_0 \in \mathbb{R}$ is the root of $q(\cdot)$ with multiplicity $m \leq 5$. Consequently,

$$q(\xi_0) = q'(\xi_0) = ... = q^{(m-1)}(\xi_0) = 0.$$

Consider the following cases:

(i) If ξ_0 has multiplicity 5, it follows that $q(\xi_0) = 0$ and $q''''(\xi_0) = -120\xi_0 = 0$, implying that $\xi_0 = 0$ and $r = 0$.

(ii) If ξ_0 has multiplicity 4, it follows that $q'''(\xi_0) = 60\xi_0^2 + 6 = 0$ and thus $\xi_0 \in i\mathbb{R}$.

(iii) If ξ_0 has multiplicity 3, it follows that $q(\xi_0) = 0$ and $q''(\xi_0) = 20\xi_0^3 + 6\xi_0 = 0$ and hence $\xi_0 = 0$ and $r = 0$ or $\xi_0 \in i\mathbb{R}$.

(iv) If ξ_0 has multiplicity 2, it follows that $q'(\xi_0) = 5\xi_0^4 + 3\xi_2 - 1 = 0$, implying that $\xi_0 \in \mathbb{C} \setminus \mathbb{R}$.

Note that in all cases, since $r \neq 0$ and $\xi_0 \in \mathbb{R}$, we get a contradiction. Consequently, $q(\cdot)$ has exactly one real root, with multiplicity 1. This means that this polynomial has two pairs of complex conjugate roots.

Now, we assume that $r = 0$. Then, we obtain that $q(\xi) = \xi(\xi^4 + \xi^2 - 1)$, whose roots are $0, \pm\rho$ and $\pm k$ where

$$\rho = \sqrt{\frac{\sqrt{5}-1}{2}} \quad \text{and} \quad k = i\sqrt{\frac{1+\sqrt{5}}{2}} \tag{5.58}$$

Thus, $q(\cdot)$ has two pairs of complex conjugate roots and three real roots, proving Claim 2.

Further to Claim 2, and to conclude the proof of Lemma 5.12, we need two additional lemmas whose proofs are given in [46, Lemmas 2.1 and 2.2].

Lemma 5.13. *Let non null $\alpha \in \mathbb{C}^4$ with $d(\alpha) = 0$ and $L > 0$ for $d(\alpha)$ defined in (5.56). Then, the set of imaginary parts of the zeros of $N_\alpha(\cdot, L)$ in (5.55) has at most two elements.*

Lemma 5.14. *For any $L > 0$, there is no Möbius transformation M, such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2\},$$

with $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$ all distinct in $\mathbb{C}$.

We are now in a position to prove the Lemma 5.12. Let us consider two cases:

i. $d(\alpha) \neq 0$;

ii. $d(\alpha) = 0$,

where $d(\alpha)$ was defined in (5.56).

First, supposing that $d(\alpha) \neq 0$, we can define the Möbius transformation. Suppose by contradiction that there exists $L > 0$ such that the function $f_a(\cdot, L)$ is entire. Then, all roots of the polynomial $q(\cdot)$ must satisfy (5.57), i.e., there exists a Möbius transformation that takes each root ξ_0 of $q(\cdot)$ into $e^{-iL\xi_0}$. However, this contradicts Lemma 5.14 and proves that if $(\mathcal{N})$ holds then $\mathcal{F}_{\alpha r} = \emptyset$ for all $r \in \mathbb{R}$. On the other hand, suppose that $d(\alpha) = 0$ and note that by using claim 2, we can conclude that the set of the imaginary parts of the polynomial $q(\cdot)$ has at least three elements, thus it follows from Lemma 5.13 that $\mathcal{F}_{\alpha r} = \emptyset$ for all $r \in \mathbb{R}$. Note that in both cases, we have that $\mathcal{F}_{\alpha r} = \emptyset$, which implies that $(\mathcal{N})$ only has the trivial solution for any $L > 0$, and the proof of Lemma 5.12 is archived. □

Proof of Proposition 5.10. Notice that (5.53) implies that the solution u can not be identically zero. However, from Lemma 5.11, one can conclude that $u = 0$, which drives us to a contradiction. □

5.4.2 Proof of Theorem 5.2

Let us consider the nonlinear Kawahara system (5.2)-(5.3), with a small initial data $\|(u_0, z_0)\|_H \leq r$, where r will be chosen later. The solution u of (5.2)-(5.3), with $p = 2$, can be written as $u = u_1 + u_2$, where u_1 is the solution of

$$\begin{cases} \partial_t u_1 - \partial_x^5 u_1 + b\partial_x^3 u_1 + a\partial_x u_1 = 0, & x \in (0, L), t > 0, \\ u_1(0, t) = u_1(L, t) = \partial_x u_1(0, t) = \partial_x u_1(L, t) = 0, & t > 0, \\ \partial_x^2 u_1(L, t) = \alpha \partial_x^2 u_1(0, t) + \beta \partial_x^2 u_1(0, t - h), & t > 0, \\ \partial_x^2 u_1(0, t) = z_0(t), & t \in (-h, 0), \\ u_1(x, 0) = u_0(x), & x \in (0, L), \end{cases}$$

and u_2 is solution of

$$\begin{cases} \partial_t u_2 - \partial_x^5 u_2 + b\partial_x^3 u_2 + a\partial_x u_2 = -u^2 \partial_x u, & x \in (0, L), t > 0, \\ u_2(0, t) = u_2(L, t) = \partial_x u_2(0, t) = \partial_x u_2(L, t) = 0, & t > 0, \\ \partial_x^2 u_2(L, t) = \alpha \partial_x^2 u_2(0, t) + \beta \partial_x^2 u_2(0, t - h), & t \in (-h, 0), \\ \partial_x^2 u_2(0, t) = 0, & x \in (0, L), \\ u_2(x, 0) = 0, & x \in (0, L), \end{cases}$$

Note that, in this case, u_1 is the solution of (5.12)-(5.13) with the initial data $(u_0, z_0) \in H$ and u_2 is solution of (5.30) with null data and right-hand side $f = u^2 \partial_x u \in L^1(0, T; L^2(0, L))$, as in Lemma 5.8.

Now, thanks to (5.49), Proposition 5.7 and Lemma 5.8, we have that

$$\begin{aligned} \|(u(T), z(T))\|_H \leq & \left\| \left(u^1(T), z^1(T)\right) \right\|_H + \left\| \left(u^2(T), z^2(T)\right) \right\|_H \\ \leq & \gamma \left\|(u_0, z_0(-h\cdot))\right\|_H + C \left\|u^p u_x\right\|_{L^1(0,T,L^2(0,L))} \\ \leq & \gamma \left\|(u_0, z_0(-h\cdot))\right\|_H + C\|u\|^2_{L^2(0,T,H^2(0,L))}, \end{aligned} \tag{5.59}$$

with $\gamma \in (0, 1)$. The goal now is to deal with the last term of the previous inequality. To this end, we use the multipliers method. First, we multiply the first equation of (5.2)-(5.3) by xu and integrate by parts to obtain

$$\begin{aligned} & \frac{1}{2} \int_0^L x|u(x, T)|^2 dx + \frac{3b}{2} \int_0^T \int_0^L |\partial_x u(x, t)|^2 \, dxdt + \frac{5}{2} \int_0^T \int_0^L \left|\partial_x^2 u(x, t)\right|^2 dxdt \\ = & \frac{1}{a} \int_0^T \int_0^L |u(x, t)|^2 dxdt + \frac{L}{2} \int_0^T (\partial_x^2 u(L, t))^2 dt + \frac{1}{2} \int_0^L x \left|u_0(x)\right|^2 dx + \frac{1}{4} \int_0^T \int_0^L |u|^4 \, dxdt. \end{aligned}$$

Consequently, using the boundary condition of (5.2)-(5.3) and (5.5), we get

$$\begin{aligned} 3b \int_0^T \int_0^L |\partial_x u(x, t)|^2 \, dxdt + 5 \int_0^T \int_0^L \left|\partial_x^2 u(x, t)\right|^2 dxdt \leq & (aT + L) \left\|(u_0, z_0)\right\|_H^2 \\ & + L \int_0^T (\alpha \partial_x^2 u(0, t) + \beta z(1, t))^2 dt + \frac{1}{2} \int_0^T \int_0^L |u|^4 \, dxdt. \end{aligned}$$

Note that Gagliardo–Nirenberg inequality ensures that

$$\begin{aligned}\int_0^T\int_0^L u^4 dxdt \leq & C\int_0^T \|u\|^3_{L^2(0,L)}\|u_x\|_{L^2(0,L)}\,dt\\ \leq & C\frac{1}{2\varepsilon}\int_0^T \|u\|^6_{L^2(0,L)}dt + C\frac{\varepsilon}{2}\int_0^T\|u_x\|^2_{L^2(0,L)}\,dt\\ \leq & C(T)\frac{1}{2\varepsilon}\|u\|^6_{L^\infty(0,T;L^2(0,L))}+C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}\\ \leq & C(T)\frac{1}{2\varepsilon}\|(u_0,z_0)\|^6_H + C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}.\end{aligned}$$

Putting together the previous inequalities, we have

$$\begin{aligned}&3b\int_0^T\int_0^L |\partial_x u(x,t)|^2\,dxdt + 5\int_0^T\int_0^L\left|\partial_x^2 u(x,t)\right|^2 dxdt \leq (aT+L)\,\|(u_0,z_0)\|^2_H\\ &+L\int_0^T(\alpha\partial_x^2u(0,t)+\beta z(1,t))^2dt + C(T)\frac{1}{2\varepsilon}\|(u_0,z_0)\|^6_H + C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}.\end{aligned} \tag{5.60}$$

Now, multiplying the first equation of (5.2) by u and integrating by parts yields that

$$\int_0^L u^2(x,T)dx - \int_0^L u_0^2(x)dx - \int_0^T (\alpha u_{xx}(0,t)+\beta z(1,t))^2\,dt + \int_0^T u_x^2(0,t)dt = 0$$

Using the same idea as in the proof of (5.19), we have that

$$\int_0^T(\partial_x^2u)^2(0,t)dt + \int_0^T z^2(1,t)dt \leq C\,\|(u_0,z_0)\|^2_H\,.$$

Consequently, the previous inequality gives

$$\int_0^T\left(\alpha\partial_x^2u(0,t)+\beta z(1,t)\right)^2 dt \leq 2C\left(\alpha^2+\beta^2\right)\|(u_0,z_0)\|^2_H\,.$$

Thus, putting the previous inequality in (5.60), and choosing $\varepsilon > 0$ sufficiently small, there exists $C > 0$ such that

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leq C\left(\|(u_0,z_0)\|^2_H + \|(u_0,z_0)\|^6_H\right). \tag{5.61}$$

Finally, gathering (5.59) and (5.61), there exists $C > 0$ such that the following holds true

$$\|(u(T),z(T))\|_H \leq \|(u_0,z_0)\|_H\left(\gamma + C\,\|(u_0,z_0)\|_H + C\,\|(u_0,z_0)\|^5_H\right),$$

which implies

$$\|(u(T),z(T))\|_H \leq \|(u_0,z_0)\|_H\left(\gamma + Cr + Cr^5\right).$$

Given $\epsilon > 0$ small enough such that $\gamma+\epsilon < 1$, we can take r small enough such that $r + r^5 < \frac{\epsilon}{C}$, in order to have

$$\|(u(T),z(T))\|_H \leq (\gamma+\epsilon)\,\|(u_0,z_0)\|_H\,,$$

with $\gamma + \epsilon < 1$. Theorem 5.2 follows using the semigroup property as in (5.49). □

5.5 Further comments

Our work presents a further step after the work [3] for a better understanding of the stabilization problem for the Kawahara equation. Indeed, a boundary time-delayed damping control is proposed to stabilize the equation in contrast to [3], where an interior damping is required and no delay is taken into consideration. We conclude our introduction with a few comments and also some open problems.

Remark 5.15. In what concerns our main results, Theorems 5.1 and 5.2, the following remarks are worth mentioning:

- Note that the rate λ of the Theorem 5.1 decreases as the delay h increases, since we have the restriction (5.9).

- A simple calculation shows that taking $\mu_1, \mu_2 \in (0,1)$ in Theorem 5.1 such that
$$\mu_2 < \min\left\{1-|\beta|-\alpha^2, \frac{(|\beta|-1)^2-\alpha^2}{1-|\beta|}, \frac{\alpha^2-\beta^2+|\beta|}{|\beta|}\right\}$$
and
$$\mu_1 < \min\left\{\frac{1-|\beta|-\mu_2-\alpha^2}{L\alpha^2}, \frac{(|\beta|-1)^2-\alpha^2-\mu_2(1-|\beta|)}{L(\alpha^2-\beta^2+|\beta|(1-\mu_2))}\right\}$$
implies that $M_{\mu_1}^{\mu_2}$ is negative definite.

- Note that the presence of the nonlinearity on the equation yields the restriction about the initial data. Hence, if we remove it, that is, by considering the linear system, it is possible to obtain the same result of the Theorem 5.1, with the same process. Nevertheless, the decay rate λ is given by
$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2+|\beta|)}, \frac{3b\pi^2-L^2a}{2L^2(1+L\mu_1)}\right\}. \tag{5.62}$$

- For sake of simplicity, we only considered the nonlinearity u^2u_x. However, Theorems 5.1 and 5.2 are still valid for u^pu_x, $p \in [1,2)$, where the proof is very similar and hence omitted.

- Recently, Zhou [104] proved the well-posedness of the following initial boundary value problem
$$\begin{cases} \partial_t u - \partial_x^5 u = c_1 u\partial_x u u + c_2 u^2 \partial_x u + b_1 \partial_x u \partial_x^2 u + b_2 u \partial_x^3 u, & x \in (0,L),\ t \in \mathbb{R}^+, \\ u(t,0) = h_1(t), \quad u(t,L) = h_2(t), \quad \partial_x u(t,0) = h_3(t), & t \in \mathbb{R}^+, \\ \partial_x u(t,L) = h_4(t), \quad \partial_x^2 u(t,L) = h(t), & t \in \mathbb{R}^+, \\ u(0,x) = u_0(x), & x \in (0,L), \end{cases} \tag{5.63}$$
Thus, due to this result, when we consider $b_1 = b_2 = 0$ and the combination $c_1 u\partial_x u + c_2 u^2 \partial_x u$ instead of $u^p \partial_x u$, for $p \in [1,2]$, in (5.2), the main results of our chapter remains valid.

- We point out that considering $a = 0$ in (5.2), Theorem 5.1 holds true. Additionally, no restriction is necessary for the length $L > 0$, and also Theorem 5.2 is still verified (see, for instance, [23, 95]).

6 Conclusion and Perspectives

6.1 Conclusion

The initial chapters of this thesis (Chapters 1 and 2) present motivations, concepts, and results obtained by the author to guide the reader to understand the main purpose of this work.

The agenda of the research of control theory for the fifth-order KdV equation is quite new and does not acknowledge many results in the literature. To fill this gap, we presented a new way to prove internal and boundary control results for this system. Precisely, in Chapter 3, we treated overdetermination control problems. The first result is concerning of the *boundary overdetermination control problem*, roughly speaking, we can find an appropriate control h, acting on the boundary term $u_{xx}(t, L)$, such that integral condition holds (see Theorem (3.1)). This result is first proved for the linear system associated and after that, using a fixed point argument extended to the nonlinear system. Theorem 3.2 follows the same idea, the main difference is related to the appropriated applications which in this case links the internal control f_0 with the overdetermination condition. Trying to extend the result of Chapter 3 to an unbounded domain, Chapter 4 shows that possibility. Precisely, the overdetermination condition is satisfied when the domain of the Kawahara equation is the real line, left half-line, and right half-line. Furthermore, we showed a type of exact control associated with the "mass" of the Kawahara equation over the right half-line, which is completely new for this type of equation.

Finally, trying to understand the stability for KdV-type systems, Chapter 5 deals with the Kawahara system in the presence of the boundary time-delayed control $\mathcal{F}(t)$ defined by (5.3). We proved in this chapter that the Kawahara system is exponentially stable using two different approaches. First, by choosing an appropriate Lyapunov functional, with some restrictions on the spatial length L and the initial data, the energy associated with the Kawahara system decays exponentially. The key idea of this analysis is the relation between the linearized system and a transport equation. Additionally of that, by combining multipliers and compactness arguments, which reduce the problem to show a unique continuation result for the state operator, we are also able to prove that the Kawahara system is exponentially stable without restriction in the initial data, but with some restriction on the spatial length L.

6.2 Perspectives

Based on the work developed in this thesis, some questions arise naturally. We will therefore describe in this section some problems that we intend to work on in the future.

6.2.1 A new controllability result

In Chapter 4, thanks to the Corollary 4.3, it is possible to obtain an exact controllability property related to the mass of the system. However, we would like to show the following new exact controllability result:

Exact control problem: *Given $u_0, u_T \in L^2(\mathbb{R}^+)$ and $g \in C(0,T;L^2(\mathbb{R}^+))$ satisfying (4.8), can we find a control $f_0 \in L^p(0,T)$ such that the solution u of (4.6) satisfies $u(T,x) = u_T(x)$?*

A possibility to answer this question is to modify the overdetermination condition (4.2). For example, if Theorem 4.1 is verified for the following integral condition

$$\widetilde{\varphi}(t) = \int_{\mathbb{R}^+} u^2(t,x)w(x)dx, \tag{6.1}$$

we can get the exact controllability in $L^2(\mathbb{R}^+)$ with internal control $f_0 \in L^2(0,T)$ by using the same argument as in Corollary 4.3. However, with the approach used here, it is not clear that the Lemma 4.10 can be replicated for the condition (6.1).

Indeed, if we consider

$$q(t) = \int_{\mathbb{R}^+} u^2(t,x)w(x)dx,$$

analyzing $q'(t)$ for $u = S(0,0,0,f_0(t)g(t,x))$ (see Lemma 4.17) we obtain

$$\begin{aligned} q'(t) = &\int_{\mathbb{R}^+} u^2(t,x)\left[\alpha w'(x) + \beta w''(x) - 2w'''''(x)\right]dx \\ &+ \int_{\mathbb{R}^+} u_x^2(t,x)\left[5w'''(x) - 3\beta w'(x) - 2w'''''(x)\right]dx \\ &- 5\int_{\mathbb{R}^+} u_{xx}^2(t,x)w'(x)dx + f_0(t)\int_{\mathbb{R}^+} g(t,x)u(t,x)w(x)dx. \end{aligned}$$

Now, introduce the operator

$$\widetilde{A} : L^p(0,T) \longrightarrow L^p(0,T)$$

defined by

$$f_0 \longmapsto \widetilde{A}(f_0) \in L^p(0,T),$$

where

$$\begin{aligned} (\widetilde{A}f_0)(t) =& \varphi'(t) - \int_{\mathbb{R}^+} u^2(t,x)\left[\alpha w'(x) + \beta w''(x) - 2w'''''(x)\right]dx \\ &- \int_{\mathbb{R}^+} u_x^2(t,x)\left[5w'''(x) - 3\beta w'(x) - 2w'''''(x)\right]dx + 5\int_{\mathbb{R}^+} u_{xx}^2(t,x)w'(x)dx. \end{aligned}$$

If we assume that $\Lambda(f_0) = \widetilde{\varphi}$, we deduce that

$$(\widetilde{A}f_0)(t) = f_0(t)\int_{\mathbb{R}^+} g(t,x)u(t,x)w(x)dx.$$

Note that this expression depends on the solution of the system (4.6), then we are not able to obtain the overdetermination control condition for $S(0,0,0,f_0(t)g(t,x))$ by using a fixed point argument for the operator

$$\left[\int_{\mathbb{R}^+} g(t,x)u(t,x)w(x)dx\right]^{-1}(\widetilde{A}f_0)(t),$$

as in the proof of Lemma 4.10. Therefore, the exact controllability with internal control does not hold. Hence, the following open question arises:

Question $\mathcal{A}$: *Is it possible to prove Theorem 4.1 for the overdetermination condition* (6.1)*?*

6.2.2 Restriction of the Lyapunov approach

From Chapter 5, specifically, Theorem 5.1, since the result is based on the appropriate choice of Lyapunov functional, we have the restriction (5.7) on the length L. This is due to the choice of the Morawetz multipliers x in the expression of V_1 defined by (5.36). Therefore, the following natural question arises.

Question $\mathcal{B}$: Can we choose another Lyapunov functional, instead of the previous one to remove the restriction over L?

6.2.2.1 Set of critical lengths

As observed in [3], considering the following initial boundary value problem for Kawahara equation

$$\begin{cases} u_t - u_x + u_{xxx} - u_{xxxxx} = 0, & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = u_x(L,t) = u_x(0,t) = u_{xx}(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{6.2}$$

it is possible to construct a nontrivial steady-state solution to (6.2) with a non-zero initial datum $u_0(x) \not\equiv 0$ and homogeneous boundary conditions upon the endpoints of the interval with a critical length. Precisely, when the authors considered the following constants

$$a = \sqrt{\sqrt{5}+1/2}, \quad b = \sqrt{\sqrt{5}-1/2}, \quad A = C_2 + C_3, \quad B = C_2 - C_3$$

$$C_2 = 1 - e^{-aL}, \quad C_3 = e^{aL} - 1, \quad C_1 = -\left(1 + \frac{a^2}{b^2}\right)A, \quad C_4 = \frac{a^2}{b^2}A, \quad C_5 = -\frac{a}{b}B,$$

they were able to define the set

$$\mathcal{N} = \left\{ L > 0 : e^{ibL} = \left(\frac{C_4 + iC_5}{|C_4 + iC_5|}\right)^2 \right\} \subset \mathbb{R}^+$$

and
$$u(x) = C_1 + C_2 e^{ax} + C_3 e^{-ax} + C_4 \cos(bx) + C_5 \sin(bx) \not\equiv 0, \quad x \in (0, L).$$
If $L \in \mathcal{N}$, then $u = u(x)$ solves $-u''''' + u''' + u' = 0$, and satisfies $u(0) = u'(0) = u''(0) = u(L) = u'(L) = u''(L) = 0$.

So, in our context, if we consider a function $N_\alpha : \mathbb{C}\times(0,\infty) \to \mathbb{C}$, with $\alpha \in \mathbb{C}^4\setminus\{0\}$, whose restriction $N_\alpha(\cdot, L)$, given by (5.10), is entire for each $L > 0$ and a family of functions $f_\alpha(\cdot, L)$, defined by (5.11), in its maximal domain, the following issue appears.

Question $\mathcal{C}$: Is it possible to find $a \in \mathbb{C}^4 \setminus \{0\}$ such that the function $f_a(\cdot, L)$ is an entire function?

Note that the proof of our result (see Theorem 5.2) heavily relies on a unique continuation property of the spectral problem associated with the space operator (see Lemma 5.12). However, due to the structure of the terms ∂_x^3 and ∂_x^5 (see again Lemma 5.12), we are unable to study the spectral problem in a direct way as in [83]. Hence, due to these two different dispersions of the third and fifth order, we believe that a new approach is needed to tackle the previous open question.

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