FORCING OPERATORS ON STAR GRAPHS APPLIED FOR THE CUBIC FOURTH ORDER SCHRÖDINGER EQUATION

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Abstract. In a recent article [16], the authors gave a starting point of the study on a series of problems concerning the initial boundary value problem and control theory of Biharmonic NLS in some non-standard domains. In this direction, this article deals to present answers for some questions left in [16] concerning the study of the cubic fourth order Schrödinger equation in a star graph structure $\mathcal{G}$. Precisely, consider $\mathcal{G}$ composed by N edges parameterized by half-lines $(0, +\infty)$ attached with a common vertex ν. With this structure the manuscript proposes to study the well-posedness of a dispersive model on star graphs with three appropriated vertex conditions by using the *boundary forcing operator approach*. More precisely, we give positive answer for the Cauchy problem in low regularity Sobolev spaces. We have noted that this approach seems very efficient, since this allows to use the tools of Harmonic Analysis, for instance, the Fourier restriction method, introduced by Bourgain, while for the other known standard methods to solve partial differential partial equations on star graphs are more complicated to capture the dispersive smoothing effect in low regularity. The arguments presented in this work have prospects to be applied for other nonlinear dispersive equations in the context of star graphs with unbounded edges.

1. Introduction.

1.1. **Quantum and metric graphs.** In mathematics and physics, a quantum graph is a linear network-shaped structure of vertices connected on edges (i.e., a graph), where a differential (or pseudo-differential) equation is posed on each edge, while in the case of each edge is equipped with a natural metric the graph is denoted as a metric graph. An example would be a power network consisting of power lines (edges) connected at transformer stations (vertices); the differential equations would

be then the voltage along each of the line and the boundary conditions for each edge equipped at the adjacent vertices ensuring that the current added over all edges adds to zero at each vertex.

Quantum graphs were first studied by Linus Pauling as models of free electrons in organic molecules in the 1930s. They also appear in a variety of mathematical contexts, e.g. as model systems in quantum chaos, in the study of waveguides, in photonic crystals and in Anderson localization - is the absence of diffusion of waves in a disordered medium, or as limit on shrinking thin wires. Quantum graphs have become prominent models in mesoscopic physics used to obtain a theoretical understanding of nanotechnology. Another, more simple notion of quantum graphs was introduced by Freedman *et al.* in [26].

Aside from actually solving the differential equations posed on a quantum graph for purposes of concrete applications, typical questions that arise are those of well-posedness, controllability (what inputs have to be provided to bring the system into a desired state, for example providing sufficient power to all houses on a power network) and identifiability (how and where one has to measure something to obtain a complete picture of the state of the system, for example measuring the pressure of a water pipe network to determine whether or not there is a leaking pipe).

1.2. **Nonlinear dispersive models on star graphs.** In the last years, the study of nonlinear dispersive models in a metric graph has attracted a lot of attention of mathematicians, physicists, chemists and engineers, see for details [9, 10, 14, 34, 35] and references therein. In particular, the framework prototype (graph-geometry) for description of these phenomena have been a *star graph* $\mathcal{G}$, namely, on metric graphs with N half-lines of the form $(0, +\infty)$ connecting at a common vertex $\nu = 0$, together with a nonlinear equation suitably defined on the edges such as the nonlinear Schrödinger equation (see Adami *et al.* [1, 2] and Angulo and Goloshchapova [5, 6]). We note that with the introduction of nonlinearities in the dispersive models, the network provides a nice field, where one can look for interesting soliton propagation and nonlinear dynamics in general. A central point that makes this analysis a delicate problem is the presence of a vertex where the underlying one-dimensional star graph should bifurcate (or multi-bifurcate in a general metric graph).

Looking at other nonlinear dispersive systems on graphs structure, we have some interesting results. For example, related with well-posedness theory, the second author in [18], studied the local well-posedness for the Cauchy problem associated to Korteweg-de Vries equation in a metric star graph with three semi-infinite edges given by one negative half-line and two positives half-lines attached to a common vertex $\nu = 0$ (the $\mathcal{Y}$-junction framework). Another nonlinear dispersive equation, the Benjamin–Bona–Mahony (BBM) equation, is treated in [11, 37]. More precisely, Bona and Cascaval [11] obtained local well-posedness in Sobolev space H^1 and Mugnolo and Rault [37] showed the existence of traveling waves for the BBM equation on graphs. Using a different approach Ammari and Crépeau [4] derived results of well-posedness and, also, stabilization for the Benjamin-Bona-Mahony equation in a star-shaped network with bounded edges.

In this aspect, regarding control theory and inverse problems, let us cite some previous works. Ignat *et al.* in [31] worked on the inverse problem for the heat equation and the Schrödinger equation on a tree. Later on, Baudouin and Yamamoto [7] proposed a unified - and simpler - method to study the inverse problem of determining a coefficient. Results of stabilization and boundary

controllability for KdV equation on star-shaped graphs was also proved by Ammari and Crépeau [3] and Cerpa *et al.* [21, 22].

We caution that this is only a small sample of the extant work on graphs structure for partial differential equations.

1.3. **Presentation of the model.** Let us now present the equation that we will study in this paper. The fourth-order nonlinear Schrödinger (4NLS) equation or biharmonic cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, \tag{1}$$

have been introduced by Karpman [32] and Karpman and Shagalov [33] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1) arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references (see [8, 24, 32, 38, 39] and references therein).

In the past twenty years such 4NLS have been deeply studied from different mathematical points of view. For example, Fibich *et al.* [25] worked various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. The well-posedness problem and existence of the solutions has been shown (see, for instance, [38, 39, 41, 42]) by means of the energy method, harmonic analysis, etc.

Recently, in [15], the first and the second authors worked with equation (1) with the purpose to obtain controllability results. More precisely, they proved that on torus $\mathbb{T}$, the solution of the associated linear system (1) is globally exponential stable, by using certain properties of propagation of compactness and regularity in Bourgain spaces. This property, together with the local exact controllability, ensures that fourth order nonlinear Schrödinger is globally exactly controllable, we suggest the reader to see [15] for more details.

Considering an different domain instead of the torus $\mathbb{T}$, the authors, in [16], considered the cubic fourth order Schrödinger equation on the right half-line

$$\begin{cases} i\partial_t u + \gamma \partial_x^4 u + \lambda |u|^2 u = 0, & (t,x) \in (0,T) \times (0,\infty), \\ u(0,x) = u_0(x), & x \in (0,\infty), \\ u(t,0) = f(t), \ u_x(t,0) = g(t) & t \in (0,T), \end{cases} \tag{2}$$

for $\gamma, \lambda \in \mathbb{R}$. When $\gamma\lambda < 0$ system (2) is so-called focusing otherwise, that is, $\gamma\lambda > 0$, is called defocusing. In [16], Capistrano-Filho *et al.* consider $\gamma = -1$ and suitable choices of $f(t)$ and $g(t)$ in the equation (2), precisely, by assuming

$$(u_0, f, g) \in H^s(\mathbb{R}^+) \times H^{\frac{2s+3}{8}}(\mathbb{R}^+) \times H^{\frac{2s+1}{8}}(\mathbb{R}^+),$$

they obtained local well-posedness on the Sobolev spaces $H^s(\mathbb{R}^+)$ for $s \in [0, \frac{1}{2})$. For $s > 1/2$, by the Sobolev embedding and the energy method one can easily show the local well-posedness in $H^s(\mathbb{R}^+)$, giving a starting point of the study on a series of problems concerning of the Biharmonic NLS on bounded domains or star graphs.

Due these results presented in this recent work, naturally, we should see what happens for the system (2) in star graph structure given by N unbounded edges $(0,\infty)$ connected with a common vertex $\nu = 0$, where a function on the graph $\mathcal{G}$ is a vector $u(t,x) = (u_1(t,x), u_2(t,x), ..., u_N(t,x))$. Thus, let us consider the fourth

order nonlinear Schödinger equation on $\mathcal{G}$, given by

$$\begin{cases} i\partial_t u_j - \partial_x^4 u_j + \lambda |u_j|^2 u_j = 0, & (t,x) \in (0,T) \times (0,\infty), \ j = 1,2,...,N \\ u_j(0,x) = u_{j0}(x), & x \in (0,\infty), \ j = 1,2,...,N \end{cases} \quad (3)$$

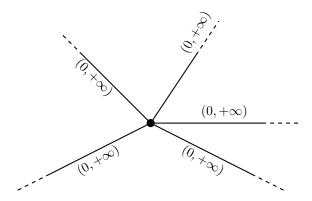with initial conditions $(u_1(0,x), u_2(0,x), ..., u_N(0,x))$.

FIGURE 1. Star graph with 5 edges

Therefore, the following natural question arise.

Problem $\mathcal{A}$. *Which are the boundary conditions that we can impose, at least mathematically acceptable, to ensure the well-posedness result for the system* (3)?

1.4. **Choosing the boundary conditions and main result.** We are interested to prove the well-posedness of (3) with appropriate boundary condition, more precisely, we will solve (3) with the following boundary conditions:

$$\text{Type } \mathcal{A}: \begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0) = \cdots = \partial_x^k u_N(t,0), \ k = 0,1 & t \in (0,T), \\ \sum_{j=1}^{N} \partial_x^k u_j(t,0) = 0, \ k = 2,3 & t \in (0,T), \end{cases} \quad (4)$$

$$\text{Type } \mathcal{B}: \begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0) = \cdots = \partial_x^k u_N(t,0), \ k = 2,3 & t \in (0,T), \\ \sum_{j=1}^{N} \partial_x^k u_j(t,0) = 0, \ k = 0,1 & t \in (0,T) \end{cases} \quad (5)$$

and

$$\text{Type } \mathcal{C}: \begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0) = \cdots = \partial_x^k u_N(t,0), \ k = 0,3 & t \in (0,T), \\ \sum_{j=1}^{N} \partial_x^k u_j(t,0) = 0, \ k = 1,2 & t \in (0,T). \end{cases} \quad (6)$$

These boundary conditions are motivated by the conservation of the mass. Let us denote the mass as

$$E(u_1(t,x), u_2(t,x), \cdots, u_N(t,x)) = \frac{1}{2} \sum_{j=1}^{N} \int_0^\infty |u_j(t,x)|^2 dx.$$

Multiplying (3) by $\overline{u}_j$, taking the imaginary part, integrating by parts and using the initial conditions of (3), we can obtain the most basic energy identity, namely

the L^2–energy, satisfying

$$E(u_1(T,x), u_2(T,x), \cdots, u_N(T,x))$$

$$= -\sum_{j=1}^{N} \int_0^T \mathrm{Im}(\partial_x^3 u_j(t,0)\overline{u}_j(t,0))dt$$

$$+ \sum_{j=1}^{N} \int_0^T \mathrm{Im}(\partial_x^2 u_j(t,0)\partial_x\overline{u}_j(t,0))dt \tag{7}$$

$$- E(u_1(0,x), u_2(0,x), \cdots, u_N(0,x)).$$

Analyzing (7), we are interested in boundary conditions to the Cauchy problem (3) such that the right hand side of (7) ensures the conservation of the mass. In this sense, the boundary conditions (4), (5) and (6) are appropriated. Assuming one of the boundary conditions (4), (5) or (6) the mass is conserved, i.e.,

$$E(u_1(t,x), u_2(t,x), \cdots, u_N(t,x)) = E(u_1(0,x), u_2(0,x), \cdots, u_N(0,x)).$$

It is important to point out that the boundary conditions of types $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ are coherent with the study of biharmonic operator on $L^2(\mathcal{G})$. More precisely, a simple calculation proves that the biharmonic operator

$$B := i\partial_x^4 : \mathcal{D}(B_i) \subset L^2(\mathcal{G}) \to L^2(\mathcal{G}), \quad i = 1, 2, 3,$$

with the following domains

$$\mathcal{D}(B_1) = \{H^4(\mathcal{G}); \, \partial_x^k u_1(0) = \partial_x^k u_2(0) = \cdots = \partial_x^k u_N(0), \, k = 0, 1$$

$$\sum_{j=1}^{N} \partial_x^k u_j(0) = 0, \, k = 2, 3 \, \},$$

$$\mathcal{D}(B_2) = \{H^4(\mathcal{G}); \, \partial_x^k u_1(0) = \partial_x^k u_2(0) = \cdots = \partial_x^k u_N(0), \, k = 2, 3$$

$$\sum_{j=1}^{N} \partial_x^k u_j(0) = 0, \, k = 0, 1 \, \}$$

or

$$\mathcal{D}(B_3) = \{H^4(\mathcal{G}); \, \partial_x^k u_1(0) = \partial_x^k u_2(0) = \cdots = \partial_x^k u_N(0), \, k = 0, 3$$

$$\sum_{j=1}^{N} \partial_x^k u_j(0) = 0, \, k = 1, 2 \, \},$$

is self-adjoint. Then, by Stone's Theorem (see e.g. [20]), B generates a linear group, denoted by $e^{it\partial_x^4}$ that solves the linear problem

$$\begin{cases} \partial_t u(x,t) = i\partial_x^4 u(x,t), \\ u(0,x) = u_0 \in \mathcal{D}(B_i), \\ u \in C(\mathbb{R}; \mathcal{D}(B_i)) \cap C^1(\mathbb{R}; L^2(\mathcal{G})) \quad i = 1, 2 \text{ or } 3. \end{cases}$$

By using the Duhamel formula together with the fact that $H^4(\mathcal{G})$ is a Banach algebra it is possible to show that problem (3) is well posed in high regularity, precisely, in $\mathcal{D}(B_i)$, $i = 1, 2$ or 3.

Remark 1. The following remarks are now in order.

- Considering the Schrödinger equation on a star graph $\mathcal{G}$, the vertex condition Type $\mathcal{A}$, when restrict on the cases $k = 0$ and $k = 1$, coincides with the classical Kirchhoff vertex condition. For this system, these conditions are rather natural in the context of water waves (and other fluids), corresponding to continuity of the flow and flux balance.

- In this direction, we cite a very interesting work of Gregorio and Mugnolo [28] that treated the bi-laplacian on star graphs and trees with bounded edges, more precisely, they given a characterization of complete graphs in terms of the Markovian property of the semigroup generated by $\mathcal{L}^2(\mathcal{G})$, the square of the discrete Laplacian acting on a connected discrete graph $\mathcal{G}$. For a complete picture about star graphs in unbounded edges, in the context of the Airy equation, we cite the work of Mugnolo *et al.* [36].

Therefore, this work gives an answer for the Problem $\mathcal{A}$, in a star graph structure, when the boundary conditions (4), (5) or (6) are considered. This problem was left as an open problem in [16]. Before to enunciate the principal result of this work, we will denote the classical Sobolev space on the star graph $\mathcal{G}$ by

$$H^s(\mathcal{G}) = \bigoplus_{i=1}^{N} H^s(0, +\infty), \quad \text{for } s \geq 0.$$

With this notation, the main result of this work can be read as follows.

Theorem 1.1. *Let $s \in [0, \frac{1}{2})$. For given initial-boundary data $(u_{10}, u_{20}, ..., u_{N0}) \in H^s(\mathcal{G})$ satisfying type $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ vertex conditions, there exist a positive time T depending on $\sum_{j=1}^{N} \|u_{j0}\|_{H^s(\mathbb{R}^+)}$ and a distributional solution $u = (u_i)_{i=1}^{N}(t, x) \in C((0, T); H^s(\mathcal{G}))$ to (3)–(4) (or (3)–(5) or (3)–(6)) satisfying*

$$u_j \in C\big(\mathbb{R}^+; \ H^{\frac{2s+3}{8}}(0, T)\big) \cap X^{s,b}(\mathbb{R}^+ \times (0, T)),$$

$$\partial_x u_j \in C\big(\mathbb{R}^+; \ H^{\frac{2s+1}{8}}(0, T)\big),$$

$$\partial_x^2 u_j \in C\big(\mathbb{R}^+; \ H^{\frac{2s-1}{8}}(0, T)\big)$$

and

$$\partial_x^3 u_j \in C\big(\mathbb{R}^+; \ H^{\frac{2s-3}{8}}(0, T)\big),$$

for some $b(s) < \frac{1}{2}$ and $j = 1, 2, ..., N$. Moreover, the map $(u_{10}, u_{20}, ..., u_{N0}) \longmapsto u$ is locally Lipschitz continuous from $H^s(\mathcal{G})$ to $C\big((0, T); H^s(\mathcal{G})\big)$.

1.5. **Heuristic of the paper and further comments.** In this work we prove the existence of solution to the problem (3)–(4) (or (3)–(5) or (3)–(6)) on star graph structure $\mathcal{G}$ composed by N unbounded edges. The proof of Theorem 1.1 will be divided in several steps. Initially, we recast the partial differential equation in each edge for a full line with a forcing term, more precisely

$$\begin{cases} i\partial_t u_j - \partial_x^4 u_j + \lambda |u_j|^2 u_j = \mathcal{T}_{1j}(x)h_{1j}(t) + \mathcal{T}_{2j}(x)h_{2j}(t), & (t, x) \in (0, T) \times \mathbb{R}, \\ u_j(0, x) = \widetilde{u}_{j0}(x), & j = 1, 2, ..., N \end{cases} \quad (8)$$

where $\mathcal{T}_{1j}$ and $\mathcal{T}_{2j}$ ($j = 1, 2, ..., N$) are distributions supported in the negative half-line $(-\infty, 0)$; the boundary forcing functions h_{1j}, h_{2j} ($j = 1, 2, ..., N$) are selected to ensure that the vertex conditions are satisfied and $\widetilde{u}_{j0}(x)$ are extensions of u_{j0} ($j = 1, 2, ..., N$) on full line satisfying

$$\|\widetilde{u}_{j0}\|_{H^s(\mathbb{R})} \leq 2\|u_{j0}\|_{H^s(\mathbb{R}^+)}.$$

Upon constructing the solution $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_N)$ of (8), we obtain the solution $u = (u_1, u_2, ..., u_N)$ of the problem (3) with appropriate boundary condition, by restriction, as

$$u = u(x,t) = (u_1, u_2, ..., u_N)\big|_{x \in \mathcal{G}, t \in (0,T)}$$
$$:= (u_1|_{x \in \mathbb{R}^+, t \in (0,T)}, u_2|_{x \in \mathbb{R}^+, t \in (0,T)}, ..., u_N|_{x \in \mathbb{R}^+, t \in (0,T)}).$$

Secondly, the solution of forced Cauchy problem (8) satisfying the vertex types $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$, is constructed using the classical Fourier restriction method due Bourgain [12, 13]. Finally, a fixed point argument ensures the proof of the Theorem 1.1.

We present some comments about the relevance of the method used in this manuscript.

   i. It is important to point out that, in our knowledge, this work is the first one in a star graphs structure $\mathcal{G}$ composed by N unbounded edges by using boundary forcing operator approach introduced first by Colliander and Kenig [23] and improved by Holmer [30].

  ii. The graph structure of this article is more complex than proposed in [18] in the following sense: To treat the extended vectorial integral equation that solves system (3), considering N unbounded edges with appropriated vertex conditions, is more delicate since the matrices associated with this problem have $2N$–order (see Section 4).

 iii. A more delicate question concerns here is the local well-posedness for the Cauchy problem (3) in low regularity. To do this we need to use a dispersive approach instead of Semigroup theory, where the principal difficulty is to use the restriction Fourier method in the context of star graphs. This motivates us to solve the problem (3) by using this approach, since the Semigroup theory does not guarantee the lower regularity to solutions of (3).

1.6. **Organization of the article.** To end our introduction, we present the outline of the manuscript. Section 2 is devoted to present the notations, more precisely, the Sobolev spaces, the Bourgain spaces, the Riemann-Liouville fractional integral operator and the Duhamel boundary forcing operator associated of (4NLS), which are paramount to prove the main result of the article. In the Section 3, we will give an overview of the main estimates proved by the authors in [16]. With these two sections in hand, we are able to prove Theorem 1.1, in several steps, in the Section 4. The Section 5 is devoted to prove an auxiliary lemma, which one is used in the proof of the main result of the article, namely, Theorem 1.1. Finally, at the end of the work, we present an Appendix A, which will we given a sketch of the proof of Theorem 1.1 with vertex conditions types $\mathcal{B}$ and $\mathcal{C}$.

2. **Preliminaries.** This section is devoted to presenting the notations, introducing the functions spaces used in this work and the Duhamel boundary forcing operator associated with the fourth order linear Schrödinger equation.

2.1. **Notations.** Let us fix a cut-off function $\psi := \psi(t)$ such that $\psi \in C_0^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$ and defined by

$$\psi \equiv 1 \ \text{ on } \ [0,1], \quad \psi \equiv 0, \text{ for } |t| \geq 2, \tag{9}$$

and, for $T > 0$, we denote $\psi_T(t) = \frac{1}{T}\psi(\frac{t}{T})$.

Now, for $s \geq 0$, define the homogeneous L^2-based Sobolev spaces $\dot{H}^s = \dot{H}^s(\mathbb{R})$ by natural norm $\|\phi\|_{\dot{H}^s} = \||\xi|^s \hat{\psi}(\xi)\|_{L^2_\xi}$ and the L^2-based inhomogeneous Sobolev

spaces $H^s = H^s(\mathbb{R})$ by the norm $\|\phi\|_{\dot{H}^s} = \|(1 + |\xi|^2)^{\frac{s}{2}}\hat{\psi}(\xi)\|_{L^2_\xi}$, where $\hat{\psi}$ denotes the Fourier transform of ψ. The function f belongs to $H^s(\mathbb{R}^+)$, if there exists $F \in H^s(\mathbb{R})$ such that $f(x) = F(x)$ for $x > 0$, in this case we set

$$\|f\|_{H^s(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}.$$

On the other hand, for $s \in \mathbb{R}$, $f \in H^s_0(\mathbb{R}^+)$ provided that there exists $F \in H^s(\mathbb{R})$ such that F is the extension of f on $\mathbb{R}$ and $F(x) = 0$ for $x < 0$. In this case, we set $\|f\|_{H^s_0(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}$. For $s < 0$, we define $H^s(\mathbb{R}^+)$ as the dual space of $H^{-s}_0(\mathbb{R}^+)$. It is well known that $H^s_0(\mathbb{R}^+) = H^s(\mathbb{R}^+)$ for $-\frac{1}{2} < s < \frac{1}{2}$.

Finally, the sets $C_0^\infty(\mathbb{R}^+) = \{f \in C^\infty(\mathbb{R}); \, \text{supp} f \subset [0, \infty)\}$ and $C_{0,c}^\infty(\mathbb{R}^+)$ are defined as the subset of $C_0^\infty(\mathbb{R}^+)$, whose members have a compact support on $(0, \infty)$. We remark that $C_{0,c}^\infty(\mathbb{R}^+)$ is dense in $H^s_0(\mathbb{R}^+)$ for all $s \in \mathbb{R}$.

2.2. **Solution spaces.** Consider $f \in \mathcal{S}(\mathbb{R}^2)$, let us denote by $\tilde{f}$ or $\mathcal{F}(f)$ the Fourier transform of f with respect to both spatial and time variables

$$\tilde{f}(\tau, \xi) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(t, x) \, dxdt.$$

Moreover, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variables, respectively (also use $\hat{}$ for both cases).

In the 90's Bourgain [12, 13] established a form to show the well-posedness of some classes of dispersive systems. Precisely, on the Sobolev spaces H^s, for smaller values of s, with these new spaces, Bourgain showed a smoothing property more suitable for solutions of these classes of dispersive equations.

In our case, considering $s, b \in \mathbb{R}$ we present below the Bourgain spaces $X^{s,b}$ associated to the linear system of (3). The space will be a completion of $\mathcal{S}'(\mathbb{R}^2)$ under the norm

$$\|f\|^2_{X^{s,b}} = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \langle \tau + \xi^4 \rangle^{2b} |\tilde{f}(\tau, \xi)|^2 \, d\xi d\tau,$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

It is important to note that $X^{s,b}$–space, with $b > \frac{1}{2}$, is well-adapted to study the IVP of dispersive equations on $\mathbb{R}^n$ or $\mathbb{T}$. However, in the study of IBVP, the standard argument cannot be applied directly. This is due to the lack of hidden regularity, more precisely, the control of (derivatives) time trace norms of the Duhamel parts requires to work in $X^{s,b}$–type spaces for $b < \frac{1}{2}$, since the full regularity range cannot be covered (see Lemma 3.6 inequality (28)).

Considering the space denoted by $Z(s, b)$ with the following norm

$$\|f\|_{Z(s,b)(\mathbb{R}^2)} = \sup_{t \in \mathbb{R}} \|f(t, \cdot)\|_{H^s(\mathbb{R})} + \sum_{j=0}^{3} \sup_{x \in \mathbb{R}} \|\partial_x^j f(\cdot, x)\|_{H^{\frac{2s+3-2j}{8}}(\mathbb{R})} + \|f\|_{X^{s,b}},$$

our goal is to find solutions of the Cauchy problem (3).

Here, in each edge of $\mathcal{G}$, we will consider the spatial and time restricted space of $Z^{s,b}(\mathbb{R}^2)$ defined in the standard way as follows

$$Z(s, b)((0, T) \times \mathbb{R}^+) = Z(s, b)\Big|_{(0,T) \times \mathbb{R}^+}$$

equipped with the norm

$$\|f\|_{Z(s,b)((0,T) \times \mathbb{R}^+)} = \inf_{g \in Z(s,b)} \{\|g\|_{Z(s,b)} : g(t, x) = f(t, x) \text{ on } (0, T) \times \mathbb{R}^+\}.$$

2.3. **Riemann-Liouville fractional integral.** Before beginning our study of the Cauchy problem (3), in this subsection, we just give a brief summary of the Riemann-Liouville fractional integral operator to make the work complete. We suggest [16, 23, 30] for the reader to see the proofs and more details.

Consider the function t_+ defined in the following way

$$t_+ = t \quad \text{if} \quad t > 0, \qquad t_+ = 0 \quad \text{if} \quad t \leq 0.$$

The tempered distribution $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function by

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \, f \right\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t) dt,$$

for Re $\alpha > 0$. By integrating by parts, we have that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k \left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)} \right), \tag{10}$$

for all $k \in \mathbb{N}$. Expression (10) allows to extend the definition of $\frac{t_+^{\alpha-1}}{\Gamma(\alpha)}$, in the sense of distributions, to all $\alpha \in \mathbb{C}$. For $f \in C_0^\infty(\mathbb{R}^+)$, define $\mathcal{I}_\alpha f$ as

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f.$$

Thus, for Re $\alpha > 0$, follows that

$$\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds \tag{11}$$

and the following properties easily holds

$$\mathcal{I}_0 f = f, \quad \mathcal{I}_1 f(t) = \int_0^t f(s) ds, \quad \mathcal{I}_{-1} f = f' \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}.$$

2.4. **Duhamel boundary forcing operator.** We present the Duhamel boundary forcing operator, which was introduced by Colliander and Kenig [23], in order to construct the solution to

$$i\partial_t u - \partial_x^4 u = 0. \tag{12}$$

For details about this subsection and for a well exposition about this topic, the authors suggest the following references [16, 17, 19, 29] .

Following [16], let us consider the oscillatory integral by

$$B(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{ix\xi} e^{-i\xi^4} \, d\xi, \tag{13}$$

which one is the key to define the Duhamel boundary forcing operator. A change of variable and contour proves that $B(0) = -\frac{i^{7/4}}{\pi} \Gamma\left(\frac{5}{4}\right)$. We will denote

$$M = \frac{1}{B(0)\Gamma(3/4)}. \tag{14}$$

For $f \in C_0^\infty(\mathbb{R}^+)$, define the boundary forcing operator $\mathcal{L}^0$ (of order 0) as

$$\mathcal{L}^0 f(t,x) := M \int_0^t e^{i(t-t')\partial_x^4} \delta_0(x) \mathcal{I}_{-\frac{3}{4}} f(t') dt', \tag{15}$$

where $e^{it\partial_x^4}$ denotes the group associated to (12) given by

$$e^{it\partial_x^4}\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}e^{-it\xi^4}\hat{\psi}(\xi)d\xi.$$

By using the following properties of the convolution operator

$$\partial_x^k(f*g) = (\partial_x^k f)*g = f*(\partial_x^k g), \quad k \in \mathbb{N}, \tag{16}$$

and the integration by parts in t' of (15), we get that

$$i\mathcal{L}^0(\partial_t f)(t,x) = iM\delta_0(x)\mathcal{I}_{-\frac{3}{4}}f(t) + \partial_x^4\mathcal{L}^0 f(t,x). \tag{17}$$

Using (13) and, again, by change of variable, we have

$$\begin{aligned}
\mathcal{L}^0 f(t,x) &= M\int_0^t e^{i(t-t')\partial_x^4}\delta_0(x)\mathcal{I}_{-\frac{3}{4}}f(t')dt' \\
&= M\int_0^t B\left(\frac{x}{(t-t')^{1/4}}\right)\frac{\mathcal{I}_{-\frac{3}{4}}f(t')}{(t-t')^{1/4}}dt'.
\end{aligned} \tag{18}$$

We are now generalize the boundary forcing operator (15). For Re $\lambda > -4$ and given $g \in C_0^\infty(\mathbb{R}^+)$, we define

$$\mathcal{L}^\lambda g(t,x) = \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}*\mathcal{L}^0\left(\mathcal{I}_{-\frac{\lambda}{4}}g\right)(t,\cdot)\right](x), \tag{19}$$

where $*$ denotes the convolution operator and $\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} = \frac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$. In particular, for Re $\lambda > 0$, we have

$$\mathcal{L}^\lambda g(t,x) = \frac{1}{\Gamma(\lambda)}\int_x^\infty (y-x)^{\lambda-1}\mathcal{L}^0\left(\mathcal{I}_{-\frac{\lambda}{4}}g\right)(t,y)dy. \tag{20}$$

By using the property (16), for $k = 4$, and (17) give us

$$\begin{aligned}
\mathcal{L}^\lambda g(t,x) &= \left[\frac{x_-^{(\lambda+4)-1}}{\Gamma(\lambda+4)}*\partial_x^4\mathcal{L}^0\left(\mathcal{I}_{-\frac{\lambda}{4}}g\right)(t,\cdot)\right](x) \\
&= iM\frac{x_-^{(\lambda+4)-1}}{\Gamma(\lambda+4)}\mathcal{I}_{-\frac{3}{4}-\frac{\lambda}{4}}g(t) \\
&\quad + i\int_x^\infty \frac{(y-x)^{(\lambda+4)-1}}{\Gamma(\lambda+4)}\mathcal{L}^0\left(\partial_t\mathcal{I}_{-\frac{\lambda}{4}}g\right)(t,y)dy,
\end{aligned} \tag{21}$$

for Re $\lambda > -4$, where M is defined as (14). From (17) and (19), we have

$$(i\partial_t - \partial_x^4)\mathcal{L}^\lambda g(t,x) = iM\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{3}{4}-\frac{\lambda}{4}}g(t),$$

in the distributional sense.

3. **Overview of the main estimates.** With all the notations and spaces defined in the Section 2, we present now the main estimates of this work, which are paramount to prove the main result of the article.

3.1. **Estimates for the function spaces.** Concerning of the $X^{s,b}$ space, we have two properties which are presented in the lemma below for the functions $\psi(t)$ and ψ_T, defined in (9). The first item can be found in [40, Lemma 2.11] and the second one in Ginibre *et al.* [27]. The lemma can be read as follows:

Lemma 3.1. *Let $\psi(t)$ be a Schwartz function in time. Then, we have*

$$\|\psi(t)f\|_{X^{s,b}} \lesssim_{\psi,b} \|f\|_{X^{s,b}}.$$

Moreover, if $-\frac{1}{2} < b' < b \leq 0$ or $0 \leq b' < b < \frac{1}{2}$, $w \in X^{s,b}$ and $s \in \mathbb{R}$, thus

$$\|\psi_T w\|_{X^{s,b'}} \lesssim T^{b-b'} \|w\|_{X^{s,b}}.$$

An result that state important properties of the Riemann-Liouville fractional integral operator is given below. The proof of this can be found in [30, Lemmas 2.1, 5.3 and 5.4].

Lemma 3.2. *If $f \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+)$, for all $\alpha \in \mathbb{C}$. Moreover, we have the following:*

  (a) *If $0 \leq \operatorname{Re} \alpha < \infty$ and $s \in \mathbb{R}$, then $\|\mathcal{I}_{-\alpha}h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)}$, where $c = c(\alpha)$.*
  (b) *If $0 \leq \operatorname{Re} \alpha < \infty$, $s \in \mathbb{R}$ and $\mu \in C_0^\infty(\mathbb{R})$, then $\|\mu\mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}$, where $c = c(\mu, \alpha)$.*

3.2. **Estimates for the boundary forcing operator.** Now, we will obtain the principal properties of the boundary forcing operator. Initially, we present the well-know properties of the spatial continuity, the decay of the $\mathcal{L}^\lambda g(t,x)$ and the explicit values for $\mathcal{L}^\lambda f(t,0)$, respectively, these results with their respective proofs can be seen in [16, Lemmas 3.2 and 3.3].

Lemma 3.3. *Let $g \in C_0^\infty(\mathbb{R}^+)$ and M be as in (14). Then, we have*

$$\mathcal{L}^{-k}g = \partial_x^k \mathcal{L}^0 \mathcal{I}_{\frac{k}{4}} g, \qquad k = 0, 1, 2, 3. \tag{22}$$

Moreover, for fixed $0 \leq t \leq 1$, $\partial_x^k \mathcal{L}^0 f(t,x)$, $k = 0, 1, 2$, is continuous in $x \in \mathbb{R}$ and $\mathcal{L}^{-3}g(t,x)$ is continuous in $x \in \mathbb{R} \setminus \{0\}$ and has a step discontinuity at $x = 0$.

Lemma 3.4. *For $\operatorname{Re} \lambda > -4$ and $f \in C_0^\infty(\mathbb{R}^+)$, we have the following value of $\mathcal{L}^\lambda f(t,0)$:*

$$\mathcal{L}^\lambda f(t,0) = \frac{M}{8}f(t)\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)}}{\sin(\frac{1-\lambda}{4}\pi)}\right). \tag{23}$$

3.3. **Energy and trilinear estimates.** In the last part, we present four lemmas related to energy and trilinear estimates for the solutions of the 4NLS equation in the Bourgain spaces $X^{s,b}$. These results and their proofs can also be found in [16, Section 4].

Lemma 3.5. *Let $s \in \mathbb{R}$ and $b \in \mathbb{R}$. If $\phi \in H^s(\mathbb{R})$, then the following estimates holds*

$$\|\psi(t)e^{it\partial_x^4}\phi(x)\|_{C_t\left(\mathbb{R}; H_x^s(\mathbb{R})\right)} \lesssim_\psi \|\phi\|_{H^s(\mathbb{R})}, \tag{24}$$

$$\|\psi(t)\partial_x^j e^{it\partial_x^4}\phi(x)\|_{C_x\left(\mathbb{R}; H_t^{\frac{2s+3-2j}{8}}(\mathbb{R})\right)} \lesssim_{\psi,s,j} \|\phi\|_{H^s(\mathbb{R})} \quad j \in \mathbb{N} \tag{25}$$

and

$$\|\psi(t)e^{it\partial_x^4}\phi(x)\|_{X^{s,b}} \lesssim_{\psi,b} \|\phi\|_{H^s(\mathbb{R})}. \tag{26}$$

Lemma 3.6. *Let $0 < b < \frac{1}{2}$ and $j = 0, 1, 2, 3$, we have the following inequalities*

$$\|\psi(t)\mathcal{D}w(t,x)\|_{C\left(\mathbb{R}_t;\,H^s(\mathbb{R}_x)\right)} \lesssim \|w\|_{X^{s,-b}}, \tag{27}$$

for $s \in \mathbb{R}$;

$$\|\psi(t)\partial_x^j\mathcal{D}w(t,x)\|_{C\left(\mathbb{R}_x;\,H^{\frac{2s+3-2j}{8}}(\mathbb{R}_t)\right)} \lesssim \|w\|_{X^{s,-b}}, \tag{28}$$

for $-\frac{3}{2} + j < s < \frac{1}{2} + j$;

$$\|\psi(t)\partial_x^j\mathcal{D}w(t,x)\|_{X^{s,b}} \lesssim \|w\|_{X^{s,-b}}, \tag{29}$$

for $s \in \mathbb{R}$.

Lemma 3.7. *Let $s \in \mathbb{R}$. Then,*

(a) *For $\frac{2s-7}{2} < \lambda < \frac{1+2s}{2}$ and $\lambda < \frac{1}{2}$ the following inequality holds*

$$\|\psi(t)\mathcal{L}^\lambda f(t,x)\|_{C\left(\mathbb{R}_t;\,H^s(\mathbb{R}_x)\right)} \lesssim \|f\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)}; \tag{30}$$

(b) *For $-4 + j < \lambda < 1 + j$, $j = 0, 1, 2, 3$, we have*

$$\|\psi(t)\partial_x^j\mathcal{L}^\lambda f(t,x)\|_{C\left(\mathbb{R}_x;\,H_0^{\frac{2s+3-2j}{8}}(\mathbb{R}_t^+)\right)} \lesssim \|f\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)}; \tag{31}$$

(c) *If $s < 4 - 4b$, $b < \frac{1}{2}$, $-5 < \lambda < \frac{1}{2}$ and $s + 4b - 2 < \lambda < s + \frac{1}{2}$ yields that*

$$\|\psi(t)\mathcal{L}^\lambda f(t,x)\|_{X^{s,b}} \lesssim \|f\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)}. \tag{32}$$

Remark 2. Let us present two remarks.

  i. The previous estimates are the so-called space traces, derivative time traces and Bourgain spaces estimates, respectively.
 ii. We observe that in [16] was obtained (25), (28) and (31), for $j = 0$ and $j = 1$, but the result for all j can be obtained directly by using the fact that

$$\partial_x^j\mathcal{L}^\lambda = \mathcal{L}^{\lambda-j}(\mathcal{I}_{-\frac{j}{4}}).$$

To close this section, let us enunciate the trilinear estimates associated to fourth order nonlinear Schrödinger equation. The proof of this estimate can be found in [42].

Lemma 3.8. *For $s \geq 0$, there exists $b = b(s) < 1/2$ such that*

$$\|u_1 u_2 \overline{u}_3\|_{X^{s,-b}} \leq c\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{s,b}}. \tag{33}$$

4. **Proof of the main result.** The aim of this section is to prove the main result announced in the introduction of this work, Theorem 1.1. Here, we only consider the vertex condition (4) (type $\mathcal{A}$) and to make the proof easy to understand, we will split it in several steps which will be divided into subsections. Additionally, the discussion of vertex conditions types $\mathcal{B}$ and $\mathcal{C}$ will be presented on Appendix A, at the end of the article.

4.1. **Obtaining a integral equation.** In this first step, we are interested in finding an extended vectorial integral equation posed in $\bigoplus_{i=1}^{N} \mathbb{R}$, such that the restrictions of this equation on $\mathcal{G}$ will solve, in the sense of distributions, the following Cauchy problem

$$\begin{cases} i\partial_t u_j - \partial_x^4 u_j + \lambda |u_j|^2 u_j = 0, & (t,x) \in (0,T) \times (0,\infty), \ j = 1,2,...,N \\ u_j(0,x) = u_{j0}(x), & x \in (0,\infty), \end{cases} \tag{34}$$

with initial conditions $(u_{10}, u_{20}, ..., u_{N0}) \in H^s(\mathcal{G})$. Let us begin rewriting the Type $\mathcal{A}$ vertex conditions (4) in terms of matrices. In this way, note that (4) is equivalent to

$$\partial_x^k u_1(t,0) = \partial_x^k u_2(t,0), \ \partial_x^k u_2(t,0) = \partial_x^k u_3(t,0), \cdots, \ \partial_x^k u_{N-1}(t,0) = \partial_x^k u_N(t,0),$$

for $k = 0, 1$, $t \in (0,T)$ and

$$\sum_{j=1}^{N} \partial_x^k u_j(t,0) = 0,$$

for $k = 2, 3$, $t \in (0,T)$. Thus, we consider the following matrices

$$[C_1]_{2N \times N} \begin{bmatrix} u_1(t,0) \\ u_2(t,0) \\ u_3(t,0) \\ \vdots \\ u_{N-1}(t,0) \\ u_N(t,0) \end{bmatrix}_{N \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2N \times 1} \; ;$$

$$[C_2]_{2N \times N} \begin{bmatrix} \partial_x u_1(t,0) \\ \partial_x u_2(t,0) \\ \partial_x u_3(t,0) \\ \vdots \\ \partial_x u_{N-1}(t,0) \\ \partial_x u_N(t,0) \end{bmatrix}_{N \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2N \times 1} \tag{35}$$

and

$$[C_3]_{2N \times N} \begin{bmatrix} \partial_x^2 u_1(t,0) \\ \partial_x^2 u_2(t,0) \\ \partial_x^2 u_3(t,0) \\ \vdots \\ \partial_x^2 u_{N-1}(t,0) \\ \partial_x^2 u_N(t,0) \end{bmatrix}_{N \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2N \times 1} \; ;$$

$$[C_4]_{2N \times N} \begin{bmatrix} \partial_x^3 u_1(t,0) \\ \partial_x^3 u_2(t,0) \\ \partial_x^3 u_3(t,0) \\ \vdots \\ \partial_x^3 u_{N-1}(t,0) \\ \partial_x^3 u_N(t,0) \end{bmatrix}_{N \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2N \times 1} , \tag{36}$$

where

$$[C_1]_{2N \times N} := \left[\begin{array}{cccccc} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} N-1 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} N+1 \text{ rows} \end{array}$$

$$\underbrace{}_{N \text{ columns}}$$

$$[C_2]_{2N \times N} := \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} N-1 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} N-1 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} 2 \text{ rows} \end{array}$$

$$\underbrace{}_{N \text{ columns}}$$

$$[C_3]_{2N \times N} := \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} 2N-2 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} 2 \text{ rows} \end{array}$$

$$\underbrace{}_{N \text{ columns}}$$

and

$$[C_4]_{2N \times N} := \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} 2N-2 \text{ rows} \\ \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} 2 \text{ rows} \end{array} \quad .$$

$$\underbrace{}_{N \text{ columns}}$$

On the other hand, let be $\widetilde{u}_{j0}$ an extension for all line $\mathbb{R}$ of u_{j0}, satisfying

$$\|\widetilde{u}_{j0}\|_{H^s(\mathbb{R})} \lesssim \|u_{j0}\|_{H^s(\mathbb{R}^+)}, \ \ j = 1, 2, ..., N,$$

respectively. Initially, we look for solutions in the form

$$u_j(t,x) = \mathcal{L}^{\lambda_{j1}}\gamma_{j1}(t,x) + \mathcal{L}^{\lambda_{j2}}\gamma_{j2}(t,x) + F_j(t,x), \quad j = 1, 2, ..., N. \tag{37}$$

Here, $\gamma_{ji}(\cdot)$, $j = 1, 2, ..., N$, $i = 1, 2$, are unknown functions and

$$F_j(t,x) = e^{it\partial_x^4}\widetilde{u}_{j0} + \mathcal{D}(\psi_T|u_j|^2 u_j)(t,x),$$

where $\mathcal{D}(w(t,x)) = -i\int_0^t e^{i(t-t')\partial_x^4}w(t',x)dt'$. By using Lemma 3.4 we see that

$$
\begin{aligned}
u_j(t,0) =& \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda_{j1})} + e^{-i\frac{\pi}{8}(1-5\lambda_{j1})}}{\sin(\frac{1-\lambda_{j1}}{4}\pi)}\right)\gamma_{j1}(t) \\
&+ \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda_{j2})} + e^{-i\frac{\pi}{8}(1-5\lambda_{j2})}}{\sin(\frac{1-\lambda_{j2}}{4}\pi)}\right)\gamma_{j2}(t) + F_j(t,0) \\
:=& a_{j1}\gamma_{j1}(t) + a_{j2}\gamma_{j2}(t) + F_j(t,0), \quad j = 1, 2, ..., N.
\end{aligned}
\tag{38}
$$

Now, let us calculate the traces of first derivative functions. Thanks to (10), (16), (20) and Lemma 3.4, we get that

$$
\begin{aligned}
\partial_x u_j(t,0) =& \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-2+3\lambda_{j1})} + e^{-i\frac{\pi}{8}(6-5\lambda_{j1})}}{\sin(\frac{2-\lambda_{j1}}{4}\pi)}\right)\mathcal{I}_{-1/4}\gamma_{j1}(t) \\
&+ \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-2+3\lambda_{j2})} + e^{-i\frac{\pi}{8}(6-5\lambda_{j2})}}{\sin(\frac{2-\lambda_{j2}}{4}\pi)}\right)\mathcal{I}_{-1/4}\gamma_{j2}(t) \\
&+ \partial_x F_j(t,0), \\
:=& b_{j1}\mathcal{I}_{-1/4}\gamma_{j1}(t) + b_{j2}\mathcal{I}_{-1/4}\gamma_{j2}(t) + \partial_x F_j(t,0), \quad j = 1, 2, ..., N.
\end{aligned}
\tag{39}
$$

In the same way, we can have the traces of second and third derivatives functions, giving us the following

$$
\begin{aligned}
\partial_x^2 u_j(t,0) =& \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-5+3\lambda_{j1})} + e^{-i\frac{\pi}{8}(11-5\lambda_{j1})}}{\sin(\frac{3-\lambda_{j1}}{4}\pi)}\right)\mathcal{I}_{-1/2}\gamma_{j1}(t) \\
&+ \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-5+3\lambda_{j2})} + e^{-i\frac{\pi}{8}(11-5\lambda_{j2})}}{\sin(\frac{3-\lambda_{j2}}{4}\pi)}\right)\mathcal{I}_{-1/2}\gamma_{j2}(t) \\
&+ \partial_x^2 F_j(t,0), \\
:=& c_{j1}\mathcal{I}_{-1/2}\gamma_{j1}(t) + c_{j2}\mathcal{I}_{-1/2}\gamma_{j2}(t) + \partial_x^2 F_j(t,0), \quad j = 1, 2, ..., N
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
\partial_x^3 u_j(t,0) =& \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-8+3\lambda_{j1})} + e^{-i\frac{\pi}{8}(16-5\lambda_{j1})}}{\sin(\frac{4-\lambda_{j1}}{4}\pi)}\right)\mathcal{I}_{-3/4}\gamma_{j1}(t) \\
&+ \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-8+3\lambda_{j2})} + e^{-i\frac{\pi}{8}(16-5\lambda_{j2})}}{\sin(\frac{4-\lambda_{j2}}{4}\pi)}\right)\mathcal{I}_{-3/4}\gamma_{j2}(t) \\
&+ \partial_x^2 F_j(t,0), \\
:=& d_{j1}\mathcal{I}_{-3/4}\gamma_{j1}(t) + d_{j2}\mathcal{I}_{-3/4}\gamma_{j2}(t) + \partial_x^3 F_j(t,0), \quad j = 1, 2, ..., N.
\end{aligned}
\tag{41}
$$

Observe that Lemmas 3.3 and 3.4 ensure these calculus are valid for $\mathrm{Re}\,\lambda > -4$. By substituting (38), (39), (40) and (41) into (35) and (36), yields that the functions γ_{ji} and indexes λ_{ji}, for $j = 1, 2, ..., N$ and $i = 1, 2$, satisfy the following equalities:

$$
[C_1]_{2N\times N}
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & a_{21} & a_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{N1} & a_{N2}
\end{bmatrix}_{N\times 2N}
\begin{bmatrix}
\gamma_{11}(t) \\
\gamma_{12}(t) \\
\gamma_{21}(t) \\
\gamma_{22}(t) \\
\vdots \\
\gamma_{N1}(t) \\
\gamma_{N2}(t)
\end{bmatrix}_{2N\times 1}
$$

$$
= -[C_1]_{2N\times N}
\begin{bmatrix}
F_1(t,0) \\
F_2(t,0) \\
\vdots \\
F_N(t,0)
\end{bmatrix}_{N\times 1},
$$

$$
[C_2]_{2N\times N}
\begin{bmatrix}
b_{11} & b_{12} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & b_{21} & b_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & b_{N1} & b_{N2}
\end{bmatrix}_{N\times 2N}
\begin{bmatrix}
\gamma_{11}(t) \\
\gamma_{12}(t) \\
\gamma_{21}(t) \\
\gamma_{22}(t) \\
\vdots \\
\gamma_{N1}(t) \\
\gamma_{N2}(t)
\end{bmatrix}_{2N\times 1}
$$

$$
= -[C_2]_{2N\times N}
\begin{bmatrix}
\partial_x \mathcal{I}_{\frac{1}{4}} F_1(t,0) \\
\partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) \\
\vdots \\
\partial_x \mathcal{I}_{\frac{1}{4}} F_N(t,0)
\end{bmatrix}_{N\times 1},
$$

$$
[C_3]_{2N\times N}
\begin{bmatrix}
c_{11} & c_{12} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & c_{21} & c_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & c_{N1} & c_{N2}
\end{bmatrix}_{N\times 2N}
\begin{bmatrix}
\gamma_{11}(t) \\
\gamma_{12}(t) \\
\gamma_{21}(t) \\
\gamma_{22}(t) \\
\vdots \\
\gamma_{N1}(t) \\
\gamma_{N2}(t)
\end{bmatrix}_{2N\times 1}
$$

$$
= -[C_3]_{2N\times N}
\begin{bmatrix}
\partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) \\
\partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) \\
\vdots \\
\partial_x^2 \mathcal{I}_{\frac{1}{2}} F_N(t,0)
\end{bmatrix}_{N\times 1}
$$

and

$$
[C_4]_{2N \times N}
\begin{bmatrix}
d_{11} & d_{12} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & d_{21} & d_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & d_{N1} & d_{N2}
\end{bmatrix}_{N \times 2N}
\begin{bmatrix}
\gamma_{11}(t) \\
\gamma_{12}(t) \\
\gamma_{21}(t) \\
\gamma_{22}(t) \\
\vdots \\
\gamma_{N1}(t) \\
\gamma_{N2}(t)
\end{bmatrix}_{2N \times 1}
$$

$$
= -[C_4]_{2N \times N}
\begin{bmatrix}
\partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) \\
\partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) \\
\vdots \\
\partial_x^3 \mathcal{I}_{\frac{3}{4}} F_N(t,0)
\end{bmatrix}_{N \times 1}.
$$

It follows that,

$$
\boldsymbol{\gamma} \times
\begin{bmatrix}
a_{11} & a_{12} & -a_{21} & -a_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{11} & a_{12} & -a_{21} & -a_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{(N-1)1} & a_{(N-1)2} & -a_{N1} & -a_{N2} \\
b_{11} & b_{12} & -b_{21} & -b_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_{11} & b_{12} & -b_{21} & -b_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & b_{(N-1)1} & b_{(N-1)2} & -b_{N1} & -b_{N2} \\
c_{11} & c_{12} & c_{21} & c_{22} & c_{31} & c_{32} & \cdots & \cdots & \cdots & c_{N1} & c_{N2} \\
d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & \cdots & \cdots & \cdots & d_{N1} & d_{N2}
\end{bmatrix}_{2N \times 2N}
$$

$$
= -
\begin{bmatrix}
F_1(t,0) - F_2(t,0) \\
\vdots \\
F_{N-1}(t,0) - F_N(t,0) \\
\partial_x \mathcal{I}_{\frac{1}{4}} F_1(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) \\
\vdots \\
\partial_x \mathcal{I}_{\frac{1}{4}} F_{N-1}(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_N(t,0) \\
\sum_{j=1}^{N} \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_j(t,0) \\
\sum_{j=1}^{N} \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_j(t,0)
\end{bmatrix}_{2N \times 1}, \quad (42)
$$

where

$$
\boldsymbol{\gamma} =
\begin{bmatrix}
\gamma_{11}(t) \\
\gamma_{12}(t) \\
\gamma_{21}(t) \\
\gamma_{22}(t) \\
\vdots \\
\gamma_{N1}(t) \\
\gamma_{N2}(t)
\end{bmatrix}_{2N \times 1}
$$

To simplify the notation, let us denote the equality (42) by

$$
\mathbf{M}(\lambda_{11}, \lambda_{12}, \cdots, \lambda_{N1}, \lambda_{N2})\boldsymbol{\gamma} = \mathbf{F}, \quad (43)
$$

where $\mathbf{M}(\lambda_{11}, \lambda_{12}, \cdots, \lambda_{N1}, \lambda_{N2})$ is the first matrix that appears in the left hand side of (42), $\boldsymbol{\gamma}$ is the matrix column given by vector $(\gamma_{11}, \gamma_{12}, \cdots, \gamma_{N1}, \gamma_{N2})$ and $\mathbf{F}$ is the matrix in the right hand side of (42).

4.2. **Choosing the appropriate parameters and functions.** In this second step, we need to choose the parameters λ_{ji} and the functions γ_{ji}, with $j = 1, 2, ..., N$, $i = 1, 2$, in such a way that we can be able to write the solution $u_j(t, x)$, in a integral form.

To do this, let us start by using the hypothesis of Lemma 3.7. We need, firstly, to fix parameters λ_{ji}, such that

$$\max\left\{\frac{2s-7}{2}, -1\right\} < \lambda_{ji}(s) < \min\left\{s + \frac{1}{2}, \frac{1}{2}\right\}, \quad j = 1, 2, ..., N, i = 1, 2. \quad (44)$$

With this restriction in hand we choose the parameters λ_{ji} as follows

$$\lambda_{11} = \lambda_{21} = \cdots = \lambda_{N1} = -\frac{1}{2} \text{ and } \lambda_{12} = \lambda_{22} = \cdots = \lambda_{N2} = \frac{1}{4}, \quad (45)$$

then, we have the equation

$$\mathbf{M}\left(-\frac{1}{2}, \frac{1}{4}, \cdots, -\frac{1}{2}, \frac{1}{4}\right)\boldsymbol{\gamma} = \mathbf{F}. \quad (46)$$

The following lemma gives us that $\mathbf{M}\left(-\frac{1}{2}, \frac{1}{4}, \cdots, -\frac{1}{2}, \frac{1}{4}\right)$ is invertible.

Lemma 4.1. *The determinant of matrix* $\mathbf{M}\left(-\frac{1}{2}, \frac{1}{4}, \cdots, -\frac{1}{2}, \frac{1}{4}\right)$ *is nonzero.*

We will prove Lemma 4.1 on the next section. Thus, these good choices of the parameters satisfying (44) together with this lemma ensures that $\mathbf{M}$ is invertible and, consequently, the following holds

$$\boldsymbol{\gamma} = \mathbf{M}^{-1}\left(-\frac{1}{2}, \frac{1}{4}, \cdots, -\frac{1}{2}, \frac{1}{4}\right)\mathbf{F}. \quad (47)$$

We empathize that γ_{ji} depends on F_1 and F_2, which depend on the unknown functions u_1 and u_2. Thus, by substituting (47) into (37), we get $u_j(t, x)$ in the integral form

$$u_j(t, x) = \mathcal{L}^{-\frac{1}{2}}\gamma_{j1}(t, x) + \mathcal{L}^{\frac{1}{4}}\gamma_{j2}(t, x) + F_j(t, x), \quad j = 1, 2, ..., N. \quad (48)$$

4.3. **Defining the truncated integral operator and functional space.** Using the previous subsection, we have the solution of the Cauchy problem (34) with Type $\mathcal{A}$ boundary condition (4) in the integral form (48). In order to use the Fourier restriction method, the third step is to define a truncated version for the integral form (48).

Pick $s \in [0, 1/2)$, we fix the parameters λ_{ji} as in (45) and define

$$\boldsymbol{\gamma} = (\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}, \cdots, \gamma_{N1}, \gamma_{N2})$$

by (47). Consider $b = b(s) < \frac{1}{2}$ and that the estimates given in Lemmas 3.5, 3.6, 3.7 and 3.8 are valid. Now, define the operator Λ by

$$\Lambda = (\Lambda_1, \Lambda_2, \cdots, \Lambda_N)$$

where

$$\Lambda_j u(t, x) = \psi(t)\mathcal{L}^{-\frac{1}{2}}\gamma_{j1}(t, x) + \psi(t)\mathcal{L}^{\frac{1}{4}}\gamma_{j2}(t, x) + F_j(t, x), \quad j = 1, 2, ..., N.$$

Here,

$$F_j(t,x) = \psi(t)(e^{it\partial_x^4}\widetilde{u}_{j0} + \lambda\mathcal{D}(\psi_T|u_j|^2 u_j)(t,x)), \quad j = 1, 2, ..., N,$$

with

$$\mathcal{D}(w(t,x)) = -i\int_0^t e^{i(t-t')\partial_x^4}w(x,t')dt'.$$

We consider Λ defined on the Banach space $Z(s,b) = \bigoplus_{j=1}^N Z_j(s,b)$ by

$$Z_j(s,b) = \left\{ w \in C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C(\mathbb{R}_x; H^{\frac{2s+3}{8}}(\mathbb{R}_t)) \cap X^{s,b}; \right.$$

$$\left. w_x \in C(\mathbb{R}_x; H^{\frac{2s+1}{8}}(\mathbb{R}_t)), w_{xx} \in C(\mathbb{R}_x; H^{\frac{2s-1}{8}}(\mathbb{R}_t)), w_{xxx} \in C(\mathbb{R}_x; H^{\frac{2s-3}{8}}(\mathbb{R}_t)) \right\},$$

for $j = 1, 2, ..., N$, with norm

$$\|(u_1, u_2, \cdots, u_N)\|_{Z(s,b)} = \|u_1\|_{Z_1(s,b)} + \|u_2\|_{Z_2(s,b)} + \cdots + \|u_N\|_{Z_N(s,b)}. \tag{49}$$

Each norm of $\|u\|_{Z_j(s,b)}$ on (49) is defined by

$$\|u\|_{Z_j(s,b)} = \|u\|_{C(\mathbb{R}_t; H^s(\mathbb{R}_x))} + \|u\|_{C\left(\mathbb{R}_x; H^{\frac{2s+3}{8}}(\mathbb{R}_t)\right)} + \|u\|_{X^{s,b}}$$

$$+ \|u_x\|_{C\left(\mathbb{R}_x; H^{\frac{2s+1}{8}}(\mathbb{R}_t)\right)} + \|u_{xx}\|_{C\left(\mathbb{R}_x; H^{\frac{2s-1}{8}}(\mathbb{R}_t)\right)} + \|u_{xxx}\|_{C\left(\mathbb{R}_x; H^{\frac{2s-3}{8}}(\mathbb{R}_t)\right)},$$

for $j = 1, 2, ..., N$.

4.4. Proving that the functions $\mathcal{L}^{-\frac{1}{2}}\gamma_{j1}$ and $\mathcal{L}^{\frac{1}{4}}\gamma_{j2}$, for $j = 1, 2, ..., N$, are well-defined.
Indeed, by using Lemma 3.7 it suffices to show that these functions are in the closure of the spaces $C_0^\infty(\mathbb{R}^+)$. By using expression (47), we see that γ_{ji} $(j = 1, 2, ..., N$ and $i = 1, 2)$ are linear combinations of the functions

$$F_1(t,0) - F_2(t,0), \ F_2(t,0) - F_3(t,0), \cdots, \ F_{N-1}(t,0) - F_N(t,0),$$

$$\partial_x\mathcal{I}_{\frac{1}{4}}F_1(t,0) - \partial_x\mathcal{I}_{\frac{1}{4}}F_2(t,0), \ \partial_x\mathcal{I}_{\frac{1}{4}}F_2(t,0) - \partial_x\mathcal{I}_{\frac{1}{4}}F_3(t,0),$$
$$\cdots, \ \partial_x\mathcal{I}_{\frac{1}{4}}F_{N-1}(t,0) - \partial_x\mathcal{I}_{\frac{1}{4}}F_N(t,0),$$

$$\partial_x^2\mathcal{I}_{\frac{1}{2}}F_1(t,0) + \partial_x^2\mathcal{I}_{\frac{1}{2}}F_2(t,0) + \cdots + \partial_x^2\mathcal{I}_{\frac{1}{2}}F_N(t,0),$$

$$\partial_x^3\mathcal{I}_{\frac{3}{4}}F_1(t,0) + \partial_x^3\mathcal{I}_{\frac{3}{4}}F_2(t,0) + \cdots + \partial_x^3\mathcal{I}_{\frac{3}{4}}F_N(t,0).$$

Thus, we need to show that the functions $F_j(t,0)$, $\partial_x\mathcal{I}_{\frac{1}{3}}F_j(t,0)$, $\partial_x^2\mathcal{I}_{\frac{2}{3}}F_j(t,0)$ are in appropriate spaces. By Lemmas 3.5, 3.7, 3.6 and 3.8 we obtain

$$\|F_j(t,0)\|_{H^{\frac{2s+3}{8}}(\mathbb{R}^+)} \leq c(\|u_{j0}\|_{H^s(\mathbb{R}^+)} + \|u_j\|_{X^{s,b}}^3). \tag{50}$$

If $0 \leq s < \frac{1}{2}$ we have that $\frac{3}{8} < \frac{2s+3}{8} < \frac{1}{2}$, then $H^{\frac{2s+3}{8}}(\mathbb{R}^+) = H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)$. It follows that $F_j(t,0) \in H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)$ for $0 \leq s < \frac{1}{2}$. Again by using Lemmas 3.5, 3.7, 3.6 and 3.8 we get

$$\|\partial_x F_j(t,0)\|_{H^{\frac{2s+1}{8}}(\mathbb{R}^+)} \leq c(\|u_{j0}\|_{H^s(\mathbb{R}^+)} + \|u_j\|_{X^{s,b}}^3).$$

Since $0 \le s < \frac{1}{2}$ we have $\frac{1}{8} \le \frac{2s+1}{8} < \frac{1}{4}$, then the functions $\partial_x F_j(t,0) \in H_0^{\frac{2s+1}{8}}(\mathbb{R}^+)$. Then, thanks to Lemma 3.2, we have that

$$\|\partial_x \mathcal{I}_{\frac{1}{4}} F_j(t,0)\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)} \le c(\|u_{j0}\|_{H^s(\mathbb{R}^+)} + \|u_j\|_{X^{s,b}}^3).$$

Therefore, this yields that

$$\partial_x \mathcal{I}_{\frac{1}{4}} F_j(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_{j+1}(t,0) \in H_0^{\frac{2s+3}{8}}(\mathbb{R}^+), \quad j = 1, 2, ..., N. \tag{51}$$

In a similar way, we can obtain

$$\|\partial_x^2 \mathcal{I}_{\frac{1}{2}} F_j(t,0)\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)} \lesssim \|u_{j0}\|_{H^s(\mathbb{R}^+)} + \|u_j\|_{X^{s,b}}^3,$$

$$\|\partial_x^3 \mathcal{I}_{\frac{3}{4}} F_i(t,0)\|_{H_0^{\frac{2s+3}{8}}(\mathbb{R}^+)} \lesssim \|u_{j0}\|_{H^s(\mathbb{R}^+)} + \|u_j\|_{X^{s,b}}^3.$$

It follows that

$$\begin{aligned}
\partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) + \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) + \cdots + \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_N(t,0) &\in H_0^{\frac{2s+3}{8}}(\mathbb{R}^+), \\
\partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) + \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) + \cdots + \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_N(t,0) &\in H_0^{\frac{2s+3}{8}}(\mathbb{R}^+).
\end{aligned} \tag{52}$$

Thus, (50), (51) and (52) imply that the functions $\mathcal{L}^{-\frac{1}{2}} \gamma_{j1}$ and $\mathcal{L}^{\frac{1}{4}} \gamma_{j2}$, for $j = 1, 2, ..., N$, are well-defined.

4.5. **Showing that Λ is a contraction in a ball of Z.** Lemmas 3.2, 3.5, 3.7, 3.6 and 3.8 guarantee that

$$\begin{aligned}
&\|\Lambda(u_1, \cdots, u_N) - \Lambda(v_1, \cdots, v_N)\|_{Z_{s,b}} \\
&\qquad \le T^\epsilon c \left(\|(u_1, \cdots, u_N)\|_Z^2 + \|(v_1, \cdots, v_N)\|_Z^2 \right) \|(u_1, \cdots, u_N) - (v_1, \cdots, v_N)\|_Z
\end{aligned}$$

and

$$\begin{aligned}
\|\Lambda(u_1, \cdots, u_N)\|_{Z_{s,b}} \le\ & c \left(\|u_{01}\|_{H^s(\mathbb{R}^+)} + \cdots + \|u_{0N}\|_{H^s(\mathbb{R}^+)} \right) \\
&+ T^\epsilon (\|u_1\|_{X^{s,b}}^3 + \cdots + \|u_N\|_{X^{s,b}}^3),
\end{aligned}$$

for ϵ adequately small.

Consider in Z the ball defined by

$$B = \{(u_1, \cdots, u_N) \in Z^{s,b}; \|(u_1, \cdots, u_N)\|_{Z^{s,b}} \le M\},$$

where

$$M = 2c \left(\|u_{01}\|_{H^s(\mathbb{R}^+)} + \cdots + \|u_{0N}\|_{H^s(\mathbb{R}^+)} \right).$$

Lastly, choosing $T = T(M)$ sufficiently small, such that

$$\|\Lambda(u_1, \cdots, u_N)\|_{Z^{s,b}} \le M$$

and

$$\|\Lambda(u_1, \cdots, u_N) - \Lambda(v_1, \cdots, v_N)\|_{Z^{s,b}} \le \frac{1}{2} \|(u_1, \cdots, u_N) - (v_1, \cdots, v_N)\|_{Z^{s,b}},$$

it follows that Λ is a contraction map on B and has a fixed point $(\widetilde{u}_1, \cdots, \widetilde{u}_N)$. Therefore, the restriction

$$(u_1, \cdots, u_N) = (\widetilde{u}_1|_{\mathbb{R}^- \times (0,T)}, \cdots, \widetilde{u}_N|_{\mathbb{R}^+ \times (0,T)})$$

solves the Cauchy problem (34) with Type $\mathcal{A}$ vertex boundary condition (4), in the sense of distributions. Thus, Theorem 1.1 is a consequence of the above steps, described in the previous subsections, finalizing so the proof. □

5. **Proof of Lemma 4.1.** First, we will prove the case $N = 2$. The vertex conditions (4), for this case, is given by

$$\begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0), \ k = 0,1 & t \in (0,T) \\ \sum_{j=1}^2 \partial_x^k u_j(t,0) = 0, \ k = 2,3 & t \in (0,T). \end{cases}$$

In this way, we consider the vertex conditions as the following matrices

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t,0) \\ u_2(t,0) \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_x u_1(t,0) \\ \partial_x u_2(t,0) \end{bmatrix} = 0 \qquad (53)$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_x^2 u_1(t,0) \\ \partial_x^2 u_2(t,0) \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \partial_x^3 u_1(t,0) \\ \partial_x^3 u_2(t,0) \end{bmatrix} = 0. \qquad (54)$$

By substituting, for $N = 2$, (38), (39), (40) and (41) into (53) and (54), yields that the functions γ_{ji} and indexes λ_{ji}, for $j = 1,2$ and $i = 1,2$, satisfy the equality of matrices:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11}(t) \\ \gamma_{12}(t) \\ \gamma_{21}(t) \\ \gamma_{22}(t) \end{bmatrix} = - \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_1(t,0) \\ F_2(t,0) \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11}(t) \\ \gamma_{12}(t) \\ \gamma_{21}(t) \\ \gamma_{22}(t) \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_x \mathcal{I}_{\frac{1}{4}} F_1(t,0) \\ \partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11}(t) \\ \gamma_{12}(t) \\ \gamma_{21}(t) \\ \gamma_{22}(t) \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) \\ \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & 0 & 0 \\ 0 & 0 & d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11}(t) \\ \gamma_{12}(t) \\ \gamma_{21}(t) \\ \gamma_{22}(t) \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) \end{bmatrix}.$$

Putting all matrices together, we have that,

$$\begin{bmatrix} a_{11} & a_{12} & -a_{21} & -a_{22} \\ b_{11} & b_{12} & -b_{21} & -b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{21} \\ \gamma_{22} \end{bmatrix} = - \begin{bmatrix} F_1(t,0) - F_2(t,0) \\ \partial_x \mathcal{I}_{\frac{1}{4}} F_1(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) \\ \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) + \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) + \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) \end{bmatrix}.$$

In the case $N = 2$, the matrix M, given by (42), can be read as follows

$$
M = \begin{bmatrix}
a_{11} & a_{12} & -a_{21} & -a_{22} \\
b_{11} & b_{12} & -b_{21} & -b_{22} \\
c_{11} & c_{12} & c_{21} & c_{22} \\
d_{11} & d_{12} & d_{21} & d_{22}
\end{bmatrix}, \tag{55}
$$

where a_{ij}, b_{ij}, c_{ij} and d_{ij} are given by (38), (39), (40) and (41), respectively.

Claim 1. M *has determinant different of zero with appropriate choice of* λ_{ji}, $j = 1, 2$ *and* $i = 1, 2$.

In fact, firstly noting that

$$
\sin\left(\frac{2-a}{4}\pi\right) = \cos\left(\frac{a\pi}{4}\right)
$$

and it is easy to see that

$$
\begin{cases}
a_{ji} = \dfrac{Me^{-\frac{i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(1-\lambda_{ji})\pi}{4}\right)}\right), \quad b_{ji} = \dfrac{Me^{\frac{2i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}-e^{\frac{5i\pi\lambda_{ji}}{8}}}{\cos\left(\frac{\lambda_{ji}\pi}{4}\right)}\right), \\[4ex]
c_{ji} = \dfrac{Me^{\frac{5i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(3-\lambda_{ji})\pi}{4}\right)}\right), \quad d_{ji} = \dfrac{Me^{\frac{8i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}-e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{\lambda_{ji}\pi}{4}\right)}\right),
\end{cases} \tag{56}
$$

for $j = 1, 2$ and $i = 1, 2$. Then, the determinant of M can be write as

$$
|M| = \frac{M^4 e^{\frac{14i\pi}{8}}}{8^4} \times \mathcal{M},
$$

where $\mathcal{M}$ is denoted by

$$
\begin{vmatrix}
\dfrac{e^{-\frac{3i\pi\lambda_{11}}{8}}+e^{\frac{5i\pi\lambda_{11}}{8}}}{\sin\left(\frac{(1-\lambda_{11})\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{12}}{8}}+e^{\frac{5i\pi\lambda_{12}}{8}}}{\sin\left(\frac{(1-\lambda_{12})\pi}{4}\right)} & -\dfrac{e^{-\frac{3i\pi\lambda_{21}}{8}}+e^{\frac{5i\pi\lambda_{21}}{8}}}{\sin\left(\frac{(1-\lambda_{21})\pi}{4}\right)} & -\dfrac{e^{-\frac{3i\pi\lambda_{22}}{8}}+e^{\frac{5i\pi\lambda_{22}}{8}}}{\sin\left(\frac{(1-\lambda_{22})\pi}{4}\right)} \\[4ex]
\dfrac{e^{-\frac{3i\pi\lambda_{11}}{8}}-e^{\frac{5i\pi\lambda_{11}}{8}}}{\cos\left(\frac{\lambda_{11}\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{12}}{8}}-e^{\frac{5i\pi\lambda_{12}}{8}}}{\cos\left(\frac{\lambda_{12}\pi}{4}\right)} & -\dfrac{e^{-\frac{3i\pi\lambda_{21}}{8}}-e^{\frac{5i\pi\lambda_{21}}{8}}}{\cos\left(\frac{\lambda_{21}\pi}{4}\right)} & -\dfrac{e^{-\frac{3i\pi\lambda_{22}}{8}}-e^{\frac{5i\pi\lambda_{22}}{8}}}{\cos\left(\frac{\lambda_{22}\pi}{4}\right)} \\[4ex]
\dfrac{e^{-\frac{3i\pi\lambda_{11}}{8}}+e^{\frac{5i\pi\lambda_{11}}{8}}}{\sin\left(\frac{(3-\lambda_{11})\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{12}}{8}}+e^{\frac{5i\pi\lambda_{12}}{8}}}{\sin\left(\frac{(3-\lambda_{12})\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{21}}{8}}+e^{\frac{5i\pi\lambda_{21}}{8}}}{\sin\left(\frac{(3-\lambda_{21})\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{22}}{8}}+e^{\frac{5i\pi\lambda_{22}}{8}}}{\sin\left(\frac{(3-\lambda_{22})\pi}{4}\right)} \\[4ex]
\dfrac{e^{-\frac{3i\pi\lambda_{11}}{8}}-e^{\frac{5i\pi\lambda_{11}}{8}}}{\sin\left(\frac{\lambda_{11}\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{12}}{8}}-e^{\frac{5i\pi\lambda_{12}}{8}}}{\sin\left(\frac{\lambda_{12}\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{21}}{8}}-e^{\frac{5i\pi\lambda_{21}}{8}}}{\sin\left(\frac{\lambda_{21}\pi}{4}\right)} & \dfrac{e^{-\frac{3i\pi\lambda_{22}}{8}}-e^{\frac{5i\pi\lambda_{22}}{8}}}{\sin\left(\frac{\lambda_{22}\pi}{4}\right)}
\end{vmatrix}.
$$

By using the identity

$$
\frac{e^{ia} - e^{ib}}{e^{ia} + e^{ib}} = i\tan\left(\frac{a-b}{2}\right),
$$

we have that

$$
|M(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22})| = \frac{M^4 e^{\frac{14i\pi}{8}}}{8^4} \times \left\{ (e^{-\frac{3\lambda_{11}\pi}{8}i} + e^{\frac{5\lambda_{11}\pi}{8}i})(e^{-\frac{3\lambda_{12}\pi}{8}i} + e^{\frac{5\lambda_{12}\pi}{8}i}) \right\} \times
$$
$$
\left\{ (e^{-\frac{3\lambda_{21}\pi}{8}i} + e^{\frac{5\lambda_{21}\pi}{8}i})(e^{-\frac{3\lambda_{22}\pi}{8}i} + e^{\frac{5\lambda_{22}\pi}{8}i}) \right\} |A|,
$$

where A is the matrix

$$A = \begin{bmatrix} \dfrac{1}{\sin\left(\frac{1-\lambda_{11}}{4}\pi\right)} & \dfrac{1}{\sin\left(\frac{1-\lambda_{12}}{4}\pi\right)} & -\dfrac{1}{\sin\left(\frac{1-\lambda_{21}}{4}\pi\right)} & -\dfrac{1}{\sin\left(\frac{1-\lambda_{22}}{4}\pi\right)} \\[2mm] -\dfrac{i\tan(\frac{\lambda_{11}}{2}\pi)}{\cos(\frac{\lambda_{11}}{4}\pi)} & -\dfrac{i\tan(\frac{\lambda_{12}}{2}\pi)}{\cos(\frac{\lambda_{12}}{4}\pi)} & \dfrac{i\tan(\frac{\lambda_{21}}{2}\pi)}{\cos(\frac{\lambda_{21}}{4}\pi)} & \dfrac{i\tan(\frac{\lambda_{22}}{2}\pi)}{\cos(\frac{\lambda_{22}}{4}\pi)} \\[2mm] \dfrac{1}{\sin\left(\frac{3-\lambda_{11}}{4}\pi\right)} & \dfrac{1}{\sin\left(\frac{3-\lambda_{12}}{4}\pi\right)} & \dfrac{1}{\sin\left(3-\frac{\lambda_{21}}{4}\pi\right)} & \dfrac{1}{\sin\left(\frac{3-\lambda_{22}}{4}\pi\right)} \\[2mm] -\dfrac{i\tan(\frac{\lambda_{11}}{2}\pi)}{\sin(\frac{\lambda_{11}}{4}\pi)} & -\dfrac{i\tan(\frac{\lambda_{12}}{2}\pi)}{\sin(\frac{\lambda_{12}}{4}\pi)} & -\dfrac{i\tan(\frac{\lambda_{21}}{2}\pi)}{\sin(\frac{\lambda_{21}}{4}\pi)} & -\dfrac{i\tan(\frac{\lambda_{22}}{2}\pi)}{\sin(\frac{\lambda_{22}}{4}\pi)} \end{bmatrix}.$$

By choosing $\lambda_{11} = \lambda_{21}$ and $\lambda_{12} = \lambda_{22}$ we have that the constant that appears before of the matrix A takes the form:

$$\frac{2M^4 e^{\frac{14}{8}\pi i}(e^{-i\lambda_{11}\pi} + 1)^2(e^{-i\lambda_{12}\pi} + 1)^2 e^{\frac{5}{4}\pi(\lambda_{11}+\lambda_{12})i}}{8^4}.$$

Note that this number is zero only in the case $\lambda_{11} = 2k + 1$ and $\lambda_{12} = 2l + 1$ for $k, l \in \mathbb{Z}$.

Let us denote the entries of the matrix A as follows:

$$A = \begin{bmatrix} a & n & -a & -n \\ f & g & -f & -g \\ c & e & c & e \\ d & m & d & m \end{bmatrix}. \tag{57}$$

Thus, its determinant is given by

$$\det A = 4(de - cm)(nf - ag). \tag{58}$$

In particular for $\lambda_{11} = \lambda_{21} = -\frac{1}{2}$ and $\lambda_{12} = \lambda_{22} = \frac{1}{4}$ matrix A can be seen as follows,

A'

$$= \begin{bmatrix} \sqrt{2}\sqrt{2-\sqrt{2}} & \sqrt{2}\sqrt{4-2\sqrt{2}+\sqrt{2(10-7\sqrt{2})}} & -\sqrt{2}\sqrt{2-\sqrt{2}} & -\sqrt{2}\sqrt{4-2\sqrt{2}+\sqrt{2(10-7\sqrt{2})}} \\[2mm] \frac{2i}{\sqrt{2+\sqrt{2}}} & \frac{i\tan(\pi/8)}{\cos(\pi/16)} & -\frac{2i}{\sqrt{2+\sqrt{2}}} & -\frac{i\tan(\pi/8)}{\cos(\pi/16)} \\[2mm] \sqrt{2}\sqrt{2+\sqrt{2}} & \sqrt{2}\sqrt{4-2\sqrt{2}-\sqrt{2(10-7\sqrt{2})}} & \sqrt{2}\sqrt{2+\sqrt{2}} & \sqrt{2}\sqrt{4-2\sqrt{2}-\sqrt{2(10-7\sqrt{2})}} \\[2mm] -i\sqrt{2(2+\sqrt{2})} & -\frac{i\tan(\pi/8)}{\sin(\frac{\pi}{16})} & -i\sqrt{2(2+\sqrt{2})} & -\frac{i\tan(\pi/8)}{\sin(\frac{\pi}{16})} \end{bmatrix}.$$

By using determinant properties the determinant of A' is equivalent of the determinant of the following matrix:

$$-(4i)(\sqrt{2}i)\begin{bmatrix} \sqrt{2-\sqrt{2}} & \sqrt{4-2\sqrt{2}+\sqrt{2(10-7\sqrt{2})}} & -\sqrt{2-\sqrt{2}} & -\sqrt{4-2\sqrt{2}+\sqrt{2(10-7\sqrt{2})}} \\[2mm] \frac{1}{\sqrt{2+\sqrt{2}}} & \frac{\tan(\pi/8)}{2\cos(\pi/16)} & -\frac{1}{\sqrt{2+\sqrt{2}}} & -\frac{\tan(\pi/8)}{2\cos(\pi/16)} \\[2mm] \sqrt{2+\sqrt{2}} & \sqrt{4-2\sqrt{2}-\sqrt{2(10-7\sqrt{2})}} & \sqrt{2+\sqrt{2}} & \sqrt{4-2\sqrt{2}-\sqrt{2(10-7\sqrt{2})}} \\[2mm] \sqrt{2+\sqrt{2}} & \frac{\tan(\pi/8)}{\sqrt{2}\sin(\frac{\pi}{16})} & \sqrt{2+\sqrt{2}} & \frac{\tan(\pi/8)}{\sqrt{2}\sin(\frac{\pi}{16})} \end{bmatrix}.$$

Therefore, we can rewrite matrix (57) as follows

$$\begin{bmatrix} a & n & -a & -n \\ \frac{1}{c} & g & -\frac{1}{c} & -g \\ c & e & c & e \\ c & m & c & m \end{bmatrix}$$

and its determinant is given by

$$4(e - m)(n - acg).$$

We only need to check that $e - m \neq 0$ and $n - acg \neq 0$. An calculation proves that

$$e - m \sim -0,6508 \quad \text{and} \quad n - acg \sim 0,9741.$$

Thus, we have that

$$det(A') = -4(2i)(\sqrt{2}i)(e - m)(n - acg) \sim -7,1722 \tag{59}$$

that is, the determinant of matrix $\mathbf{M}\left(-\frac{1}{2}, \frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$ given by (55) is nonzero, proving the Claim 1 and Lemma 4.1, for the case $N = 2$.

For a better understanding of the reader, before to do the general case, we will present briefly also the proof of Lemma 4.1 considering $N = 3$. For instance, vertex conditions (4), in this case, are given in the matrices form as follows:

$$
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_1(t,0) \\
u_2(t,0) \\
u_3(t,0)
\end{bmatrix} = 0,
\qquad
\begin{bmatrix}
0 & 0 & 0 \\
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\partial_x u_1(t,0) \\
\partial_x u_2(t,0) \\
\partial_x u_3(t,0)
\end{bmatrix} = 0,
$$

and

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\partial_x^2 u_1(t,0) \\
\partial_x^2 u_2(t,0) \\
\partial_x^2 u_3(t,0)
\end{bmatrix} = 0,
\qquad
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\partial_x^3 u_1(t,0) \\
\partial_x^3 u_2(t,0) \\
\partial_x^3 u_3(t,0)
\end{bmatrix} = 0.
$$

Thus, combining the above matrices and the integral form of solution (37), as in the case $N = 2$, we obtain

$$
\begin{bmatrix}
a_{11} & a_{12} & -a_{21} & -a_{22} & 0 & 0 \\
0 & 0 & a_{21} & a_{22} & -a_{31} & -a_{32} \\
b_{11} & b_{12} & -b_{21} & -b_{22} & 0 & 0 \\
0 & 0 & b_{21} & b_{22} & -b_{31} & -b_{32} \\
c_{11} & c_{12} & c_{21} & c_{22} & c_{31} & c_{32} \\
d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32}
\end{bmatrix}
\begin{bmatrix}
\gamma_{11} \\
\gamma_{12} \\
\gamma_{21} \\
\gamma_{22} \\
\gamma_{31} \\
\gamma_{32}
\end{bmatrix}
$$

$$= - \begin{bmatrix} F_1(t,0) - F_2(t,0) \\ \\ F_2(t,0) - F_3(t,0) \\ \\ \partial_x \mathcal{I}_{\frac{1}{4}} F_1(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) \\ \\ \partial_x \mathcal{I}_{\frac{1}{4}} F_2(t,0) - \partial_x \mathcal{I}_{\frac{1}{4}} F_3(t,0) \\ \\ \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) + \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) + \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_3(t,0) \\ \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) + \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) + \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_3(t,0) \end{bmatrix}.$$

Let us consider M the following matrix

$$M = \begin{bmatrix} a_{11} & a_{12} & -a_{21} & -a_{22} & 0 & 0 \\ \\ 0 & 0 & a_{21} & a_{22} & -a_{31} & -a_{32} \\ \\ b_{11} & b_{12} & -b_{21} & -b_{22} & 0 & 0 \\ \\ 0 & 0 & b_{21} & b_{22} & -b_{31} & -b_{32} \\ \\ c_{11} & c_{12} & c_{21} & c_{22} & c_{31} & c_{32} \\ \\ d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} \end{bmatrix}. \tag{60}$$

Claim 2. *M has determinant different of zero with appropriate choice of λ_{ji}, $j = 1, 2, 3$ and $i = 1, 2$.*

Indeed, similarly as in the case $N = 2$ and by using the identities (56), yields that

$$\begin{cases} a_{ji} = \dfrac{Me^{-\frac{i\pi}{8}}}{8} \left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(1-\lambda_{ji})\pi}{4}\right)} \right), & b_{ji} = \dfrac{Me^{\frac{2i\pi}{8}}}{8} \left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}} - e^{\frac{5i\pi\lambda_{ji}}{8}}}{\cos\left(\frac{\lambda_{ji}\pi}{4}\right)} \right), \\ \\ c_{ji} = \dfrac{Me^{\frac{5i\pi}{8}}}{8} \left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(3-\lambda_{ji})\pi}{4}\right)} \right), & d_{ji} = \dfrac{Me^{\frac{8i\pi}{8}}}{8} \left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}} - e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{\lambda_{ji}\pi}{4}\right)} \right), \end{cases}$$

for $j = 1, 2, 3$ and $i = 1, 2$. By determinant properties, we can get the determinant of M as

$$|M| = \left(\frac{Me^{-\frac{i\pi}{8}}}{8} \right)^2 \left(\frac{Me^{\frac{2i\pi}{8}}}{8} \right)^2 \left(\frac{Me^{\frac{5i\pi}{8}}}{8} \right) \left(\frac{Me^{\frac{8i\pi}{8}}}{8} \right) \prod_{i=1,2, j=1,2,3} \left(e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}} \right) |A|$$

$$= \frac{M^6 e^{\frac{15i\pi}{8}}}{8^6} \prod_{i=1,2, j=1,2,3} \left(e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}} \right) |A|,$$

where A is the following matrix

$$A' = \begin{bmatrix} \frac{1}{\sin(\theta_1)} & \frac{1}{\sin(\theta_2)} & -\frac{1}{\sin(\theta_3)} & -\frac{1}{\sin(\theta_4)} & 0 & 0 \\[2mm] 0 & 0 & \frac{1}{\sin(\theta_3)} & \frac{1}{\sin(\theta_4)} & -\frac{1}{\sin(\theta_5)} & -\frac{1}{\sin(\theta_6)} \\[2mm] -\frac{i\tan(\frac{\lambda_{11}}{2}\pi)}{\cos(\frac{\lambda_{11}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{12}}{2}\pi)}{\cos(\frac{\lambda_{12}}{4}\pi)} & \frac{i\tan(\frac{\lambda_{21}}{2}\pi)}{\cos(\frac{\lambda_{21}}{4}\pi)} & \frac{i\tan(\frac{\lambda_{22}}{2}\pi)}{\cos(\frac{\lambda_{22}}{4}\pi)} & 0 & 0 \\[2mm] 0 & 0 & -\frac{i\tan(\frac{\lambda_{21}}{2}\pi)}{\cos(\frac{\lambda_{21}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{22}}{2}\pi)}{\cos(\frac{\lambda_{22}}{4}\pi)} & \frac{i\tan(\frac{\lambda_{31}}{2}\pi)}{\cos(\frac{\lambda_{31}}{4}\pi)} & \frac{i\tan(\frac{\lambda_{32}}{2}\pi)}{\cos(\frac{\lambda_{32}}{4}\pi)} \\[2mm] \frac{1}{\sin(\beta_1)} & \frac{1}{\sin(\beta_2)} & \frac{1}{\sin(\beta_3)} & \frac{1}{\sin(\beta_4)} & \frac{1}{\sin(\beta_5)} & \frac{1}{\sin(\beta_6)} \\[2mm] -\frac{i\tan(\frac{\lambda_{11}}{2}\pi)}{\sin(\frac{\lambda_{11}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{12}}{2}\pi)}{\sin(\frac{\lambda_{12}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{21}}{2}\pi)}{\sin(\frac{\lambda_{21}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{22}}{2}\pi)}{\sin(\frac{\lambda_{22}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{31}}{2}\pi)}{\sin(\frac{\lambda_{31}}{4}\pi)} & -\frac{i\tan(\frac{\lambda_{32}}{2}\pi)}{\sin(\frac{\lambda_{32}}{4}\pi)} \end{bmatrix},$$

where

$$\theta_1 = \frac{1-\lambda_{11}}{4}\pi, \ \theta_2 = \frac{1-\lambda_{12}}{4}\pi, \ \theta_3 = \frac{1-\lambda_{21}}{4}\pi,$$

$$\theta_4 = \frac{1-\lambda_{22}}{4}\pi, \ \theta_5 = \frac{1-\lambda_{31}}{4}\pi, \ \theta_6 = \frac{1-\lambda_{32}}{4}\pi,$$

$$\beta_1 = \frac{3-\lambda_{11}}{4}\pi, \ \beta_2 = \frac{3-\lambda_{12}}{4}\pi, \ \beta_3 = \frac{3-\lambda_{21}}{4}\pi,$$

and

$$\beta_4 = \frac{3-\lambda_{22}}{4}\pi, \ \beta_5 = \frac{3-\lambda_{31}}{4}\pi, \ \beta_6 = \frac{3-\lambda_{32}}{4}\pi.$$

By choosing $\lambda_{11} = \lambda_{21} = \lambda_{31}$ and $\lambda_{12} = \lambda_{22} = \lambda_{32}$, we have that the constant that appears before of the matrix A takes the form:

$$\frac{M^6 e^{\frac{15i\pi}{8}}}{8^6}(e^{-i\pi\lambda_{11}} + 1)^3(e^{-i\pi\lambda_{12}} + 1)^3 e^{\frac{15}{4}\pi(\lambda_{11}+\lambda_{12})i}.$$

Note that this number is zero only in the case $\lambda_{11} = 2n + 1$ and $\lambda_{12} = 2m + 1$ for $n, m \in \mathbb{Z}$. Let us rewrite the entries of matrix A as follows:

$$A = \begin{bmatrix} a & n & -a & -n & 0 & 0 \\ 0 & 0 & a & n & -a & -n \\ f & g & -f & -g & 0 & 0 \\ 0 & 0 & f & g & -f & -g \\ c & e & c & e & c & e \\ d & m & d & m & d & m \end{bmatrix}.$$

Thus, its determinant is given by

$$|A'| = 9(de - cm)(ag - nf)^2.$$

Finally, considering $\lambda_{11} = \lambda_{21} = \lambda_{31} = -\frac{1}{2}$ and $\lambda_{12} = \lambda_{22} = \lambda_{32} = \frac{1}{4}$, thanks to the case $N = 2$, we have that $(ag - fn) \neq 0$ and $(de - cm) \neq 0$, thus $|A| \neq 0$. Claim 2 is thus proved and Lemma 4.1 is achieved, when $N = 3$.

Let us now deal with the general situation, that is, when $N > 3$. Consider

$$\mathbf{M} = \mathbf{M}(\lambda_{11}, \lambda_{12}, \cdots, \lambda_{N1}, \lambda_{N2})$$

defined by (42), namely,

$$\mathbf{M}_{2N\times 2N} =$$

$$\left[\begin{array}{ccccccccccc}
a_{11} & a_{12} & -a_{21} & -a_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{11} & a_{12} & -a_{21} & -a_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_{(N-1)1} & a_{(N-1)2} & -a_{N1} & -a_{N2} \\
b_{11} & b_{12} & -b_{21} & -b_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & b_{11} & b_{12} & -b_{21} & -b_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & b_{(N-1)1} & b_{(N-1)2} & -b_{N1} & -b_{N2} \\
c_{11} & c_{12} & c_{21} & c_{22} & c_{31} & c_{32} & \cdots & \cdots & \cdots & c_{N1} & c_{N2} \\
d_{11} & d_{12} & d_{21} & d_{22} & d_{31} & d_{32} & \cdots & \cdots & \cdots & d_{N1} & d_{N2}
\end{array}\right],$$

where a_{ij}, b_{ij}, c_{ij} and d_{ij} are given by (38), (39), (40) and (41), respectively. As we noted in the cases $N = 2, 3$, this function of λ_{ji} can be take the form

$$\begin{cases}
a_{ji} = \dfrac{Me^{-\frac{i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(1-\lambda_{ji})\pi}{4}\right)}\right), \quad b_{ji} = \dfrac{Me^{\frac{2i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}-e^{\frac{5i\pi\lambda_{ji}}{8}}}{\cos\left(\frac{\lambda_{ji}\pi}{4}\right)}\right), \\[4mm]
c_{ji} = \dfrac{Me^{\frac{5i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{(3-\lambda_{ji})\pi}{4}\right)}\right), \quad d_{ji} = \dfrac{Me^{\frac{8i\pi}{8}}}{8}\left(\dfrac{e^{-\frac{3i\pi\lambda_{ji}}{8}}-e^{\frac{5i\pi\lambda_{ji}}{8}}}{\sin\left(\frac{\lambda_{ji}\pi}{4}\right)}\right),
\end{cases}$$

for $j = 1, 2, ..., N$ and $i = 1, 2$. Thus, by using the determinant properties, we have that the determinant $|\mathbf{M}|$ is

$$\left(\frac{Me^{-\frac{i\pi}{8}}}{8}\right)^{N-1}\left(\frac{Me^{\frac{2i\pi}{8}}}{8}\right)^{N-1}\left(\frac{Me^{\frac{5i\pi}{8}}}{8}\right)\left(\frac{Me^{\frac{8i\pi}{8}}}{8}\right)\prod_{i=1,2,j=1,\cdots,N}\left(e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}\right)|\mathbf{M}'|$$

$$= \frac{M^{2N}e^{\frac{(12+N)i\pi}{8}}}{8^{2N}}\prod_{i=1,2,j=1,\cdots,N}\left(e^{-\frac{3i\pi\lambda_{ji}}{8}}+e^{\frac{5i\pi\lambda_{ji}}{8}}\right)|\mathbf{M}'|,$$

where M' is a matrix, depending only of λ_{ji}, given by

$$\left[\begin{array}{ccccccccccc}
\bar{a}_{11} & \bar{a}_{12} & -\bar{a}_{21} & -\bar{a}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{a}_{11} & \bar{a}_{12} & -\bar{a}_{21} & -\bar{a}_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \bar{a}_{(N-1)1} & \bar{a}_{(N-1)2} & -\bar{a}_{N1} & -\bar{a}_{N2} \\
\bar{b}_{11} & \bar{b}_{12} & -\bar{b}_{21} & -\bar{b}_{22} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{b}_{11} & \bar{b}_{12} & -\bar{b}_{21} & -\bar{b}_{22} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \bar{b}_{(N-1)1} & \bar{b}_{(N-1)2} & -\bar{b}_{N1} & -\bar{b}_{N2} \\
\bar{c}_{11} & \bar{c}_{12} & \bar{c}_{21} & \bar{c}_{22} & \bar{c}_{31} & \bar{c}_{32} & \cdots & \cdots & \cdots & \bar{c}_{N1} & \bar{c}_{N2} \\
\bar{d}_{11} & \bar{d}_{12} & \bar{d}_{21} & \bar{d}_{22} & \bar{d}_{31} & \bar{d}_{32} & \cdots & \cdots & \cdots & \bar{d}_{N1} & \bar{d}_{N2}
\end{array}\right]_{2N\times 2N}.$$

Here, the coefficients of matrix $\mathbf{M}'$ are given by

$$\begin{cases}
\bar{a}_{ji} = \dfrac{1}{\sin\left(\frac{(1-\lambda_{ji})\pi}{4}\right)}, \quad \bar{b}_{ji} = -\dfrac{i\tan\left(\frac{\lambda_{ji}\pi}{2}\right)}{\cos\left(\frac{\lambda_{ji}\pi}{4}\right)}, \\[5mm]
\bar{c}_{ji} = \dfrac{1}{\sin\left(\frac{(3-\lambda_{ji})\pi}{4}\right)}, \quad \bar{d}_{ji} = -\dfrac{i\tan\left(\frac{\lambda_{ji}\pi}{2}\right)}{\sin\left(\frac{\lambda_{ji}\pi}{4}\right)},
\end{cases} \tag{61}$$

for $j = 1, 2, ..., N$ and $i = 1, 2$. By choosing $\lambda_{11} = \lambda_{21} = \cdots = \lambda_{N1}$ and $\lambda_{12} = \lambda_{22} = \cdots = \lambda_{N2}$, we have that the constant that appears before of the matrix $\mathbf{M}'$ takes

the form:

$$\frac{M^{2N}e^{\frac{(12+N)i\pi}{8}}}{8^{2N}}(e^{-i\pi\lambda_{11}}+1)^N(e^{-i\pi\lambda_{12}}+1)^N e^{\frac{5N}{4}\pi(\lambda_{11}+\lambda_{12})i}.$$

Note that this number is zero only in the case $\lambda_{11} = 2n+1$ and $\lambda_{12} = 2m+1$ for $n, m \in \mathbb{Z}$. Let us denote the entries of matrix $\mathbf{M}'$ as follows:

$$\mathbf{M}' = \begin{bmatrix} a & n & -a & -n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & n & -a & -n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & n & -a & -n \\ f & g & -f & -g & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & f & g & -f & -g & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & f & g & -f & -g \\ c & e & c & e & c & e & \cdots & \cdots & \cdots & c & e \\ d & m & d & m & d & m & \cdots & \cdots & \cdots & d & m \end{bmatrix}_{2N\times 2N}.$$

Moreover, by using the determinant properties, it yields that

$$|\mathbf{M}'| = \begin{vmatrix} a & n & -a & -n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ f & g & -f & -g & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & n & -a & -n & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & f & g & -f & -g & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & n & -a & -n \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & f & g & -f & -g \\ c & e & c & e & c & e & \cdots & \cdots & \cdots & c & e \\ d & m & d & m & d & m & \cdots & \cdots & \cdots & d & m \end{vmatrix}_{2N\times 2N}.$$

Considering the matrix

$$A = \begin{bmatrix} a & n \\ f & g \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & e \\ d & m \end{bmatrix},$$

the determinant of $\mathbf{M}'$ can be write as a block matrices, namely,

$$|\mathbf{M}'| = \begin{vmatrix} A_{2\times 2} & -A_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & A_{2\times 2} & -A_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \cdots & A_{2\times 2} & -A_{2\times 2} \\ B_{2\times 2} & B_{2\times 2} & B_{2\times 2} & \cdots & B_{2\times 2} & B_{2\times 2} \end{vmatrix}_{2N\times 2N}. \tag{62}$$

From now on, we denote $0_{n\times n}$ and $I_{n\times n}$ the null and identity matrices, respectively.

Let us introduce the properties of determinants that helped us to prove Lemma 4.1 in general form. Consider a block matrix N of size $(n+m) \times (n+m)$ of the form

$$N = \begin{bmatrix} C & D \\ F & G \end{bmatrix},$$

where C, D, F and G are of size $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively. If G is invertible, then

$$\det N = \det(C - DG^{-1}F)\det(G). \tag{63}$$

In fact, this property follows immediately from the following identity

$$\begin{bmatrix} C & D \\ F & G \end{bmatrix} \begin{bmatrix} I & 0 \\ -G^{-1}F & I \end{bmatrix} = \begin{bmatrix} C - DG^{-1}F & D \\ 0 & G \end{bmatrix}.$$

Finally, recall that the determinant of a block triangular matrix is the product of the determinants of its diagonal blocks.

With these two properties in hand, define C, D, F and G, respectively, by

$$C = \begin{bmatrix} A_{2\times 2} & -A_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & A_{2\times 2} & -A_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & 0_{2\times 2} & \cdots & A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2(N-2)}, \quad D = \begin{bmatrix} 0_{2\times 2} \\ 0_{2\times 2} \\ \vdots \\ -A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2}$$

and

$$F = \begin{bmatrix} B_{2\times 2} & B_{2\times 2} & B_{2\times 2} & \cdots & B_{2\times 2} \end{bmatrix}_{2\times 2(N-2)}, \quad G = B_{2\times 2}.$$

Thanks to the case $N = 2$, we already know that

$$\det G = \det B_{2\times 2} = cm - de \neq 0, \tag{64}$$

which implies that G is invertible. Thus, the determinant (62) takes the form

$$|\mathbf{M}'| = \begin{vmatrix} C_{2(N-2)\times 2(N-2)} & D_{2\times 2(N-2)} \\ F_{2\times 2(N-2)} & B_{2\times 2} \end{vmatrix}_{2N\times 2N}$$

and by using the property (63), it yields that

$$\det \mathbf{M}' = \det\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1} F_{2\times 2(N-2)}\right) \det B_{2\times 2}. \tag{65}$$

Claim 3. $\mathbf{M}'$ *has determinant different of zero with appropriate choice of* λ_{ji}, $j = 1, 2, ..., N$ *and* $i = 1, 2.$

From (64) is enough to prove that

$$\det\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1} F_{2\times 2(N-2)}\right)$$

is nonzero. In order to analyze the above determinant, note that

$$\left(B_{2\times 2}\right)^{-1} F_{2\times 2(N-2)} = \left(B_{2\times 2}\right)^{-1} \begin{bmatrix} B_{2\times 2} & B_{2\times 2} & B_{2\times 2} & \cdots & B_{2\times 2} \end{bmatrix}_{2\times 2(N-2)}$$

$$= \begin{bmatrix} I_{2\times 2} & I_{2\times 2} & I_{2\times 2} & \cdots & I_{2\times 2} \end{bmatrix}_{2\times 2(N-2)}$$

and

$$D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}$$

$$=\begin{bmatrix} 0_{2\times 2} \\ 0_{2\times 2} \\ \vdots \\ -A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2}\begin{bmatrix} I_{2\times 2} & I_{2\times 2} & I_{2\times 2} & \cdots & I_{2\times 2} \end{bmatrix}_{2\times 2(N-2)}$$

$$=\begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \cdots & \vdots \\ 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ -A_{2\times 2} & -A_{2\times 2} & \cdots & -A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2(N-2)}.$$

Therefore, we get

$$C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}$$

$$=\begin{bmatrix} A_{2\times 2} & -A_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & A_{2\times 2} & -A_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{2\times 2} & A_{2\times 2} & A_{2\times 2} & \cdots & 2A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2(N-2)}.$$

Then,

$$C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}$$

only depends of $A_{2\times 2}$ and, consequently, if

$$\det\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}\right) = 0,$$

we have that

$$\dim \operatorname{Ker}\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}\right) > 0. \qquad (66)$$

Thus, (66) implies that there exists a vector

$$X_{2(N-2)\times 1} \in \operatorname{Ker}\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}\right)$$

such that

$$X_{2(N-2)\times 1} = (x_1, x_2, x_3, x_4 \cdots, x_{2(N-3)}, x_{2(N-2)})^T \neq 0_{2(N-2)\times 1} \qquad (67)$$

and

$$\left(C_{2(N-2)\times 2(N-2)} - D_{2\times 2(N-2)}\left(B_{2\times 2}\right)^{-1}F_{2\times 2(N-2)}\right)\cdot X_{2(N-2)\times 1} = 0_{2(N-2)\times 1},$$

or equivalent,

$$\begin{bmatrix} A_{2\times 2} & -A_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & A_{2\times 2} & -A_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{2\times 2} & A_{2\times 2} & A_{2\times 2} & \cdots & 2A_{2\times 2} \end{bmatrix}_{2(N-2)\times 2(N-2)}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{2(N-3)} \\ x_{2(N-2)} \end{bmatrix}_{2(N-2)\times 1} = 0_{2(N-2)\times 1}.$$

$$(68)$$

To finalize the proof of the Claim 3, denote

$$H_{2\times1}^1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad H_{2\times1}^2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \quad ..., \quad H_{2\times1}^{(N-2)} = \begin{bmatrix} x_{(2(N-3))} \\ x_{(2(N-2))} \end{bmatrix}.$$

Therefore, the product (68) can be write in the form

$$\begin{bmatrix} A_{2\times2} & -A_{2\times2} & 0_{2\times2} & \cdots & 0_{2\times2} \\ 0_{2\times2} & A_{2\times2} & -A_{2\times2} & \cdots & 0_{2\times2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{2\times2} & A_{2\times2} & A_{2\times2} & \cdots & 2A_{2\times2} \end{bmatrix}_{2(N-2)\times2(N-2)} \begin{bmatrix} H_{2\times1}^1 \\ H_{2\times1}^2 \\ \vdots \\ H_{2\times1}^{(N-2)} \end{bmatrix}_{2(N-2)\times1} = 0_{2(N-2)\times1}.$$

Thus, we have that

$$\begin{bmatrix} A_{2\times2}\left(H_{2\times1}^1 - H_{2\times1}^2\right) \\ \\ A_{2\times2}\left(H_{2\times1}^2 - H_{2\times1}^3\right) \\ \\ \vdots \\ \\ A_{2\times2}\left(H_{2\times1}^{(N-1)} - H_{2\times1}^{(N-2)}\right) \\ \\ A_{2\times2}\left(H_{2\times1}^1 + H_{2\times1}^2 + \cdots + 2H_{2\times1}^{(N-2)}\right) \end{bmatrix}_{2(N-2)\times1} = 0_{2(N-2)\times1}.$$

Now, let us now argue by contradiction. If there exists $k \in \{1, 2, \cdots N-1\}$ such that

$$H_{2\times1}^k - H_{2\times1}^{(k+1)} \neq 0_{2\times1},$$

we obtain

$$A_{2\times2}\left(H_{2\times1}^k - H_{2\times1}^{(k+1)}\right) = 0_{2\times1},$$

which implies that $\dim \ker A_{2\times2} > 0$, it means that $\det A_{2\times2} = 0$. However, from the case $N = 2$, we known that

$$\det A_{2\times2} = ag - fn \neq 0,$$

and hence we obtain a contradiction. On the other hand, suppose that

$$H_{2\times1}^j - H_{2\times1}^{(j+1)} = 0_{2\times1}, \quad \forall j = 1, 2, ..., N-2. \tag{69}$$

Thus, from (67) and (69), we deduce that $H_{2\times1}^j \neq 0_{2\times1}$ for some $j \in \{1, 2, ..., N-2\}$ and

$$A_{2\times2}\left(H_{2\times1}^1 + H_{2\times1}^2 + \cdots + 2H_{2\times1}^{(N-2)}\right) = (N-1)A_{2\times2}H_{2\times1}^j = 0_{2(N-2)\times1}.$$

Which is again a contradiction, by using the case $N = 2$. Hence, in the two cases, we only have that

$$\det\left(C_{2(N-2)\times2(N-2)} - D_{2\times2(N-2)}\left(B_{2\times2}\right)^{-1} F_{2\times2(N-2)}\right) \neq 0,$$

it implies that $\det \mathbf{M}' \neq 0$. Consequently, the determinant of $\mathbf{M}$ is nonzero, implying that the matrix $\mathbf{M}$ is invertible and the Claim 3 follows. Thus, Lemma 4.1 is proved. □

Appendix A. **Vertex conditions types $\mathcal{B}$ and $\mathcal{C}$.** In this appendix, we will outline how to prove that matrices associated with vertex conditions (5) (type $\mathcal{B}$) and (6) (type $\mathcal{C}$) are invertible. We will consider the vertex conditions

Type $\mathcal{B}$:
$$\begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0) = \cdots = \partial_x^k u_N(t,0), \ k = 2,3 & t \in (0,T), \\ \sum_{j=1}^N \partial_x^k u_j(t,0) = 0, \ k = 0,1 & t \in (0,T), \end{cases}$$

and

Type $\mathcal{C}$:
$$\begin{cases} \partial_x^k u_1(t,0) = \partial_x^k u_2(t,0) = \cdots = \partial_x^k u_N(t,0), \ k = 0,3 & t \in (0,T), \\ \sum_{j=1}^N \partial_x^k u_j(t,0) = 0, \ k = 1,2 & t \in (0,T), \end{cases}$$

which may be expressed in matrices form as follows

$$\mathbf{M}_{\mathcal{B}} \ \boldsymbol{\gamma_{\mathcal{B}}} = \mathbf{F}_{\mathcal{B}}, \quad \text{and} \quad \mathbf{M}_{\mathcal{C}} \ \boldsymbol{\gamma_{\mathcal{C}}} = \mathbf{F}_{\mathcal{C}},$$

respectively. Here $\mathbf{F}_{\mathcal{B}}$ and $\mathbf{F}_{\mathcal{C}}$ are the function vectors defined by

$$\mathbf{F}_{\mathcal{B}} := - \begin{bmatrix} \sum_{j=1}^N F_j(t,0) \\ \sum_{j=1}^N \partial_x \mathcal{I}_{\frac{1}{4}} F_j(t,0) \\ \\ \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_1(t,0) - \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_2(t,0) \\ \vdots \\ \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_{N-1}(t,0) - \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_N(t,0) \\ \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) - \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) \\ \vdots \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_{N-1}(t,0) - \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_N(t,0) \end{bmatrix}_{2N \times 1}$$

and

$$\mathbf{F}_{\mathcal{C}} := - \begin{bmatrix} F_1(t,0) - F_2(t,0) \\ \vdots \\ F_{N-1}(t,0) - F_N(t,0) \\ \\ \sum_{j=1}^N \partial_x \mathcal{I}_{\frac{1}{4}} F_j(t,0) \\ \sum_{j=1}^N \partial_x^2 \mathcal{I}_{\frac{1}{2}} F_j(t,0) \\ \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_1(t,0) - \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_2(t,0) \\ \vdots \\ \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_{N-1}(t,0) - \partial_x^3 \mathcal{I}_{\frac{3}{4}} F_N(t,0) \end{bmatrix}_{2N \times 1},$$

$\boldsymbol{\gamma_{\mathcal{B}}}$ and $\boldsymbol{\gamma_{\mathcal{C}}}$ are the matrices column given by vectors $(\gamma_{11}^{\mathcal{B}}, \gamma_{12}^{\mathcal{B}}, \cdots, \gamma_{N1}^{\mathcal{B}}, \gamma_{N2}^{\mathcal{B}})$ and $(\gamma_{11}^{\mathcal{C}}, \gamma_{12}^{\mathcal{C}}, \cdots, \gamma_{N1}^{\mathcal{C}}, \gamma_{N2}^{\mathcal{C}})$, respectively.

Note that choosing $\lambda_{11} = \lambda_{21} = \cdots = \lambda_{N1}$ and $\lambda_{12} = \lambda_{22} = \cdots = \lambda_{N2}$ and arguing as in the Section 5, the determinants of the matrices

$$\mathbf{M}_{\mathcal{B}} = \mathbf{M}_{\mathcal{B}}(-1/2, 1/4, \cdots, -1/2, 1/4) \quad \text{and} \quad \mathbf{M}_{\mathcal{C}} = \mathbf{M}_{\mathcal{C}}(-1/2, 1/4, \cdots, -1/2, 1/4)$$

are given by

$$|\mathbf{M}_{\mathcal{B}}| = \frac{M^{2N} e^{\frac{(12+N)i\pi}{8}}}{8^{2N}} \prod_{i=1,2,j=1,\cdots,N} \left(e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}} \right) |\mathbf{M}'_{\mathcal{B}}|,$$

and

$$|\mathbf{M}_{\mathcal{C}}| = \frac{M^{2N} e^{\frac{(12+N)i\pi}{8}}}{8^{2N}} \prod_{i=1,2,j=1,\cdots,N} \left(e^{-\frac{3i\pi\lambda_{ji}}{8}} + e^{\frac{5i\pi\lambda_{ji}}{8}} \right) |\mathbf{M}'_{\mathcal{C}}|,$$

where

$$|\mathbf{M}'_{\mathcal{B}}| = \begin{vmatrix}
a & n & a & n & a & n & \cdots & a & n & a & n \\
f & g & f & g & f & g & \cdots & f & g & f & g \\
c & e & -c & -e & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c & e & -c & -e & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & e & -c & -e \\
d & m & -d & -m & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & d & m & -d & -m & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & m & -d & -m
\end{vmatrix}_{2N \times 2N},$$

and

$$|\mathbf{M}'_{\mathcal{C}}| = \begin{vmatrix}
a & n & -a & -n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a & n & -a & -n & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & a & n & -a & -n \\
f & g & f & g & f & g & \cdots & f & g & f & g \\
c & e & c & e & c & e & \cdots & c & e & c & e \\
d & m & -d & -m & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & d & m & -d & -m & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & m & -d & -m
\end{vmatrix}_{2N \times 2N}.$$

As in the case vertex type $\mathcal{A}$, we need to study the determinants of the matrices $\mathbf{M}'_{\mathcal{B}}$ and $\mathbf{M}'_{\mathcal{C}}$. In order to see the determinant of $\mathbf{M}'_{\mathcal{B}}$ and $\mathbf{M}'_{\mathcal{C}}$ are nonzero, we use the determinant properties together with (63) to observe that

$$\det \mathbf{M}'_{\mathcal{B}} = \det \mathbf{M}''_{\mathcal{B}} \det \begin{bmatrix} a & n \\ f & g \end{bmatrix} \quad \text{and} \quad \det \mathbf{M}'_{\mathcal{C}} = \det \mathbf{M}''_{\mathcal{C}} \det \begin{bmatrix} c & e \\ f & g \end{bmatrix}.$$

So, these two matrices, namely, $\mathbf{M}'_{\mathcal{B}}$ and $\mathbf{M}'_{\mathcal{C}}$, have the following two following properties:

(i) If $det \begin{bmatrix} c & e \\ d & m \end{bmatrix} = cm - de \neq 0$, then $\det \mathbf{M}''_{\mathcal{B}} \neq 0$.

(ii) If $det \begin{bmatrix} a & n \\ d & m \end{bmatrix} = am - dn \neq 0$, then $\det \mathbf{M}''_{\mathcal{C}} \neq 0$.

Claim 4. *The relations*

$$ag - fn \neq, \quad cg - fe \neq 0$$

and

$$cm - de \neq 0, \quad am - dn \neq 0$$

are valid.

In fact, choosing

$$\lambda_{11} = \lambda_{21} = \cdots = \lambda_{N1} = -\frac{1}{2} \quad \text{and} \quad \lambda_{11} = \lambda_{21} = \cdots = \lambda_{N1} = -\frac{1}{4},$$

together with (58), (59) and (61), we already now that $ag - fn$ and $cm - de$ are nonzero. Finally, easy calculations yield that

$$cg - fe = \left(\frac{1}{\sin\left(\frac{7}{8}\pi\right)} \right) \left(-\frac{\tan\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{16}\right)} \right) - \left(\frac{1}{\sin\left(\frac{11}{16}\pi\right)} \right) \left(-\frac{\tan\left(-\frac{\pi}{4}\right)}{\cos\left(-\frac{\pi}{8}\right)} \right)$$
$$\sim -2.4053 \neq 0,$$

$$am - dn = \left(\frac{1}{\sin\left(\frac{3}{8}\pi\right)} \right) \left(-\frac{\tan\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{16}\right)} \right) - \left(\frac{1}{\sin\left(\frac{5}{16}\pi\right)} \right) \left(-\frac{\tan\left(-\frac{\pi}{4}\right)}{\sin\left(-\frac{\pi}{8}\right)} \right)$$
$$\sim 0.8446 \neq 0,$$

showing the Claim 4, and thus the matrices $\mathbf{M}_{\mathcal{B}}$ and $\mathbf{M}_{\mathcal{C}}$ are invertible.

A.1. **Proof of Theorem 1.1: Vertex conditions type $\mathcal{B}$ and $\mathcal{C}$.** The analysis developed above give us the following representations for $\boldsymbol{\gamma_{\mathcal{B}}}$ and $\boldsymbol{\gamma_{\mathcal{C}}}$

$$\boldsymbol{\gamma_{\mathcal{B}}} = \mathbf{M}_{\mathcal{B}}{}^{-1} \mathbf{F}_{\mathcal{B}} \quad \text{and} \quad \boldsymbol{\gamma_{\mathcal{C}}} = \mathbf{M}_{\mathcal{C}}{}^{-1} \mathbf{F}_{\mathcal{C}},$$

respectively. Therefore, the solution $u_j(t, x)$ of the Cauchy problem (34) with vertex conditions type $\mathcal{B}$ and $\mathcal{C}$ can be express in a integral forms

$$u_j^{\mathcal{B}}(t, x) = \mathcal{L}^{-\frac{1}{2}} \gamma_{j1}^{\mathcal{B}}(t, x) + \mathcal{L}^{\frac{1}{4}} \gamma_{j2}^{\mathcal{B}}(t, x) + F_j(t, x), \quad j = 1, 2, ..., N \qquad (70)$$

and

$$u_j^{\mathcal{C}}(t, x) = \mathcal{L}^{-\frac{1}{2}} \gamma_{j1}^{\mathcal{C}}(t, x) + \mathcal{L}^{\frac{1}{4}} \gamma_{j2}^{\mathcal{C}}(t, x) + F_j(t, x), \quad j = 1, 2, ..., N. \qquad (71)$$

Finally, in order to establish Theorem 1.1 with boundary conditions type $\mathcal{B}$ and $\mathcal{C}$, we closely follow the same steps of subsection 4.3, 4.4 and 4.5, for use the Fourier restriction method to define a truncated version for (70) and (71), proving thus that $\mathcal{L}^{-\frac{1}{2}}\gamma_{j1}^{\mathcal{B}}$, $\mathcal{L}^{-\frac{1}{2}}\gamma_{j1}^{\mathcal{C}}$, $\mathcal{L}^{\frac{1}{4}}\gamma_{j2}^{\mathcal{B}}$ and $\mathcal{L}^{-\frac{1}{2}}\gamma_{j1}^{\mathcal{C}}$ for $j = 1, 2, ..., N$, are well-defined. With this in hand, a contraction mapping argument gives us the result desired. □

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