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Two stability results for the Kawahara equation with a time-delayed boundary control

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Abstract. In this paper, we consider the Kawahara equation in a bounded interval and with a delay term in one of the boundary conditions. Using two different approaches, we prove that this system is exponentially stable under a condition on the length of the spatial domain. Specifically, the first result is obtained by introducing a suitable energy functional and using Lyapunov's approach, to ensure that the energy of the Kawahara system goes to 0 exponentially as $t \to \infty$. The second result is achieved by employing a compactness–uniqueness argument, which reduces our study to prove an observability inequality. Furthermore, the novelty of this work is to characterize the critical lengths phenomenon for this equation by showing that the stability results hold whenever the spatial length is related to the Möbius transformations.

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Keywords. Nonlinear Kawahara equation, Boundary time-delay, exponential stability, Critical set.

1. Introduction

1.1. Physical motivation and goal

It is well-known that the following fifth-order nonlinear dispersive equation

$$\pm 2\partial_t u + 3u\partial_x u - \nu\partial_x^3 u + \frac{1}{45}\partial_x^5 u = 0, \tag{1.1}$$

models numerous physical phenomena. In fact, considering suitable assumptions on the amplitude, wavelength, wave steepness and so on, the properties of the asymptotic models for water waves have been extensively studied in the last years, through (1.1), to understand the full water wave system [1].

In some situations, we can formulate the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form with at least two (non-dimensional) parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wavelength and the amplitude of the free surface are parameterized as h, λ and a, respectively. In turn, if we introduce another non-dimensional parameter μ, so-called the Bond number, which measures the importance of gravitational forces compared to surface tension forces, then the physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves. On the other hand, there are several long wave approximations depending on the relations between ε and δ. For instance, if we consider $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{\frac{1}{2}}$, and in connection with the critical Bond number $\mu = \frac{1}{3}$, we have the so-called Kawahara equation, represented by (1.1), and derived by Hasimoto and Kawahara in [26,31].

[1] See for instance [1,6,32] and references therein, for a rigorous justification of various asymptotic models for surface and internal waves.

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In the last years, there has been an extensive mathematical endeavor that focuses on the analytical and numerical methods for solving the Kawahara equation (1.1). These methods include the tanh-function method [4], extended tanh-function method [5], sine-cosine method [44], Jacobi elliptic functions method [27], direct algebraic method [37] as well as the variational iterations and homotopy perturbations methods [29]. These approaches deal, as a rule, with soliton-like solutions obtained while one considers problems posed on a whole real line. For numerical simulations, however, there appears the question of cutting-off the spatial domain [7,8]. This motivates the detailed qualitative analysis of the problem (1.1) in bounded regions [24].

In this spirit, the main concern of this paper is to deal with the Kawahara equation in a bounded domain under the action of time-delayed boundary control, namely

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) + u^p(t,x)\partial_x u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t > 0, \\ \partial_x^2 u(t,L) = \mathcal{F}(t,h), & t > 0, \\ \partial_x^2 u(t,0) = z_0(t), & t \in \mathcal{T}, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

In (1.2), $\Omega = (0,L)$, where $L > 0$, while $a > 0$ and $b > 0$ are physical parameters. Moreover, $p \in [1,2]$ and $\mathcal{F}(t,h)$ is the delayed control given by

$$\mathcal{F}(t) = \alpha\partial_x^2 u(t,0) + \beta\partial_x^2 u(t-h,0), \tag{1.3}$$

in which $h > 0$ is the time delay, α and β are two feedback gains satisfying the restriction

$$|\alpha| + |\beta| < 1. \tag{1.4}$$

Finally, $\mathcal{T} = (-h,0)$, while u_0 and z_0 are initial conditions.

Thereafter, the functional energy associated to the system (1.2) and (1.3) is

$$E(t) = \int_0^L u^2(t,x)\mathrm{d}x + h|\beta| \int_0^1 (\partial_x^2 u(t-h\rho,0))^2 \mathrm{d}\rho, \ t \geq 0. \tag{1.5}$$

Now, recall that if $\alpha = \beta = 0$, then the term $\partial_x^2 u(t,0)$ represents a feedback damping mechanism (see for instance [2], where $a = 1$ and [42], where $a = 0$) but an extra internal damping $a(x)u(t,x)$ is required to achieve the stability of the solutions. Note that $a(x)$ in is a nonnegative function and positive only on an open subset of $(0,L)$. Therefore, taking into account the action of the time-delayed boundary control (1.3) in (1.2), the following issue will be addressed in this article

Does $E(t) \longrightarrow 0$, as $t \to \infty$? If it is the case, can we provide a decay rate?

It is also worth noting that the answer to the above question is crucial in the understanding of the behavior of the solutions to the Kawahara equation when it is subject to a delayed boundary control $\mathcal{F}(t,h)$. In other words, are the solutions to our problem stable despite the action of the delay? If yes, then how robust is the stability property of the solutions?

Of course, a time delay is inevitable in practical systems for several reasons and may appear from different sources. It is particularly abundant for controlled systems, where sensors and actuators are involved. It is therefore primordial to investigate the impact of a time-delay on the behavior of the solutions to our Kawahara problem (1.2) and (1.3).

1.2. Historical background

Let us first present a review of the main results available in the literature for the analysis of the Kawahara equation in a bounded interval. A pioneer work is due by Silva and Vasconcellos [40,41], where the authors studied the stabilization of global solutions of the linear Kawahara equation in a bounded interval under the effect of a localized damping mechanism. The second endeavor, in this line, is completed by the same authors in [42], where the problem (1.2) and (1.3) is considered with $a = \alpha = \beta = 0, b = p = 1$ and under the action of the a localized interior control $a(x)u(t,x)$. Then, exponential stability results are obtained. Subsequently, Capistrano-Filho et al. [2] considered the generalized Kawahara equation in a bounded domain $Q_T = (0,T) \times (0,L)$:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u - \partial_x^5 u + u^p \partial_x u + a(x)u = 0, & \text{in } Q_T, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = \partial_x^2 u(t,L) = 0, & \text{on } [0,T], \\ u(0,x) = u_0(x), & \text{in } [0,L], \end{cases} \tag{1.6}$$

with $p \in [1,4)$. It is proven that the solutions of the above system decay exponentially.

The internal controllability problem has been tackled by Chen [15] for the Kawahara equation with homogeneous boundary conditions. Using Carleman estimates associated to the linear operator of the Kawahara equation with an internal observation, a null controllable result is shown when the internal control is effective in a subdomain $\omega \subset (0,L)$. In [10], considering the system (1.6) with an internal control $f(t,x)$ and homogeneous boundary conditions, the equation is shown to be controllable in L^2-weighted Sobolev spaces and, additionally, controllable by regions in L^2-Sobolev space.

Recently, a new tool for the control properties for the Kawahara operator was proposed in [12,13]. First, in [12], the authors showed a new type of controllability for the Kawahara equation, what they called *overdetermination control problem.* Boundary control is designed so that the solution of the problem under consideration satisfies an integral condition. Furthermore, when the control acts internally in the system, instead of the boundary, the authors proved that this integral condition is also satisfied. After that, in [13], the authors extend this idea to the internal control problem for the Kawahara equation on unbounded domains. Precisely, under certain hypotheses over the initial and boundary data, the internal control input is designed so that the solutions of the Kawahara equation satisfy an integral overdetermination condition, whether the Kawahara equation is posed in the real line, left half-line, or right half-line. We also note that the existence and uniqueness of solutions as well as their stability are investigated for the Kawahara-type equation posed in the whole real line [16–19,28], the half-line [20,34], a periodic domain [25,30], and a non-periodic bounded domain [21,22,33,34]. We conclude the literature review by mentioning the last works on the stabilization of the Kawahara equation with a localized time-delayed interior control. In [11,14], under suitable assumptions on the time delay coefficients, the authors are able to prove that solutions of the Kawahara system are exponentially stable. The results are obtained using either the Lyapunov approach or a compactness-uniqueness argument.

1.3. Novel contribution of this work

Now, after providing an overview of the results previously obtained in the literature, let us highlight the novelty and contribution of the present work.

Among the new contributions of this article, we provide a systematic study of the well-posedness and stability results for the Kawahara equation with a delayed boundary control. To the authors' best knowledge, no attempt has been made in this direction. To be more specific, the present work shows that the existence, uniqueness and stability properties of the solutions of the Kawahara equation with a boundary delayed control remain "robust" with respect to the presence of a time-delay in the boundary control.

Not only that, we manage to show that the presence of a time-delayed term in the boundary control (1.3) may play a dissipation role in the system. This can be explained by the fact although it might be possible to take $\alpha = 0$ and $\beta > 0$ in (1.4), the solutions to the Kawahara problem (1.2) and (1.3) remain exponentially stable.

Concerning the main contributions, we have:

(i) The results obtained in this article do not require the presence of localized interior damping control, which constitutes an improvement of the results in [40–42];

(ii) Contrary to the works [40–42], where the nonlinearity has the simple form $u\partial_x u$, we can extend the results of these works to the case where the more general nonlinearity term $u^p\partial_x u$, $p \in [1,2]$. Additionally, unlike these works, where the boundary conditions are all homogeneous, a boundary delayed control is present in one of our boundary conditions;

(iii) The results of [2] are complemented by taking into consideration a delayed boundary control. Specifically, the stability of the solutions to the Kawahara equations is conserved despite the presence of a time delay in one of the boundary conditions;

(iv) Our stability result is obtained via two different approaches, namely the energy method and a compactness argument;

(v) We give a relation between the spatial length L and the Möbius transform (see Sect. 1.5 for more details about this point).

1.4. Notations and main results

First of all, let us introduce the following notations that we will use throughout this manuscript.

(i) We consider the space of solutions

$$X(Q_T) = C(0,T;L^2(0,L)) \cap L^2(0,T;H_0^2(0,L))$$

equipped with the norm

$$\|v\|_{X(Q_T)} = \max_{t\in(0,T)} \|v(t,\cdot)\|_{L^2(0,L)} + \left(\int_0^T \|v(t,\cdot)\|^2_{H_0^2(0,L)} \mathrm{d}t\right)^{\frac{1}{2}}.$$

(ii) Denote by

$$\tilde{H} = L^2(0,L) \times L^2(-h,0)$$

the Hilbert space equipped with the inner product

$$\langle (u_1,z_1),(u_2,z_2)\rangle_{\tilde{H}} = \int_0^L u_1u_2 \mathrm{d}x + |\beta| \int_{-h}^0 z_1(s)z_2(s)\ \mathrm{d}s,$$

which yields the following norm

$$\|(u,z)\|^2_{\tilde{H}} = \int_0^L u^2(x)\mathrm{d}x + |\beta| \int_{-h}^0 z^2(\rho)\mathrm{d}\rho.$$

(iii) Throughout all the manuscript, $(\cdot,\cdot)_{\mathbb{R}^2}$ denotes the canonical inner product of $\mathbb{R}^2$.

With the above notations in hand, let us state our first main result in this article:

Theorem 1.1. *Let α and β be two real constants satisfying (1.4) and suppose that the spatial length L fulfills*

$$0 < L < \sqrt{\frac{3b}{a}}\pi. \tag{1.7}$$

Then, there exists $r > 0$ sufficiently small, such that for every $(u_0, z_0) \in H$ with $\|(u_0, z_0)\|_H < r$, the energy of system (1.2) and (1.3), denoted by E and defined by (1.5) exponentially decays, that is, there exist two positive constants κ and λ such that

$$E(t) \leq \kappa E(0)e^{-2\lambda t},\ t > 0. \tag{1.8}$$

Here,

$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2+|\beta|)}, \frac{3b\pi^2 - r^2L - L^2a}{2L^2(1+L\mu_1)}\mu_1\right\} \tag{1.9}$$

and

$$\kappa \leq \left(1+\max\left\{L\mu_1, \frac{\mu_2}{|\beta|}\right\}\right),$$

for $\mu_1, \mu_2 \in (0,1)$ sufficiently small.

The second main result gives another answer to the question presented in this introduction. Indeed, using a different approach based on an observability inequality, we can highlight the critical lengths phenomenon observed in [2] for the Kawahara equation.

Theorem 1.2. *Assume that α and β satisfy (1.4), whereas $L > 0$ is taken so that the problem $(\mathcal{N})$ (see Lemma 4.3) has only the trivial solution. Then, there exists $r > 0$ such that for every $(u_0, z_0) \in H$ satisfying*

$$\|(u_0, z_0)\|_H \leq r,$$

the energy of system (1.2) and (1.3), denoted by E and defined by (1.5), decays exponentially. More precisely, there exist two positive constants ν and κ such that

$$E(t) \leq \kappa E(0)e^{-\nu t}, \quad t > 0.$$

1.5. Heuristic of the article and its structure

In this article, we prove that the Kawahara system (1.2) is exponentially stable despite the presence of the boundary time-delayed control $\mathcal{F}(t)$ defined by (1.3).

In order to show Theorem 1.1, we use the idea of the work that treated the delayed wave systems [43] (see also [35]). More precisely, choosing an appropriate Lyapunov functional associated to the solutions of (1.2) and (1.3) and with some restrictions on the spatial length L and an appropriate size of the initial data, that is, L bounded as in (1.7) and

$$\|(u_0, z_0)\|_H < \frac{2}{\pi}\sqrt{\frac{3b\pi^2 - L^2a}{L}},$$

the energy (1.5) exponentially decays. The key idea of this analysis is the relation between the linearized system associated to (1.2) and (1.3) and a transport equation (see Sect. 2 for more details). Let us mention that such an approach is also used for the Korteweg–de Vries (KdV) with a boundary delayed control in [3] and for the Kawahara equation with a localized time-delayed interior control [14]. However, the nonlinearity in [3,14] is $u\partial_x u$, which becomes a special case in our study, that is, $p = 1$ in (1.2).

Note that the KdV equation studied in [3] is of order three, while the Kawahara equation is a fifth-order equation. Furthermore, in this work, the boundary conditions are two homogeneous Dirichlet boundary conditions and one Neumann right-end control $\partial_x u(t, L)$. This means, unlike in our case, that no second-order derivative is involved in the boundary conditions. Moreover, the reader can notice that the difference between the order of the derivative in the equation and the highest order of the derivatives in the boundary conditions is two, whereas it is three in our problem. These points are the main differences between our work and [3] although there are similarities in the proof of Theorem 1.1 and [3, Theorem 1].

Concerning the proof of Theorem 1.2, we proceed as in [38], i.e., combining multipliers and compactness arguments which reduces the problem to show a unique continuation result for the state operator. To prove the latter, we extend the solution under consideration by zero in $\mathbb{R} \setminus [0, L]$ and take the Fourier transform. However, due to the complexity of the system, after taking the Fourier transform of the extended solution u, it is not possible to adopt the same techniques used in [38]. Thus, to prove our main result, we invoke the result due Santos et al. [23]. Specifically, after taking the Fourier transform, the issue is to establish when a certain quotient of entire functions still turns out to be an entire function. We then pick a polynomial function $q : \mathbb{C} \to \mathbb{C}$ and a family of functions

$$N_\alpha : \mathbb{C} \times (0, \infty) \to \mathbb{C}, \tag{1.10}$$

with $\alpha \in \mathbb{C}^4 \setminus \{0\}$, whose restriction $N_\alpha(\cdot, L)$ is entire for each $L > 0$. Next, we consider a family of functions $f_\alpha(\cdot, L)$, defined by

$$f_\alpha(\mu, L) = \frac{N_\alpha(\mu, L)}{q(\mu)}, \tag{1.11}$$

in its maximal domain. The problem is then reduced to determine $L > 0$, for which there exists $\alpha \in \mathbb{C}^4\setminus\{0\}$ such that $f_\alpha(\cdot, L)$ is entire. In contrast with the analysis developed in [38], this approach does not provide an explicit characterization of the set of critical lengths, if it exists, but only ensures that the roots of f have a relation with the Möbius transform (see the proof of Lemma 4.3 above). It is also worth recalling that the proof of Theorem 1.2 is inspired by [38], which in turn has been used in [3]. However, the result obtained in the last work directly relies on [38]. This is not the case for our Theorem 1.2 as we explained above. On the other hand, the so-called set of critical lengths of the KdV problem is explicitly known in [38], and this made the task easier for the exponential stability of the KdV case [3]. It is also important to point out that the derivation of the set of critical lengths for the Kawahara problem is more challenging, and we only manage to derive a relation between the length of L and the Möbius transformation, while an explicit deduction of the critical set phenomena remains an open problem. This happens because the roots of the function

$$N_\alpha(\xi, L) = \alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L}$$

in (1.11) cannot be found explicitly as in the KdV case. Hence, due to these facts, mentioned above, our problem is more challenging than that of [3,38].

Finally, let us present the outline of our work: First, in Sect. 2, we prove the regularity properties of the solutions to the linear system associated to (1.2) and (1.3) and then show that the well-posedness of the problem (1.2) and (1.3). Section 3 is devoted to the proof of the first main result of this article, namely Theorem 1.1. In Sect. 4, with the help of the result established in [23], we show our second stability outcome stated in Theorem 1.2. Finally, in Sect. 5, we present some additional comments and open questions.

2. Well-posedness results

The goal of this section is to prove that the full nonlinear Kawahara system (1.2) and (1.3) is well-posed. The proof is divided into four parts by using the strategy due to Rosier [38]:

(1) Well-posedness to the linear system associated to (1.2) and (1.3).
(2) Regularity properties of the linear system associated to (1.2) and (1.3).
(3) Well-posedness of the linear system associated to (1.2) and (1.3) with a source term.
(4) Well-posedness of the system (1.2) and (1.3).

2.1. Well-posedness: linear system

We begin by proving the well-posedness of the linearized system

$$\begin{cases} \partial_t u(t,x) + a\partial_x u(t,x) + b\partial_x^3 u(t,x) - \partial_x^5 u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,0) = u(t,L) = \partial_x u(t,0) = \partial_x u(t,L) = 0, & t > 0, \\ \partial_x^2 u(t,L) = \alpha\partial_x^2 u(t,0) + \beta\partial_x^2 u(t-h,0), & t > 0, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

In order to investigate (2.1), let $z(t,\rho)=\partial_x^2 u(t-\rho h,0)$, which satisfies the transport equation [43] (see also [35])

$$\begin{cases} h\partial_t z(t,\rho)+\partial_\rho z(t,\rho)=0, \ \rho\in(0,1),\ t\ >0, \\ z(t,0)=\partial_x^2 u(t,0), \qquad t>0, \\ z(0,\rho)=z_0(-h\rho), \qquad \rho\in(0,1). \end{cases} \tag{2.2}$$

Next, we consider the Hilbert space $H=L^2(0,L)\times L^2(0,1)$ equipped with the following inner product:

$$\langle (u_1,z_1),(u_2,z_2)\rangle_H=\int_0^L u_1u_2\mathrm{d}x+|\beta|h\int_0^1 z_1z_2\mathrm{d}\rho.$$

Subsequently, one can rewrite (2.1) and (2.2) as follows:

$$\begin{cases} U_t(t)=AU(t), \quad t>0, \\ U(0)=U_0\in H, \end{cases} \tag{2.3}$$

where

$$A=\begin{bmatrix} -a\partial_x-b\partial_x^3+\partial_x^5 & 0 \\ 0 & -\frac{1}{h}\partial_\rho \end{bmatrix},$$
$$U(t)=\begin{bmatrix} u(t,\cdot) \\ z(t,\cdot) \end{bmatrix}, U_0=\begin{bmatrix} u_0(\cdot) \\ z_0(-h(\cdot)) \end{bmatrix},$$

and

$$D(A)=\{(u,z)\in H^5(0,L)\cap H_0^2(0,L)\times H^1(0,1);\ \partial_x^2u(0)=z(0),\partial_x^2u(L)=\alpha\partial_x^2u(0)+\beta z(1)\}.$$

The next result ensures the well-posedness for the problem (2.1).

Proposition 2.1. *Assume that the constants α and β satisfy (1.4) and that $U_0\in H$. Then, there exists a unique mild solution $U\in C([0,+\infty),H)$ for the system (2.1). Additionally, considering $U_0\in D(A)$, we have a classical solution with the following regularity:*

$$U\in C([0,+\infty),D(A))\cap C^1([0,+\infty),H).$$

Proof. As the proof uses standard arguments, only a sketch of it will be provided. Let $U=(u,z)\in D(A)$. Then, integrating by parts and using the boundary conditions of (2.1) and (2.2), we obtain

$$\begin{aligned} \langle AU(t),U(t)\rangle_H =&\frac{1}{2}\left(\alpha^2(\partial_x^2u(t,0))^2+2\alpha\beta\partial_x^2u(t,0)\partial_x^2u(t-h,0)\right) \\ &+\frac{1}{2}\left(\beta^2(\partial_x^2u(t-h,0))^2-(\partial_x^2u(t,0))^2\right) \\ &+\frac{1}{2}\left(-|\beta|(\partial_x^2u(t-h,0))^2+|\beta|(\partial_x^2u(t,0))^2\right)=\frac{1}{2}(M\eta(t),\eta(t))_{\mathbb{R}^2}, \end{aligned} \tag{2.4}$$

where

$$\eta=\begin{bmatrix} \partial_x^2u(t,0) \\ \partial_x^2u(t-h,0)) \end{bmatrix} \ \text{ and } \ M=\begin{bmatrix} \alpha^2-1+|\beta| & \alpha\beta \\ \alpha\beta & \beta^2-|\beta| \end{bmatrix}. \tag{2.5}$$

Now, observe that adjoint of A, denoted by A^*, is defined by

$$A^*=\begin{bmatrix} a\partial_x+b\partial_x^3-\partial_x^5 & 0 \\ 0 & \frac{1}{h}\partial_\rho \end{bmatrix}$$

with

$$D(A^*)=\{(\varphi,\psi)\in H^5(0,L)\cap H_0^2(0,L)\times H^1(0,1);\ \psi(1)=\frac{\beta}{|\beta|}\partial_x^2\varphi(L),\ \partial_x^2\varphi(0)=\alpha\partial_x^2\varphi(L)+|\beta|\psi(0)\}.$$

Similarly, we have, for $V = (\varphi, \psi) \in D(A^*)$, that

$$\begin{aligned}\langle A^*V, V\rangle_H =&\frac{1}{2}\left[(\alpha^2 - 1 + |\beta|^2)\partial_x^2\varphi(L) + 2\alpha|\beta|\partial_x^2\varphi(L)\psi(0) + (|\beta|^2 - |\beta|)\psi(0)^2\right]\\ =&\frac{1}{2}(M^*\eta^*, \eta^*)_{\mathbb{R}^2},\end{aligned}$$

where

$$\eta^* = \begin{bmatrix}\partial_x^2\varphi(L)\\ \psi(0)\end{bmatrix} \quad \text{and} \quad M^* = \begin{bmatrix}\alpha^2 - 1 + |\beta| & \alpha|\beta|\\ \alpha|\beta| & \beta^2 - |\beta|\end{bmatrix}. \tag{2.6}$$

Now, let us check that M and M^* are negative definite. For this, we will use the following lemma:

Lemma 2.2. *Let $M = (m_{ij})_{i,j} \in \mathbb{M}_{2\times 2}(\mathbb{R})$ be a symmetric matrix. If $m_{11} < 0$ and $\det(M) > 0$, then M is negative definite.*

Proof. It is sufficient to note that for all $u = (x\ y) \neq (0\ 0)$, we have

$$uMu^\top = m_{11}x^2 + 2xym_{12} + m_{22}y^2 = m_{11}\left(x + \frac{m_{12}}{m_{11}}y\right)^2 + \left(\frac{m_{11}m_{22} - m_{12}^2}{m_{11}}\right)y^2 < 0,$$

which completes the proof. □

Now, we are in position to finish the proof of Proposition 2.1. From (2.5), (2.6) and the condition (1.4), we see that $m_{11} = m_{11}^* = \alpha^2 - 1 + |\beta| < 0$ and

$$\det M = \det M^* = |\beta|((|\beta| - 1)^2 - \alpha^2) > 0,$$

where $M = (m_{ij})_{i,j\in\{1,2\}}$ and $M^* = (m_{i,j}^*)_{i,j\in\{1,2\}}$. Therefore, by virtue of Lemma 2.2, it follows that M and M^* are negative definite and hence both A and A^* are dissipative in view of (2.4) and (2.6).

Finally, since A and A^* are densely defined closed linear operators and both A and A^* are dissipative, one can use the semigroups theory of linear operators [36] to claim that A is a generator of a C_0–semigroups of contractions on H, together with the statements of Proposition 2.1. □

Remark 2.3. It is important to point out that considering $\alpha = \beta = 0$ or $\alpha \neq 0$ and $\beta = 0$, the well posedness of (2.1) is easily obtained. Indeed, if $\alpha = \beta = 0$, the result follows from [2, Lemma 2.1]. In the case when $\alpha \neq 0$ and $\beta = 0$, we have $Au = -a\partial_x - b\partial_x^3 u + \partial_x^5 u$ with domain

$$D(A) = \{u \in H^5(0, L) : u(0) = u(L) = \partial_x u(0) = \partial_x u(L) = 0, \partial_x^2 u(L) = \alpha\partial_x^2 u(0)\}.$$

One can see that $A^*v = a\partial_x v + b\partial_x^3 v - \partial_x^5 v$ with domain

$$D(A^*) = \{v \in H^5(0, L) : v(0) = v(L) = \partial_x v(0) = \partial_x v(L) = 0, \partial_x^2 v(0) = \alpha\partial_x^2 v(L)\},$$

and we easily verify that

$$(Au, u)_{L^2(0,L)} = \frac{(\alpha^2 - 1)}{2}(\partial_x^2 u(0))^2 \quad \text{and} \quad (A^*v, v)_{L^2(0,L)} = \frac{(\alpha^2 - 1)}{2}(\partial_x^2 v(L))^2,$$

so in this case, it is necessary to take $|\alpha| < 1$ in order to obtain the well-posedness result.

2.2. Regularity estimates: linear system

In the sequel, let $\{S(t)\}_{t\geq 0}$ be the semigroup of contractions for the operator A. We have some *a priori* estimates and regularity estimates for the linear systems (2.1) and (2.2).

Proposition 2.4. *Suppose that (1.4) holds. Then, the application*

$$\begin{aligned} \mathcal{S}: H &\longrightarrow X(Q_T)\times C(0,T;L^2(0,1)) \\ (u_0, z_0(-h(\cdot))) &\longmapsto S(\cdot)(u_0, z_0(-h(\cdot))) \end{aligned} \tag{2.7}$$

is well-defined and continuous. Moreover, for every $(u_0(\cdot), z_0(-h(\cdot)))\in H$, *we have*

$$(\partial_x^2 u(\cdot,0), z(\cdot,1))\in L^2(0,T)\times L^2(0,T)$$

and the following estimates hold:

$$\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)} \le C\big(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\big), \tag{2.8}$$

$$\|u_0\|^2_{L^2(0,L)} \le \frac{1}{T}\|u\|^2_{L^2(0,T;L^2(0,L))} + \|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)}, \tag{2.9}$$

and

$$\|z_0(-h(\cdot))\|^2_{L^2(0,1)} \le \|z(T,\cdot)\|^2_{L^2(0,1)} + \frac{1}{h}\|z(\cdot,1)\|^2_{L^2(0,T)}, \tag{2.10}$$

for some constant $C>0$ *that may depend of* $a, b, \alpha, \beta, L, T$ *and* h.

Proof. We split the proof into several steps.
Step 1 Main identities.

For every $(u_0, z_0(-h(\cdot)))\in H$, the semigroups theory gives that

$$S(\cdot)(u_0, z_0(-h(\cdot)))\in C(0,T;H),$$

and due to the fact that A generates a C_0-semigroup of contractions, we have that

$$\|u(t)\|^2_{L^2(0,L)} + h|\beta|\|z(t)\|^2_{L^2(0,1)} \le \|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}, \forall t\in[0,T]. \tag{2.11}$$

Now, let $\phi\in C^\infty([0,1]\times[0,T])$, $\psi\in C^\infty([0,L]\times[0,T])$ and $(u,z)\in D(A)$. Then, multiplying (2.2) by ϕz and (2.1) by ψu, using integrations by parts and the initial conditions, we have

$$\begin{aligned} \int_0^1 [\phi(T,\rho)z(T,\rho)^2 - \phi(0,\rho)z_0(-h\rho)^2]\mathrm{d}\rho - \frac{1}{h}\int_0^T\int_0^1 [h\partial_t\phi(t,\rho) + \partial_\rho\phi(t,\rho)]z(t,\rho)^2\mathrm{d}\rho\mathrm{d}t \\ + \frac{1}{h}\int_0^T [\phi(t,1)z(t,1)^2 - \phi(t,0)(\partial_x^2 u(t,0))^2]\mathrm{d}t = 0 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &-\int_0^T\int_0^L [\partial_t\psi(t,x) + a\partial_x\psi(t,x) + b\partial_x^3\psi(t,x) - \partial_x^5\psi(t,x)]u^2(t,x)\mathrm{d}x\mathrm{d}t \\ &+ 3b\int_0^T\int_0^L \partial_x\psi(t,x)(\partial_x u(t,x))^2\mathrm{d}x\mathrm{d}t + \int_0^L [\psi(T,x)u^2(T,x) - \psi(0,x)u_0(x)^2]\mathrm{d}x \\ &+ 5\int_0^T\int_0^L [\partial_x\psi(t,x)(\partial_x^2 u(t,x))^2 - \partial_x^3\psi(t,x)(\partial_x u(t,x))^2]\mathrm{d}x\mathrm{d}t \\ &- \int_0^T \psi(t,L)[\alpha\partial_x^2 u(t,0) + \beta z(t,1)]^2\mathrm{d}t + \int_0^T \psi(t,0)(\partial_x^2 u(t,0))^2\mathrm{d}t = 0. \end{aligned} \tag{2.13}$$

Step 2 Proof of (2.8).

Let us pick $\phi(t,\rho)=\rho$ in (2.12) to get

$$\int_0^1 (z(T,\rho)^2 - z_0(-\rho h)^2)\rho \mathrm{d}\rho - \frac{1}{h}\int_0^T\int_0^1 z(t,\rho)^2 \mathrm{d}\rho \mathrm{d}t + \frac{1}{h}\int_0^T z(t,1)^2 \mathrm{d}t = 0.$$

Owing to (2.11), the latter gives

$$\|z(\cdot,1)\|^2_{L^2(0,T)} \le (T+1)\left(1+\frac{1}{h|\beta|}\right)\left(\|u_0\|^2_{L^2(0,L)} + \|z_0(-h(\cdot))\|^2_{L^2(0,1)}\right). \tag{2.14}$$

Now, choosing $\psi(t,x)=1$ in (2.13) yields

$$\int_0^L [u^2(t,x)-u_0(x)^2]\mathrm{d}x + \int_0^T (\partial_x^2 u(t,0))^2 \mathrm{d}t - \int_0^T [\alpha\partial_x^2 u(t,0)^2 + \beta z(t,1)]^2 \mathrm{d}x = 0,$$

which implies

$$\int_0^T (\partial_x^2 u(t,0))^2 \mathrm{d}t \le \int_0^T (\alpha\partial_x^2 u(t,0) + \beta z(t,1))^2 \mathrm{d}t + \|u_0\|^2_{L^2(0,L)}. \tag{2.15}$$

Since

$$(\alpha\partial_x^2 u(t,0) + \beta z(t,1))^2 \le (\alpha^2+\beta^2)((\partial_x^2 u(t,0))^2 + (z(t,1))^2), \tag{2.16}$$

it follows from (2.15) and (2.16) that

$$\int_0^T \big(1-(\alpha^2+\beta^2)\big)(\partial_x^2 u(t,0))^2 \mathrm{d}t \le \int_0^T (\alpha^2+\beta^2) z(t,1)^2 \mathrm{d}t + \|u_0\|^2_{L^2(0,L)}.$$

In view of (2.14) and (1.4), the last estimate yields

$$\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} \le (T+1)\frac{1}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\left(\|u_0\|^2_{L^2(0,L)} + \|z(-h(\cdot)))\|^2_{L^2(0,1)}\right). \tag{2.17}$$

Combining (2.17) and (2.14), the estimate (2.8) follows.
Step 3 The map (2.7) is well-defined and continuous.

Letting $\psi(t,x)=x$ in (2.13) gives

$$\begin{aligned} -a\int_0^T\int_0^L u^2(t,x)\mathrm{d}x\mathrm{d}t + 3b\int_0^T\int_0^L (\partial_x u(t,x))^2 \mathrm{d}x\mathrm{d}t + 5\int_0^T\int_0^L (\partial_x^2 u(t,x))^2 \mathrm{d}x\mathrm{d}t \\ + \int_0^L x[u^2(T,x)-u_0(x)^2]\mathrm{d}x - L\int_0^T [\alpha\partial_x^2 u(t,0) + \beta z(t,1)]^2 \mathrm{d}t = 0. \end{aligned}$$

which implies, using (2.11) and (2.16), that

$$\begin{aligned} 3b\int_0^T\int_0^L (\partial_x u(t,x))^2 \mathrm{d}x\mathrm{d}t + 5\int_0^T\int_0^L (\partial_x^2 u(t,x))^2 \mathrm{d}x\mathrm{d}t \le a\big(\|u_0\|^2_{L^2(0,L)} + h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\big) \\ + L\|u_0\|^2_{L^2(0,L)} + L(\alpha^2+\beta^2)\big(\|\partial_x^2 u(\cdot,0)\|^2_{L^2(0,T)} + \|z(\cdot,1)\|^2_{L^2(0,T)}\big). \end{aligned}$$

In light of (2.8), we deduce that

$$\begin{aligned}\|\partial_x u\|^2_{L^2(0,T;L^2(0,L))}+\|\partial_{xx}u\|^2_{L^2(0,T,L^2(0,L))}\leq&\frac{a}{\min\{3b,5\}}\left(\|u_0\|^2_{L^2(0,L)}+h|\beta|\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\right)\\&+(T+1)\frac{2-(\alpha^2+\beta^2)}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\frac{L}{\min\{3b,5\}}(\alpha^2+\beta^2)\\&\times\left(\|u_0\|^2_{L^2(0,L)}+\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\right)+\frac{L}{\min\{3b,5\}}\|u_0\|^2_{L^2(0,L)}\\\leq&C_0(T+1)\left(\|u_0\|^2_{L^2(0,L)}+\|z_0(-h(\cdot))\|^2_{L^2(0,1)}\right),\end{aligned}\tag{2.18}$$

where

$$C_0=\max\left\{\frac{a}{\min\{3b,5\}},\frac{a}{\min\{3b,5\}}|\beta|h,\left(\frac{2-(\alpha^2+\beta^2)}{1-(\alpha^2+\beta^2)}\left(1+\frac{1}{h|\beta|}\right)\frac{L}{\min\{3b,5\}}(\alpha^2+\beta^2)\right)\right\}.$$

Combining (2.18) and (2.11), we obtain the desired result.
Step 4 Proof of (2.9) and (2.10).

In order to show these inequalities, choose $\psi=T-t$ in (2.13) and $\phi(t,\rho)=1$ in (2.12), respectively. Performing similar computations as we did in step 2, the result follows. Moreover, owing to the density of $D(A)$ in H, the proof of Proposition 2.4 is achieved. $\square$

2.3. Well-posedness: linear system with a source term

Now, we consider the linear system with a source term

$$\begin{cases}\partial_t u(t,x)+a\partial_x u(t,x)+b\partial_x^3u(t,x)-\partial_x^5u(t,x)=f(t,x), & (t,x)\in\mathbb{R}^+\times\Omega,\\ u(t,0)=u(t,L)=\partial_x u(t,0)=\partial_x u(t,L)=0, & t>0,\\ \partial_x^2u(t,L)=\alpha\partial_x^2u(t,0)+\beta\partial_x^2u(t-h,0), & t>0,\\ \partial_x^2u(t,0)=z_0(t), & t>0,\\ u(0,x)=u_0(x), & x\in\Omega.\end{cases}\tag{2.19}$$

Then, we have the following result.

Proposition 2.5. *Let $|\alpha|$ and $|\beta|$ satisfying (1.4). For every $(u_0,z_0)\in H$ and $f\in L^2(0,T;L^2(0,L))$, there exists a unique mild solution $(u,\partial_x^2u(t-h.,0))\in X(Q_T)\times C(0,T;L^2(0,1))$ to (2.19). Moreover, there exists a constant $C>0$ such that*

$$\|(u,z)\|_{C(0,T;H)}\leq C\Bigg(\|(u_0,z_0(-h(\cdot)))\|_H+\|f\|_{L^1(0,T;L^2(0,L))}\Bigg)\tag{2.20}$$

and

$$\|\partial_x^2u\|^2_{L^2(0,T;L^2(0,L))}\leq C\Bigg(\|(u_0,z_0(-h(\cdot)))\|^2_H+\|f\|^2_{L^1(0,T;L^2(0,L))}\Bigg).\tag{2.21}$$

Proof. This proof is analogous to that of [3, Proposition 2] and hence we omit it. $\square$

2.4. Well-posedness of the nonlinear system (1.2) and (1.3)

Let us now prove that system (1.2) and (1.3) is well-posed. To do so, we first deal with the properties of the nonlinearities through the following lemma.

Lemma 2.6. *Let $u \in L^2(0,T;H^2(0,L)) = L^2(H^2)$. Then, $u\partial_x u$ and $u^2\partial_x u$ belong to $L^1(0,T;L^2(0,L))$. Besides, there exist positives constants C_0 and C_1, depending of L, such that for every $u, v \in L^2(0,T;H^2(0,L))$, one has*

$$\int_0^T \|u\partial_x u - v\partial_x v\|_{L^2(0,L)}\mathrm{d}t \leq C_0(\|u\|_{L^2(H^2)} + \|v\|_{L^2(H^2)})\|u-v\|_{L^2(H^2)} \tag{2.22}$$

and

$$\int_0^T \|u^2\partial_x u - v^2\partial_x v\|_{L^2(0,L)}\mathrm{d}t \leq C_0(1+T^{\frac{1}{2}})\left(\|u\|^2_{X(Q_T)} + \|v\|^2_{X(Q_T)}\right)\|u-v\|_{X(Q_T)}. \tag{2.23}$$

Proof. Observe that (2.22) follows from [42, Lemma 2.1, p. 106]. Concerning (2.23), note that

$$\sup_{x\in(0,L)} |u(x)^2| \leq \|u\|^2_{L^2(0,L)} + \|u\|_{L^2(0,L)}\|\partial_x u\|_{L^2(0,L)},$$

for $u \in H^1(0,L)$. Let $u, z \in X(Q_T)$, then

$$\begin{aligned}\|u^2(\partial_x u - \partial_x v)\|_{L^1(0,T;L^2(0,L))} &= \int_0^T \|u(t,\cdot)\|^2_{L^\infty(0,L)}\|(\partial_x u - \partial_x v)(t,\cdot)\|_{L^2(0,L)}\mathrm{d}t\\ &\leq T^{\frac{1}{2}}\|u\|^2_{L^\infty(0,T;L^2(0,L))}\|u-v\|_{L^2(0,T;H^2(0,L))}\\ &\quad + \|u\|_{L^\infty(0,T;L^2(0,L))}\|u\|_{L^2(0,T;H^2(0,L))}\|u-v\|_{L^2(0,T;H^2(0,L))}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|(u^2-v^2)\partial_x v\|_{L^1(0,T;L^2(0,L))} &= \int_0^T \left(\int_0^L |u+v|^2|u-v|^2|\partial_x v|^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t\\ &\leq \int_0^T \left(\|(u+v)(t,\cdot)\|^2_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|^2_{L^\infty(0,L)}\int_0^L |\partial_x v|^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}t\\ &= \int_0^T \|(u+v)(t,\cdot)\|_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|_{L^\infty(0,L)}\|\partial_x v(t,\cdot)\|_{L^2(0,L)}\mathrm{d}t.\end{aligned}$$

Now, observe that

$$\begin{aligned}&\|(u+v)(t,\cdot)\|_{L^\infty(0,L)}\|(u-v)(t,\cdot)\|_{L^\infty(0,L)}\\ &\quad\leq \left(\|(u+v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\|(\partial_x u+\partial_x v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\right)\\ &\qquad\times\left(\|(u-v)(t,\cdot)\|_{L^2(0,L)} + \|(u-v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\|(\partial_x u-\partial_x v)(t,\cdot)\|^{\frac{1}{2}}_{L^2(0,L)}\right)\\ &\quad\leq \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)}\\ &\qquad + \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u-\partial_x v)(t,\cdot)\|_{L^2(0,L)} + \|(u-v)(t,\cdot)\|_{L^2(0,L)}\|(u+v)(t,\cdot)\|_{L^2(0,L)}\\ &\qquad + \|(u-v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u+\partial_x v)(t,\cdot)\|_{L^2(0,L)} + \|(u+v)(t,\cdot)\|_{L^2(0,L)}\|(\partial_x u-\partial_x v)(t,\cdot)\|_{L^2(0,L)}\\ &\qquad + \|(\partial_x u+\partial_x v)(t,\cdot)\|_{L^2(0,L)}\|(u-v)(t,\cdot)\|_{L^2(0,L)}.\end{aligned}$$

Hence,

$$\|u^2\partial_x u - v^2\partial_x v\|_{L^1(0,T;L^2(0,L))} \leq (1+T^{\frac{1}{2}})\left(\|u\|^2_{X(Q_T)} + \|v\|^2_{X(Q_T)}\right)\|u-v\|_{X(Q_T)},$$

and thus (2.23) is proved. $\square$

Note that the arguments used for the proof of Proposition 2.1, one can show after a simple calculation that the energy E defined by (1.5) is decreasing, that is,

$$E'(t) = (M\eta(t), \eta(t))_{\mathbb{R}^2} \leq 0, \quad t > 0, \tag{2.24}$$

where M and η are defined by (2.5). Combining this fact with Lemma 2.6 and Proposition 2.5, one can use a classical fixed-point argument (see, for instance, [2]) to obtain the following well-posedness result.

Theorem 2.7. *Let $L > 0$, $a, b > 0$ and $\alpha, \beta \in \mathbb{R}$ satisfying (1.4). Assume $p \in [1, 2]$ and $h > 0$. If $u_0 \in L^2(0, L)$ and $z_0 \in L^2(0, 1)$ are sufficient small, then system (1.2) and (1.3) admits a unique solution $u \in X(Q_T)$.*

3. A stabilization result via Lyapunov approach

This part of the work aims to prove our first main result presented in Theorem 1.1. Precisely, we will prove the case $p = 2$, that is, when the nonlinearity takes the form $u^2\partial_x u$. The case $u\partial_x u$ can be shown similarly, therefore, we will omit its proof.

Proof of Theorem 1.1. First, we choose the following Lyapunov functional:

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t).$$

Here, $\mu_1, \mu_2 \in (0, 1)$, V_1 is defined by

$$V_1(t) = \int_0^L x u^2(t, x)\mathrm{d}x \tag{3.1}$$

and V_2 is given by

$$V_2(t) = h\int_0^1 (1-\rho)(\partial_x^2 u(t - h\rho, 0))^2 \mathrm{d}\rho,$$

for any regular solution of (1.2) and (1.3). Clearly, we have the following

$$E(t) \leq V(t), \tag{3.2}$$

for all $t \geq 0$. On the other hand, we have

$$\begin{aligned}
\mu_1 V_1(t) + \mu_2 V_2(t) &= \mu_1 \int_0^L x u^2(t, x)\mathrm{d}x + h\mu_2 \int_0^1 (1-\rho)(\partial_x^2 u(t - h\rho, 0))^2 \mathrm{d}\rho \\
&\leq \mu_1 L \int_0^L u^2(t, x)\mathrm{d}x + \mu_2 \frac{h}{|\beta|}|\beta| \int_0^1 (1-\rho)(\partial_x^2 u(t - h\rho, 0))^2 \mathrm{d}\rho \\
&\leq \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\} E(t),
\end{aligned}$$

that is,

$$E(t) \leq V(t) \leq \left(1 + \max\left\{\mu_1 L, \frac{\mu_2}{|\beta|}\right\}\right) E(t), \tag{3.3}$$

for all $t \geq 0$.

Now, consider a sufficiently regular solution u of (1.2) and (1.3). Differentiating $V_1(t)$, using integration by parts and the boundary conditions of (1.2) and (1.3), it follows that

$$\begin{aligned}\frac{d}{\mathrm{d}t}V_1(t) &= -2\int_0^L xu(t,x)\big[a\partial_x u + b\partial_x^3 u - \partial_x^5 u + u^2\partial_x u\big](t,x)\mathrm{d}x\\
&= a\int_0^L u^2(t,x)\mathrm{d}x - 3b\int_0^L(\partial_x u(t,x))^2\mathrm{d}x - 5\int_0^L(\partial_x^2 u(t,x))^2\mathrm{d}x + \frac{1}{2}\int_0^L u^4(t,x)\mathrm{d}x\\
&\quad + L\Big[\alpha^2(\partial_x^2 u(t,0))^2 + 2\alpha\beta\partial_x^2 u(t,0)\partial_x^2 u(t-h,0) + \beta^2(\partial_x^2 u(t-h,0))^2\Big]. \qquad (3.4)\end{aligned}$$

Similarly, in view of (2.2), we have

$$\begin{aligned}\frac{d}{\mathrm{d}t}V_2(t) &= 2h\int_0^1(1-\rho)\partial_x^2 u(t-\rho h,0)\frac{d}{\mathrm{d}t}\partial_x^2 u(t-\rho h,0)\mathrm{d}\rho\\
&= \partial_x^2 u(t,0)^2 - \int_0^1(\partial_x^2 u(t-\rho h,0))^2\mathrm{d}\rho. \qquad (3.5)\end{aligned}$$

Consequently, (3.4) and (3.5) imply that for any $\lambda > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\lambda V(t) =&\Big(\alpha^2 - 1 + |\beta| + L\mu_1\alpha^2 + \mu_2\Big)(\partial_x^2 u(t,0))^2 + \Big(\beta^2 - |\beta| + L\mu_1\beta^2\Big)(\partial_x^2 u(t-h,0))^2\\
&+ 2\alpha\beta\Big(1 + L\mu_1\Big)\partial_x^2 u(t,0)\partial_x^2 u(t-h,0) + (2\lambda h|\beta| - \mu_2)\int_0^1(\partial_x^2 u(t-\rho h,0))^2\mathrm{d}\rho\\
&+ 2\lambda\mu_2 h\int_0^1(1-\rho)(\partial_x^2 u(t-\rho h,0))^2\mathrm{d}\rho + 2\lambda\mu_1\int_0^L xu^2(t,x)\mathrm{d}x + \frac{\mu_1}{2}\int_0^L u^4(t,x)\mathrm{d}x\\
&+ (\mu_1 a + 2\lambda)\int_0^L u^2(t,x)\mathrm{d}x - 3b\mu_1\int_0^L(\partial_x u(t,x))^2\mathrm{d}x - 5\mu_1\int_0^L(\partial_x^2 u(t,x))^2\mathrm{d}x,\end{aligned}$$

or equivalently, by reorganizing the terms

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\lambda V(t) \leq&\big(M_{\mu_1}^{\mu_2}\ \eta(t),\eta(t)\big)_{\mathbb{R}^2} - 3b\mu_1\int_0^L(\partial_x u(t,x))^2\mathrm{d}x - 5\mu_1\int_0^L(\partial_x^2 u(t,x))^2\mathrm{d}x\\
&+ \big(2\lambda h(\mu_2 + |\beta|) - \mu_2\big)\int_0^1(\partial_x^2 u(t-\rho h,0))^2\mathrm{d}\rho\\
&+ \big(\mu_1 a + 2\lambda(1 + L\mu_1)\big)\int_0^L u^2(t,x)\mathrm{d}x + \frac{\mu_1}{2}\int_0^L u^4(t,x)\mathrm{d}x, \qquad (3.6)\end{aligned}$$

where $\eta(t) = (\partial_x^2 u(t,0), \partial_x^2 u(t-h,0))$ and

$$M_{\mu_1}^{\mu_2} = \begin{bmatrix}(1 + L\mu_1)\alpha^2 - 1 + |\beta| + \mu_2 & \alpha\beta(1 + L\mu_1)\\ \alpha\beta(1 + L\mu_1) & \beta^2 - |\beta| + L\mu_1\beta^2\end{bmatrix}.$$

Observe that

$$M_{\mu_1}^{\mu_2} = M + L\mu_1 \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where M is defined by (2.5). Since M is negative definite (see the proof of Proposition 2.1 and by virtue of the continuity of the determinant and the trace, one can claim that for μ_1 and $\mu_2 > 0$ small enough, the matric $M_{\mu_1}^{\mu_2}$ can also be made negative definite.

Finally, taking into account μ_1 and $\mu_2 > 0$ are small enough and using Poincaré inequality[2], we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\lambda V(t) \leq \big(2\lambda h(\mu_2 + |\beta|) - \mu_2\big) \int_0^1 (\partial_x^2 u(t-\rho h, 0))^2 \mathrm{d}\rho \\ -5\mu_1 \int_0^L (\partial_x^2 u(t,x))^2 \mathrm{d}x + \frac{\mu_1}{2} \int_0^L u^4(t,x)\mathrm{d}x \\ + \left(\frac{L^2}{\pi^2}(\mu_1 a + 2\lambda(1 + L\mu_1)) - 3b\mu_1 \right) \int_0^L (\partial_x u^2(t,x))^2 \mathrm{d}x. \end{aligned} \tag{3.7}$$

Additionally, applying Cauchy–Schwarz inequality and using the facts that the energy E defined by (1.5) is nonincreasing, together with $H_0^1(0,L) \hookrightarrow L^\infty(0,L)$, we have

$$\begin{aligned} \frac{\mu_1}{2} \int_0^L u^4(t,x)\mathrm{d}x &\leq \frac{\mu_1}{2} \|u(t,\cdot)\|^2_{L^\infty(0,L)} \int_0^L u^2(t,x)\mathrm{d}x \\ &\leq \frac{\mu_1}{2} L \|\partial_x u(t,\cdot)\|^2_{L^2(0,L)} \|u(t,x)\|^2_{L^2(0,L)} \\ &\leq \frac{L\mu_1}{2} \left(\|u_0\|^2_{L^2(0,L)} + h|\beta| \|z_0(-h(\cdot))\|^2_{L^2(0,1)} \right) \|\partial_x u(t,\cdot)\|^2_{L^2(0,L)} \\ &\leq \frac{L\mu_1}{2} \|(u_0, z_0)\|^2_H \|\partial_x u(t,\cdot)\|^2_{L^2(0,L)} \leq r^2 \frac{L\mu_1}{2} \|\partial_x u(t,\cdot)\|^2_{L^2(0,L)}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\lambda V(t) \leq \Xi \|\partial_x^2 u(t,x)\|^2_{L^2(0,L)} + \big(2\lambda h(\mu_2 + |\beta|) - \mu_2\big) \|\partial_x^2 u(t-\rho h, 0)\|^2_{L^2(0,1)}, \tag{3.9}$$

where

$$\Xi = \frac{L\mu_1}{2} r^2 + \frac{L^2}{\pi^2} \big(\mu_1 a + 2\lambda(1 + L\mu_1)\big) - 3b\mu_1.$$

In view of the constraint (1.7) on the length L, one can choose r small enough to get

$$0 < r < \frac{2}{\pi} \sqrt{\frac{3b\pi^2 - L^2 a}{L}}.$$

Then, we pick $\lambda > 0$ such that (1.9) holds to ensure that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) + 2\lambda V(t) \leq 0, \tag{3.10}$$

for all $t > 0$. Therefore, integrating (3.10) over $(0,t)$, and thanks to (3.2) and (3.3), yields that

$$E(t) \leq \left(1 + \max\left\{ \mu_1 L, \frac{\mu_2}{|\beta|} \right\} \right) E(0) e^{-2\lambda t}, \tag{3.11}$$

[2] $\|u\|^2_{L^2(0,L)} \leq \frac{L^2}{\pi^2} \|\partial_x u\|_{L^2(0,L)}$ for $u \in H_0^2(0,L)$.

for all $t > 0$, which completes the proof. □

4. Second stability result *via* compactness-uniqueness argument

The second part of this manuscript is devoted to the proof of another stability result of (1.2) and (1.3) stated in Theorem 1.2. To be more precise, we shall show a generic exponential stability result of the solutions to (1.2) and (1.3) by attempting to study the phenomenon of critical lengths of the system.

4.1. Stability of the linear system

We first prove that the following observability inequality ensures that the linear system (2.1) is exponentially stable.

Proposition 4.1. *Assume that α and β satisfies (1.4) and $L > 0$. Thus, there exists a constant $C > 0$, such that for all $(u_0, z_0) \in H$*

$$\int_0^L u_0^2(x)dx + |\beta|h \int_0^1 z_0^2(-h\rho)d\rho \leq C \int_0^T \left((\partial_x^2 u(0,t))^2 + z^2(1,t)\right) dt \tag{4.1}$$

where $(u, z) = S(.)\,(u_0, z_0(-h\cdot))$ is the solution of the system (2.1) and (2.2).

Indeed, if (4.1) is true, we get

$$E(T) - E(0) \leq -\frac{E(0)}{C} \Rightarrow E(T) \leq E(0) - \frac{E(0)}{C} \leq E(0) - \frac{E(T)}{C},$$

where $E(t)$ is defined by (1.5). Thus,

$$E(T) \leq \gamma E(0), \quad \text{where} \quad \gamma = \frac{C}{1+C} < 1. \tag{4.2}$$

Now, the same argument used on the interval $[(m-1)T, mT]$ for $m = 1, 2, \ldots$ yields that

$$E(mT) \leq \gamma E((m-1)T) \leq \cdots \leq \gamma^m E(0).$$

Thus, we have

$$E(mT) \leq e^{-\nu mT} E(0) \quad \text{with} \quad \nu = \frac{1}{T}\ln\left(1 + \frac{1}{C}\right) > 0.$$

For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$, and by the non-increasing property of the energy, we conclude that

$$E(t) \leq E((m-1)T) \leq e^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} e^{-\nu t} E(0),$$

showing the exponential stability result for the linear system.

For sake of clarity, the proof of Proposition 4.1 will be achieved by steps. Moreover, to be simple and without loss of generality, we will take $a = b = 1$.

Step 1: Compactness–uniqueness argument

We argue by contradiction. Suppose that (4.1) does not hold and hence there exists a sequence $((u_0^n, z_0^n(-h\cdot)))_n \subset H$ such that

$$\int_0^L \left(u_0^n\right)^2(x)dx + |\beta|h \int_0^1 \left(z_0^n\right)^2(-h\rho)d\rho = 1 \tag{4.3}$$

and

$$\left\|\partial_x^2 u^n(0,.)\right\|_{L^2(0,T)}^2+\left\|z^n(1,.)\right\|_{L^2(0,T)}^2 \rightarrow 0 \text{ as } n \rightarrow +\infty, \tag{4.4}$$

where $(u^n, z^n)=S\left(u_0^n, z_0^n(-h\cdot)\right)$.

Owing to Proposition 2.1, $\left(u^n\right)_n$ is a bounded sequence in $L^2\left(0, T, H^2(0, L)\right)$, and consequently

$$\partial_t u^n=-\partial_x u^n-\partial_x^3 u^n+\partial_x^5 u \quad \text{is bounded in} \quad L^2\left(0, T, H^{-3}(0, L)\right).$$

Thanks to a result of [39], $\left(u^n\right)_n$ is relatively compact in $L^2\left(0, T, L^2(0, L)\right)$ and we may assume that $\left(u^n\right)_n$ is convergent in $L^2\left(0, T, L^2(0, L)\right)$. Moreover, using (2.9) and (4.4), we have that $\left(u_0^n\right)_n$ is a Cauchy sequence in $L^2(0, L)$.

Claim 1. *If $T>h$, then $\left(z_0^n(-h\cdot)\right)_n$ is a Cauchy sequence in $L^2(0,1)$.*

In fact, since $z^n(\rho, T)=u_{xx}^n(0, T-\rho h)$, if $T>h$, we have

$$\int_0^1\left(z^n(\rho, T)\right)^2 d\rho=\int_0^1\left(\partial_x^2 u^n(0, T-\rho h)\right)^2 d\rho \leq \frac{1}{h} \int_0^T\left(\partial_x^2 u^n(0, t)\right)^2 \mathrm{d}t.$$

Using (2.10), for $T>h$ yields that

$$\left\|z_0^n(-h\cdot)\right\|_{L^2(0,1)}^2 \leq \frac{1}{h}\left\|\partial_x^2 u^n(0, \cdot)\right\|_{L^2(0, T)}^2+\frac{1}{h}\left\|z^n(1, \cdot)\right\|_{L^2(0, T)}^2 .$$

Thus, $\left(z_0^n(-h\cdot)\right)_n$ is a Cauchy sequence in $L^2(0,1)$ by means of (4.4) and hence the Claim 1 is ascertained.

Now, let us pick $\left(u_0, z_0(-h\cdot)\right)=\lim_{n \rightarrow \infty}\left(u_0^n, z_0^n(-h\cdot)\right)$ in H. This, together with (4.3), yields that

$$\int_0^L u_0^2(x) dx+|\beta| h \int_0^1 z_0^2(-h \rho) d\rho=1.$$

Furthermore, let $(u, z)=S(\cdot)\left(u_0, z_0(-h\cdot)\right)$, which implies, thanks to Proposition 2.1, that

$$\left(\partial_x^2 u(0, \cdot), z(1, \cdot)\right)=\lim_{n \rightarrow \infty}\left(\partial_x^2 u^n(0, \cdot), z^n(1, \cdot)\right)$$

in $L^2(0, T)$. Combining the latter with (4.4) gives $\left(\partial_x^2 u(0, \cdot), z(1, .)\right)=0$. As we have $z(1, t)=\partial_x^2 u(0, t-h)=0$, we deduce that $z_0=0$ and $z=0$. Consequently, u is solution of

$$\begin{cases}\partial_t u+\partial_x u+\partial_x^3 u-\partial_x^5 u=0, & x \in(0, L), t>0, \\ u(0, t)=u(L, t)=\partial_x u(L, t)=\partial_x u(0, t)=\partial_x^2 u(L, t)=\partial_x^2 u(0, t)=0, & t>0, \\ u(x, 0)=u_0(x), & x \in(0, L),\end{cases} \tag{4.5}$$

with

$$\left\|u_0\right\|_{L^2(0, L)}=1. \tag{4.6}$$

Step 2: Reduction to a spectral problem

Lemma 4.2. *For any $T>0$, let N_T denote the space of the initial state $u_0 \in L^2(0, L)$, such that the solution of the Kawahara system $u(t)=S(t) u_0$ satisfies (4.5). Then, $N_T=\{0\}$.*

Proof. We argue as in [38, Theorem 3.7]. If $N_T \neq\{0\}$, then the map $u_0 \in \mathbb{C} N_T \rightarrow A\left(N_T\right) \subset \mathbb{C} N_T$ ($\mathbb{C} N_T$ denotes the complexification of N_T) has (at least) one eigenvalue. Hence, there exists a pair $\left(\lambda, u_0\right) \in \mathbb{C} \times H^5(0, L) \backslash\{0\}$ such that

$$\begin{cases}\lambda u_0+u_0^{\prime}+u_0^{\prime \prime \prime}-u_0^{\prime \prime \prime \prime \prime}=0, & \text{in } (0, L), \\ u_0(0)=u_0(L)=u_0^{\prime}(0)=u_0^{\prime}(L)=u_0^{\prime \prime}(0)=u_0^{\prime \prime}(L)=0 .\end{cases}$$

To obtain the contradiction, it remains to prove that such a pair $\left(\lambda, u_0\right)$ does not exist. This will be done in the next step. ∎

Step 3: Möbius transformation

To simplify the notation, henceforth, we denote $u_0 := u$. Moreover, the notation $\{0, L\}$ means that the function is applied to 0 and L, respectively.

Lemma 4.3. *Let $L > 0$ and consider the assertion*

$$(\mathcal{N}): \quad \exists \lambda \in \mathbb{C}, \exists u \in H_0^2(0,L) \cap H^5(0,L) \text{ such that } \begin{cases} \lambda u + u' + u''' - u''''' = 0, & \text{on } (0,L), \\ u(x) = u'(x) = u''(x) = 0, & \text{in } \{0,L\}. \end{cases}$$

If $(\lambda, u) \in \mathbb{C} \times H_0^2(0,L) \cap H^5(0,L)$ is a solution of $(\mathcal{N})$, then $u = 0$.

Proof. Consider the following system:

$$\begin{cases} \lambda u + u' + u''' - u''''' = 0, & \text{on } (0,L), \\ u(x) = u'(x) = u''(x) = 0, & \text{in } \{0,L\}. \end{cases} \tag{4.7}$$

Multiplying equation (4.7) by $\overline{u}$ and integrating in $[0, L]$, we have that λ is purely imaginary, i.e., $\lambda = ir$, for $r \in \mathbb{R}$. Now, extending the function u to $\mathbb{R}$ by setting $u = 0$ for $x \notin [0, L]$, we have that the extended function satisfies

$$\lambda u + u' + u''' - u''''' = -u''''(0)\delta_0' + u''''(L)\delta_L' - u'''(0)\delta_0 + u'''(L)\delta_L,$$

in $\mathcal{S}'(\mathbb{R})$, where δ_ζ denotes the Dirac measure at $x = \zeta$ and the derivatives $u''''(0)$, $u''''(L)$, $u'''(0)$ and $u'''(L)$ are those of the function u when restricted to $[0, L]$. Taking the Fourier transform of each term in the above system and integrating by parts, we obtain

$$\lambda\hat{u}(\xi) + i\xi\hat{u}(\xi) + (i\xi)^3\hat{u}(\xi) - (i\xi)^5\hat{u}(\xi) = -(i\xi)u'''(0) + (i\xi)u'''(L)e^{-iL\xi} - u''''(0) + u''''(L)e^{-iL\xi}.$$

Take $\lambda = -ir$ and let $f_\alpha(\xi, L) = i\hat{u}(\xi)$. The latter gives

$$f_\alpha(\xi, L) = \frac{N_\alpha(\xi, L)}{q(\xi)},$$

where $N_\alpha(\cdot, L)$ is defined by

$$N_\alpha(\xi, L) = \alpha_1 i\xi - \alpha_2 i\xi e^{-i\xi L} + \alpha_3 - \alpha_4 e^{-i\xi L} \tag{4.8}$$

and

$$q(\xi) = \xi^5 + \xi^3 - \xi + r,$$

where α_i, for $i = 1, 2, 3, 4$, are the traces of u''' and u''''.

For each $r \in \mathbb{R}$ and $\alpha \in \mathbb{C}^4 \setminus \{0\}$, let $\mathcal{F}_{\alpha r}$ be the set of $L > 0$ values, for which the function $f_\alpha(\cdot, L)$ is entire. Now, let us recall the equivalent following statements:

- A1 $f_\alpha(\cdot, L)$ is entire;
- A2 All zeros, taking the respective multiplicities into account, of the polynomial q are zeros of $N_\alpha(\cdot, L)$;
- A3 The maximal domain of $f_\alpha(\cdot, L)$ is $\mathbb{C}$.

Whereupon, the function $f_\alpha(\cdot, L)$ is entire, due to the equivalence between statement A1 and A2, if the following holds

$$\frac{\alpha_1 i\xi_i + \alpha_3}{\alpha_2 i\xi_i + \alpha_4} = e^{-iL\xi_i},$$

where ξ_i denotes the zeros of $q(\xi)$, for $i = 1, 2, 3, 4, 5$. Thereafter, let us define, for $\alpha \in \mathbb{C}^4 \setminus \{0\}$, the following discriminant

$$d(\alpha) = \alpha_1\alpha_3 - \alpha_2\alpha_4. \tag{4.9}$$

Then, for $\alpha \in \mathbb{C}^4 \setminus \{0\}$, such that $d(\alpha) \neq 0$ the Möbius transformations can be introduced by

$$M(\xi_i) = e^{-iL\xi_i}, \tag{4.10}$$

for each zero ξ_i of the polynomial $q(\xi)$.

The next claim describes the behavior of the roots of polynomial $q(\cdot)$:

Claim 2. *The polynomial $q(\cdot)$ has exactly one real root with multiplicity 1 and two pairs of complex conjugate roots.*

Indeed, we suppose that $r \neq 0$ (the case $r = 0$ will be discussed later). Note that the derivative of q is given by

$$q'(\xi) = 5\xi^4 + 3\xi^2 - 1,$$

and its zeros are $\pm z_1$ and $\pm z_2$, where

$$z_1 = \sqrt{\frac{-3-\sqrt{29}}{10}} \quad \text{and} \quad z_2 = \sqrt{\frac{-3+\sqrt{29}}{10}}.$$

It is easy to see that z_1 belongs to $\mathbb{C} \setminus \mathbb{R}$ and z_2 belongs to $\mathbb{R}$. Hence, the polynomial $q(\cdot)$ does not have critical points, which means that $q(\cdot)$ has exactly one real root. Suppose that $\xi_0 \in \mathbb{R}$ is the root of $q(\cdot)$ with multiplicity $m \leq 5$. Consequently,

$$q(\xi_0) = q'(\xi_0) = ... = q^{(m-1)}(\xi_0) = 0.$$

Consider the following cases:

(i) If ξ_0 has multiplicity 5, it follows that $q(\xi_0) = 0$ and $q''''(\xi_0) = -120\xi_0 = 0$, implying that $\xi_0 = 0$ and $r = 0$.
(ii) If ξ_0 has multiplicity 4, it follows that $q'''(\xi_0) = 60\xi_0^2 + 6 = 0$ and thus $\xi_0 \in i\mathbb{R}$.
(iii) If ξ_0 has multiplicity 3, it follows that $q(\xi_0) = 0$ and $q''(\xi_0) = 20\xi_0^3 + 6\xi_0 = 0$ and hence $\xi_0 = 0$ and $r = 0$ or $\xi_0 \in i\mathbb{R}$.
(iv) If ξ_0 has multiplicity 2, it follows that $q'(\xi_0) = 5\xi_0^4 + 3\xi_2 - 1 = 0$, implying that $\xi_0 \in \mathbb{C} \setminus \mathbb{R}$.

Note that in all cases, since $r \neq 0$ and $\xi_0 \in \mathbb{R}$, we get a contradiction. Consequently, $q(\cdot)$ has exactly one real root, with multiplicity 1. This means that this polynomial has two pairs of complex conjugate roots.

Now, we assume that $r = 0$. Then, we obtain that $q(\xi) = \xi(\xi^4 + \xi^2 - 1)$, whose roots are $0, \pm\rho$ and $\pm k$ where

$$\rho = \sqrt{\frac{\sqrt{5}-1}{2}} \quad \text{and} \quad k = i\sqrt{\frac{1+\sqrt{5}}{2}} \tag{4.11}$$

Thus, $q(\cdot)$ has two pairs of complex conjugate roots and three real roots, proving Claim 2.

Further to Claim 2, and in order to conclude the proof of Lemma 4.3, we need two additional lemmas whose proofs are given in [23] (see Lemmas 2.1 and 2.2).

Lemma 4.4. *Let non null $\alpha \in \mathbb{C}^4$ with $d(\alpha) = 0$ and $L > 0$ for $d(\alpha)$ defined in (4.9). Then, the set of the imaginary parts of the zeros of $N_\alpha(\cdot, L)$ in (4.8) has at most two elements.*

Lemma 4.5. *For any $L > 0$, there is no Möbius transformation M, such that*

$$M(\xi) = e^{-iL\xi}, \quad \xi \in \{\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2\},$$

with $\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2$ all distinct in $\mathbb{C}$.

We are now in position to prove Lemma 4.3. Let us consider two cases:

(i) $d(\alpha) \neq 0$;
(ii) $d(\alpha) = 0$,

where $d(\alpha)$ was defined in (4.9).

First, supposing that $d(\alpha) \neq 0$, we can define the Möbius transformation. In fact, suppose by contradiction that there exists $L > 0$ such that the function $f_a(\cdot, L)$ is entire. Then, all roots of the polynomial $q(\cdot)$ must satisfy (4.10), i.e., there exists a Möbius transformation that takes each root ξ_0 of $q(\cdot)$ into $e^{-iL\xi_0}$. However, this contradicts Lemma 4.5 and proves that if $(\mathcal{N})$ holds, then $\mathcal{F}_{\alpha r} = \emptyset$ for all $r \in \mathbb{R}$. On the other hand, suppose that $d(\alpha) = 0$ and note that by using claim 2, we can conclude that the set of the imaginary parts of the polynomial $q(\cdot)$ has at least three elements, thus it follows from Lemma 4.4 that $\mathcal{F}_{\alpha r} = \emptyset$ for all $r \in \mathbb{R}$. Note that in both cases, we have that $\mathcal{F}_{\alpha r} = \emptyset$, which implies that $(\mathcal{N})$ has only the trivial solution for any $L > 0$, and the proof of Lemma 4.3 is archived. □

Proof of Proposition 4.1. Notice that (4.6) implies that the solution u cannot be identically zero. However, from Lemma 4.2, one can conclude that $u = 0$, which drives us to a contradiction. □

4.2. Proof of theorem 1.2

Let us consider the nonlinear Kawahara system (1.2) and (1.3), with a small initial data $\|(u_0, z_0)\|_H \leq r$, where r will be chosen later. The solution u of (1.2) and (1.3), with $p = 2$, can be written as $u = u_1 + u_2$, where u_1 is the solution of

$$\begin{cases} \partial_t u_1 - \partial_x^5 u_1 + b\partial_x^3 u_1 + a\partial_x u_1 = 0, & x \in (0, L), t > 0, \\ u_1(0, t) = u_1(L, t) = \partial_x u_1(0, t) = \partial_x u_1(L, t) = 0, & t > 0, \\ \partial_x^2 u_1(L, t) = \alpha\partial_x^2 u_1(0, t) + \beta\partial_x^2 u_1(0, t - h), & t > 0, \\ \partial_x^2 u_1(0, t) = z_0(t), & t \in (-h, 0), \\ u_1(x, 0) = u_0(x), & x \in (0, L), \end{cases}$$

and u_2 is solution of

$$\begin{cases} \partial_t u_2 - \partial_x^5 u_2 + b\partial_x^3 u_2 + a\partial_x u_2 = -u^2\partial_x u, & x \in (0, L), t > 0, \\ u_2(0, t) = u_2(L, t) = \partial_x u_2(0, t) = \partial_x u_2(L, t) = 0, & t > 0, \\ \partial_x^2 u_2(L, t) = \alpha\partial_x^2 u_2(0, t) + \beta\partial_x^2 u_2(0, t - h), & t \in (-h, 0), \\ \partial_x^2 u_2(0, t) = 0, & x \in (0, L), \\ u_2(x, 0) = 0, & x \in (0, L), \end{cases}$$

Note that, in this case, u_1 is the solution of (2.1) and (2.2) with the initial data $(u_0, z_0) \in H$ and u_2 is solution of (2.19) with null data and right-hand side $f = u^2\partial_x u \in L^1(0, T; L^2(0, L))$, as in Lemma 2.6.

Now, thanks to (4.2), Proposition 2.5 and Lemma 2.6, we have that

$$\begin{aligned} \|(u(T), z(T))\|_H \leq& \left\|\left(u^1(T), z^1(T)\right)\right\|_H + \left\|\left(u^2(T), z^2(T)\right)\right\|_H \\ \leq& \gamma \left\|(u_0, z_0(-h\cdot))\right\|_H + C \left\|u^p u_x\right\|_{L^1(0,T,L^2(0,L))} \\ \leq& \gamma \left\|(u_0, z_0(-h\cdot))\right\|_H + C\|u\|^2_{L^2(0,T,H^2(0,L))}, \end{aligned} \tag{4.12}$$

with $\gamma \in (0, 1)$. The goal now is to deal with the least term of the previous inequality. To this end, we use the multipliers method. First, we multiply the first equation of (1.2) and (1.3) by xu and integrate by parts to obtain

$$\begin{aligned} &\frac{1}{2}\int_0^L x|u(x, T)|^2 dx + \frac{3b}{2}\int_0^T\int_0^L |\partial_x u(x, t)|^2 \, dxdt + \frac{5}{2}\int_0^T\int_0^L \left|\partial_x^2 u(x, t)\right|^2 dx\mathrm{d}t \\ &= \frac{1}{a}\int_0^T\int_0^L |u(x, t)|^2 dx\mathrm{d}t + \frac{L}{2}\int_0^T (\partial_x^2 u(L, t))^2 \mathrm{d}t + \frac{1}{2}\int_0^L x\left|u_0(x)\right|^2 dx + \frac{1}{4}\int_0^T\int_0^L |u|^4 \, \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Consequently, using the boundary conditions of (1.2), (1.3) and (2.24), we get

$$3b\int_0^T\int_0^L |\partial_x u(x,t)|^2\,dxdt+5\int_0^T\int_0^L \left|\partial_x^2 u(x,t)\right|^2 dxdt\le (aT+L)\left\|(u_0,z_0)\right\|_H^2$$
$$+L\int_0^T(\alpha\partial_x^2u(0,t)+\beta z(1,t))^2\mathrm{d}t+\frac{1}{2}\int_0^T\int_0^L |u|^4\,\mathrm{d}x\mathrm{d}t.$$

Note that Gagliardo–Nirenberg inequality ensures that

$$\begin{aligned}\int_0^T\int_0^L u^4dxdt&\le C\int_0^T \|u\|^3_{L^2(0,L)}\|u_x\|_{L^2(0,L)}\,\mathrm{d}t\\&\le C\frac{1}{2\varepsilon}\int_0^T\|u\|^6_{L^2(0,L)}\mathrm{d}t+C\frac{\varepsilon}{2}\int_0^T\|u_x\|^2_{L^2(0,L)}\,\mathrm{d}t\\&\le C(T)\frac{1}{2\varepsilon}\|u\|^6_{L^\infty(0,T;L^2(0,L))}+C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}\\&\le C(T)\frac{1}{2\varepsilon}\left\|(u_0,z_0)\right\|_H^6+C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}.\end{aligned}$$

Putting together the previous inequalities, we have

$$\begin{aligned}3b\int_0^T\int_0^L |\partial_x u(x,t)|^2\,\mathrm{d}x\mathrm{d}t+5\int_0^T\int_0^L \left|\partial_x^2 u(x,t)\right|^2 \mathrm{d}x\mathrm{d}t\le (aT+L)\left\|(u_0,z_0)\right\|_H^2\\+L\int_0^T(\alpha\partial_x^2u(0,t)+\beta z(1,t))^2\mathrm{d}t+C(T)\frac{1}{2\varepsilon}\left\|(u_0,z_0)\right\|_H^6+C\frac{\varepsilon}{2}\|u\|^2_{L^2(0,T;H^2(0,L))}.\end{aligned}\tag{4.13}$$

Now, multiplying the first equation of (1.2) by u and integrating by parts yields that

$$\int_0^L u^2(x,T)\mathrm{d}x-\int_0^L u_0^2(x)\mathrm{d}x-\int_0^T\left(\alpha u_{xx}(0,t)+\beta z(1,t)\right)^2\mathrm{d}t+\int_0^T u_x^2(0,t)\mathrm{d}t=0$$

Using the same idea as in the proof of (2.8), we have that

$$\int_0^T(\partial_x^2u)^2(0,t)\mathrm{d}t+\int_0^T z^2(1,t)dt\le C\left\|(u_0,z_0)\right\|_H^2.$$

Consequently, the previous inequality gives

$$\int_0^T\left(\alpha\partial_x^2u(0,t)+\beta z(1,t)\right)^2\mathrm{d}t\le 2C\left(\alpha^2+\beta^2\right)\left\|(u_0,z_0)\right\|_H^2.$$

Thus, putting the previous inequality in (4.13), and choosing $\varepsilon>0$ sufficiently small, there exists $C>0$ such that

$$\|u\|^2_{L^2(0,T;H^2(0,L))}\le C\left(\left\|(u_0,z_0)\right\|_H^2+\left\|(u_0,z_0)\right\|_H^6\right).\tag{4.14}$$

Finally, gathering (4.12) and (4.14), there exists $C > 0$ such that the following holds true

$$\|(u(T), z(T))\|_H \leq \|(u_0, z_0)\|_H \left(\gamma + C\,\|(u_0, z_0)\|_H + C\,\|(u_0, z_0)\|_H^5\right),$$

which implies

$$\|(u(T), z(T))\|_H \leq \|(u_0, z_0)\|_H \left(\gamma + Cr + Cr^5\right).$$

Given $\epsilon > 0$ small enough such that $\gamma + \epsilon < 1$, we can take r small enough such that $r + r^5 < \frac{\epsilon}{C}$, in order to have

$$\|(u(T), z(T))\|_H \leq (\gamma + \epsilon)\,\|(u_0, z_0)\|_H,$$

with $\gamma + \epsilon < 1$. Theorem 1.2 follows using the semigroup property as in (4.2). □

5. Further comments and open problems

Our work presents a further step after the work [2] for a better understanding of the stabilization problem for the Kawahara equation. Indeed, a boundary time-delayed damping control is proposed to stabilize the equation in contrast to [2], where an interior damping is required and no delay is taken into consideration. We conclude our paper with a few comments and also some open problems.

Remark 5.1. In what concerns our main results, Theorems 1.1 and 1.2, the following remarks are worth mentioning:

- Note that the rate λ of the Theorem 1.1 decreases as the delay h increases, since we have the restriction (1.9).
- A simple calculation shows that taking $\mu_1, \mu_2 \in (0, 1)$ in Theorem 1.1 such that
$$\mu_2 < \min\left\{1 - |\beta| - \alpha^2, \frac{(|\beta| - 1)^2 - \alpha^2}{1 - |\beta|}, \frac{\alpha^2 - \beta^2 + |\beta|}{|\beta|}\right\}$$
and
$$\mu_1 < \min\left\{\frac{1 - |\beta| - \mu_2 - \alpha^2}{L\alpha^2}, \frac{(|\beta| - 1)^2 - \alpha^2 - \mu_2(1 - |\beta|)}{L(\alpha^2 - \beta^2 + |\beta|(1 - \mu_2))}\right\}$$
implies that $M_{\mu_1}^{\mu_2}$ is negative definite.
- Note that the presence of nonlinearity on the equation yields the restriction about the initial data. Hence, if we remove it, that is, by considering the linear system, it is possible to obtain the same result of the Theorem 1.1, with the same process. Nevertheless, the decay rate λ is given by
$$\lambda \leq \min\left\{\frac{\mu_2}{2h(\mu_2 + |\beta|)}, \frac{3b\pi^2 - L^2 a}{2L^2(1 + L\mu_1)}\right\}. \tag{5.1}$$
- For sake of simplicity, we only considered in this article the nonlinearity $u^2 u_x$. However, Theorems 1.1 and 1.2 are still valid for $u^p u_x$, $p \in [1, 2)$, where the proof is very similar and hence omitted.
- Recently, Zhou [45] proved the well-posedness of the following initial boundary value problem
$$\begin{cases} \partial_t u - \partial_x^5 u = c_1 u \partial_x u u + c_2 u^2 \partial_x u + b_1 \partial_x u \partial_x^2 u + b_2 u \partial_x^3 u, & x \in (0, L),\ t \in \mathbb{R}^+, \\ u(t, 0) = h_1(t), \quad u(t, L) = h_2(t), \quad \partial_x u(t, 0) = h_3(t), & t \in \mathbb{R}^+, \\ \partial_x u(t, L) = h_4(t), \quad \partial_x^2 u(t, L) = h(t), & t \in \mathbb{R}^+, \\ u(0, x) = u_0(x), & x \in (0, L), \end{cases} \tag{5.2}$$
Thus, due to this result, when we consider $b_1 = b_2 = 0$ and the combination $c_1 u \partial_x u + c_2 u^2 \partial_x u$ instead of $u^p \partial_x u$, for $p \in [1, 2]$, in (1.2), the main results of our article remain valid.
- We point out that considering $a = 0$ in (1.2), Theorem 1.1 holds true. Additionally, no restriction is necessary in the length $L > 0$, and also Theorem 1.2 is still verified (see, for instance, [9, 42]).

5.1. Open problems

Based on the outcomes of this paper on the dispersive Kawahara equation, some interesting open problems appear.

5.1.1. Restriction of the Lyapunov approach. Observe that in our first result, Theorem 1.1, since the outcome is based on the appropriate choice of Lyapunov functional, we have a restriction (1.7) on the length L. This is due to the choice of the Morawetz multipliers x in the expression of V_1 defined by (3.1). Therefore, the following natural question arises.
Question A: Can we choose another Lyapunov functional, instead of the previous one to remove the restriction over L?

5.1.2. Set of critical lengths. As observed in [2], considering the following initial boundary value problem for the Kawahara equation

$$\begin{cases} u_t - u_x + u_{xxx} - u_{xxxxx} = 0, & x \in (0,L), t > 0 \\ u(0,t) = u(L,t) = u_x(L,t) = u_x(0,t) = u_{xx}(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{5.3}$$

it is possible to construct a nontrivial steady-state solution to (5.3) with a nonzero initial datum $u_0(x) \not\equiv 0$ and homogeneous boundary conditions upon the endpoints of the interval with a critical length. Precisely, when the authors considered the following constants

$$a = \sqrt{\sqrt{5} + 1/2}, \quad b = \sqrt{\sqrt{5} - 1/2}, \quad A = C_2 + C_3, \quad B = C_2 - C_3$$
$$C_2 = 1 - e^{-aL}, \quad C_3 = e^{aL} - 1, \quad C_1 = -\left(1 + \frac{a^2}{b^2}\right) A, \quad C_4 = \frac{a^2}{b^2} A, \quad C_5 = -\frac{a}{b} B,$$

they were able to define the set

$$\mathcal{N} = \left\{ L > 0 : e^{ibL} = \left(\frac{C_4 + iC_5}{|C_4 + iC_5|} \right)^2 \right\} \subset \mathbb{R}^+$$

and

$$u(x) = C_1 + C_2 e^{ax} + C_3 e^{-ax} + C_4 \cos(bx) + C_5 \sin(bx) \not\equiv 0, \quad x \in (0,L).$$

If $L \in \mathcal{N}$, then $u = u(x)$ solves $-u''''' + u''' + u' = 0$, and satisfies $u(0) = u'(0) = u''(0) = u(L) = u'(L) = u''(L) = 0$.

So, in our context, if we consider a function $N_\alpha : \mathbb{C} \times (0,\infty) \to \mathbb{C}$, with $\alpha \in \mathbb{C}^4 \setminus \{0\}$, whose restriction $N_\alpha(\cdot, L)$, given by (1.10), is entire for each $L > 0$ and a family of functions $f_\alpha(\cdot, L)$, defined by (1.11), in its maximal domain, the following issue appears.
Question B: Is it possible to find $a \in \mathbb{C}^4 \setminus \{0\}$ such that the function $f_a(\cdot, L)$ is an entire function?

Note that the proof of Theorem 1.2 heavily relies on a unique continuation property of the spectral problem associated to the space operator (see Lemma 4.3). However, due to the structure of the terms ∂_x^3 and ∂_x^5 (see Lemma 4.3), we are unable to study the spectral problem in a direct way as in [38]. Hence, due to these two different dispersions of third and fifth order, we believe that a new approach is needed to tackle the previous open question.

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Declaration

Conflict of interests The authors declare that they have no conflict of interest regarding the publication of this paper.

References

[1] Alvarez-Samaniego, B., Lannes, D.: Large time existence for 3D water-waves and asymptotics. Invent. Math. **171**, 485–541 (2008)
[2] Araruna, F.D., Capistrano-Filho, R.A., Doronin, G.G.: Energy decay for the modified Kawahara equation posed in a bounded domain. J. Math. Anal. Appl. **385**, 743–756 (2012)
[3] Baudouin, L., Crépeau, E., Valein, J.: Two approaches for the stabilization of nonlinear KdV equation with boundary time-delay feedback. IEEE Trans. Automat. Control. **64**, 1403–1414 (2019)
[4] Berloff, N., Howard, L.: Solitary and periodic solutions of nonlinear nonintegrable equations. Stud. Appl. Math. **99**(1), 1–24 (1997)
[5] Biswas, A.: Solitary wave solution for the generalized Kawahara equation. Appl. Math. Lett. **22**, 208–210 (2009)
[6] Bona, J.L., Lannes, D., Saut, J.-C.: Asymptotic models for internal waves. J. Math. Pures Appl. **89**(9), 538–566 (2008)
[7] Bona, J.L., Sun, S.M., Zhang, B.-Y.: A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. Commun. Partial Differ. Equ. **28**, 1391–1436 (2003)
[8] Bona, J.L., Sun, S.M., Zhang, B.Y.: A non-homogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain II. J. Differ. Equ. **247**(9), 2558–2596 (2009)
[9] Capistrano-Filho, R. A., de Jesus, I. M.: Massera's theorems for a higher order dispersive system, arXiv:2205.12200 [math.AP] (2022)
[10] Capistrano-Filho, R.A., de Gomes, M.M.S.: Well-posedness and controllability of Kawahara equation in weighted Sobolev spaces. Nonlinear Anal. **207**, 1–24 (2011)
[11] Capistrano-Filho, R. A., Gonzalez Martinez, V. H.: Stabilization results for delayed fifth order KdV-type equation in a bounded domain, arXiv:2112.14854 [math.AP] (2021)
[12] Capistrano-Filho, R.A., de Sousa, L.S.: Control results with overdetermination condition for higher order dispersive system. J. Math. Anal. Appl. **506**, 1–22 (2022)
[13] Capistrano-Filho, R. A., de Sousa, L. S., Gallego, F. A.: Control of Kawahara equation with overdetermination condition: the unbounded cases, arXiv:2110.08803 [math.AP] (2021)
[14] Chentouf, B.: Well-posedness and exponential stability of the Kawahara equation with a time-delayed localized damping. Math. Methods Appl. Sci. **45**, 10312–10330 (2022). https://doi.org/10.1002/mma.8369
[15] Chen, M.: Internal controllability of the Kawahara equation on a bounded domain. Nonlinear Anal. **185**, 356–373 (2019)
[16] Coclite, G.M., di Ruvo, L.: On the classical solutions for a Rosenau-Korteweg-deVries-Kawahara type equation. Asymptot. Anal. **129**, 51–73 (2022)
[17] Coclite, G.M., di Ruvo, L.: Wellposedness of the classical solutions for a Kawahara-Korteweg-de Vries type equation. J. Evol. Equ. **21**, 625–651 (2021)
[18] Coclite, G.M., di Ruvo, L.: Convergence results related to the modified Kawahara equation. Boll. Unione Mat. Ital. **8**, 265–286 (2016)
[19] Cui, S., Tao, S.: Strichartz estimates for dispersive equations and solvability of the Kawahara equation. J. Math. Anal. Appl. **304**, 683–702 (2005)

[20] Doronin, G.G., Larkin, N.A.: Kawahara equation in a quarter-plane and in a finite domain. Bol. Soc. Parana. Mat. **25**, 9–16 (2007)
[21] Doronin, G.G., Larkin, N.A.: Boundary value problems for the stationary Kawahara equation. Nonlinear Anal **69**, 1655–1665 (2008)
[22] Doronin, G.G., Larkin, N.A.: Kawahara equation in a bounded domain. Discrete Contin. Dyn. Syst. Ser. B **10**, 783–799 (2008)
[23] dos Santos, A.L.C., da Silva, P.N., Vasconcellos, C.F.: Entire functions related to stationary solutions of the Kawahara equation. Electron. J. Differ. Equ. **43**, 13 (2016)
[24] Faminskii, A.V., Larkin, N.A.: Initial-boundary value problems for quasilinear dispersive equations posed on a bounded interval. Electron. J. Differ. Equ. **2010**(1), 1–20 (2010)
[25] Hirayama, H.: Local well-posedness for the periodic higher order KdV type equations. NoDEA Nonlinear Differ. Equ. Appl. **19**, 677–693 (2012)
[26] Hasimoto, H.: *Water waves*, their dispersion and steeping, Kagaku (Science). **40**, 401–408 [Japanese] (1970)
[27] Hunter, J.K., Scheurle, J.: Existence of perturbed solitary wave solutions to a model equation for water waves. Phys. D **32**, 253–268 (1998)
[28] Isaza, P., Linares, F., Ponce, G.: Decay properties for solutions of fifth order nonlinear dispersive equations. J. Differ. Equ. **258**, 764–795 (2015)
[29] Jin, L.: Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation. Math. Comput. Model. **49**, 573–578 (2009)
[30] Kato, T.: Low regularity well-posedness for the periodic Kawahara equation. Differ. Integr. Equ. **25**, 1011–1036 (2012)
[31] Kawahara, T.: Oscillatory solitary waves in dispersive media. J. Phys. Soc. Jpn. **33**, 260–264 (1972)
[32] Lannes, D.: *The water waves problem. Mathematical analysis and asymptotics*. In Mathematical Surveys and Monographs, 188. American Mathematical Society, Providence, pp. 321 (2013)
[33] Larkin, N.A.: Correct initial boundary value problems for dispersive equations. J. Math. Anal. Appl. **344**, 1079–1092 (2008)
[34] Larkin, N.A., Simoes, M.H.: The Kawahara equation on bounded intervals and on a half-line. Nonlinear Anal. TMA **127**, 397–412 (2015)
[35] Nicaise, S., Pignotti, C.: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. **45**, 1561–1585 (2006)
[36] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York (1983)
[37] Polat, N., Kaya, D., Tutalar, H.I.: An analytic and numerical solution to a modified Kawahara equation and a convergence analysis of the method. Appl. Math. Comput. **179**, 466–472 (2006)
[38] Rosier, L.: Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. **2**, 33–55 (1997)
[39] Simon, J.: Compact sets in the space $L^p(0,T;B)$. Ann. Mat. Pura Appl. **146**, 65–96 (1986)
[40] Vasconcellos, C.F., Silva, P.N.: Stabilization of the linear Kawahara equation with localized damping. Asymptot. Anal. **58**, 229–252 (2008)
[41] Vasconcellos, C.F., Silva, P.N.: Stabilization of the linear Kawahara equation with localized damping. Asymptot. Anal. **66**, 119–124 (2010)
[42] Vasconcellos, C.F., Silva, P.N.: Stabilization of the Kawahara equation with localized damping. ESAIM Control Optim. Calc. Var. **17**, 102–116 (2011)
[43] Xu, G.Q., Yung, S.P., Li, L.K.: Stabilization of wave systems with input delay in the boundary control. ESAIM Control Optim. Calc. Var. **12**, 770–785 (2006)
[44] Yusufoglu, E., Bekir, A., Alp, M.: Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using Sine-Cosine method. Chaos Solitons Fractals **37**, 1193–1197 (2008)
[45] Zhou, D.: Non-homogeneous initial-boundary-value problem of the fifth-order Korteweg-de Vries equation with a nonlinear dispersive term. J. Math. Anal. Appl. **497**, Paper 124848, 1–28 (2021)

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