On the control issues for higher-order nonlinear dispersive equations on the circle

Roberto de A. Capistrano–Filho [a], Chulkwang Kwak [b],*,
Francisco J. Vielma Leal [c]

[a] *Departamento de Matemática, Universidade Federal de Pernambuco, S/N Cidade Universitária, 50740-545, Recife (PE), Brazil*
[b] *Department of Mathematics, Ewha Womans University, Seoul 03760, Republic of Korea*
[c] *Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica (IMECC), 13083-859, Campinas (SP), Brazil*

A R T I C L E   I N F O

A B S T R A C T

The local and global control results for a general higher-order KdV-type operator posed on the unit circle are presented. Using spectral analysis, we are able to prove local results, that is, the equation is locally controllable and exponentially stable. To extend the local results to the global one we captured the smoothing properties of the Bourgain spaces, the so-called *propagation of regularity*, which are proved with a new perspective. These propagation, together with the *Strichartz estimates*, are the key to extending the local control properties to the global one, precisely, higher-order KdV-type equations are globally controllable and exponentially stabilizable in the Sobolev space $H^s(\mathbb{T})$ for any $s \geq 0$. Our results recover previous results in the literature for the KdV and Kawahara equations and extend, for a general higher-order operator of KdV-type, the Strichartz estimates as well as the propagation results, which are the main novelties of this work.

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1. Introduction

1.1. Model description

The full water wave system is too complex to allow to easily derive and rigorously from it relevant qualitative information on the dynamics of the waves. Alternatively, under suitable assumption on amplitude, wavelength, wave steepness and so on, the study on asymptotic models for water waves has been extensively investigated to understand the full water wave system, see, for instance, [1–6] and references therein for a rigorous justification of various asymptotic models for surface and internal waves.

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* Corresponding author.
 E-mail addresses: roberto.capistranofilho@ufpe.br (R.de A. Capistrano–Filho), ckkwak@ewha.ac.kr (C. Kwak), vielma@ime.unicamp.br (F.J. Vielma Leal).

Particularly, formulating the waves as a free boundary problem of the incompressible, irrotational Euler equation in an appropriate non-dimensional form, one has two non-dimensional parameters $\delta := \frac{h}{\lambda}$ and $\varepsilon := \frac{a}{h}$, where the water depth, the wave length and the amplitude of the free surface are parameterized as h, λ and a, respectively. Moreover, another non-dimensional parameter μ is called the Bond number, which measures the importance of gravitational forces compared to surface tension forces. The physical condition $\delta \ll 1$ characterizes the waves, which are called long waves or shallow water waves, but there are several long wave approximations according to relations between ε and δ, specially,

(1) Korteweg–de Vries (KdV): $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$.
(2) Kawahara: $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu \varepsilon^{\frac{1}{2}}$.

Under the regime for ε, δ, μ given in Item (1), Korteweg and de Vries [7][1] derived the following equation well-known as a central equation among other dispersive or shallow water wave models called the KdV equation from the equations for capillary–gravity waves:

$$\pm 2u_t + 3uu_x + \left(\frac{1}{3} - \mu \right) u_{xxx} = 0.$$

In connection with the critical Bond number $\mu = \frac{1}{3}$, Hasimoto [9] derived a fifth-order KdV equation of the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45} u_{xxxxx} = 0$$

in the regime for ε, δ, μ given in Item (2), which is nowadays called the Kawahara equation.

Our main focus is to investigate the higher-order extension of KdV and Kawahara equations. Consider the Cauchy problem for the following higher-order KdV-type equation posed on the unit circle $\mathbb{T}$:

$$\begin{cases} \partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + \frac{1}{2} \partial_x (u^2) = 0, \\ u(0, x) = u_0(x) \in H^s(\mathbb{T}), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{T}, \qquad (1.1)$$

for $j \in \mathbb{N}$ and u is a real-valued function. Especially, (1.1) is called KdV and Kawahara equation when $j = 1$ and $j = 2$, respectively. These types of equations have conservation laws such as

$$M[u] = \int_{\mathbb{T}} u \, dx, \quad \text{(Mass)}$$

$$E[u] = \int_{\mathbb{T}} u^2 \, dx, \qquad (1.2)$$

$$H[u] = \int_{\mathbb{T}} \frac{1}{2} \left(\partial_x^j u \right)^2 - \frac{1}{6} u^3 \, dx, \quad \text{(Hamiltonian)}.$$

Furthermore, (1.1) is the Hamiltonian equation with respect to $H[u]$ defined in (1.2). In other words, we can rewrite (1.1) as follows:

$$u_t = \partial_x \nabla_u H (u(t)) = \nabla_\omega H (u(t))$$

where ∇_u is the L^2 gradient and $\nabla_\omega = \nabla_{\omega_{-\frac{1}{2}}}$ is the symplectic gradient

$$\omega_{-\frac{1}{2}} (u, v) := \int_{\mathbb{T}} u \partial_x^{-1} v \, dx.$$

These three conservation laws play various roles (particularly to determine the global behavior of solutions and the global control properties of Eq. (1.1)) in the study of the partial differential equations.

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[1] This equation indeed firstly introduced by Boussinesq [8], and Korteweg and de Vries rediscovered it twenty years later.

1.2. Problems under consideration

In this paper, we prove that the higher-order KdV-type equation[2]

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = f(t,x), \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}, \qquad (1.3)$$

posed on periodic domain $\mathbb{T}$ is globally controllable in H^s, for $s \geq 0$, when we introduce a forcing term $f = f(t,x)$ added to the equation as a control input. Here, f is assumed to be supported in a given open set $\omega \subset \mathbb{T}$. The following control problems are considered:

Exact control problem: *Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that Eq. (1.3) admits a solution u which satisfies $u|_{t=T} = u_1$?*

Stabilization problem: *Can one find a feedback control law $f = Ku$ so that the resulting closed-loop system*

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = Ku, \quad (t,x) \in \mathbb{R} \times \mathbb{T},$$

is asymptotically stable at an equilibrium point as $t \to +\infty$?

The higher-order KdV-type equations keep its mass conserved, see for instance (1.2), thus

$$\frac{d}{dt}\int_{\mathbb{T}} u(t,x)\,dx = 0,$$

for any $t \in \mathbb{R}$ when no control is in action ($f \equiv 0$). In applications, one would also like to keep the mass conserved while conducting control. For that purpose, a natural constraint on our control input f is as follows:

$$\int_{\mathbb{T}} f(t,x)\,dx = 0, \forall t \in \mathbb{R}.$$

Thus, as in [10], the natural control input $f(t,x)$ is chosen to be of the form

$$f(t,x) = [Gh](t,x) := g(x)\left(h(t,x) - \int_{\mathbb{T}} g(y)h(t,y)\,dy\right), \qquad (1.4)$$

where h is considered as a new control input, and $g(x)$ is a given nonnegative smooth function such that

$$2\pi\,[g] = \int_{\mathbb{T}} g(x)\,dx = 1.$$

Here, we denote ω by the set $\omega := \{g > 0\}$, where the control function is effectively acting.

1.3. Review of the results in the literature

The local and global well-posedness of (1.1) were widely studied. For the local well-posedness result, Gorsky and Himonas [11] firstly proved this problem for $s \geq -\frac{1}{2}$ and Hirayama [12] improved for $s \geq -\frac{j}{2}$. Both works are based on the standard Fourier restriction norm method. Hirayama improved the bilinear estimate by using the factorization of the resonant function.

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[2] One may generalize Eq. (1.3) as

$$\partial_t u + \sum_{m=0}^{j} \alpha_m \partial_x^{2m+1} u + \frac{1}{2}\partial_x(u^2) = f,$$

where $\alpha_m \in \mathbb{R}$. However, main analyses in the paper are almost analogous without additional difficulties, thus Eq. (1.3) does not lose the generality in a sense of the aim in this paper. See Remark 2.

The results of the global well-posedness for (1.1), when $j = 1, 2$, were proved by Colliander et al. [13] and Kato [14], respectively, via "I-method". In [15] the authors extend the results of [13] and [14] for $j \geq 3$. The method basically follows the argument in [13] for periodic KdV equation, while some estimates are slightly different. More precisely, they showed that for $j \geq 3$ and $s \geq -\frac{j}{2}$, the IVP (1.1) is globally well-posed in $H^s(\mathbb{T})$.

Regarding the control theory, when $j = 1$, the system (1.3) has good control properties. The study of the controllability and stabilization to the KdV equation started with the work of Russell and Zhang [16] for the linear system

$$u_t + u_{xxx} = f, \tag{1.5}$$

with periodic boundary conditions and an internal control f. Since then, both controllability and stabilization problems have been intensively studied.

It is well-known that (1.5) with $f = -uu_x$ allows an infinite set of conserved integral quantities, for instance, $M[u]$ and $E[u]$, defined in (1.2). From the historical origins of the KdV equation involving the behavior of water waves in a shallow channel [7,8,17], it is natural to think of $M[u]$ and $E[u]$ as expressing conservation of *volume (or mass)* and *energy*, respectively.

The Russell and Zhang's work [16] is purely linear. In fact, until Bourgain [18] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain, no results of the nonlinear problems were solved. This novelty, discovered by Bourgain, has played a crucial role in the proof of the results in [10].

Specifically, in [10] the authors studied the nonlinear equation associated to (1.5) from a control point of view with a forcing term $f = f(t, x)$ added to the equation as a control input:

$$u_t + u_{xxx} + uu_x = f, \quad (t, x) \in \mathbb{R} \times \mathbb{T}. \tag{1.6}$$

With this in hand, Russell and Zhang were able to show the local exact controllability and local exponential stabilizability for the system (1.6). Indeed, the results presented in [10] are essentially linear; they are more or less small perturbations of the linear results. After these works, Laurent, Rosier and Zhang [19] show that still it is possible to guide the system (1.6) from a given initial state u_0 to a given terminal state u_1 when u_0 and u_1 have large amplitude by choosing an appropriate control input. Furthermore, they showed that the large amplitude solutions of the closed-loop system (1.6) decay exponentially as $t \to \infty$. Hence, the authors in [19] proved global exact controllability and global exponential stabilizability extending the results obtained by Russell and Zhang in [10]. These global results are established with the aid of certain of propagation of compactness and regularity in Bourgain spaces for the solutions of the associated linear system of (1.6).

Considering $j = 2$ the system (1.3) is the so-called Kawahara equation

$$\partial_t u - \partial_x^5 u + \frac{1}{2}\partial_x(u^2) = f, \qquad (t, x) \in \mathbb{R} \times \mathbb{T}. \tag{1.7}$$

Recently, the first author, in [20], studied the stabilization problem and conjectured a critical set phenomenon for Kawahara equations as occurs with the KdV equation [21,22] and Boussinesq KdV–KdV system [23], for example. Moreover, as far as we know, the control problem was, first, studied in [24,25] when the authors considered the Kawahara equation on a periodic domain $\mathbb{T}$ with a distributed control of the form (1.4). First, the authors were able to prove the local controllability results for this equation in [24]. Aided by smoothing properties of the system in Bourgain spaces, they were able to show that the Kawahara equation is globally exactly controllable and globally exponentially stabilizable (see [25]).

We caution that this is only a small sample of the extant equations with the similar structure to the system (1.3), (1.5) and (1.7). For an extensive review of the physical meanings of these equations, as well as well-posedness and controllability results the authors suggest the following nice Refs. [12,26–28] and the references therein.

1.4. Notation and main results

Let us introduce some notation and present the main results of the manuscript. For $x, y \in \mathbb{R}_+$, $x \lesssim y$ denotes $x \leq Cy$ for some $C > 0$ and $x \sim y$ means $x \lesssim y$ and $y \lesssim x$. Also, $x \ll y$ denotes $x \leq cy$ for a small positive constant c. Let us consider the Fourier and inverse Fourier transforms with respect to the spatial variable $x \in \mathbb{T}$,

$$\mathcal{F}_x(f)(k) = \widehat{f}(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} e^{-ikx} f(x) \, dx \quad \text{and} \quad \mathcal{F}_x^{-1}(f)(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} e^{ikx} f(k),$$

respectively. Additionally, the space–time Fourier and inverse Fourier transforms are

$$\mathcal{F}(f)(\tau, k) = \widetilde{f}(\tau, k) := \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{T}} e^{-it\tau} e^{-ikx} f(x) \, dx \, dt$$

and

$$\mathcal{F}^{-1}(f)(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} e^{it\tau} e^{ikx} f(\tau, k) \, d\tau,$$

respectively.

Consider now the $H^s(\mathbb{T}) := H^s$ space with the inner product as

$$(f, g)_{H^s} = (f, g)_s := \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \widehat{f}(k) \overline{\widehat{g}(k)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We simply denote the $H^0 := L^2$ inner product by $(\cdot, \cdot)$. It naturally defines H^s norm as $\|f\|_{H^s} = \sqrt{(f, f)_{H^s}}$. We will use $H_0^s(\mathbb{T})$ as the subspace of $H^s(\mathbb{T})$ whose elements obey the mean zero condition, i.e.,

$$H_0^s(\mathbb{T}) = \left\{ f \in H^s(\mathbb{T}) : \int_{\mathbb{T}} f = 0 \right\}.$$

The aim of this manuscript is to address the control and stabilization (particularly global) issues. However, before presenting the global results, let us present a theorem that shows the exact control result.

Theorem 1.1 (*[29]*). *Let $T > 0$ and $s \geq 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with*

$$\|u_0\|_{H^s(\mathbb{T})} \leq \delta \quad \text{and} \quad \|u_1\|_{H^s(\mathbb{T})} \leq \delta,$$

one can find a control function $h \in L^2([0,T]; H^s(\mathbb{T}))$ such that the system

$$\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + \frac{1}{2} \partial_x(u^2) = Gh, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \tag{1.8}$$

where G is defined by (1.4), admits a solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u|_{t=0} = u_0, \quad u|_{t=T} = u_1.$$

Now, due to the advantage of the results proved in [30,31], the local exponential result in $H^s(\mathbb{T})$, for any $s \geq 0$, can be established.

Theorem 1.2 (*[29]*). *Let $s \geq 0$ and $\lambda > 0$ be given. There exists a bounded linear operator*

$$K_\lambda : H^s(\mathbb{T}) \to H^s(\mathbb{T})$$

such that if one chooses the feedback control $h = K_\lambda u$ in (1.8), then the resulting closed-loop system

$$\begin{cases} \partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + \frac{1}{2} \partial_x (u^2) = G K_\lambda u, \\ u(0, x) = u_0(x), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{T}, \qquad (1.9)$$

is locally exponentially stable in the space $H^s(\mathbb{T})$, for $s \geq 0$, that is, there exists a $\delta > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s(\mathbb{T})} \leq \delta$, the corresponding solution u of (1.9) satisfies

$$\|u(\cdot, t) - [u_0]\|_{H^s(\mathbb{T})} < C e^{-\lambda t} \|u_0 - [u_0]\|_{H^s(\mathbb{T})}, \quad \forall t > 0.$$

Remark 1. We point out that Theorems 1.1 and 1.2 have already been proved by Zhao and Bai [29]. For self-containedness, we will also give rigorous proofs of them in Appendices A and C.

These results shown that one can always find an appropriate control input h to guide the system (1.8) from a given initial state u_0 to a given terminal state u_1 as long as *their amplitudes are small* and $[u_0] = [u_1]$. However, some natural questions arise.

Question $\mathcal{A}$. Can one still guide the system (1.8) by choosing an appropriate control input h (defined on a sufficiently long time interval) from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have large amplitude?

According to Theorem 1.2, solutions of system (1.8) issued from initial data close to their mean values converge at a uniform exponential rate to their mean values in the space $H^s(\mathbb{T})$ as $t \to +\infty$. One may ask the following issue:

Question $\mathcal{B}$. Does any solution of the closed-loop system (1.9) converge exponentially to its mean value as $t \to +\infty$?

Thus, additionally to the local results, presented in Theorems 1.1 and 1.2, our work gives a positive answer to these questions that have a global character. This is possible due to the celebrated results obtained by Bourgain [18]. One of the main results in this work gives an answer to the Question $\mathcal{A}$, the result ensures that the system (1.8) is *globally* exactly controllable.

Theorem 1.3. *Let $s \geq 0$, $R > 0$ and $\mu \in \mathbb{R}$ be given. There exists a time $T > 0$ such that if u_0, $u_1 \in H^s(\mathbb{T})$, with $[u_0] = [u_1] = \mu$, satisfies*

$$\|u_0\|_{H^s(\mathbb{T})} \leq R, \quad \|u_1\|_{H^s(\mathbb{T})} \leq R,$$

then one can find a control input $h \in L^2(0, T; H^s(\mathbb{T}))$ such that the system (1.8) admits a solution $u \in C([0, T], H^s(\mathbb{T}))$ satisfying

$$u|_{t=0} = u_0, \quad u|_{t=T} = u_1.$$

As for Question $\mathcal{B}$, we have the following affirmative answer.

Theorem 1.4. *Let $s \geq 0$ and $\mu \in \mathbb{R}$. There exists a constant $\gamma > 0$ such that for any $u_0 \in H^s(\mathbb{T})$ with $[u_0] = \mu$, the corresponding solution u of the system (1.8), with $h(x, t) = -G^* u(x, t)$, satisfies*

$$\|u(\cdot, t) - [u_0]\|_{H^s(\mathbb{T})} < \alpha_{s,\mu} \left(\|u_0 - [u_0]\|_{L^2(\mathbb{T})} \right) C e^{-\gamma t} \|u_0 - [u_0]\|_{H^s(\mathbb{T})}, \forall t \geq 0,$$

where $\alpha_{s,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s and μ.

1.5. Heuristic of the article

In this manuscript our goal is to give answers for two global control problems mentioned in the previous section. Observe that the results obtained so far are concentrated in a single KdV Eq. (1.6), see e.g. [10,16], and Kawahara Eq. (1.7), see for instance [24,25]. Moreover, even higher-order KdV type (1.3) has been studied in the sense of local controllability and stabilization in [29], while global control results for the generalized higher-order KdV type Eq. (1.3) are still open, so, under this direction, our work is a generalization of the previous result for KdV and Kawahara equations. Let us describe briefly the main arguments of the proof of our theorem and give consideration of the importance of the work in the study of the control theory for general dispersive operators.

The first two results are local, that is, Theorems 1.1 and 1.2. In fact, first, due to the properties of the operator

$$Aw = -(-1)^{j+1}\partial_x^{2j+1}w, \tag{1.10}$$

we can use classical theorems of Ingham and Beurling [32,33] to ensure that the linear system associated to (1.8) has control and stabilization properties. To extend these results for the nonlinear case, one has to control one derivative in the nonlinear term. However, it is well-known that linear solutions have no dispersive and no smoothing effect under periodic boundary condition. Due to Bourgain [18,34], by regarding the linear estimate as multilinear interactions in L^2, now, one can recover derivative loss occurring in Sobolev inequality, thus the nonlinearity can be controlled. Note that this is not the only way to handle the nonlinearity, compare [12] with [15]. Here, the main point is to prove the following *Strichartz estimates*

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \|f\|_{X^{0,b}}, \quad b > \frac{j+1}{2(2j+1)}, \quad \text{for all } j \in \mathbb{N}.$$

After that we are able to extend the local solutions for the global one and prove the nonlinear (local) control results as a perturbation of the linear one. Note that the arguments here are purely linear. In addition, we emphasize that the exact controllability and stabilizability results of the linear system associated to (1.8) are valid in $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$.

It is important to point out that Theorems 1.3 and 1.4 have global character. Precisely, the control result for large data (Theorem 1.3) will be a combination of a global stabilization result (Theorem 1.4) and the local control result (Theorem 1.1). Indeed, given the initial state $u_0 \in H^s(\mathbb{T})$ to be controlled, by means of the damping term $Ku = -GG^*u$ supported in $\omega \subset \mathbb{T}$, i.e., solving the IVP (1.8) with $h = -G^*u$, we drive u_0 to a state $\tilde{u}_0$ close enough to the mean value μ in a sufficiently large time T_1, by Theorem 1.4. Again, using this result, we do the same with the final state $u_1 \in H^s(\mathbb{T})$ by solving the system backwards in time, due to the time reversibility of the higher-order KdV-type equation. This process produces two states $\tilde{u}_0$ and $\tilde{u}_1$ which are close enough to μ so that the local controllability result (Theorem 1.1) can be applied around the state $u(x) = \mu$. We can see this mechanism illustrated in Fig. 1.

Lastly, the proof of Theorem 1.4 is equivalent to prove an *observability inequality*, which one, by using contradiction arguments, relies on to prove a unique continuation property for the system (1.8). This property is achieved due to the propagation results using again smooth properties of solution in Bourgain spaces. The main difficulties to prove the propagation results arise from the fact that the system (1.8) has a general structure. To overcome this difficulty the Strichartz estimates, for the solution of our problem in Bourgain Spaces, are essential.

We finish this introduction by mentioning that the global results presented in this article, even the local results, are not a consequence of the previous results for the KdV and Kawahara equations. Indeed, taking $j = 1$ and 2 in our operator A, defined in (1.10), we can recover the previous results in the literature for these equations, nevertheless, the necessary estimates to treat the operator A as well as the propagation of regularity are the main novelties of this work. In summary, the key ingredients of this work are:
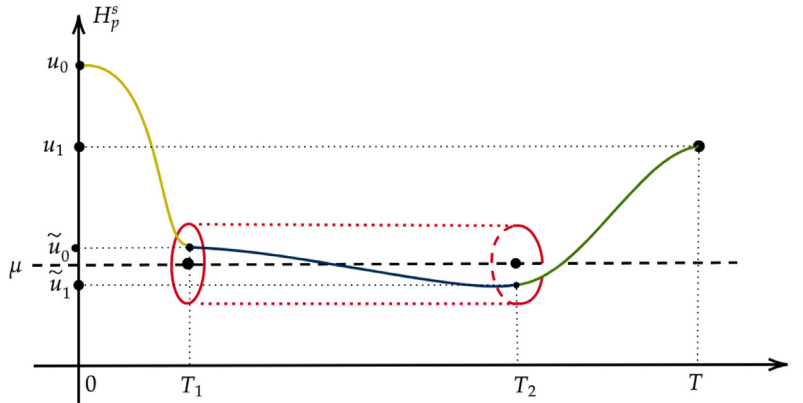
Fig. 1. The constructive approach of the proof of Theorem 1.3.

1. *Strichartz estimates* associated to the solution of the problem under consideration;
2. *Microlocal analysis* to prove propagation of the regularity and compactness;
3. *Unique continuation property* for the operator A.

1.6. Structure of the paper

Some preliminaries are given in Section 2, particularly, spectral property of the operator A in (1.10) is studied and Bourgain spaces are introduced. In Section 3, we give a rigorous proof of Strichartz estimate. In Section 4, we investigate propagation of regularity and unique continuation property. The global stabilization result (Theorem 1.4) is proved in Section 5, and in Section 6, some comments and open questions are presented. In Appendices, as mentioned, some analyses for local results are given. The linear system is studied in Appendix A, particularly, we present the linear control problems, which are a consequence of the spectral analysis. A brief proof of the global well-posedness of the closed loop system is presented in Appendix B. Finally the proofs of Theorems 1.1 and 1.2 are presented in Appendix C.

2. Preliminaries

2.1. Spectral property

Consider the operator A denoted by

$$Aw = -(-1)^{j+1}\partial_x^{2j+1}w \tag{2.1}$$

with domain $\mathcal{D}(A) = H^{2j+1}(\mathbb{T})$. The operator A generates a continuous unitary operator group $W(t)$ on the space $L^2(\mathbb{T})$, precisely,

$$W(t)f(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{ikx}e^{itk^{2j+1}}\widehat{f}(k).$$

Remark that

$$A^* = -A \quad \text{and} \quad W^*(-t) = W(t). \tag{2.2}$$

One immediately knows that eigenfunctions of the operator A are the orthonormal Fourier bases of $L^2(\mathbb{T})$,

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}, \quad k \in \mathbb{Z},$$

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and its corresponding eigenvalues are

$$\lambda_k = ik^{2j+1}, \qquad k \in \mathbb{Z}. \tag{2.3}$$

We now prove a *gap condition* which will be used to prove a local controllability result in Appendix A. The result can be read as follows.

Lemma 2.1 (*Gap Condition*). *Let $j \geq 1$ and $k \in \mathbb{Z}$. For λ_k defined as in (2.3), if $|k| \geq j + 1$, we have*

$$|\lambda_{k+1} - \lambda_k| \geq k^2, \tag{2.4}$$

which implies

$$\lim_{|k| \to +\infty} |\lambda_{k+1} - \lambda_k| = \infty. \tag{2.5}$$

Proof. When $j = 1$, it is easy to see that

$$-i(\lambda_{k+1} - \lambda_k) = 3k^2 + 3k + 1 = \frac{3}{2}k^2 + \frac{3}{2}(k^2 + 2k + 1) - \frac{1}{2}$$
$$= k^2 + \frac{3}{2}(k + 1)^2 + \frac{1}{2}(k^2 - 1) \geq k^2,$$

for $|k| \geq 1$, which satisfies (2.4), and thus (2.5) follows.

Now, fix $j \geq 2$. A straightforward computation yields

$$2\,mk^2 + (2j + 1 - 2m + 1)k > 0, \quad m = 1, 2, \ldots, j,$$

whenever $k > 0$ or $k < -\frac{j-m+1}{m}$, which implies

$$\binom{2j+1}{2m-1}k^2 + \binom{2j+1}{2m}k$$
$$= \left(2\,mk^2 + (2j + 1 - 2m + 1)k\right)\frac{(2j+1)(2j)\cdots(2j+1-(2m-1)+1)}{(2\,m)!}$$
$$> \frac{(j+1)}{m}\binom{2j+1}{2m-1}.$$

Thus, we conclude for $|k| \geq j + 1$ that

$$|\lambda_{k+1} - \lambda_k| = \left|\sum_{\ell=1}^{2j+1}\binom{2j+1}{\ell}k^{2j+1-\ell}\right|$$
$$= \left|1 + \sum_{m=1}^{j}\left(\binom{2j+1}{2m-1}k^2 + \binom{2j+1}{2m}k\right)k^{2(j-m)}\right|$$
$$> (2j+1)(j+1)k^{2(j-1)}$$
$$> k^2,$$

which completes the proof. $\square$

Remark 2. Lemma 2.1 ensures that the gap condition is still valid for the (generalized) linear operator

$$\tilde{A} = \sum_{m=0}^{j}(-1)^m\alpha_m\partial_x^{2m-1},$$

9

where $\alpha_m \geq 0$ with $\alpha_j \neq 0$. Indeed, we have eigenvalues associated to $\tilde{A}$ as

$$\tilde{\lambda}_k = i \sum_{m=0}^{j} \alpha_m k^{2m-1},$$

and the gap condition

$$|\tilde{\lambda}_{k+1} - \tilde{\lambda}_k| \geq k^2 \max_{m=0,\ldots,j} \{\alpha_m\},$$

for $|k| > j$, which gives $|\tilde{\lambda}_{k+1} - \tilde{\lambda}_k| \to \infty$ when $|k| \to \infty$. Additionally, it is not necessary to restrict $\alpha_m \geq 0$, however we do not further discuss about it here.

2.2. Fourier restriction spaces

The function space equipped with the Fourier restriction norm, which is the so-called $X^{s,b}$ spaces, has been proposed by Bourgain [18,34] to solve the periodic NLS and generalized KdV. Since then, it has played a crucial role in the theory of dispersive equations, and has been further developed by many researchers, in particular, Kenig, Ponce and Vega [35] and Tao [36]. In the following, to ensure global control results, the space $X^{s,b}$ will be of paramount importance.

Let f be a Schwartz function, i.e., $f \in \mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$. $\widetilde{f}$ or $\mathcal{F}(f)$ denotes the space–time Fourier transform of f defined by

$$\widetilde{f}(\tau, n) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{0}^{2\pi} e^{-ixn} e^{-it\tau} f(t,x) \, dx dt.$$

Then, it is known that the (space–time) inverse Fourier transform is naturally defined as

$$f(t,x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} e^{ixn} e^{it\tau} \widetilde{f}(\tau, n) \, dt.$$

Moreover, we use $\mathcal{F}_x$ (or $\widehat{}$) and $\mathcal{F}_t$ to denote the spatial and temporal Fourier transform, respectively.

For given $s, b \in \mathbb{R}$, we define the space $X^{s,b}$ associated to (1.3) as the closure of $\mathcal{S}_{t,x}(\mathbb{R} \times \mathbb{T})$ under the norm

$$\|f\|_{X^{s,b}}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \langle k \rangle^{2s} \left\langle \tau - k^{2j+1} \right\rangle^{2b} \left| \widetilde{f}(\tau, k) \right|^2 \, d\tau,$$

which is equivalent to the expression $\|W(-t)f(t,x)\|_{H_t^b H_x^s}$. Note that the definition of $X^{s,b}$ ensures the trivial nesting property

$$X^{s,b} \subset X^{s',b'}, \quad \text{whenever} \quad s' \leq s, \ b' \leq b, \tag{2.6}$$

and this immersion is continuous. Moreover, it is known (see, for instance, [37, Lemma 2.11]) that $X^{s,b}$ space is stable with respect to time localization, that is,

$$\|\eta(t)u\|_{X^{s,b}} \lesssim_{b,\psi} \|u\|_{X^{s,b}}, \tag{2.7}$$

for any time cutoff function $\eta \in \mathcal{S}_t(\mathbb{R})$.

According to [35, Theorem 1.2], it is necessary to fix the exponent $b = \frac{1}{2}$ in $X^{s,b}$ for the study of the periodic KdV equation. Otherwise, one cannot, indeed, obtain one-derivative gain in the *high-low* non-resonant interactions to kill the derivative in the nonlinearity. Thereafter, it becomes natural to fix $b = \frac{1}{2}$ even for the other periodic problems, but it is not necessary. For instance, in the higher-order KdV-type case, in particular, $j \geq 2$, one can obtain $\min\{2bj, (1-b)j\}$-derivative gains from the *high-low* non-resonant interactions, which is enough to remove the one derivative in the nonlinearity, whenever $\frac{1}{2j} \leq b \leq 1 - \frac{1}{2j}$. However, the present paper covers the KdV and Kawahara cases as well, we, thus, fix $b = \frac{1}{2}$, throughout the paper.

On the other hand, the space $X^{s,\frac{1}{2}}$ is not embedded in the classical solution space $C_t H^s \equiv C(\mathbb{R}, H^s(\mathbb{T}))$. Therefore, this space is not enough to develop the well-posedness theory. Nevertheless, the lack of embedding property can be overcome by defining the space Y^s of solutions with the following norm

$$\|f\|_{Y^s} := \|f\|_{X^{s,\frac{1}{2}}} + \left\| \langle k \rangle^s \widetilde{f} \right\|_{\ell_k^2 L_\tau^1}.$$

For a given time interval I, let $X_I^{s,b}$ (resp. Y_I^s) denote the time localization of $X^{s,b}$ (resp. Y^s) on the interval I with the norm

$$\|f\|_{X_I^{s,b}} = \inf \left\{ \|g\|_{X^{s,b}} : g = f \ \text{ on } \ I \times \mathbb{T} \right\}$$

$$\left(\text{resp. } \|f\|_{Y_I^s} = \inf \left\{ \|g\|_{Y^s} : g = f \ \text{ on } \ I \times \mathbb{T} \right\} \right).$$

For simplicity, we denote $X_I^{s,b}$ (resp. Y_I^s) by $X_T^{s,b}$ (resp. Y_T^s), if $I = (0, T)$.

2.3. Estimates for higher-order KdV-type equation

We summarize well-known estimates, already established in the literature, which will play important roles in establishing the exact controllability and stabilizability of the system (1.3). For this, we introduce a cut-off function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta = 1$, if $t \in [-1, 1]$ and $\eta = 0$, if $t \notin (-2, 2)$.

Lemma 2.2 ($X^{s,b}$ *Estimates, [12,38]*)**.** *Let $0 < T < \infty$ be given. Then,*

(1) *For all $s \in \mathbb{R}$, we have for $u \in Y_T^s$*

$$\|u\|_{C_T H^s} \lesssim \|u\|_{Y_T^s}.$$

(2) *For all $s \in \mathbb{R}$, we have for $f \in H^s$*

$$\|W(t)f\|_{Y_T^s} \lesssim_{T,\eta} \|f\|_{H^s}.$$

If $T \leq 1$, then the constant on the right-hand side does not depend on T.

(3) *For all $s \in \mathbb{R}$, we have for $F \in Y_T^s$*

$$\left\| \int_0^t W(t - \tau) F(\tau) \, d\tau \right\|_{Y_T^s} \lesssim_{T,\eta} \left\| \mathcal{F}^{-1} \left(\langle \tau - k^{2j+1} \rangle^{-1} \widetilde{F} \right) \right\|_{Y_T^s}. \tag{2.8}$$

If $T \leq 1$, then the positive constant involved in (2.8) does not depend on T.

(4) *For $s \in \mathbb{R}$ with $s \geq -\frac{j}{2}$, we have for $u, v \in Y_T^s$*

$$\left\| \mathcal{F}^{-1} \left(\langle \tau - k^{2j+1} \rangle^{-1} \widetilde{\partial_x (uv)} \right) \right\|_{Y_T^s} \lesssim \|u\|_{Y_T^s} \|v\|_{Y_T^s}$$

(5) *For all $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leq b < \frac{1}{2}$ and $0 < T < 1$, we have for $u \in X_T^{s,b}$*

$$\|u\|_{X_T^{s,b'}} \lesssim T^{b-b'} \|u\|_{X_T^{s,b}}.$$

Remark 3. The right-hand side of (2.8) is simply dominated by $\|F\|_{L^2(0,T;H^s)}$ due to the definition of Y_T^s norm, the nesting property (2.6) and the weight $\langle \tau - k^{2j+1} \rangle^{-1}$.

3. L^4-Strichartz estimate

In this section, we provide a rigorous proof of L^4-Strichartz estimate for higher-order KdV equation.

Lemma 3.1 (*Strichartz Estimates*)**.** *The following estimates hold:*

$$\|f\|_{L^4(\mathbb{R}\times\mathbb{T})} \lesssim \|f\|_{X^{0,b}}, \quad b > \frac{j+1}{2(2j+1)}, \quad \forall j \in \mathbb{N}, \tag{3.1}$$

where the implicit constant depends on b and j.

Remark 4. As well-known, the intuition of Lemma 3.1 is as follows: By Sobolev embedding (in both time and spatial variables), one has

$$\|f\|_{L^4_{t,x}} \lesssim \|S(-t)f\|_{H_t^{\frac{1}{4}} H_x^{\frac{1}{4}}}.$$

On the other hand, from (1.1), one roughly guesses that $\partial_t \sim \partial_x^{2j+1}$, which transfers spatial derivatives to temporal derivatives ($\partial_x^{\frac{1}{4}} \mapsto \partial_t^{\frac{1}{4(2j+1)}}$). Hence, one can guess

$$\|f\|_{L^4_{t,x}} \lesssim \|S(-t)f\|_{H_t^{\frac{1}{4}} H_x^{\frac{1}{4}}} \lesssim \|S(-t)f\|_{H_t^{\frac{1}{4}+\frac{1}{4(2j+1)}} L_x^2} \lesssim \|f\|_{X^{0,b}}, \quad b > \frac{j+1}{2(2j+1)}.$$

The equality $b = \frac{j+1}{2(2j+1)}$ can also be obtained, but we do not attempt to give it here, in order to avoid complicated computations.

The following lemma plays an essential role to prove Lemma 3.1.

Lemma 3.2. *For $j \in \mathbb{Z}$ and $c \in \mathbb{R}$ with $j \geq 0$ and $c > 0$, let*

$$h_j(x) = x^{2j+1} + (c-x)^{2j+1}. \tag{3.2}$$

Then h_j satisfies

(1) h_j *is a symmetry about $x = \frac{c}{2}$.*
(2) $h_j(x) > 0$ *for all $x \in \mathbb{R}$.*
(3) $h_j'(\frac{c}{2}) = 0$.

If $j \geq 1$,

(4) $h_j''(x) > 0$ *for all $x \in \mathbb{R}$.*
(5) h_j *has only one absolute minimum value at $x = \frac{c}{2}$.*
(6) h_j *can be written as*

$$h_j(x) = \left(x - \frac{c}{2} + \alpha\right)\left(x - \frac{c}{2} - \alpha\right) \sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right) \left(\sum_{\ell=0}^{n-1} \left(x - \frac{c}{2}\right)^{2n-2-2\ell} \alpha^{2\ell}\right)$$
$$+ \sum_{n=0}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right) \alpha^{2n}, \tag{3.3}$$

for any $\alpha \in \mathbb{R}$.

Proof. It is easy to see that

$$h_j\left(\frac{c}{2} + x\right) = \left(\frac{c}{2} + x\right)^{2j+1} + \left(\frac{c}{2} - x\right)^{2j+1} = h_j\left(\frac{c}{2} - x\right),$$

which satisfies Item (1).

For Item (2), it suffices to show $h_j(x) > 0$ for all $x \geq \frac{c}{2}$ due to Item (1). Obviously, $h_j(\frac{c}{2}) = \frac{c^{2j+1}}{2^{2j}} > 0$. Since

$$a^{2j+1} + b^{2j+1} = (a+b)\left(\sum_{n=0}^{2j}(-1)^n a^{2j-n}b^n\right) = (a+b)\left(\sum_{m=0}^{j-1}(a-b)a^{2j-2m-1}b^{2m} + b^{2j}\right),$$

for $x > \frac{c}{2}$, we have

$$h_j(x) = c\left(\sum_{m=0}^{j-1}(2x-c)x^{2j-2m-1}(c-x)^{2m} + (c-x)^{2j}\right).$$

Note that when $x = c$, we have $h_j(c) = c^{2j+1} > 0$. Thus, for all $x > \frac{c}{2}$ with $x \neq c$, all terms are strictly positive, which proves Item (2).

A direct computation gives

$$h_j'(x) = (2j+1)\left(x^{2j} - (c-x)^{2j}\right),$$

thus $h'(\frac{c}{2}) = 0$. This proves Item (3).

In what follows, we fix $j \geq 1$. Item (4) follows immediately from Item (2) due to

$$h_j''(x) = (2j+1)(2j)\left(x^{2j-1} + (c-x)^{2j-1}\right).$$

Item (5) immediately follows from Items (3) and (4).

For Item (6), we first show

$$h_j(x) = \sum_{n=0}^{j}\binom{2j+1}{2n}h_{j-n}\left(\frac{c}{2}\right)\left(x-\frac{c}{2}\right)^{2n}. \tag{3.4}$$

When $j = 1$, we see that

$$h_1(x) = x^3 + (c-x)^3 = 3c(x^2 - cx) + c^3 = 3c\left(x-\frac{c}{2}\right)^2 + \frac{c^3}{4}.$$

Since

$$\binom{3}{0}h_1\left(\frac{c}{2}\right) = 2\cdot\left(\frac{c}{2}\right)^3 = \frac{c^3}{4} \quad \text{and} \quad \binom{3}{2}h_0\left(\frac{c}{2}\right) = 3c,$$

(3.4) is true for $j = 1$. Assume that (3.4) is true for $j = m-1$. Then, by the induction hypothesis, we have

$$h_m''(x) = (2m+1)(2\,m)h_{m-1}(x) = (2m+1)(2\,m)\sum_{n=0}^{m-1}\binom{2m-1}{2n}h_{m-1-n}\left(\frac{c}{2}\right)\left(x-\frac{c}{2}\right)^{2n}.$$

Since $h_m'(\frac{c}{2}) = 0$, by integrating from $\frac{c}{2}$ to x twice, we obtain

$$\begin{aligned}
h_m(x) &= \sum_{n=0}^{m-1}\frac{(2m+1)(2\,m)}{(2n+2)(2n+1)}\binom{2m-1}{2n}h_{m-1-n}\left(\frac{c}{2}\right)\left(x-\frac{c}{2}\right)^{2n+2} + h_m\left(\frac{c}{2}\right)\\
&= \sum_{n=0}^{m}\binom{2m+1}{2n}h_{m-n}\left(\frac{c}{2}\right)\left(x-\frac{c}{2}\right)^{2n}.
\end{aligned}$$

Thus, by the mathematical induction, we prove (3.4) for all $j \geq 1$.

In order to derive (3.3) from (3.4), it suffices to show that for each $n \in \mathbb{N}$

$$\left(x-\frac{c}{2}\right)^{2n} = \left(x-\frac{c}{2}+\alpha\right)\left(x-\frac{c}{2}-\alpha\right)\sum_{\ell=0}^{n-1}\left(x-\frac{c}{2}\right)^{2n-2-2\ell}\alpha^{2\ell} + \alpha^{2n}. \tag{3.5}$$

Indeed, if (3.5) is true, then we reduce (3.4) as

$$
\begin{aligned}
h_j(x) &= \sum_{n=0}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right)\left(x - \frac{c}{2}\right)^{2n} \\
&= \sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right)\left(\left(x - \frac{c}{2} + \alpha\right)\left(x - \frac{c}{2} - \alpha\right)\sum_{\ell=0}^{n-1}\left(x - \frac{c}{2}\right)^{2n-2-2\ell}\alpha^{2\ell} + \alpha^{2n}\right) \\
&\quad + \binom{2j+1}{0}h_j\left(\frac{c}{2}\right) \\
&= \left(x - \frac{c}{2} + \alpha\right)\left(x - \frac{c}{2} - \alpha\right)\sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right)\left(\sum_{\ell=0}^{n-1}\left(x - \frac{c}{2}\right)^{2n-2-2\ell}\alpha^{2\ell}\right) \\
&\quad + \sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{c}{2}\right)\alpha^{2n} + \binom{2j+1}{0}h_j\left(\frac{c}{2}\right),
\end{aligned}
$$

which proves (3.3).

We use, again, the mathematical induction to prove (3.5). When $n = 1$, we obtain

$$
\left(x - \frac{c}{2}\right)^2 = \left(x - \frac{c}{2} - \alpha\right)\left(x - \frac{c}{2} + \alpha\right) + \alpha^2. \tag{3.6}
$$

Assume that (3.5) is true for $n = m - 1$. Then, using (3.6) and the induction hypothesis, we have

$$
\begin{aligned}
\left(x - \frac{c}{2}\right)^{2m} &= \left(\left(x - \frac{c}{2} - \alpha\right)\left(x - \frac{c}{2} + \alpha\right) + \alpha^2\right)\left(x - \frac{c}{2}\right)^{2m-2} \\
&= \left(x - \frac{c}{2} - \alpha\right)\left(x - \frac{c}{2} + \alpha\right)\left(x - \frac{c}{2}\right)^{2m-2} \\
&\quad + \alpha^2\left(\left(x - \frac{c}{2} + \alpha\right)\left(x - \frac{c}{2} - \alpha\right)\sum_{\ell=0}^{m-2}\left(x - \frac{c}{2}\right)^{2m-4-2\ell}\alpha^{2\ell} + \alpha^{2m-2}\right) \\
&= \left(x - \frac{c}{2} - \alpha\right)\left(x - \frac{c}{2} + \alpha\right)\sum_{\ell=0}^{m-1}\left(x - \frac{c}{2}\right)^{2m-2-2\ell}\alpha^{2\ell} + \alpha^{2m},
\end{aligned}
$$

which proves (3.5). $\square$

Remark 5. Collecting all information in Lemma 3.2, one can roughly sketch the shape of $h_j(x)$ defined by (3.2) as in Fig. 2:

Proof of Lemma 3.1. The proof basically follows the proof of Lemma 2.1 in [39] associated with the Airy flow. Moreover, we refer to [28] for the case when $j = 2$. Thus, in the proof below, we fix $j \geq 3$.

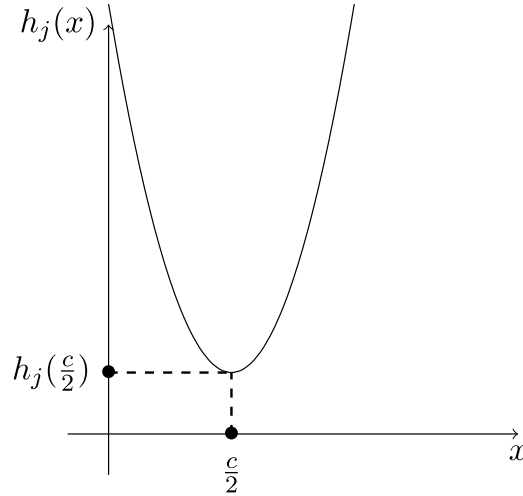Let $f = f_1 + f_2$, where

$$
\widehat{f}_1(k) = 0, \quad \text{if} \quad |k| > 1.
$$

Note that $|\{k \in \mathbb{Z} : k \in \text{supp}(\widehat{f}_1)\}| = 3$. Since $f^2 \leq 2f_1^2 + 2f_2^2$, it suffices to treat $\left\|f_1^2\right\|_{L^2(\mathbb{R}\times\mathbb{T})}$ and $\left\|f_2^2\right\|_{L^2(\mathbb{R}\times\mathbb{T})}$ separately.

f_1^2 **case.** A computation gives

$$
\left\|f_1^2\right\|_{L^2(\mathbb{R}\times\mathbb{T})}^2 \leq \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}|\widetilde{f}_1(\tau_1,k_1)||\widetilde{f}_1(\tau-\tau_1,k-k_1)|\,d\tau_1\right|^2 d\tau. \tag{3.7}
$$

Fig. 2. Lemma 3.2 describes that $h_j(x)$ has a convex and symmetric form, and its slope is strictly increasing if $x > \frac{c}{2}$, while strictly decreasing otherwise.

From the support property, the right-hand side of (3.7) vanishes unless $|k| \leq 2j$. Let

$$\widetilde{F}_1(\tau, k) = \left\langle \tau - k^{2j+1} \right\rangle^b |\widetilde{f}_1(\tau, k)|.$$

The Cauchy–Schwarz inequality and the Minkowski inequality, we see that for $b > \frac{1}{4}$,

$$
\text{RHS of (3.7)} \lesssim \sum_{\substack{k \in \mathbb{Z} \\ |k| \leq 2}} \int_{\mathbb{R}} \left(\sum_{k_1 \in \mathbb{Z}} \left(\int_{\mathbb{R}} \left\langle \tau - \tau_1 - (k - k_1)^{2j+1} \right\rangle^{-2b} \left\langle \tau_1 - k_1^{2j+1} \right\rangle^{-2b} d\tau_1 \right) \right.
$$
$$
\left. \times \left(\int_{\mathbb{R}} |\widetilde{F}_1(\tau_1, k_1)|^2 |\widetilde{F}_1(\tau - \tau_1, k - k_1)|^2 d\tau_1 \right)^{\frac{1}{2}} \right)^2 d\tau
$$
$$
\lesssim \sum_{\substack{k \in \mathbb{Z} \\ |k| \leq 2}} \left(\sum_{k_1 \in \mathbb{Z}} \left(\int_{\mathbb{R}^2} |\widetilde{F}_1(\tau_1, k_1)|^2 |\widetilde{F}_1(\tau - \tau_1, k - k_1)|^2 d\tau_1 d\tau \right)^{\frac{1}{2}} \right)^2
$$
$$
\lesssim \|f_1\|_{X^{0,b}}^4 \lesssim \|f\|_{X^{0,b}}^4 \,.
$$

f_2^2 **case.** Analogous to (3.7), we have

$$
\left\| f_2^2 \right\|_{L^2(\mathbb{R} \times \mathbb{T})}^2 \leq \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left| \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} |\widetilde{f}_2(\tau_1, k_1)| |\widetilde{f}_2(\tau - \tau_1, k - k_1)| \, d\tau_1 \right|^2 d\tau
$$
$$
= \sum_{\substack{k \in \mathbb{Z} \\ |k| \leq 1}} \int_{\mathbb{R}} \left| \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} |\widetilde{f}_2(\tau_1, k_1)| |\widetilde{f}_2(\tau - \tau_1, k - k_1)| \, d\tau_1 \right|^2 d\tau
$$
$$
+ \sum_{\substack{k \in \mathbb{Z} \\ |k| > 1}} \int_{\mathbb{R}} \left| \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} |\widetilde{f}_2(\tau_1, k_1)| |\widetilde{f}_2(\tau - \tau_1, k - k_1)| \, d\tau_1 \right|^2 d\tau
$$
$$
=: I_1 + I_2.
$$

The term I_1 can be treated similarly as f_1^2 **case**. For the term I_2, we may assume that $k_1 > 1$ and $k - k_1 > 1$ (thus, $k > 1$). Indeed, let $f_2 = f_{2,1} + f_{2,2}$, where

$$\widehat{f_{2,1}}(k) = 0 \quad \text{if} \quad k > 1,$$

then $\left\| f_2^2 \right\|_{L^2}^2 \leq 2 \left\| f_{2,1}^2 \right\|_{L^2}^2 + 2 \left\| f_{2,2}^2 \right\|_{L^2}^2$ and $\left\| f_{2,1}^2 \right\|_{L^2} = \left\| \overline{f_{2,1}}^2 \right\|_{L^2} = \left\| f_{2,2}^2 \right\|_{L^2}$. Similarly as (3.7), we have

$$I_2 \lesssim \sum_{\substack{k \in \mathbb{Z} \\ k > 1}} \int_{\mathbb{R}} \left(\left(\sum_{\substack{k_1 \in \mathbb{Z} \\ k_1, k-k_1 > 1}} \int_{\mathbb{R}} \left\langle \tau - \tau_1 - (k - k_1)^{2j+1} \right\rangle^{-2b} \left\langle \tau_1 - k_1^{2j+1} \right\rangle^{-2b} d\tau_1 \right)^{\frac{1}{2}} \right.$$

$$\left. \times \left(\sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \left| \widetilde{F}_2(\tau_1, k_1) \right|^2 \left| \widetilde{F}_2(\tau - \tau_1, k - k_1) \right|^2 d\tau_1 \right)^{\frac{1}{2}} \right)^2 d\tau$$

$$\lesssim M \left\| f_2 \right\|_{X^{0,b}}^4,$$

where

$$M = \sup_{\substack{\tau \in \mathbb{R}, k \in \mathbb{Z} \\ k > 1}} \sum_{\substack{k_1 \in \mathbb{Z} \\ k_1, k-k_1 > 1}} \int_{\mathbb{R}} \left\langle \tau - \tau_1 - (k - k_1)^{2j+1} \right\rangle^{-2b} \left\langle \tau_1 - k_1^{2j+1} \right\rangle^{-2b} d\tau_1.$$

Thus, it is enough to show that $M \lesssim 1$ whenever $b > \frac{j+1}{2(2j+1)}$.

A direct computation

$$\int_{\mathbb{R}} \langle a \rangle^{-\alpha} \langle b - a \rangle^{-\alpha} \, da \lesssim \langle b \rangle^{1-2\alpha},$$

for $\frac{1}{2} < \alpha < 1$, yields

$$M \lesssim \sup_{\substack{\tau \in \mathbb{R}, k \in \mathbb{Z} \\ k > 1}} \sum_{\substack{k_1 \in \mathbb{Z} \\ k_1, k-k_1 > 1}} \left\langle \tau - k_1^{2j+1} - (k - k_1)^{2j+1} \right\rangle^{1-4b}.$$

For each $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $k > 1$, let $h(x) := h_j(x) - \tau$, for h_j as in (3.2) with $c = k$. From (3.4), we know

$$h(x) = \sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n} \left(\frac{k}{2} \right) \left(x - \frac{k}{2} \right)^{2n} + h_j \left(\frac{k}{2} \right) - \tau.$$

From Lemma 3.2 Item (5), we know $h_j \left(\frac{k}{2} \right) - \tau$ is the absolute minimum value of h. If $h_j \left(\frac{k}{2} \right) - \tau \geq 0$, we know

$$\langle h(k_1) \rangle^{1-4b} \leq \left((2j+1) h_0 \left(\frac{k}{2} \right) \left(k_1 - \frac{k}{2} \right)^{2j} \right)^{1-4b} = \left((2j+1) k \left(k_1 - \frac{k}{2} \right)^{2j} \right)^{1-4b} \quad \text{on } A^c,$$

where the set

$$A = \left\{ k_1 \in \mathbb{Z} : \left| k_1 - \frac{k}{2} \right| \leq 1 \right\}. \tag{3.8}$$

Note that $|A| \leq 3$. Thus,

$$\sum_{\substack{k_1 \in \mathbb{Z} \\ k_1, k-k_1 > 1}} \left\langle \tau - k_1^{2j+1} - (k - k_1)^{2j+1} \right\rangle^{1-4b} \lesssim \sum_{k_1 \in A} 1 + \sum_{\substack{k_1 \in A^c \\ k_1, k-k_1 > 1}} \left(k \left(k_1 - \frac{k}{2} \right)^{2j} \right)^{1-4b}$$

$$\lesssim 1 + \sum_{k > 1} k^{(2j+1)(1-4b)} + \sum_{|k_1 - \frac{k}{2}| > 1} \left| k_1 - \frac{k}{2} \right|^{(2j+1)(1-4b)}$$

$$\lesssim 1,$$

provided that $b > \frac{j+1}{2(2j+1)}$.

On the other hand, if $h_j\left(\frac{k}{2}\right) - \tau < 0$, since h is symmetry about $x = \frac{k}{2}$ and has the absolute minimum value at $\frac{k}{2}$, there is $\alpha > 0$ such that $\frac{k}{2} + \alpha$ and $\frac{k}{2} - \alpha$ are the only roots of h, i.e., $h(\frac{k}{2} + \alpha) = h(\frac{k}{2} - \alpha) = 0$. A direct computation gives

$$0 = h\left(\frac{k}{2} + \alpha\right) = h\left(\frac{k}{2} - \alpha\right) = \sum_{n=0}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{k}{2}\right) \alpha^{2n} - \tau.$$

With this, by Lemma 3.2 Item (6), we know

$$h(x) = \left(x - \frac{k}{2} + \alpha\right)\left(x - \frac{k}{2} - \alpha\right) \sum_{n=1}^{j} \binom{2j+1}{2n} h_{j-n}\left(\frac{k}{2}\right) \left(\sum_{\ell=0}^{n-1} \left(x - \frac{k}{2}\right)^{2n-2-2\ell} \alpha^{2\ell}\right).$$

Let $\Omega_\pm$ be a set of k_1 defined by

$$\Omega_\pm = \left\{k_1 \in \mathbb{Z} : \left|k_1 - \frac{k}{2} \pm \alpha\right|\right\}.$$

Note that $|\Omega_\pm| \le 3$. Then, similarly as before, we have

$$\langle h(k_1)\rangle^{1-4b} \le \left((2j+1)h_0\left(\frac{k}{2}\right)\left(k_1 - \frac{k}{2} + \alpha\right)\left(k_1 - \frac{k}{2} - \alpha\right)\left(k_1 - \frac{k}{2}\right)^{2j-2}\right)^{1-4b}$$

$$= \left((2j+1)k\left|k_1 - \frac{k}{2} + \alpha\right|\left|k_1 - \frac{k}{2} - \alpha\right|\left(k_1 - \frac{k}{2}\right)^{2j-2}\right)^{1-4b}$$

on $(\Omega_+ \cup \Omega_- \cup A)^c$, where the set A is as in (3.8). Therefore, we conclude that

$$\sum_{\substack{k_1 \in \mathbb{Z} \\ k_1, k-k_1 > 1}} \left\langle \tau - k_1^{2j+1} - (k-k_1)^{2j+1}\right\rangle^{1-4b}$$

$$\lesssim \sum_{k>1} k^{(2j+1)(1-4b)} + \sum_{|k_1 - \frac{k}{2} + \alpha| > 1} \left|k_1 - \frac{k}{2} + \alpha\right|^{(2j+1)(1-4b)}$$

$$+ \sum_{|k_1 - \frac{k}{2} - \alpha| > 1} \left|k_1 - \frac{k}{2} - \alpha\right|^{(2j+1)(1-4b)} + \sum_{|k_1 - \frac{k}{2}| > 1} \left|k_1 - \frac{k}{2}\right|^{(2j+1)(1-4b)}$$

$$\lesssim 1,$$

provided that $b > \frac{j+1}{2(2j+1)}$. This completes the proof. $\square$

4. Propagation of regularity and unique continuation property

In this section, we provide necessary basic tools that we use to demonstrate the main results of this work. In what follows, $[\cdot, \cdot]$ denotes the standard commutator operator defined by $[A, B] = AB - BA$.

4.1. Auxiliary lemmas

We first recall three useful lemmas for our analyses below.

Lemma 4.1 ([40, Lemma A.1]). *A function $\phi \in C^\infty(\mathbb{T})$ can be written in the form $\partial_x \varphi$ for some function $\varphi \in C^\infty(\mathbb{T})$ if and only if $\int_\mathbb{T} \phi(x)\, dx = 0$.*

Lemma 4.2 ([41, Lemma A.1.]). *Let $s, r \in \mathbb{R}$. Let f denote the operator of multiplication by $f \in C^\infty(\mathbb{T})$. Then, $[D^r, f]$ maps any H^s into H^{s-r+1}, where D^r operator is defined on distributions $\mathcal{D}'(\mathbb{T})$ by*

$$\widehat{D^r f}(n) = \begin{cases} sgn(n)|n|^r \hat{f}(n), & \text{if } n \neq 0, \\ \hat{f}(0), & \text{if } n = 0. \end{cases}$$

Lemma 4.3 ([41, Lemma A.3.]). *Let $s \in \mathbb{R}$. Let $f \in C^\infty(\mathbb{T})$ and $\rho_\varepsilon = e^{\varepsilon^2 \partial_x^2}$ with $0 \leq \varepsilon \leq 1$. Then, $[\rho_\varepsilon, f]$ is uniformly bounded as an operator from H^s into H^{s+1}.*

We end this subsection with the multiplication property of $X^{s,b}$ spaces.

Lemma 4.4 (*Multiplication Property*). *Let $-1 \leq b \leq 1$, $s \in \mathbb{R}$ and $\varphi \in C^\infty(\mathbb{T})$. Then, $\varphi u \in X^{s-2j|b|,b}$, for any $u \in X^{s,b}$. Moreover, the map $u \mapsto \varphi u$ from $X_T^{s,b}$ into $X_T^{s-2j|b|,b}$ is bounded.*

Proof. Since $X^{s,b}$ space is stable with respect to time localization (see (2.7)), it is enough to prove the first part (without time localization). When $b = 0$, it is obvious due to $X^{s,0} = L^2(\mathbb{R}; H^s)$ (see [40, Theorem 4.3]) and

$$\|\varphi u\|_{H^s} \lesssim \|u\|_{H^s}, \tag{4.1}$$

where the implicit constant depends on s and φ.

We now take $b = 1$, then it is known that

$$u \in X^{s,1} \iff u \in L^2(\mathbb{R}; H^s) \quad \text{and} \quad \left(\partial_t + (-1)^{j+1}\partial_x^{2j+1}\right) u \in L^2(\mathbb{R}; H^s), \tag{4.2}$$

due to the definition of $X^{s,b}$ and $\langle \cdot \rangle \sim 1 + |\cdot|$. A computation gives

$$\left(\partial_t + (-1)^{j+1}\partial_x^{2j+1}\right)(\varphi u) = \varphi \left(\partial_t + (-1)^{j+1}\partial_x^{2j+1}\right) u - [\varphi, (-1)^{j+1}\partial_x^{2j+1}]u. \tag{4.3}$$

Due to (4.3) and (4.1) in the definition (4.2), it suffices to show

$$\left\|[\varphi, (-1)^{j+1}\partial_x^{2j+1}]u\right\|_{H^{s-2j}} \lesssim \|u\|_{H^s}. \tag{4.4}$$

Observe that

$$[\varphi, (-1)^{j+1}\partial_x^{2j+1}]u = (-1)^j \sum_{\ell=0}^{2j} \binom{2j+1}{\ell} \partial_x^{2j+1-\ell}\varphi \, \partial_x^\ell u.$$

This, in addition to (4.1), immediately implies (4.4). Thus, by the complex interpolation theorem of Stein–Weiss for weighted L^p spaces (see [42, p. 114]), we complete the proof for the case when $0 \leq b \leq 1$.

The case when $-1 \leq b \leq 0$ can be proved via the duality argument. Precisely, the duality argument ensures the map $u \mapsto \varphi u$ from $X^{-s+2jb,-b}$ to X^{-s-b} is bounded for $0 \leq b \leq 1$. Since the spatial regularity is arbitrary, by replacing $-s + 2jb$ by s, we conclude that multiplication map is bounded from $X^{s,-b}$ to $X^{s-2jb,-b}$, which implies the desired result for $-1 \leq b \leq 0$, we thus complete the proof. □

4.2. Propagation of compactness

In this section, we present the properties of propagation of compactness for the linear differential operator $L = \partial_t + (-1)^{j+1}\partial_x^{2j+1}$ associated with the higher-order KdV type equation. The main ingredient is basically pseudo-differential analysis.

Proposition 4.5. *Let $T > 0$ and $0 \leq b' \leq b \leq 1$ be given, with $b > 0$. Suppose that $u_n \in X_T^{0,b}$ and $f_n \in X_T^{-2j+2jb,-b}$ satisfy*

$$\partial_t u_n + (-1)^{j+1}\partial_x^{2j+1}u_n = f_n,$$

for $n \in \mathbb{N}$. Assume that there exists a constant $C > 0$ such that

$$\|u_n\|_{X_T^{0,b}} \leq C \tag{4.5}$$

and that

$$\|u_n\|_{X_T^{-2j+2jb,-b}} + \|f_n\|_{X_T^{-2j+2jb,-b}} + \|u_n\|_{X_T^{-1+2jb',-b'}} \to 0, \quad as\ n \to +\infty. \tag{4.6}$$

In addition, assume that for some nonempty open set $\omega \subset \mathbb{T}$ it holds

$$u_n \to 0\ strongly\ in\ L^2\left(0,T; L^2\left(\omega\right)\right). \tag{4.7}$$

Then,

$$u_n \to 0\ strongly\ in\ L_{loc}^2\left((0,T); L^2\left(\mathbb{T}\right)\right), \quad as\ n \to +\infty.$$

Proof. For any compact interval $I \subset (0,T)$, we choose a cut-off function $\psi \in C_c^\infty((0,T))$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in I. Then, a simple computation yields

$$\|u_n\|_{L^2(I;L^2)}^2 \leq \int_0^T \psi(t)(u_n, u_n)\,dt.$$

On the other hand, since $\mathbb{T}$ is compact, there exist a finite number of open interval of the length less than the size of ω centered at x_0^m, $m = 1, 2, \ldots, M$ for some M. For an appropriate $\chi \in C_c^\infty(\omega)$, we can construct a partition of unity as

$$0 \leq \chi(x - x_0^m) \leq 1 \quad \text{and} \quad \sum_{m=1}^M \chi(x - x_0^m) \equiv 1, \tag{4.8}$$

for all $x \in \mathbb{T}$ and $m = 1, \ldots, M$. Then,

$$\int_0^T \psi(t)(u_n, u_n)\,dt \leq \sum_{m=1}^M \int_0^T \psi(t)(\chi(\cdot - x_0^m)u_n, u_n)\,dt,$$

which reduces our problem to proving that for any $\chi \in C_c^\infty(\omega)$ and $x_0 \in \mathbb{T}$

$$(\psi(t)\chi(\cdot - x_0)u_n, u_n)_{L^2(0,T;L^2)} \to 0, \quad n \to \infty. \tag{4.9}$$

Once proving that

$$\lim_{n\to\infty} (\psi(t)(\partial_x\varphi)u_n, u_n)_{L^2(0,T;L^2)} = 0, \tag{4.10}$$

for some $\varphi \in C^\infty(\mathbb{T})$, we immediately obtain (4.9) by putting

$$\partial_x\varphi = \chi(x) - \chi(x - x_0). \tag{4.11}$$

Indeed, a direct computation gives

$$(\psi(t)\chi(x - x_0)u_n, u_n)_{L^2(0,T;L^2)} = (\psi(t)\chi(x)u_n, u_n)_{L^2(0,T;L^2)} - (\psi(t)(\partial_x\varphi)u_n, u_n)_{L^2(0,T;L^2)},$$

and the right-hand side goes to zero due to (4.7) and (4.10), which implies (4.9). Note that Lemma 4.1 ensures to find $\varphi \in C^\infty(\mathbb{T})$ satisfying (4.11). Thus, we are now further reduced to proving (4.10).

On the other hand, one knows from Plancherel's theorem that

$$\int_{\mathbb{T}} \partial_x \varphi(x) u_n(x) u_n(x) \, dx = \widehat{u_n}(0) \int_{\mathbb{T}} \partial_x \varphi(x) u_n(x) \, dx$$
$$+ \int_{\mathbb{T}} (-1)^j \partial_x^{2j} D^{-2j} u_n(x) \partial_x \varphi(x) u_n(x) \, dx.$$

Therefore, the proof of Proposition 4.5 is completed from

$$\lim_{n \to \infty} \left| \left(\psi(t)(\partial_x \varphi) \partial_x^{2j} D^{-2j} u_n, u_n \right)_{L^2(0,T;L^2)} \right| = 0 \tag{4.12}$$

and

$$\lim_{n \to \infty} \left| \left(\psi(t)(\partial_x \varphi) \widehat{u_n}(t,0), u_n \right)_{L^2(0,T;L^2)} \right| = 0. \tag{4.13}$$

Proof of (4.12). Let take real valued $\varphi \in C^\infty(\mathbb{T})$ (satisfying (4.11)) and $\psi \in C_0^\infty((0,T))$, and let set

$$\mathcal{B} := \varphi(x) D^{-2j} \quad \text{and} \quad \mathcal{A} := \psi(t)\mathcal{B}.$$

It is straightforward to know $\mathcal{A}^* = \psi(t) D^{-2j} \varphi(x)$. We denote by $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ the regularization of $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{A}_\varepsilon := \mathcal{A} e^{\varepsilon \partial_x^2} \quad \text{and} \quad \mathcal{B}_\varepsilon := \mathcal{B} e^{\varepsilon \partial_x^2},$$

respectively, and set $\alpha_{n,\varepsilon} = ([\mathcal{A}_\varepsilon, \mathcal{L}]u_n, u_n)_{L^2(0,T;L^2)}$, where $\mathcal{L} = \partial_t + (-1)^{j+1} \partial_x^{2j+1}$. Note that $\mathcal{A}_\varepsilon = \psi(t)\mathcal{B}_\varepsilon$. From $\mathcal{L}u_n = f_n$ and $\mathcal{L}^* = -\mathcal{L}$, one has

$$\alpha_{n,\varepsilon} = (f_n, \mathcal{A}_\varepsilon^* u_n)_{L^2(0,T;L^2)} + (\mathcal{A}_\varepsilon u_n, f_n)_{L^2(0,T;L^2)}.$$

Using (2.7) and Lemma 4.4, the Cauchy–Schwarz inequality yields

$$\left| (f_n, \mathcal{A}_\varepsilon^* u_n)_{L^2(0,T;L^2)} \right| \leq \|f_n\|_{X_T^{-2j+2jb,-b}} \|\mathcal{A}_\varepsilon^* u_n\|_{X_T^{2j-2jb,b}} \lesssim \|f_n\|_{X_T^{-2j+2jb,-b}} \|u_n\|_{X_T^{0,b}}.$$

Similarly, we show

$$\left| (\mathcal{A}_\varepsilon u_n, f_n)_{L^2(0,T;L^2)} \right| \lesssim \|u_n\|_{X_T^{0,b}} \|f_n\|_{X_T^{-2j+2jb,-b}}.$$

Thus, the assumption (4.6) ensures

$$\lim_{n \to \infty} \sup_{0 < \varepsilon \leq 1} \alpha_{n,\varepsilon} = 0.$$

On the other hand, the fact $\partial_t \mathcal{A}_\varepsilon = \psi'(t)\mathcal{B}_\varepsilon + \mathcal{A}_\varepsilon \partial_t$ enables us to rewrite

$$\alpha_{n,\varepsilon} = \left([\mathcal{A}_\varepsilon, (-1)^{j+1} \partial_x^{2j+1}] u_n, u_n \right)_{L^2(0,T;L^2)} - (\psi'(t) \mathcal{B}_\varepsilon u_n, u_n)_{L^2(0,T;L^2)}.$$

Then, the analogous argument shows

$$\left| (\psi'(t) \mathcal{B}_\varepsilon u_n, u_n)_{L^2(0,T;L^2)} \right| \lesssim \|u_n\|_{X_T^{-2j+2jb,-b}} \|u_n\|_{X_T^{0,b}},$$

which implies

$$\lim_{n \to \infty} \sup_{0 < \varepsilon \leq 1} \left| (\psi'(t) \mathcal{B}_\varepsilon u_n, u_n)_{L^2(0,T;L^2)} \right| = 0.$$

Hence, we conclude that

$$\lim_{n \to \infty} \sup_{0 < \varepsilon \leq 1} \left([\mathcal{A}_\varepsilon, (-1)^{j+1} \partial_x^{2j+1}] u_n, u_n \right)_{L^2(0,T;L^2)} = 0.$$

In particular,

$$\lim_{n \to \infty} \left([\mathcal{A}, (-1)^{j+1} \partial_x^{2j+1}] u_n, u_n \right)_{L^2(0,T;L^2)} = 0.$$

Since ∂_x commutes with D^{-1}, a straightforward computation gives

$$[\mathcal{A}, (-1)^{j+1}\partial_x^{2j+1}] = (-1)^j \psi(t) \sum_{\ell=1}^{2j+1} \binom{2j+1}{\ell} \partial_x^\ell \varphi(x) \partial_x^{2j+1-\ell} D^{-2j}. \tag{4.14}$$

Analogously, we show for $\ell = 2, \ldots, 2j+1$ that[3]

$$\left| \left((-1)^j \psi(t) \binom{2j+1}{\ell} \partial_x^\ell \varphi(x) \partial_x^{2j+1-\ell} D^{-2j} u_n, u_n \right)_{L^2(0,T;L^2)} \right|$$

$$\leq \left\| \psi(t) \binom{2j+1}{\ell} \partial_x^\ell \varphi(x) \partial_x^{2j+1-\ell} D^{-2j} u_n \right\|_{X_T^{0,-b'}} \|u_n\|_{X_T^{0,b'}}$$

$$\lesssim \|u_n\|_{X_T^{1-\ell+2jb',-b'}} \|u_n\|_{X_T^{0,b'}},$$

which implies

$$\lim_{n\to\infty} \left| \left((-1)^j \psi(t) \binom{2j+1}{\ell} \partial_x^\ell \varphi(x) \partial_x^{2j+1-\ell} D^{-2j} u_n, u_n \right)_{L^2(0,T;L^2)} \right| = 0. \tag{4.15}$$

Collecting (4.14) and (4.15), we complete the proof of (4.12).

Proof of (4.13). A straightforward computation in addition to (4.5) yields

$$\|\widehat{u_n}(t,0)\|_{L^2((0,T))} \lesssim \|u_n\|_{X_T^{0,b}} \leq C.$$

Thus, the sequence $\widehat{u_n}(\cdot,0)$ is bounded in $H^b(0,T)$, which is compactly embedded in $L^2(0,T)$, by the Rellich Theorem. Therefore, there exists a subsequence that converges strongly in $L^2(0,T)$. Next, it can be seen that the only weak limit of a subsequence in $L^2(0,T)$ is zero, so that the whole sequence tends strongly to 0 in $L^2(0,T)$. Hence,

$$\widehat{u_n}(t,0) \to 0, \text{ (strongly) in } L^2(0,T), \text{ as } n \to +\infty,$$

and (4.13) holds. Consequently, Proposition 4.5 is proved. $\square$

4.3. Propagation of regularity

We now present the properties of propagation of regularity for the linear differential operator $\mathcal{L} = \partial_t + (-1)^{j+1}\partial_x^{2j+1}$ associated with the higher-order KdV-type equation.

Proposition 4.6. *Let $T > 0$, $r \in \mathbb{R}$, $0 \leq b \leq 1$ and $f \in X_T^{r,-b}$ be given. Let $u \in X_T^{r,b}$ be a solution of*

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1} u = f.$$

If there exists a nonempty $\omega \subset \mathbb{T}$ such that $u \in L^2_{loc}((0,T), H^{r+\rho}(\omega))$ for some ρ with

$$0 < \rho \leq \min\left\{ j(1-b), \frac{1}{2} \right\},$$

then

$$u \in L^2_{loc}((0,T), H^{r+\rho}(\mathbb{T})).$$

---

[3] It suffices to choose $b' = b$ when $\ell = 2j+1$.

Proof. The strategy of the proof is analogous to the proof of Proposition 4.5. Let $s := r + \rho$. For any compact interval $I \subset (0, T)$ and $\psi \in C_c^\infty((0, T))$ as in the proof of Proposition 4.5, we have

$$\|u\|_{L^2(I;H^s)}^2 \leq \int_0^T \psi(t)(u,u)_{H^s}\, dt$$
$$\lesssim \|u\|_{L^2(0,T;L^2)}^2 + \left(\psi(t)D^{2s-2j}(-\partial_x^2)^j u, u\right)_{L^2(0,T;L^2)},$$

Hence, we are reduced to proving

$$\left|\left(\psi(t)D^{2s-2j}\partial_x^{2j}u, u\right)_{L^2(0,T;L^2)}\right| \lesssim 1.$$

On the other hand, using a partition of unity as in (4.8) (but χ^2 instead of χ^4), it is enough to show that for any $\chi \in C_c^\infty(\omega)$ and $x_0 \in \mathbb{T}$, we have

$$\left|\left(\psi(t)D^{2s-2j}\chi^2(x - x_0)\partial_x^{2j}u, u\right)_{L^2(0,T;L^2)}\right| \lesssim 1.$$

Moreover, by taking $\partial_x\varphi = \chi^2(x) - \chi^2(x - x_0)$, we are finally reduced to proving

$$\left|\left(\psi(t)D^{2s-2j}(\partial_x\varphi)\partial_x^{2j}u, u\right)_{L^2(0,T;L^2)}\right| \lesssim 1, \tag{4.16}$$

for some $\varphi \in C^\infty(\mathbb{T})$, and

$$\left|\left(\psi(t)D^{2s-2j}\chi^2(x)\partial_x^{2j}u, u\right)_{L^2(0,T;L^2)}\right| \lesssim 1. \tag{4.17}$$

Proof of (4.16). For $n \in \mathbb{N}$, set

$$u_n := e^{\frac{1}{n}\partial_x^2}u \quad \text{and} \quad f_n := e^{\frac{1}{n}\partial_x^2}f.$$

Note that $\mathcal{L}u_n = f_n$. Then, there exists $C > 0$ such that

$$\|u_n\|_{X_T^{r,b}},\ \ \|f_n\|_{X_T^{r,-b}} \leq C$$

for all $n \in \mathbb{N}$. Define operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\mathcal{B} := D^{2s-2j}\varphi(x) \quad \text{and} \quad \mathcal{A} := \psi(t)\mathcal{B},$$

then we know similarly as in the proof of Proposition 4.5 that

$$\left([\mathcal{A}, (-1)^{j+1}\partial_x^{2j+1}]u_n, u_n\right)_{L^2(0,T;L^2)} - \left(\psi'(t)\mathcal{B}u_n, u_n\right)_{L^2(0,T;L^2)} = \left(f_n, \mathcal{A}^*u_n\right)_{L^2(0,T;L^2)}$$
$$+ \left(\mathcal{A}u_n, f_n\right)_{L^2(0,T;L^2)}.$$

Using (2.7) and Lemma 4.4, one shows

$$\left|\left(\mathcal{A}u_n, f_n\right)_{L^2(0,T;L^2)}\right| \leq \|\mathcal{A}u_n\|_{X_T^{-r,b}}\|f_n\|_{X_T^{r,-b}}$$
$$\lesssim \|u_n\|_{X_T^{-r+2jb+2s-2j,b}}\|f_n\|_{X_T^{r,-b}}$$
$$\lesssim \|u_n\|_{X_T^{r,b}}\|f_n\|_{X_T^{r,-b}}$$
$$\lesssim 1,$$

since $-r + 2jb + 2s - 2j = r + 2\rho - 2j(1 - b) \leq r$. Analogously, we show

$$\left|\left(\psi'(t)\mathcal{B}u_n, u_n\right)_{L^2(0,T;L^2)}\right|,\ \ \left|\left(f_n, \mathcal{A}^*u_n\right)_{L^2(0,T;L^2)}\right| \lesssim 1,$$

---

[4] It is possible if we simply take $\bar{\chi} = \sqrt{\chi}$.

which says

$$\left|\left([\mathcal{A},(-1)^{j+1}\partial_x^{2j+1}]u_n,u_n\right)_{L^2(0,T;L^2)}\right| \lesssim 1.$$

Note that the implicit constant, here, does not depend on $n \in \mathbb{N}$. Similarly as in (4.14), we know

$$[\mathcal{A},(-1)^{j+1}\partial_x^{2j+1}] = (-1)^j\psi(t)D^{2s-2j}\sum_{\ell=1}^{2j+1}\binom{2j+1}{\ell}\partial_x^\ell\varphi(x)\partial_x^{2j+1-\ell}.$$

For $\ell = 2, \ldots, 2j+1$, a direct computation gives

$$\left|\left((-1)^j\psi(t)D^{2s-2j}\binom{2j+1}{\ell}\partial_x^\ell\varphi(x)\partial_x^{2j+1-\ell}u_n,u_n\right)_{L^2(0,T;L^2)}\right|$$
$$\lesssim \left\|\psi(t)D^{2s-2j}\partial_x^\ell\varphi(x)\partial_x^{2j+1-\ell}u_n\right\|_{L^2(0,T;H^{-r})}\|u_n\|_{L^2(0,T;H^r)}$$
$$\lesssim \|u_n\|_{L^2(0,T;H^{-r+2s+1-\ell})}\|u_n\|_{L^2(0,T;H^r)}.$$

Since $-r+2s+1-\ell = r+2\rho-1+(2-\ell) \leq r$, for all $\ell = 2, \ldots, 2j+1$, whenever $\rho \leq \frac{1}{2}$, we conclude that

$$\left|\left((-1)^j\psi(t)D^{2s-2j}\binom{2j+1}{\ell}\partial_x^\ell\varphi(x)\partial_x^{2j+1-\ell}u_n,u_n\right)_{L^2(0,T;L^2)}\right| \lesssim 1,$$

for $\ell = 2, \ldots, 2j+1$ and $n \in \mathbb{N}$. Consequently, we obtain

$$\left|\left((-1)^j\psi(t)D^{2s-2j}(\partial_x\varphi)\partial_x^{2j}u_n,u_n\right)_{L^2(0,T;L^2)}\right| \lesssim 1.$$

Taking the limit on n, we conclude (4.16).

Proof of (4.17). A straightforward computation gives

$$\begin{aligned}
\left(\psi(t)D^{2s-2j}\chi^2\partial_x^{2j}u_n,u_n\right)_{L^2(0,T;L^2)} &= \left(\psi(t)[D^{s-2j},\chi]\chi\partial_x^{2j}u_n,D^su_n\right)_{L^2(0,T;L^2)} \\
&+ \left(\psi(t)D^{s-2j}\chi\partial_x^{2j}u_n,[D^s,\chi]u_n\right)_{L^2(0,T;L^2)} \\
&+ \left(\psi(t)D^{s-2j}\chi\partial_x^{2j}u_n,D^s\chi u_n\right)_{L^2(0,T;L^2)} \\
&=: I + II + III.
\end{aligned}$$

Lemmas 4.2 and 4.3 ensure for $u \in X_T^{r,b} \cap L^2_{loc}(0,T;H^s(\omega))$ that

$$\|\chi u_n\|_{H^s} \leq \left\|e^{\frac{1}{n}\partial_x^2}\chi u\right\|_{H^s} + \left\|[\chi,e^{\frac{1}{n}\partial_x^2}]u\right\|_{H^s} \lesssim \|\chi u\|_{H^s} + \|u\|_{H^{s-1}}$$

and

$$\left\|\chi\partial_x^{2j}u_n\right\|_{H^{\sigma-2j}} \leq \|\chi u_n\|_{H^\sigma} + \left\|[\chi,\partial_x^{2j}]u_n\right\|_{H^{\sigma-2j}} \lesssim \|\chi u_n\|_{H^\sigma} + \|u_n\|_{H^{\sigma-1}},$$

for $\sigma \in \mathbb{R}$. The Cauchy–Schwarz inequality and Lemmas 4.2 and 4.3, in addition to above estimates, yield

$$\begin{aligned}
|I| &\leq \|u_n\|_{L^2(0,T;H^r)}\left\|\psi(t)D^\rho[D^{s-2j},\chi]\chi\partial_x^{2j}u_n\right\|_{L^2(0,T;L^2)} \\
&\leq \|u_n\|_{L^2(0,T;H^r)}\left\|\psi(t)\chi\partial_x^{2j}u_n\right\|_{L^2(0,T;H^{s+\rho-1-2j})} \\
&\lesssim \|u_n\|_{X_T^{r,b}}\left(\|\psi(t)\chi u\|_{L^2(0,T;H^s)} + \|u\|_{X_T^{r,b}} + \|u_n\|_{X_T^{r,b}}\right) \\
&\lesssim 1, \\
|II| &\leq \left\|\psi(t)\chi\partial_x^{2j}u_n\right\|_{L^2(0,T;H^{s-2j-\rho})}\left\|D^\rho[D^s,\chi]u_n\right\|_{L^2(0,T;L^2)} \\
&\leq \left\|\psi(t)\chi\partial_x^{2j}u_n\right\|_{L^2(0,T;H^{s-2j-\rho})}\|u_n\|_{L^2(0,T;H^{\rho+s-1})} \\
&\lesssim \|u_n\|_{X_T^{r,b}}\left(\|\psi(t)\chi u\|_{L^2(0,T;H^s)} + \|u\|_{X_T^{r,b}} + \|u_n\|_{X_T^{r,b}}\right) \\
&\lesssim 1
\end{aligned}$$

and

$$
\begin{aligned}
|III| &\leq \int_0^T \psi(t) \left\| \chi \partial_x^{2j} u_n \right\|_{H^{s-2j}} \left\| \chi u_n \right\|_{H^s} \, dt \\
&\lesssim \int_0^T \psi(t) \left\| \chi u_n \right\|_{H^s} \left(\left\| \chi u_n \right\|_{H^s} + \left\| u_n \right\|_{H^{s-1}} \right) \, dt \\
&\lesssim \| u \|_{L^2(I;H^s(\omega))}^2 + \| u \|_{X_T^{r,b}}^2 + \left(\| \psi(t) \chi u \|_{L^2(0,T;H^s)} + \| u \|_{X_T^{r,b}} \right) \| u_n \|_{X_T^{r,b}} \\
&\lesssim 1.
\end{aligned}
$$

Thus, we obtain

$$
\left| \left(\psi(t) D^{2s-2j} \chi^2 \partial_x^{2j} u_n, u_n \right)_{L^2(0,T;L^2)} \right| \lesssim 1,
$$

which implies (4.17) by taking limit on n, and the proof of Proposition 4.6 is achieved. □

4.4. Unique continuation property

As a consequence of the propagation of regularity, we prove the following unique continuation property for the higher-order KdV type. First, let us prove the auxiliary lemma.

Lemma 4.7. *Let $u \in X_T^{0,\frac{1}{2}}$ be a solution of*

$$
\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + u \partial_x u = 0 \quad \text{on} \ (0,T) \times \mathbb{T}. \tag{4.18}
$$

Assume that $u \in C^\infty((0,T) \times \omega)$, where $\omega \subset \mathbb{T}$ nonempty set. Then, $u \in C^\infty((0,T) \times \mathbb{T})$.

Proof. Recall that the mean value $[u]$ is conserved. Changing a into $a + [u]$ if needed, we may assume that $[u] = 0$. Using Lemma 3.1 (or from Lemma 2.2), we have that $u \partial_x u \in X_T^{0,-\frac{1}{2}}$. It follows from Proposition 4.6 with $f = -u \partial_x u$ that

$$
u \in L^2_{loc}(0,T; H^{\frac{1}{2}}(\mathbb{T})).
$$

Choose t_0 such that $u(t_0) \in H^{\frac{1}{2}}(\mathbb{T})$. We can then solve (4.18) in $X_T^{\frac{1}{2},\frac{1}{2}}$ with the initial data $u(t_0)$. By uniqueness of solution in $X_T^{0,\frac{1}{2}}$, we conclude that $u \in X_T^{\frac{1}{2},\frac{1}{2}}$. Applying Proposition 4.6 iteratively, we obtain

$$
u \in L^2(0,T; H^r(\mathbb{T})), \ \forall r \geq 0,
$$

and, hence $u \in C^\infty((0,T) \times \mathbb{T})$. □

As a consequence of the previous result, we have the following unique continuation property.

Corollary 4.8. *Let ω be a nonempty open set in $\mathbb{T}$ and let $u \in X_T^{0,\frac{1}{2}}$ be a solution of*

$$
\begin{cases}
\partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + u \partial_x u = 0 & \text{on} \ (0,T) \times \mathbb{T}, \\
u = c & \text{on} \ (0,T) \times \omega,
\end{cases}
$$

where $c \in \mathbb{R}$ denotes some constant. Then, $u(t,x) = c$ on $(0,T) \times \mathbb{T}$. Furthermore, if the mean $[u] = 0$, then $u(t,x) = 0$ on $(0,T) \times \mathbb{T}$.

Proof. Using Lemma 4.7, we infer that $u \in C^\infty((0,T) \times \mathbb{T})$. It follows that $u \equiv c$ on $(0,T) \times \mathbb{T}$ by the unique property proved by Saut and Scheurer in [43]. □

5. Global stability: Proof of Theorem 1.4

In this section, we can establish the global results for the higher-order nonlinear dispersive equation (while Theorem 1.2 has a local aspect). The main ingredients are the propagation of regularity and the unique continuation property shown in the previous section. Accurately, we are concerned with the stability properties of the closed loop system

$$
\begin{cases}
\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + u\partial_x u = -K_\lambda u & \text{in } \{t > 0\} \times \mathbb{T}, \\
u(0, x) = u_0(x) & \text{on } \mathbb{T},
\end{cases}
\tag{5.1}
$$

where $\lambda \geq 0$ is a given number, $u_0 \in H_0^s(\mathbb{T})$, for any $s \geq 0$ and K_λ is defined by $K_\lambda u(t, x) \equiv GG^* L_\lambda^{-1} u(t, x)$ for the operator G defined as in (1.4). It is known (see, for e.g. [19]) that G is a linear bounded operator from $L^2(0, T; H_0^s(\mathbb{T}))$ into itself. Moreover, G is a self-adjoint positive operator on $L_0^2(\mathbb{T})$.

5.1. Proof of Theorem 1.4 for $s = 0$

Consider $\lambda = 0$ in K_λ and remember that $K_0 = GG^*$, so Theorem 1.4 in L^2-level is a direct consequence of the following *observability inequality*:

Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\mu > 1$ such that for any $u_0 \in L_0^2(\mathbb{T})$ satisfying

$$
\|u_0\|_{L^2} \leq R_0,
$$

the corresponding solution u of (5.1), *with $\lambda = 0$, satisfies*

$$
\|u_0\|_{L^2}^2 \leq \mu \int_0^T \|Gu(t)\|_{L^2}^2 \, dt.
\tag{5.2}
$$

Indeed, suppose that (5.2) holds. Assuming that $\lambda = 0$, the energy estimate gives us

$$
\|u(T, \cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - \int_0^T \|Gu(t)\|_{L^2}^2 \, dt,
\tag{5.3}
$$

which jointly with (5.2) insures,

$$
\|u(T, \cdot)\|_{L^2}^2 \leq (\mu - 1) \int_0^T \|Gu(t)\|_{L^2}^2 \, dt
$$

or equivalently,

$$
\|u(T, \cdot)\|_{L^2}^2 \leq (\mu - 1)\left(\|u_0\|_{L^2}^2 - \|u(T, \cdot)\|_{L^2}^2\right).
$$

Thus,

$$
\|u(T, \cdot)\|_{L^2}^2 \leq \frac{\mu - 1}{\mu} \|u_0\|_{L^2}^2.
$$

In this way, we inductively obtain that

$$
\|u(kT, \cdot)\|_{L^2}^2 \leqslant \left(\frac{\mu - 1}{\mu}\right)^k \|u_0\|_{L^2}^2,
$$

for all $k \geqslant 0$. Finally, analogously to (5.3), we know

$$
\|u(t, \cdot)\|_{L^2} \leqslant \|u(kT, \cdot)\|_{L^2},
$$

for $kT \leqslant t \leqslant (k+1)T$, thus

$$
\|u(t, \cdot)\|_{L^2(\mathbb{T})} \leqslant ce^{-\gamma t}\|u_0\|_{L^2(\mathbb{T})}, \quad \forall t \geqslant 0,
\tag{5.4}
$$

where $c = \frac{\mu}{\mu - 1}$ and $\gamma = \frac{\log\left(\frac{\mu}{\mu - 1}\right)}{T}$, and Theorem 1.4 holds true for $s = 0$. $\quad\square$

Let us now turn to prove inequality (5.2). To do that we argue by contradiction. Suppose not, there exist a sequence $\{u_n\}_{n\in\mathbb{N}} = u_n$, such that $u_n \in Y_T^0$ is solution of (5.1) satisfying

$$\|u_{0,n}\|_{L^2} \leq R_0$$

but

$$\int_0^T \|Gu_n(t)\|_{L^2}^2 \ dt < \frac{1}{n} \|u_{0,n}\|_{L^2}^2 , \tag{5.5}$$

where $u_{0,n} = u_n(0)$. Let $\xi_n := \|u_{0,n}\|_{L^2} \leq R_0$. Then, one can choose a subsequence of $\xi_n = \{\xi_n\}_{n\in\mathbb{N}}$, still denote by ξ_n, such that,

$$\lim_{n\to\infty} \xi_n = \xi.$$

There are two possible cases: (a) $\xi > 0$ and (b) $\xi = 0$.

Case (a): $\xi > 0$. Since the sequence u_n is bounded in Y_T^0, by Lemma 2.2 (see particularly [15, Lemma 2.4]), the sequence $\{\partial_x(u_n^2)\}_{n\in\mathbb{N}}$ is bounded in $X_T^{0,-\frac{1}{2}}$. By the compactness of embedding (taking subsequences if needed, but still denote by u_n), we know

$$u_n \to u \quad \text{in } X_T^{-1,0} \quad \text{and} \quad -\frac{1}{2}\partial_x(u_n^2) \to f \quad \text{in } X_T^{0,-\frac{1}{2}},$$

where $u \in X_T^{0,\frac{1}{2}}$ and $f \in X_T^{0,-\frac{1}{2}}$. Moreover, from (3.1), we obtain

$$X_T^{0,\frac{1}{2}} \hookrightarrow L^4((0,T)\times\mathbb{T}),$$

which ensures that u_n^2 is bounded in $L^2((0,T)\times\mathbb{T})$. Therefore, its follows that $\partial_x(u_n^2)$ is bounded in

$$X_T^{-1,0} = L^2\left(0,T;H^{-1}(0,L)\right).$$

From interpolation of the space $X_T^{0,-\frac{1}{2}}$ and $X_T^{-1,0}$, we obtain that $\partial_x(u_n^2)$ is bounded in $X_T^{-\theta,-\frac{1}{2}+\frac{\theta}{2}}$, for $\theta \in [0,1]$. Again using the compactness of embedding we conclude that

$$-\frac{1}{2}\partial_x(u_n^2) \to f \text{ in } X_T^{-1,-\frac{1}{2}}.$$

On the other hand, it follows from (5.5) that

$$\int_0^T \|Gu_n\|_{L^2}^2 \, dt \to \int_0^T \|Gu\|_{L^2}^2 \, dt = 0, \tag{5.6}$$

which implies from the definition of the operator G as in (1.4) that

$$u(t,x) = \int_{\mathbb{T}} g(y)u(t,y) \, dy =: c(t) \quad \text{on } (0,T)\times\omega,$$

where $\omega = \{x \in \mathbb{T} : g > 0\}$. Thus, taking $n \to \infty$, we obtain from (5.1) that

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1}u = f & \text{on } (0,T)\times\mathbb{T}, \\ u(t,x) = c(t) & \text{on } (0,T)\times\omega. \end{cases}$$

We prove now that $f = -\frac{1}{2}\partial_x(u^2)$. Indeed, pick $w_n = u_n - u$ and $f_n = -\frac{1}{2}\partial_x(u_n^2) - f - K_0 u_n$. Remark that from (5.6),

$$\int_0^T \|Gw_n\|_{L^2}^2 \, dt = \int_0^T \|Gu_n\|_{L^2}^2 \, dt + \int_0^T \|Gu\|_{L^2}^2 \, dt - 2\int_0^T (Gu_n, Gu) \, dt \to 0. \tag{5.7}$$

Since $w_n \rightharpoonup 0$ in $X_T^{0,\frac{1}{2}}$ we infer from Rellich theorem that

$$\int_{\mathbb{T}} g(y) w_n(t,y) \, dy \to 0 \text{ in } L^2(0,T).$$

Combined with (5.7), this yields

$$\|gw_n\|_{L^2(0,T;L^2)} \le \|Gw_n\|_{L^2(0,T;L^2)} + \left\|\int_{\mathbb{T}} g(y)w_n(\cdot,y) \, dy\right\|_{L^2(0,T)} \to 0.$$

Then, w_n and f_n satisfy

$$\partial_t w_n + (-1)^{j+1} \partial_x^{2j+1} w_n = f_n$$

and

$$f_n \to 0 \text{ in } X_T^{-1,-\frac{1}{2}} \quad \text{and} \quad w_n \to 0 \text{ in } L^2\left(0,T;L^2(\tilde{\omega})\right),$$

where $\tilde{\omega} := \left\{g > \frac{\|g\|_{L^\infty}}{2}\right\}$. Applying Proposition 4.5 with $b = \frac{1}{2}$ and $b' = 0$, we conclude that

$$w_n \to 0 \text{ in } L_{loc}^2\left(0,T;L^2(\mathbb{T})\right).$$

Consequently, u_n^2 tends to u^2 in $L_{loc}^1\left(0,T;L^2(\mathbb{T})\right)$ and $\partial_x(u_n^2)$ tends to $\partial_x(u^2)$ in distributional sense. Therefore, $f = -\frac{1}{2}\partial_x(u^2)$ and $u \in X_T^{0,\frac{1}{2}}$ satisfies

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + \frac{1}{2}\partial_x(u^2) = 0 & \text{on } (0,T) \times \mathbb{T}, \\ u(t,x) = c(t) & \text{on } (0,T) \times \omega. \end{cases}$$

The first equation gives $c'(t) = 0$ which, combined with the unique continuation property (Corollary 4.8) ensures that $u(t,x) = c$, for some $c \in \mathbb{R}$. Since $[u] = 0$, then $c = 0$ and

$$u_n \to 0 \text{ in } L_{loc}^2\left((0,T),L^2(\mathbb{T})\right).$$

To end the proof of **Case (a)**, we take a particular time $t_0 \in [0,T]$ such that

$$u_n(t_0) \to 0 \text{ in } L^2(\mathbb{T}).$$

A direct computation gives

$$\|u_{0,n}\|_{L^2}^2 = \|u_n(t_0)\|_{L^2}^2 + \int_0^{t_0} \|Gu_n(t)\|_{L^2}^2 \, dt,$$

and this makes a contradiction, since the right-hand side converges to 0 while the left-hand side does not by the hypothesis.

Case (b): $\xi = 0$. Note from (5.5) that $\xi_n > 0$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, set $v_n = \frac{u_n}{\xi_n}$. Then v_n satisfies

$$\partial_t v_n + (-1)^{j+1}\partial_x^{2j+1}v_n + K_0 v_n + \frac{\xi_n}{2}\partial_x(v_n^2) = 0$$

with

$$\int_0^T \|Gv_n\|_{L^2}^2 \, dt < \frac{1}{n} \tag{5.8}$$

and

$$\|v_{0,n}\|_{L^2} = 1.$$

Analogously as above, by the compactness of embedding, v_n (by extracting the subsequences if needed, but still denote by v_n) satisfies

$$v_n \to v \text{ in } X_T^{-1,-\frac{1}{2}} \cap X_T^{-1,0} \quad \text{and} \quad \xi_n \partial_x(v_n^2) \to 0 \text{ in } X_T^{0,-\frac{1}{2}}.$$

From (5.8), we have

$$\int_0^T \|Gv\|_{L^2}^2 \, dt = 0,$$

which ensures that v solves

$$\begin{cases} \partial_t v + (-1)^{j+1} \partial_x^{2j+1} v = 0 & \text{on } (0, T) \times \mathbb{T}, \\ v(t, x) = c(t) & \text{on } (0, T) \times \omega. \end{cases} \tag{5.9}$$

Due to Holmgren's uniqueness theorem (see e.g. [44]), we conclude $c(t) = c \in \mathbb{R}$. Moreover, as $[v] = 0$ then $c = 0$. According to (5.8), Gv_n converges to 0 in $L^2(0, T; L^2)$, thus so

$$K_0 v_n \to 0 \text{ in } X_T^{-1, -\frac{1}{2}}.$$

Applying Proposition 4.5 as in **Case (a)**, we have

$$v_n \to 0 \text{ in } L_{loc}^2 \left(0, T; L^2(\mathbb{T})\right),$$

thus we achieve the same conclusion, showing the result. □

Remark 6. In view of the hypotheses in Proposition 4.5, the proof above is still valid, even if f_n converges to 0 only in larger class (e.g., $X^{-j, -\frac{1}{2}}$). In other words, the property of propagation of compactness (Proposition 4.5) established in this work enables one to extend Theorem 1.4 for rougher solutions.

5.2. Proof of Theorem 1.4

Once again, consider $\lambda = 0$ in K_λ. Now we prove that the solution u of (5.1) decays exponentially in H^s-level for any $s > 0$. We first prove it when $s = 2j + 1$. Then, interpolating with the result in Section 5.1, we obtain the conclusion for $0 \leq s \leq 2j + 1$. The similar argument can be applied for $s = (2j + 1)\mathbb{N}$, and thus we complete the proof.

Fix $s = 2j + 1$, $j \in \mathbb{N}$, and pick any $R > 0$ and any $u_0 \in H^{2j+1}(\mathbb{T})$ with $\|u_0\|_{L^2(\mathbb{T})} \leq R$. Let u be solution of (5.1) with initial condition u_0, and set $v = u_t$. Then, v satisfies

$$\begin{cases} \partial_t v + (-1)^{j+1} \partial_x^{2j+1} v + \partial_x(uv) = -K_0 v & \text{in } \{t > 0\} \times \mathbb{T}, \\ v(0, x) = v_0(x) & \text{on } \mathbb{T}, \end{cases} \tag{5.10}$$

where

$$v_0 = -K_0 u_0 - u_0 u_0' - (-1)^{j+1} \partial_x^{2j+1} u_0. \tag{5.11}$$

According to Theorem B.3 and the exponential decay (5.4), for any $T > 0$ there exist constants $C > 0$ and $\gamma > 0$ depending only on R and T such that

$$\|u(\cdot, t)\|_{Y_{[t, t+T]}^0} \leq C e^{-\gamma t} \|u_0\|_{L^2} \quad \text{for all } t \geq 0.$$

Thus, for any $\epsilon > 0$, there exists a $t^* > 0$ such that if $t \geq t^*$, one has

$$\|u(\cdot, t)\|_{Y_{[t, t+T]}^s} \leq \epsilon.$$

At this point we need an exponential stability result for the linearized system

$$\begin{cases} \partial_t w + (-1)^{j+1} \partial_x^{2j+1} w + \partial_x(aw) = -K_0 w & \text{in } \{t > 0\} \times \mathbb{T}, \\ w(0, x) = w_0(x) & \text{on } \mathbb{T}, \end{cases} \tag{5.12}$$

where $a \in Y_T^s$ is a given function.

Lemma 5.1. *Let $s \geq 0$ and $a \in Y_T^s$ for all $T > 0$. Then for any $\gamma' \in (0, \gamma)$ there exist $T > 0$ and $\beta > 0$ such that if*

$$\sup_{n \geq 1} \|a\|_{Y_{[nT,(n+1)T]}^s} \leq \beta \tag{5.13}$$

then

$$\|w(\cdot, t)\|_{H^s} \lesssim e^{-\gamma' t} \|w_0\|_{H^s} \quad \text{for all } t \geq 0.$$

Remark that the implicit constant depends on $\|w_0\|_{H^s}$, but not w_0.

Suppose Lemma 5.1 is valid. Choose $\epsilon < \beta$, and then apply Lemma 5.1 to (5.10) to obtain

$$\|v(\cdot, t)\|_{L^2} \lesssim e^{-\gamma'(t-t^*)} \|v(\cdot, t^*)\|_{L^2},$$

for any $t \geq t^*$, or equivalently

$$\|v(\cdot, t)\|_{L^2} \lesssim e^{-\gamma' t} \|v_0\|_{L^2},$$

for any $t \geq 0$. From

$$(-1)^{j+1} \partial_x^{2j+1} u = -K_0 u - u \partial_x u - v,$$

a direct computation gives

$$\|u(t)\|_{H^s} \lesssim \|u(t)\|_{L^2} + \|u(t)\|_{L^2} \|u(t)\|_{H^s} + \|v(t)\|_{L^2}.$$

Applying Theorem 1.4 for $s = 0$ established in Section 5.1, Theorem B.3 and Lemma 5.1 with (5.11) to the right-hand side, we obtain

$$\|u(\cdot, t)\|_{H^s} \lesssim C e^{-\gamma' t} \|u_0\|_{H^s},$$

for any $t \geq 0$. Note that the implicit constant here depends only on R. This proves Theorem 1.4 for $s = 2j+1$. Moreover applying Lemma 5.1 for $w = u_1 - u_2$ and $a = u_1 + u_2$ when u_1, u_2 are two different solutions, we obtain the Lipschitz stability estimate, which is required for interpolation:

$$\|(u_1 - u_2)(\cdot, t)\|_0 \leq C e^{-\gamma' t} \|(u_1 - u_2)(\cdot, 0)\|_0.$$

Thus, it remains to prove Lemma 5.1. $\quad\square$

Proof of Lemma 5.1. Let $T > 0$ and $s \geq 0$ be given, and $a \in Y_T^s$. Similarly as the proof of Theorem B.3, we can show that the system (5.12) admits a unique solution $Y_T^s \cap C_T H^s$, and the solution u satisfies

$$\|u\|_{Y_T^s} \leq \alpha_{T,s}(\|a\|_{Y_T^s}) \|u_0\|_{H^s}, \tag{5.14}$$

where $\alpha_{T,s}$ is positive nondecreasing continuous function. By Duhamel's principle, the solution u to (5.12) is equivalent to the following integral form:

$$w(t) = W_0(t) w_0 - \int_0^t W_0(t-s) \partial_x(aw)(s) \, ds,$$

where $W_0(t) = e^{-t\left((-1)^{j+1} \partial_x^{2j+1} + K_0\right)}$. Then, due to Proposition A.4, Lemma B.1 and (5.14), we get

$$\|w(\cdot, T)\|_{H^s} \leq C_1 e^{-\gamma T} \|w_0\|_{H^s} + C_2 \|a\|_{Y_T^s} \alpha_{T,s}\left(\|a\|_{Y_T^s}\right) \|w_0\|_{H^s}, \tag{5.15}$$

where $C_1 > 0$ is independent of T while C_2 may depend on T. Let

$$y_n = w(\cdot, nT) \quad \text{for } n \in \mathbb{N}.$$

Then, similarly as (5.15), we obtain for each $n \in \mathbb{N}$ that

$$\|y_{n+1}\|_{H^s} \leq C_1 e^{-\gamma T} \|y_n\|_{H^s} + C_{2,n} \|a\|_{Y^s_{[nT,(n+1)T]}} \alpha_{T,s} \left(\|a\|_{Y^s_{[nT,(n+1)T]}} \right) \|y_n\|_{H^s}.$$

By choosing appropriate $T > 0$ large enough and $\beta > 0$ small enough such that

$$C_1 e^{-\gamma T} + C_2 \beta \alpha_{T,s}(\beta) = e^{-\gamma' T},$$

we conclude that

$$\|y_{n+1}\|_{H^s} \leq e^{-\gamma' T} \|y_n\|_{H^s} \tag{5.16}$$

for any $n \geq 1$ as long as (5.13) is assumed. By using (5.16) inductively, we obtain

$$\|y_n\|_{H^s} \leq e^{-n\gamma' T} \|y_0\|_{H^s}$$

for any $n \geq 1$, which implies that

$$\|w(\cdot, t)\|_{H^s} \leq C e^{-\gamma' t} \|w_0\|_{H^s}$$

for all $t \geq 0$. This completes the proof. $\square$

6. Concluding remarks and open issues

In this work we treat the global control issues for the general higher-order KdV type equation on periodic domain

$$\begin{cases} \partial_t u + (-1)^{j+1} \partial_x^{2j+1} u + u \partial_x u = Gh, \\ u(0, x) = u_0, \end{cases} \qquad (t, x) \in (0, T) \times \mathbb{T}, \tag{6.1}$$

where, Gh is defined as (1.4) and can take also the form $Gh := K_\lambda u(t, x) \equiv GG^* L_\lambda^{-1} u(t, x)$. The results presented in the manuscript recovered previous global control problems for the KdV and Kawahara equations, when $j = 1$ and 2, respectively. Nevertheless, presents global control results for a general KdV type equation, which is more complex than the studies previously presented.

Precisely, due to the smoothing properties of solutions in Bourgain spaces we are able to prove the *Strichartz estimates* and *propagation of regularity* associated with the solution of the linear system of (6.1). With this in hand, we prove an *observability inequality* for the solutions of the system(6.1). This helps us prove the main results of the article. Even though it has a generalist character, the work presents interesting problems from the mathematical point of view, which we will detail below.

6.1. Time-varying feedback law

A natural question that arises is related to global stabilization with an arbitrary large decay rate. This can be obtained by using a time-varying feedback law. As for the KdV and Kawahara equations, the time-varying feedback control law for the higher-order KdV type equation can be found. Precisely, it is possible to construct a continuous time-varying feedback law $K \equiv K(u, t)$ such that a semi-global stabilization holds with an arbitrary large decay rate in the Sobolev space $H^s(\mathbb{T})$ for any $s \geq 0$. In fact, K has the following form

$$K(u, t) := \rho \left(\|u\|_{H^s(\mathbb{T})}^2 \right) \left[\theta \left(\frac{t}{T} \right) K_\lambda(u) + \theta \left(\frac{t}{T} - T \right) GG^* u \right] + \left(1 - \rho \left(\|u\|_{H^s(\mathbb{T})}^2 \right) \right) GG^* u,$$

where $\rho \in C^\infty(\mathbb{R}^+; [0, 1])$ is a function such that for some $r_0 \in (0, 1)$, we have

$$\rho(r) := \begin{cases} 1, & \text{for } r \leq r_0, \\ 0, & \text{for } r \geq 1 \end{cases}$$

and $\theta \in C^\infty(\mathbb{R}; [0, 1])$ is a function with the following properties: $\theta(t + 2) = \theta(t)$, for all $t \in \mathbb{R}$ and

$$\theta(t) := \begin{cases} 1, & \text{for } \delta \le t \le 1 - \delta, \\ 0, & \text{for } 1 \le t \le 2, \end{cases}$$

for some $\delta \in (0, \frac{1}{10})$. Then, the following result holds true.

Theorem 6.1. *Let $\lambda > 0$ and let $K = K(u, t)$ be as above. Pick any $\lambda' \in (0, \lambda)$ and any $\lambda'' \in (\lambda'/2, (k + \lambda')/2)$. Then there exists a time $T_0 > 0$ such that for $T > T_0$, $t_0 \in \mathbb{R}$ and $u_0 \in H^s(\mathbb{T})$, the unique solution of the closed-loop system*

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + u\partial_x u = -K(u, t), \quad (t, x) \in \mathbb{R} \times \mathbb{T}$$

satisfies

$$\|u(\cdot, t) - [u_0]\|_{H^s} \le \gamma_{s,\mu}(\|u_0 - [u_0]\|_{H^s}) e^{-\lambda''(t-t_0)}\|u_0 - [u_0]\|_{H^s}, \text{ for all } t > t_0,$$

where γ_s is a nondecreasing continuous function.

6.2. Low regularity control results

Observe that the results presented in this work are verified in $H^s(\mathbb{T})$, when $s \ge 0$. However, by comparing with [15], a natural question appears.

Problem $\mathcal{A}$: Is it possible to prove control results for the system (6.1) with $-j/2 \le s < 0$?

The answer for this question may be very technical and the well-posedness result probably will be the biggest challenge. Additionally, the unique continuation property needs to be proved and appears to be also a hard problem. Thus, the following open issues naturally appear.

Problem $\mathcal{B}$: Is the system (6.1) globally well-posed in $H^s(\mathbb{T})$, for $-j/2 \le s < 0$?

Problem $\mathcal{C}$: Is the unique continuation property, presented in Lemma 4.7, true for $-j/2 \le s < 0$?

Acknowledgments

Appendix A. Controllability and stability results: Linear problems

Let us consider the linear open loop control system

$$\begin{cases} \partial_t u + (-1)^{j+1}\partial_x^{2j+1}u = Gh, & (t, x) \in \mathbb{R} \times \mathbb{T}, \\ u(0, x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{A.1}$$

where the operator G is defined as in (1.4) and $h = h(t, x)$ is the control input.

Consider the L^2-basis $\{\phi_k\}_{k\in\mathbb{Z}}$, thus the solution u of (A.1) can be expressed in the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} \left(e^{\lambda_k t} u_{0,k} + \int_0^t e^{\lambda_k(t-\tau)} G_k[h](\tau)\,d\tau \right) \phi_k(x), \tag{A.2}$$

where $u_{0,k}$ are the Fourier coefficients of u_0 and $G[h]$ are

$$u_{0,k} = (u_0, \phi_k) \quad \text{and} \quad G_k[h] = (Gh, \phi_k) = (h, G\phi_k) \tag{A.3}$$

for $k \in \mathbb{Z}$, respectively. Moreover, for given $s \in \mathbb{R}$, if $u_0 \in H^s(\mathbb{T})$ and $h \in L^2(0,T; H^s(\mathbb{T}))$, the function given by (A.2) belongs to the space $C([0,T]; H^s(\mathbb{T}))$. Now, we are in position to prove control results for the system (A.1).

A.1. Controllability result

The first result means that system (A.1) is exactly controllable in time $T > 0$ and can be read as follows.

Theorem A.1. *Let $T > 0$ and $s \in \mathbb{R}$ be given. There exists a bounded linear operator*

$$\Phi : H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to L^2(0,T; H^s(\mathbb{T}))$$

such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$, if one chooses $h = \Phi(u_0, u_1)$ in (A.1), then the system (A.1) admits a solution $u \in C([0,T]; H^s(\mathbb{T}))$ satisfying

$$u|_{t=0} = u_0, \quad u|_{t=T} = u_1.$$

Moreover, we have

$$\|\Phi(u_0, u_1)\|_{L^2(0,T;H^s)} \lesssim (\|u_0\|_{H^s} + \|u_1\|_{H^s}), \tag{A.4}$$

here the implicit constant depends only on T, $\|g\|_{H^s}$ and $\|g\|_{H^{-s}}$.

Remark 7. The proof of Theorem A.1 is standard provided that the eigenvalues of the associated linear operator satisfy Lemma 2.1, precisely, the proof relies only on the fact that the dual basis of $\{e^{\lambda_k t}\}$ is a Riesz sequence, which follows from a classical theorem of Ingham and Beurling [32,33].

Proof of Theorem A.1. As mentioned in Remark 7, the proof of Theorem A.1 is now standard, and can be found in the literature, for instance [10,19,25]. However, we also give a proof for the sake of self-containedness.

Since the solution $u(t,x)$ can be expressed as in (A.2), it suffices to find $h \in L^2(0,T; H^s(\mathbb{T}))$ such that

$$u_1(x) = \sum_{k\in\mathbb{Z}} \left(e^{\lambda_k T} u_{0,k} + \int_0^T e^{\lambda_k(T-\tau)} G_k[h](\tau)\,d\tau \right) \phi_k(x),$$

which follows from

$$e^{-\lambda_k T} u_{1,k} - u_{0,k} = \int_0^T e^{-\lambda_k \tau} G_k[h](\tau)\,d\tau, \quad \text{for each } k \in \mathbb{Z}, \tag{A.5}$$

where $u_{0,k}$, $u_{1,k}$ and $G_k[h]$ are the Fourier coefficients defined as in (A.3).

Note that $\mathcal{P} := \{p_k = e^{\lambda_k t} : k \in \mathbb{Z}\}$ forms a Riesz basis for its closed span $\mathcal{P}_T$ in $L^2(0,T)$, and there uniquely exists the dual basis $\mathcal{Q} = \{q_k : k \in \mathbb{Z}\}$ in $\mathcal{P}_T$ such that

$$\int_0^T q_j(t)\overline{p_k(t)}\,dt = \delta_{jk}, \quad -\infty < j,k < \infty. \tag{A.6}$$

We take the control input h of the form

$$h(t, x) = \sum_{j \in \mathbb{Z}} h_j q_j(t)(G\phi_j)(x), \tag{A.7}$$

where the coefficients h_j are to be precisely determined later, depending on given v_0, v_1 and g.

Inserting (A.7) into (A.5) and using (A.6) and the fact that G is a self-adjoin operator, one has

$$e^{-\lambda_k T} u_{1,k} - u_{0,k} = \sum_{j \in \mathbb{Z}} \int_0^T e^{-\lambda_k t} h_j q_j(t)(G(G\phi_j), \phi_k) dt = h_k \|G\phi_k\|_{L^2}^2. \tag{A.8}$$

We set $\beta_k := \|G\phi_k\|_{L^2}^2$. The definitions of ϕ_k and $G\phi_k$ ensure that $\beta_k > 0$ for all $k \neq 0$. Moreover, a direct computation in (1.4) gives

$$\begin{aligned}
\beta_k &= \frac{1}{2\pi} \int_{\mathbb{T}} g(x)^2 \, dx - 2\mathrm{Re}\left(\left(\int_{\mathbb{T}} g(x)\phi_k(x) \, dx\right) \int_{\mathbb{T}} g(x)^2 \phi_{-k}(x) \, dx\right) \\
&\quad + \left|\int_{\mathbb{T}} g(x)\phi_k(x) \, dx\right|^2 \int_{\mathbb{T}} g(x)^2 \, dx,
\end{aligned}$$

which, in addition to Riemann–Lebesgue lemma, ensures

$$\lim_{|k| \to \infty} \beta_k = \frac{1}{2\pi} \int_{\mathbb{T}} g(x)^2 \, dx > 0.$$

From above observation, it follows that there exists $\delta > 0$ such that

$$\beta_k > \delta > 0, \quad k \neq 0.$$

Thus, from (A.8), h_k is naturally defined as

$$h_0 = 0 \quad \text{and} \quad h_k = \frac{e^{-\lambda_k T} u_{1,k} - u_{0,k}}{\beta_k}, \quad k \neq 0.$$

The rest of proof is to show that h is in $L^2(0, T; H^s(\mathbb{T}))$ for all $u_0, u_1 \in H^s$. We write $G\phi_j$ and g with the standard basis $\{\phi_n\}$ as

$$G\phi_j(x) = \sum_{n \in \mathbb{Z}} \mathcal{G}_{j,n} \phi_n(x) \quad \text{and} \quad g(x) = \sum_{n \in \mathbb{Z}} g_n \phi_n(x),$$

where $\mathcal{G}_{j,n} = (G\phi_j, \phi_n)$ and $g_n = (g, \phi_n)$, for all $j, n \in \mathbb{Z}$. Then, h in (A.7) can be rewritten as

$$h(t, x) = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} h_j q_j(t) \mathcal{G}_{j,n} \phi_n(x).$$

By using a classical theorem of Ingham and Beruling [32,33], a computation with the property of the Riesz basis $\mathcal{Q}$ gives

$$\begin{aligned}
\|h\|_{L^2(0,T:H^s)}^2 &\lesssim \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \sum_{j \in \mathbb{Z} \setminus \{0\}} |h_j \mathcal{G}_{j,n}|^2 \\
&= \sum_{j \in \mathbb{Z} \setminus \{0\}} |h_j|^2 \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\mathcal{G}_{j,n}|^2.
\end{aligned}$$

Using definitions of $\mathcal{G}_{j,n}$ and $G\phi_j$, one knows

$$\mathcal{G}_{j,n} = (G\phi_j, \phi_n) = \sum_{k \in \mathbb{Z}} g_k(\phi_k\phi_j, \phi_n) - g_{-j}g_n = \frac{1}{\sqrt{2\pi}} g_{n-j} - g_{-j}g_n,$$

hence we obtain

$$|\mathcal{G}_{j,n}|^2 \lesssim |g_{n-j}|^2 + |g_{-j}|^2 |g_n|^2,$$

and conclude

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\mathcal{G}_{j,n}|^2 \lesssim \sum_{n\in\mathbb{Z}} \langle n+j\rangle^{2s} |g_n|^2 + |g_{-j}|^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |g_n|^2.$$

When $s \geq 0$, a direct computation yields

$$\begin{aligned}
\|h\|_{L^2(0,T:H^s)}^2 &\lesssim \sum_{j\in\mathbb{Z}\setminus\{0\}} |h_j|^2 \left(\sum_{n\in\mathbb{Z}} \langle j+n\rangle^{2s} |g_n|^2 + |g_{-j}|^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |g_n|^2 \right) \\
&\lesssim \|g\|_{H^s}^2 \sum_{j\in\mathbb{Z}\setminus\{0\}} (\langle j\rangle^{2s} + |g_{-j}|^2)|h_j|^2 \\
&\lesssim \|g\|_{H^s}^2 \sum_{j\in\mathbb{Z}\setminus\{0\}} (\langle j\rangle^{2s} + |g_{-j}|^2)\beta_j^{-2} |e^{-\lambda_j T} u_{1,j} - u_{0,j}|^2 \\
&\lesssim \max_{j\in\mathbb{Z}\setminus\{0\}} \beta_j^{-2} \|g\|_{H^s}^2 (1+\|g\|_{L^2}^2)\left(\|u_0\|_{H^s}^2 + \|u_1\|_{H^s}^2\right).
\end{aligned}$$

On the other hand, when $s < 0$, due to the crude estimate

$$\langle a+b\rangle^c \lesssim \langle a\rangle^{|c|} \langle b\rangle^c, \quad a,b,c \in \mathbb{R},$$

we have similarly as before

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \langle j\rangle^{-2s} \langle n\rangle^{2s} |\mathcal{G}_{j,n}|^2 &\lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^{-2s} |g_n|^2 + \langle -j\rangle^{-2s} |g_{-j}|^2 \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |g_n|^2 \\
&\lesssim (1+\|g\|_{H^s}^2) \|g\|_{H^{-s}}^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\|h\|_{L^2(0,T:H^s)}^2 &\lesssim \sum_{j\in\mathbb{Z}\setminus\{0\}} \langle j\rangle^{2s} |h_j|^2 \sum_{n\in\mathbb{Z}} \langle j\rangle^{-2s} \langle n\rangle^{2s} |\mathcal{G}_{j,n}|^2 \\
&\lesssim \max_{j\in\mathbb{Z}\setminus\{0\}} \beta_j^{-2}(1+\|g\|_{H^s}^2) \|g\|_{H^{-s}}^2 \left(\|u_0\|_{H^s}^2 + \|u_1\|_{H^s}^2\right).
\end{aligned}$$

Therefore, we complete the proof. $\square$

As a consequence of Theorem A.1, we have the following property for the unitary operator group W.

Corollary A.2. *Let $T > 0$ be given. Then, there exists $\delta > 0$ such that*

$$\int_0^T \|GW(t)f\|_{L^2}^2 \ dt \geq \delta \|f\|_{L^2}^2,$$

for any $f \in L^2$.

A.2. Feedback stabilization

This part of the work gives a positive answer to the stabilization problem. Let us remember that if $s \in \mathbb{R}$, for any $\lambda > 0$, we define a bounded linear operator from $H^s(\mathbb{T})$ to itself by

$$L_\lambda f = \int_0^1 e^{-2\lambda\tau} W(-\tau) G G^* W^*(-\tau) f \ d\tau,$$

for any $f \in H^s(\mathbb{T})$. Note that $L_0 \equiv I$. It is known that L_λ is a self-adjoint positive operator on $H_0^s(\mathbb{T})$ and so is its inverse L_λ^{-1}, for all $s \geq 0$ (see, for instance, [19, Lemma 2.4]). This fact enables us to take the control function $h(t,x) = -G^*L_\lambda^{-1}u(t,x)$, and by employing the following feedback control law

$$K_\lambda u(t,x) \equiv GG^*L_\lambda^{-1}u(t,x)$$

we obtain the closed-loop system from (A.1), namely

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u = -K_\lambda u. \tag{A.9}$$

We will give a stabilizability result, to see this we rewrite system (A.9) as an abstract control system in the Hilbert space $\mathcal{V}$:

$$\partial_t u = Au + Bh, \quad u(0) = u_0, \tag{A.10}$$

where A is the operator defined by (2.1) which corresponds to the continuous unitary operator group $W(t)$ on the space $L^2(\mathbb{T})$ satisfying (2.2). The following theorem is derived from Theorem A.1 and a classical principle *exact controllability implies exponential stabilizability* for conservative control systems. For details, we suggest for the reader the Refs. [30,31].

Theorem A.3.   *Assume that the assumptions of Theorem A.1 are satisfied. Then*

(i) *There exist a $T > 0$ and $\delta > 0$ such that*

$$\int_0^T \|B^*W^*(t)u_0\|_{L^2(\mathbb{T})}^2 \, dt \geq \delta \, \|u_0\|_{H^s(\mathbb{T})}^2,$$

for any $u_0 \in H^s$.

(ii) *For any given $\lambda > 0$, there exists an operator $K \in \mathcal{L}(H^s(\mathbb{T}), L^2(\mathbb{T}))$ such that if one chooses $h = Ku$ in (A.10), then the resulting closed-loop system*

$$\partial_t u = Au + BKu, \quad u(0) = u_0,$$

has the property that its solution satisfies

$$\|u(t)\|_{H^s} \lesssim e^{-\lambda t} \|u_0\|_{H^s},$$

In our context, by using the result due [30,31], the following proposition presents that the closed-loop system (A.9) is exponentially stable:

Proposition A.4.   *Let $s \geq 0$ and $\lambda > 0$ be given. Then for any $u_0 \in H^s(\mathbb{T})$, the linear closed-loop system (A.9) admits a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$. Moreover, the solution u obeys the following decay property:*

$$\|u(t) - [u_0]\|_{H^s} \lesssim e^{-\lambda t} \|u_0 - [u_0]\|_{H^s},$$

for any $t > 0$. The implicit constant depends only on s.

Proof.   The proof is the same as that of Theorem A.3.   □

Appendix B. Global well-posedness for the closed loop system

Recall the system (A.9) with the nonlinearity uu_x

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u + uu_x = -K_\lambda u. \tag{B.1}$$

and its integral formulas

$$
\begin{aligned}
u(t) &= W(t)u_0 - \int_0^t W(t-\tau)(K_\lambda u)(\tau)d\tau - \int_0^t W(t-\tau)\,(uu_x)\,(\tau)d\tau \\
&=: W_\lambda(t)u_0 - \int_0^t W_\lambda(t-\tau)\,(uu_x)\,(\tau)d\tau,
\end{aligned}
\tag{B.2}
$$

where $W_\lambda(T)$ is the linear propagator associated to (A.9). Using Lemma 2.2 (5) (with small modification) and the boundedness of G, G^* and L_λ^{-1}, we immediately obtain

$$\left\|\int_a^t W(t-\tau)(K_\lambda u)(\tau)\,d\tau\right\|_{Y_I^s} \lesssim |I|^{1-\epsilon}\,\|u\|_{Y_I^s}, \tag{B.3}$$

for $I = [a,b]$ with $0 < b-a < 1$ and $0 < \epsilon < \frac{1}{2}$.

We now establish a similar result to Lemma 2.2 (2) and (3), associated to the propagator W_λ but in Y^s.

Lemma B.1. *Let $T > 0$ be given.*

(1) *For all $s \in \mathbb{R}$, we have for $f \in H^s$*

$$\|W_\lambda(t)f\|_{Y_T^s} \lesssim \|f\|_{H^s}.$$

(2) *For all $s \in \mathbb{R}$, we have for $F \in Y_T^s$*

$$\left\|\int_0^t W_\lambda(t-\tau)F(\tau)\,d\tau\right\|_{Y_T^s} \lesssim \left\|\mathcal{F}^{-1}\left(\langle\tau - k^{2j+1}\rangle^{-1}\,\widetilde{F}\right)\right\|_{Y_T^s}.$$

The implicit constants depend on T and s.

Proof. For given $f \in H^s$ and $F \in Y_T^s$, set

$$u(t,x) = W_\lambda(t)f + \int_0^t W_\lambda(t-\tau)F(\tau)\,d\tau.$$

Then, u solves

$$\partial_t u + (-1)^{j+1}\partial_x^{2j+1}u = -K_\lambda u + F$$

with $u(0) = f$, equivalently,

$$u(t,x) = W(t)f - \int_0^t W(t-\tau)(K_\lambda u)(\tau)\,d\tau + \int_0^t W(t-\tau)F(\tau)\,d\tau.$$

Using (B.3), one has

$$\|u\|_{Y_I^s} \lesssim \left(\|f\|_{H^s} + \left\|\mathcal{F}^{-1}\left(\langle\tau - k^{2j+1}\rangle^{-1}\,\widetilde{F}\right)\right\|_{Y_T^s}\right) + |I|^{1-\epsilon}\,\|u\|_{Y_I^s},$$

which implies

$$\|u\|_{Y_I^s} \leq C(I)\left(\|f\|_{H^s} + \left\|\mathcal{F}^{-1}\left(\langle\tau - k^{2j+1}\rangle^{-1}\,\widetilde{F}\right)\right\|_{Y_T^s}\right),$$

for a proper non-empty $I \subset [0, T]$ with $|I| < 1$. Let $I = [0, t_0]$, then we divide $[0, T]$ into $\left[\frac{T}{t_0}\right] + 1$ the subintervals, denoted by I_j, precisely, let $t_* := \left[\frac{T}{t_0}\right]$, set

$$I_0 = [0, t_0], \quad I_j = [jt_0, (j+1)t_0], \quad j = 1, 2, \ldots, t_* - 1, \quad \text{and} \quad I_{t_*} = [t_* t_0, T].$$

Therefore, we have[5]

$$\|u\|_{Y_T^s} \lesssim (t_* + 1) C(t_0) \left(\|f\|_{H^s} + \left\| \mathcal{F}^{-1} \left(\langle \tau - k^{2j+1} \rangle^{-1} \widetilde{F} \right) \right\|_{Y_T^s} \right),$$

which completes the proof. $\square$

Using Lemmas B.1 and 2.2 Item (4), one proves the local well-posedness of (B.1).

Lemma B.2 (*Local Well-Posedness of Nonlinear Closed Loop System*)**.** *Let $s \geq 0$ and $T > 0$ be given. Let define a map $\Gamma : H^s \to C_T H^s$ as in the second part of the right-hand side of (B.2) (again denoted by Γu). Then, there exists $\delta = \delta(T) > 0$ such that if*

$$\|u_0\|_{H^s} \leq \delta,$$

then the map Γ is a contraction map on a suitable ball. Moreover the map is locally uniformly continuous.

Proof. The proof is analogous to the proof of Lemma C.1. Taking Y_T^s norm to the map Γu and applying Lemmas B.1 and 2.2 Item (4), one has

$$\|\Gamma u\|_{Y_T^s} \leq C \|u_0\|_{H^s} + C \|u\|_{Y_T^s}^2$$

and for $u - \underline{u}$ with $u(0) = \underline{u}(0)$,

$$\|\Gamma u - \Gamma \underline{u}\|_{Y_T^s} \leq C \left(\|u\|_{Y_T^s} + \|\underline{u}\|_{Y_T^s} \right) \|u - \underline{u}\|_{Y_T^s},$$

where the constant C be the maximum one among constants appearing in Lemma 2.2 (4) and Lemma B.1. By taking $\delta > 0$ satisfying $8C^2\delta < 1$, we claim the map Γ is contractive on a ball

$$\{v \in Y_T^s : \|v\|_{Y_T^s} \leq 2C\delta\}.$$

One similarly proves that the map is Lipschitz continuous, thus we complete the proof. $\square$

The local solution constructed in Lemma B.2 can be extended to the global one.

Theorem B.3 (*Global Well-Posedness*)**.** *Let $s \geq 0$ and $T > 0$ be given. For any $u_0 \in H^s$, there exists a unique solution u to (B.1) in $Y_T^s \cap C_T H^s$ such that the following estimate holds true:*

$$\|u\|_{Y_T^s} \leq \alpha_{T,s}(\|u_0\|_{L^2}) \|u_0\|_{H^s},$$

where $\alpha_{T,s}$ is positive nondecreasing continuous function depending on T and s.

---

[5] One may use time cut-off functions supported on each I_j so that $u = \sum_{j=0}^{t_*} \eta_{I_j}(t) u$ on $[0, T]$. Then Y_T^s norm of each part is bounded by $Y_{I_j}^s$ norm of the same one.

Proof. A direct computation, in addition to the fact that G is self-adjoint in L^2, yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} u^2 = (-1)^j \int_{\mathbb{T}} u\partial_x^{2j+1} u - \int_{\mathbb{T}} u(uu_x) - \int_{\mathbb{T}} uK_\lambda u = -(GL_\lambda^{-1}u, Gu). \tag{B.4}$$

Since G and L_λ^{-1} is bounded in L^2, the Cauchy–Schwarz and the Grönwall's inequalities ensure

$$\|u(t)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2\, e^{ct}, \tag{B.5}$$

for some $c > 0$ depending on G and L_λ, and $t > 0$. Together with Lemma B.2 and the standard continuity argument, we complete the global well-posedness of (B.1) in L^2.

Let $v = u_t$ for a smooth solution to (B.1). Then, v solves

$$\partial_t v + (-1)^{j+1}\partial_x^{2j+1} v + (uv)_x = -K_\lambda v \tag{B.6}$$

where

$$v_0 = (-1)^j \partial_x^{2j+1} u_0 - u_0 \partial_x u_0 - K_\lambda u_0. \tag{B.7}$$

Note that u is a solution to (B.1), thus we can take $T_0 > 0$ such that

$$\|u\|_{Y_{T_0}^0} \lesssim \|u_0\|_{L^2}.$$

Then, analogously as in the proof of Lemma B.2, we also have

$$\|v\|_{L^\infty(0,T_1;L^2)} \lesssim \|v\|_{Y_{T_1}^0} \lesssim 2\|v_0\|_{L^2},$$

here $0 < T_1 < T_0$ is chosen appropriately.

On the other hand, a direct computation gives

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{T}}(\partial_x^j u)^2 - \frac{1}{6}\int_{\mathbb{T}} u^3\right) = -(GL_\lambda^{-1}\partial_x^j u, G\partial_x^j u) + \frac{1}{2}\int_{\mathbb{T}} u^2 K_\lambda u.$$

Using boundedness of G and L_λ^{-1}, and thus K_λ, and Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
\|\partial_x^j u(t)\|_{L^2}^2 &\le \frac{1}{3}\int_T u^3 + \|\partial_x^j u_0\|_{L^2}^2 + \frac{1}{3}\int_{\mathbb{T}} u_0^3 \\
&\quad + C\int_0^t \|\partial_x^j u(s)\|_{L^2}^2\, ds + \int_0^t \frac{1}{4}\|\partial_x^j u(s)\|_{L^2}^2 + \frac{1}{4}\|u(s)\|_{L^2}^6 + \frac{1}{2}\|u(s)\|_{L^2}^2\, ds \\
&\le \frac{1}{12}\|\partial_x^j u(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^2}^{\frac{10}{3}} + \frac{13}{12}\|\partial_x^j u_0\|_{L^2}^2 + \frac{1}{4}\|u_0\|_{L^2}^{\frac{10}{3}} \\
&\quad + C\int_0^t \|\partial_x^j u(s)\|_{L^2}^2\, ds + \int_0^t \frac{1}{4}\|\partial_x^j u(s)\|_{L^2}^2 + \frac{1}{4}\|u(s)\|_{L^2}^6 + \frac{1}{2}\|u(s)\|_{L^2}^2\, ds.
\end{aligned}$$

With (B.5), we claim from Grönwall's inequality that $\|\partial_x^j u(t)\|_{L^2}$ does not blow up in finite time, thus so $\|\partial_x u(t)\|_{L^\infty}$.

A similar computation as in (B.4) yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} v^2 = -\frac{1}{2}\int_{\mathbb{T}} u_x v^2 - (GL_\lambda^{-1}v, Gv),$$

which ensures that

$$\|v(t)\|_{L^2} \le e^{Ct + \int_0^t \|\partial_x u(s)\|_{L^\infty}}\|v_0\|_{L^2}$$

for all $t > 0$, due to global boundedness of $\|\partial_x u(u)\|_{L^\infty}$. Finally, for given $T > 0$, a direct computation with $v = u_t = -(-1)^{j+1}\partial_x^{2j+1}u - uu_x - K_\lambda u$ gives

$$\|\partial_x^{2j+1}u\|_{L^2} \lesssim \|u\|_{L^2} + \|v\|_{L^2} + \|u\|_{L^2}\|u_x\|_{L^\infty}$$
$$\leq C(\|u_0\|_{L^2} + \|v_0\|_{L^2} + \|u_0\|_{L^2}^3) + \frac{1}{2}\|\partial_x^{2j+1}u\|_{L^2},$$

for some $C > 0$ depending on T, which in addition to (B.7) implies

$$\|u\|_{L^\infty(0,T;H^{2j+1})} \leq \alpha_{T,2j+1}(\|u_0\|_{L^2})\|u_0\|_{H^{2j+1}}.$$

For $s \in (2j+1)\mathbb{N}$, one can show the global well-posedness similarly, and for $(2j+1)(n-1) < s < (2j+1)n$, $n \in \mathbb{N}$, it follows from the interpolation argument. Therefore, we complete the proof. $\square$

Appendix C. Local controllability and stability: Nonlinear results

This section devotes to proving Theorems 1.1 and 1.2. The proofs can be found in [29], but we just provide here the proof for the self-containedness.

C.1. Proof of Theorem 1.1

Rewrite the system (1.8) in its equivalent integral equation form:

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)(Gh)(\tau)d\tau - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau. \tag{C.1}$$

Define

$$\omega(T,u) := \int_0^T W(T-\tau)(uu_x)(\tau)d\tau.$$

Then, Lemma 2.2 (3) and (4) yield

$$\|\omega(T,u)\|_{H^s} \lesssim \left\|\langle k\rangle^s \mathcal{F}\left(\int_0^t W(t-\tau)(uu_x)(\tau)d\tau\right)\right\|_{\ell_k^2 L_\tau^1} \lesssim \|u\|_{Y_T^s}^2 < \infty, \tag{C.2}$$

provided that $u \in Y_T^s$. Choose $h = \Phi(u_0, u_1 + \omega(T,u))$ in Eq. (C.1) for $u \in Y_T^s$. From Theorem A.1, we have that for given u_0 and u_1

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)(G\Phi(u_0, u_1 + \omega(T,u)))(\tau)d\tau - \int_0^t W(t-\tau)(uu_x)(\tau)d\tau \tag{C.3}$$

with $u|_{t=0} = u_0$ and $u|_{t=T} = u_1$. Then, the following lemma proves Theorem 1.1.

Lemma C.1. *Let $s \geq 0$ and $T > 0$ be given. Let define a map $\Gamma : H^s \to C_T H^s$ as in (C.3) (denoted by Γu). Then, there exists $\delta = \delta(T) > 0$ such that if $\|u_0\|_{H^s} \leq \delta$ and $\|u_1\|_{H^s} \leq \delta$, then the map Γ is a contraction map on a suitable ball.*

Remark 8. The standard Picard iteration argument ensures the uniqueness of the fixed point, hence the condition $u(T,x) = u_1$ is guaranteed.

Proof of Lemma C.1. We denote the maximum implicit constant among ones appearing in Lemma 2.2, (A.4) and (C.2) by $C > 0$. Note that here the constant C depends on time $T > 0$, precisely, C is increasing when T grows up.

So, using Lemma 2.2 with Remark 3, (A.4) and (C.2), one has

$$\|\Gamma u\|_{Y_T^s} \leq C \left(\|u_0\|_{H^s} + \|u\|_{Y_T^s}^2 + \left(\|u_0\|_{H^s} + \|u_1\|_{H^s} + \|u\|_{Y_T^s}^2 \right) \right).$$

Analogously, for solutions u and $\underline{u}$ with $u(0) = \underline{u}(0)$ and $u(T) = \underline{u}(T)$, we have

$$\|\Gamma u - \Gamma \underline{u}\|_{Y_T^s} \leq 2C \left(\|u\|_{Y_T^s} + \|\underline{u}\|_{Y_T^s} \right) \|u - \underline{u}\|_{Y_T^s}.$$

Taking $\delta > 0$ satisfying $48C^2\delta \leq 1$,[6] we conclude that the map Γ is contractive in a ball

$$\{v \in Y_T^s : \|v\|_{Y_T^s} \leq 6C\delta\},$$

thus this completes the proof. □

C.2. Proof of Theorem 1.2

We are now ready to prove Theorem 1.2. For given $s \geq 0$ and $\lambda > 0$, we have from Proposition A.4 that

$$\|W_\lambda(t)u_0\|_{H^s} \leq Ce^{-\lambda t} \|u_0\|_{H^s},$$

where the implicit constant $C > 0$ depends only on s. For any $0 < \lambda' < \lambda$, take $T = T(\lambda') > 0$ such that

$$2Ce^{-\lambda T} \leq e^{-\lambda' T}.$$

Let us consider solution u to the integral Eq. (B.2) as a fixed point of the map

$$\Gamma u(t) = W_\lambda(t)u_0 - \int_0^t W_\lambda(t - \tau) (uu_x)(\tau)d\tau$$

in some closed ball $B_R(0)$ in the function space Y_T^s. This will be done provided that $\|u_0\|_{H^s} \leq \delta$ where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and R will be chosen in such a way that

$$\|u(T)\|_{H^s} \leq e^{-\lambda' T} \|u_0\|_{H^s}.$$

Applying Lemmas B.1 and 2.2 (4), there exist some positive constant C_1, C_2 (independent of δ and R) such that

$$\|\Gamma u\|_{Y_T^s} \leq C_1 \|u_0\|_{H^s} + C_2 \|u\|_{Y_T^s}^2$$

and for $u - \underline{u}$ with $u(0) = \underline{u}(0)$,

$$\|\Gamma u - \Gamma \underline{u}\|_{Y_T^s} \leq C_2 \left(\|u\|_{Y_T^s} + \|\underline{u}\|_{Y_T^s} \right) \|u - \underline{u}\|_{Y_T^s}.$$

On the other hand, we have for some constant $C' > 0$ and all $u \in B_R(0)$

$$\|\Gamma(u)(T)\|_{Y_T^s} \leq C_1 \|W_\lambda(T)u_0\|_{Y_T^s} + C_2 \left\| \int_0^T W_\lambda(T - \tau) (uu_x)(\tau)d\tau \right\|_{Y_T^s}$$

$$\leq e^{-\lambda T}\delta + C'R^2.$$

Pick $\delta = C_4 R^2$, where C_4 and R are chosen so that

$$\frac{C'}{C_4} \leq Ce^{-\lambda T}, \quad (C_1 C_4 + C_2) R^2 \leq R \quad \text{and} \quad 2C_2 R \leq \frac{1}{2}.$$

---

[6] Here δ depends on T since the constant relies on T.

Then we have

$$\|\Gamma(u)\|_{Y_T^s} \leq R, \quad \forall u \in B_R(0)$$

and

$$\|\Gamma(u_1) - \Gamma(u_2)\|_{Y_T^s} \leq \frac{1}{2} \|u_1 - u_2\|_{\mathbb{Z}_{T,\sigma}^T}, \quad \forall u_1, u_2 \in B_R(0).$$

Therefore, Γ is a contraction in $B_R(0)$. Furthermore, its unique fixed point $u \in B_R(0)$ fulfills

$$\|u(T)\|_{H^s} \leq \|\Gamma(u)(T)\|_{Y_T^s} \leq e^{-\lambda'T}\delta$$

Assume now that $0 < \|u_0\|_0 < \delta$. Changing δ into $\delta' \equiv \|u_0\|_s$ and R into $R' \equiv (\delta'/\delta)^{\frac{1}{2}} R$, we infer that

$$\|u(T)\|_{H^s} \leq e^{-\lambda'T} \|u_0\|_{H^s}$$

and an obvious induction yields

$$\|u(nT)\|_{H^s} \leq e^{-\lambda'nT} \|u_0\|_{H^s}$$

for any $n \geq 0$. We infer by the semigroup property that there exists some positive constant $C > 0$ such that

$$\|u(t)\|_{H^s} \leq Ce^{-\lambda t} \|u_0\|_{H^s},$$

if $\|u_0\|_{H^s} \leq \delta$. The proof is complete. $\square$

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