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We deal with the initial-boundary value problem of the biharmonic cubic nonlinear Schrödinger equation in a quarter plane with inhomogeneous Dirichlet–Neumann boundary data. We prove local well-posedness in the low regularity Sobolev spaces by introducing Duhamel boundary forcing operator associated to the linear equation in order to construct solutions in the whole line. With this in hand, the energy and nonlinear estimates allow us to apply the Fourier restriction method, introduced by J. Bourgain, to obtain our main result. Additionally, we discuss adaptations of this approach for the biharmonic cubic nonlinear Schrödinger equation on star graphs.

1. Introduction

1A. ***Presentation of the model.*** The fourth-order nonlinear Schrödinger (4NLS) equation or biharmonic cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, \tag{1-1}$$

was introduced in [Karpman 1996; Karpman and Shagalov 2000] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1-1) arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references (see [Ben-Artzi et al. 2000; Cui and Guo 2007; Karpman 1996; Pausader 2007; 2009a]).

The past twenty years such 4NLS equations have been deeply studied from different mathematical viewpoints. For example, Fibich et al. [2002] worked on various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. The well-posedness and existence of solutions for different domains have been shown (see, for instance, [Capistrano-Filho et al. 2019; Kwak 2018; Özsarı and Yolcu 2019; Pausader 2007;

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2009a; Tsutsumi 2014; Oh and Tzvetkov 2017; Wen et al. 2014]) by means of the Fourier restriction method, energy method, forcing boundary operators, Laplace transform, harmonic analysis, Fokas method, etc.

It is interesting to point out that there are many works related to (1-1) not only dealing with well-posedness theory. For example, Natali and Pastor [2015] considered the fourth-order dispersive cubic nonlinear Schrödinger equation on the line with mixed dispersion. They proved the orbital stability, in the $H^2(\mathbb{R})$-energy space, by constructing a suitable Lyapunov function. Considering (1-1) on the circle, Oh and Tzvetkov [2017] showed that the mean-zero Gaussian measures on Sobolev spaces $H^s(\mathbb{T})$, for $s > \frac{3}{4}$, are quasi-invariant under the flow. There has been significant progress over recent years; see for instance [Burq et al. 2002; 2013] for the nonlinear Schrödinger equation.

In addition to these works, two of us worked recently with the intent of proving controllability results for the 4NLS equation. More precisely, we proved that the solutions of the associated linear system (1-1) is globally exponentially stable in a periodic domain $\mathbb{T}$, by using certain properties of propagation of compactness and regularity in Bourgain spaces. Theses properties together with the local exact controllability ensure that fourth order nonlinear Schrödinger is globally exactly controllable; for details, see [Capistrano-Filho and Cavalcante 2019].

Özsarı and Yolcu [2019] proposed (1-1) without the term $\partial_x^2 u$. This system has an interesting physical point of view, precisely, the model corresponds to a situation in which wave is generated from a fixed source such that it moves into the medium in one specific direction.

1B. ***Setting of the problem.*** We mainly consider the biharmonic Schrödinger equation on the right half-line

$$
\begin{cases} i\partial_t u - \partial_x^4 u + \lambda|u|^2 u = 0, & (t,x) \in (0,T) \times (0,\infty), \\ u(0,x) = u_0(x), & x \in (0,\infty), \\ u(t,0) = f(t),\ u_x(t,0) = g(t), & t \in (0,T). \end{cases} \tag{1-2}
$$

With suitable choices of $f(t)$ and $g(t)$ in (1-2), we are interested on the following initial-boundary value problem (IBVP):

Is the IBVP (1-2) *local well-posed in the low regularity Sobolev space, more precisely, in $H^s(\mathbb{R}^+)$ for $0 \le s < \frac{1}{2}$?*

Before presenting the answer for this question, let us present some brief comments on the techniques to solve IBVPs on the half-line.

1C. ***Comments about the techniques to solve IBVPs on the half-line.*** Different techniques have been developed in the last years in order to solve IBVPs associated to some dispersive models on the half-line. Fokas [2008] introduced an approach to solve IBVPs associated to integrable nonlinear evolution equations, which is known as the unified transform method (UTM) or as Fokas transform method. The UTM provides a generalization of the inverse scattering transform method from initial value problems (IVP) to IBVPs. The classical method based on the Laplace transform was used successfully in [Bona et al. 2006; 2018; Erdoğan and Tzirakis 2017; Compaan and Tzirakis 2017]. A new approach was introduced by Colliander and Kenig [2002] by recasting the IBVP on the half-line by a forced IVP defined in the line $\mathbb{R}$. To see other applications of this technique, we refer the results established in [Cavalcante 2017; Cavalcante and Corcho 2019; Holmer 2005; 2006]. On the other hand, Faminskii [2019] used an approach based on the investigation of special solutions of a "boundary potential" type for solution of linearized Korteweg–de Vries (KdV) equation in order to obtain global results for the IBVP associated to the KdV equation on the half-line with more general boundary conditions. Fokas et al. [2016] introduced a method which combines the UTM with a contraction mapping principle. We caution that this is only a small sample of the extant works on these techniques.

1D. ***Biharmonic NLS equation.*** As mentioned in the beginning of this introduction, the 4NLS equation or biharmonic NLS equation

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, \tag{1-3}$$

was introduced in [Karpman 1996; Karpman and Shagalov 2000]. Huo and Jia [2005] studied the Cauchy problem of one-dimensional fourth-order nonlinear Schrödinger equation related to the vortex filament. They proved the local well-posedness for initial data in $H^s(\mathbb{R})$ for $s \geq \frac{1}{2}$ by using the Fourier restriction norm method under certain coefficient condition. Concerning local well-posedness of the nonlinear fourth order Schrödinger equations, we cite [Hao et al. 2006; Segata 2004]. With respect of the global well-posedness, in the one-dimensional case with some restriction in the initial data for various nonlinearities, we refer to [Hayashi and Naumkin 2015a; 2015b; 2015c; 2015d] and, finally, for the study n-dimensional case the reader can see [Pausader 2009b; Pausader and Shao 2010].

Lastly, in a recent work of IBVP for biharmonic Schrödinger equation on the half-line

$$i\partial_t u + \partial_x^4 u = \lambda |u|^p u, \tag{1-4}$$

Özsarı and Yolcu [2019], proved local well-posedness on the high regularity function spaces $H^s(\mathbb{R}^+)$, for $\frac{1}{2} < s < \frac{9}{2}$, with $s \neq \frac{3}{2}$. The authors used the Fokas method [1997; 2008] combined with contraction arguments to achieve the result.

1E. ***Main result.*** Now, let us present the main result of this article. Consider the biharmonic Schrödinger equation on the right half-line

$$
\begin{cases} i\partial_t u + \gamma \partial_x^4 u + \lambda |u|^2 u = 0, & (t,x) \in (0,T) \times (0,\infty), \\ u(0,x) = u_0(x), & x \in (0,\infty), \\ u(t,0) = f(t),\ u_x(t,0) = g(t), & t \in (0,T), \end{cases} \tag{1-5}
$$

for $\gamma, \lambda \in \mathbb{R}$. We say that system (1-5) is focusing if $\gamma\lambda < 0$ and defocusing when $\gamma\lambda > 0$. In this paper we will study the case when $\gamma = -1$, however the approach used here can be applied when $\gamma \in \mathbb{R} \setminus \{0\}$.

The presence of two boundary conditions in (1-5) can be motivated by integral identities on smooth decaying solutions for the linear equation

$$
i\partial_t u - \partial_x^4 u = 0. \tag{1-6}
$$

Indeed, for a smooth decaying solution u of (1-6) and $T > 0$, we have

$$
\begin{aligned} \int_0^\infty |u(T,x)|^2\,dx = \int_0^\infty |u(0,x)|^2\,dx &- \int_0^T \operatorname{Im}(\partial_x^3 u(t,0)\bar{u}(t,0))\,dt \\ &+ \int_0^T \operatorname{Im}(\partial_x^2 u(t,0)\partial_x \bar{u}(t,0))\,dt. \end{aligned} \tag{1-7}
$$

Thus, from (1-7) we can conclude that if we assume $u(0,x) = u(t,0) = u_x(t,0) = 0$ the linear solution for (1-6) is the trivial one.

It is well-known by [Kenig et al. 1991] that the *local smoothing effect* for the fourth-order linear group operator $e^{it\partial_x^4}$

$$
\|\partial_x^j e^{it\partial_x^4}\phi\|_{L_x^\infty \dot{H}^{\frac{1}{8}(2s+3-2j)}(\mathbb{R}_t)} \le c\|\phi\|_{H^s(\mathbb{R})} \quad \text{for } j = 0, 1 \text{ and } s \in \mathbb{R}, \tag{1-8}
$$

which motivates the relation of regularities among initial and boundary data.

Thus, we are able to present the main goal in the paper: to answer the problem cited in the beginning of this introduction, that is, to show the local well-posedness of (1-5) in the low regularity Sobolev space $H^s(\mathbb{R}^+)$, for $0 \le s < \frac{1}{2}$.

We state the main theorem for IBVP (1-5) as follows.

Theorem 1.1. *Let $s \in \big[0, \frac{1}{2}\big)$. For given initial-boundary data*

$$
(u_0, f, g) \in H^s(\mathbb{R}^+) \times H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+) \times H^{\frac{1}{8}(2s+1)}(\mathbb{R}^+),
$$

there exist a positive time

$$
T := T\big(\|u_0\|_{H^s(\mathbb{R}^+)}, \|f\|_{H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+)}, \|g\|_{H^{\frac{1}{8}(2s+1)}(\mathbb{R}^+)}\big),
$$

and unique solution $u(t,x) \in C((0,T); H^s(\mathbb{R}^+))$ *of the IBVP* (1-5), *when* $\gamma = -1$, *satisfying*

$$u \in C\big(\mathbb{R}^+; H^{\frac{1}{8}(2s+3)}(0,T)\big) \cap X^{s,b}\big((0,T)\times\mathbb{R}^+\big) \quad \text{and} \quad \partial_x u \in C\big(\mathbb{R}^+; H^{\frac{1}{8}(2s+1)}(0,T)\big),$$

for some $b(s) < \frac{1}{2}$. *Moreover, the map* $(u_0, f, g) \mapsto u$ *is analytic from* $H^s(\mathbb{R}^+) \times H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+) \times H^{\frac{1}{8}(2s+1)}(\mathbb{R}^+)$ *to* $C\big((0,T);\ H^s(\mathbb{R}^+)\big)$.

Remarks. Finally, the following comments are now in order:

1. The proof of Theorem 1.1 is based on the Fourier restriction method for a suitable extension of solutions. We first convert the IBVP of (1-5) posed in $\mathbb{R}^+ \times \mathbb{R}^+$ to the initial value problem (IVP) of (1-5) (integral equation formula) in the whole space $\mathbb{R} \times \mathbb{R}$ (see Section 3) by using the Duhamel boundary forcing operator. The energy and nonlinear estimates (established in Section 4) allow us to apply the Picard iteration method for IVP of (1-5), and hence we can complete the proof. The new tools used here are the Duhamel boundary forcing operator for the fourth-order linear equation and its analysis.

2. Note that Theorem 1.1 give us the local well-posedness in low regularity for the biharmonic nonlinear Schrödinger equation. However, in [Özsarı and Yolcu 2019], the authors showed the local well-posedness in the Sobolev spaces, by using Fokas approach. We point out that the low regularity in our main result is obtained using the boundary forcing operator, proposed by Holmer, which has been obtained in an independent way and with a different approach to that of [Özsarı and Yolcu 2019].

3. The approach used in our result, together with some extension as it was done in [Cavalcante 2017; Cavalcante and Kwak 2019; Holmer 2005; 2006] also guarantee the local well-posedness result in high regularity.

1F. ***Notations.*** In all this paper, we will consider $\mathbb{R}^+$ as $(0,\infty)$. Moreover, for positive real numbers $x, y \in \mathbb{R}^+$, we mean $x \lesssim y$ by $x \le Cy$ for some $C > 0$. Also, denote $x \sim y$ by $x \lesssim y$ and $y \lesssim x$. Similarly, $\lesssim_s$ and $\sim_s$ can be defined, where the implicit constants depend on s.

Our work is outlined in the following way: In Section 2, we introduce some function spaces defined on the half-line and construct the solution spaces. Section 3 is devoted to the introduction of the boundary forcing operator for the biharmonic Schrödinger equation. In Section 4, we show the energy estimates and present the trilinear estimates, respectively. The main result of this article, Theorem 1.1, is proved in Section 5. Finally, in Section 6, we present some open problems which seem to be of interest from the mathematical point of view.

2. Preliminaries

Throughout the paper, we fix a cut-off function $\psi(t) := \psi$,

(2-1) $\psi \in C_0^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $[0,1]$, $\psi \equiv 0$ for $|t| \ge 2$,

and for $T > 0$ we denote $\psi_T(t) = \frac{1}{T}\psi\left(\frac{t}{T}\right)$.

2A. ***Sobolev spaces on the half-line.*** For $s \ge 0$, we define the homogeneous L^2-based Sobolev spaces $\dot{H}^s = \dot{H}^s(\mathbb{R})$ by the norm $\|\phi\|_{\dot{H}^s} = \big\| |\xi|^s \hat{\psi}(\xi)\big\|_{L^2_\xi}$ and the L^2-based inhomogeneous Sobolev spaces $H^s = H^s(\mathbb{R})$ by the norm $\|\phi\|_{H^s} = \|(1+|\xi|^2)^{s/2}\hat{\psi}(\xi)\|_{L^2_\xi}$, where $\hat{\phi}$ denotes the Fourier transform of ϕ. Moreover, we say that $f \in H^s(\mathbb{R}^+)$ if there exists $F \in H^s(\mathbb{R})$ such that $f(x) = F(x)$ for $x > 0$, in this case we set

$$\|f\|_{H^s(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}.$$

On the other hand for $s \in \mathbb{R}$, we have $f \in H_0^s(\mathbb{R}^+)$ provided that there exists $F \in H^s(\mathbb{R})$ such that F is the extension of f on $\mathbb{R}$ and $F(x) = 0$ for $x < 0$. In this case, we set $\|f\|_{H_0^s(\mathbb{R}^+)} = \inf_F \|F\|_{H^s(\mathbb{R})}$. For $s < 0$, we define $H^s(\mathbb{R}^+)$ as the dual space of $H_0^{-s}(\mathbb{R}^+)$.

Let us also define the sets $C_0^\infty(\mathbb{R}^+) = \{f \in C^\infty(\mathbb{R});\ \operatorname{supp} f \subset [0,\infty)\}$ and $C_{0,c}^\infty(\mathbb{R}^+)$ as the subset of $C_0^\infty(\mathbb{R}^+)$, whose members have a compact support on $(0,\infty)$. We remark that $C_{0,c}^\infty(\mathbb{R}^+)$ is dense in $H_0^s(\mathbb{R}^+)$ for all $s \in \mathbb{R}$.

We finish this subsection with some elementary properties of the Sobolev spaces.

Lemma 2.1 [Jerison and Kenig 1995, Lemma 3.5]. *For* $-\frac{1}{2} < s < \frac{1}{2}$ *and* $f \in H^s(\mathbb{R})$,

$$\|\chi_{(0,\infty)} f\|_{H^s(\mathbb{R})} \le c\|f\|_{H^s(\mathbb{R})}. \tag{2-2}$$

Lemma 2.2 [Colliander and Kenig 2002, Lemma 2.8]. *If* $0 \le s < \frac{1}{2}$, *then, for the cut-off function* ψ *defined in* (2-1), $\|\psi f\|_{H^s(\mathbb{R})} \le c\|f\|_{\dot{H}^s(\mathbb{R})}$ *and* $\|\psi f\|_{\dot{H}^{-s}(\mathbb{R})} \le c\|f\|_{H^{-s}(\mathbb{R})}$, *where the constant c depends only on s and* ψ.

Remark. Lemma 2.2 is equivalent to

$$\|f\|_{H^s(\mathbb{R})} \sim \|f\|_{\dot{H}^s(\mathbb{R})},$$

for $-\frac{1}{2} < s < \frac{1}{2}$, where $f \in H^s(\mathbb{R})$ with $\operatorname{supp} f \subset [0,1]$.

The following two auxiliaries lemmas can be found in [Colliander and Kenig 2002] and their proofs will be omitted.

Lemma 2.3 [Colliander and Kenig 2002, Proposition 2.4]. *If* $\frac{1}{2} < s < \frac{3}{2}$, *the following statements are valid*:

(a) $H_0^s(\mathbb{R}^+) = \{f \in H^s(\mathbb{R}^+);\ f(0) = 0\}$.

(b) *If* $f \in H^s(\mathbb{R}^+)$ *with* $f(0) = 0$, *then* $\|\chi_{(0,\infty)} f\|_{H_0^s(\mathbb{R}^+)} \le c\|f\|_{H^s(\mathbb{R}^+)}$.

Lemma 2.4 [Colliander and Kenig 2002, Proposition 2.5]. *Let* $-\infty < s < \infty$ *and* $f \in H_0^s(\mathbb{R}^+)$. *For the cut-off function* ψ *defined in* (2-1), *we have* $\|\psi f\|_{H_0^s(\mathbb{R}^+)} \le c\|f\|_{H_0^s(\mathbb{R}^+)}$.

2B. *Solution spaces.* For $f \in \mathcal{S}(\mathbb{R}^2)$, we denote by $\tilde{f}$ or $\mathcal{F}(f)$ the Fourier transform of f with respect to both spatial and time variables

$$\tilde{f}(\tau,\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(t,x)\,dx\,dt.$$

Moreover, we use $\mathcal{F}_x$ and $\mathcal{F}_t$ to denote the Fourier transform with respect to space and time variable respectively (also we use $\widehat{\ }$ for both cases).

Bourgain [1993a; 1993b] established a way to prove the well-posedness of a classes of dispersive systems. More precisely, on the Sobolev spaces H^s, for smaller values of s, Bourgain found a yet more suitable smoothing property for solutions of the Korteweg–de Vries equation.

In this spirit, for $s, b \in \mathbb{R}$, we introduce the classical Bourgain spaces $X^{s,b}$ associated to (1-2) as the completion of $\mathcal{S}'(\mathbb{R}^2)$ under the norm

$$\|f\|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} \langle\xi\rangle^{2s} \langle\tau+\xi^4\rangle^{2b} |\tilde{f}(\tau,\xi)|^2\,d\xi\,d\tau,$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

One basic property of $X^{s,b}$ can be read as follows:

Lemma 2.5 [Tao 2006, Lemma 2.11]. *Let* $\psi(t)$ *be a Schwartz function in time. Then, we have*

$$\|\psi(t) f\|_{X^{s,b}} \lesssim_{\psi,b} \|f\|_{X^{s,b}}.$$

Ginibre et al. [1997], while establishing local well-posedness results for the Zakharov system, showed the following important estimate:

Lemma 2.6. *Let* $-\frac{1}{2} < b' < b \le 0$ *or* $0 \le b' < b < \frac{1}{2}$, $w \in X^{s,b}(\phi)$ *and* $s \in \mathbb{R}$. *Then*

$$\|\psi_T w\|_{X^{s,b'}(\phi)} \le cT^{b-b'} \|w\|_{X^{s,b}(\phi)}.$$

As is well-known, the space $X^{s,b}$ with $b > \frac{1}{2}$ is well-adapted to study the IVP of dispersive equations. However, in the study of IBVP, the standard argument cannot be applied directly. This is due to the lack of hidden regularity, more precisely, the control of (derivatives) time trace norms of the Duhamel boundary operator requires to work in $X^{s,b}$-type spaces for $b < \frac{1}{2}$, since the full regularity range cannot be covered (see Lemma 4.2 inequality (4-5)).

Therefore, to treat the solution of our problem, set the solution space denoted by $Z^{s,b}$ with the norm

$$\|f\|_{Z^{s,b}(\mathbb{R}^2)} = \sup_{t\in\mathbb{R}} \|f(t,\cdot)\|_{H^s(\mathbb{R})} + \sum_{j=0}^{1} \sup_{x\in\mathbb{R}} \|\partial_x^j f(\cdot,x)\|_{H^{\frac{1}{8}(2s+3-2j)}(\mathbb{R})} + \|f\|_{X^{s,b}}.$$

The spatial and time restricted space of $Z^{s,b}(\mathbb{R}^2)$ is defined in the standard way:

$$Z^{s,b}((0,T)\times\mathbb{R}^+) = Z^{s,b}\big|_{(0,T)\times\mathbb{R}^+}$$

equipped with the norm

$$\|f\|_{Z^{s,b}((0,T)\times\mathbb{R}^+)} = \inf_{g\in Z^{s,b}} \{\|g\|_{Z^{s,b}} : g(t,x) = f(t,x) \text{ on } (0,T)\times\mathbb{R}^+\}.$$

2C. ***Riemann–Liouville fractional integral.*** Before we begin our study of the IBVP for (1-2), we give a brief summary of the Riemann–Liouville fractional integral operator; see [Colliander and Kenig 2002; Holmer 2006] for more details.

Let us define the function t_+ as follows:

$$t_+ = \begin{cases} t & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

The tempered distribution $t_+^{\alpha-1}/\Gamma(\alpha)$ is defined like a locally integrable function for $\operatorname{Re}\alpha > 0$ by

$$\left\langle \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, f \right\rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(t)\,dt.$$

It follows that

$$\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} = \partial_t^k \left(\frac{t_+^{\alpha+k-1}}{\Gamma(\alpha+k)} \right), \tag{2-3}$$

for all $k \in \mathbb{N}$. Expression (2-3) can be used to extend the definition of $t_+^{\alpha-1}/\Gamma(\alpha)$ for all $\alpha \in \mathbb{C}$ in the sense of distributions. In fact, a change of contour shows that the Fourier transform of $t_+^{\alpha-1}/\Gamma(\alpha)$ is

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^{\widehat{}}(\tau) = e^{-\frac{1}{2}\pi i\alpha}(\tau - i0)^{-\alpha}, \tag{2-4}$$

where

$$(\tau - i0)^{-\alpha} = |\tau|^{-\alpha}\chi_{(0,\infty)} + e^{\alpha\pi i}|\tau|^{-\alpha}\chi_{(-\infty,0)} \tag{2-5}$$

is the distributional limit. For $\alpha \notin \mathbb{Z}$, by using (2-5), we rewrite (2-4) in the following way:

$$\left(\frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \right)^{\widehat{}}(\tau) = e^{-\frac{1}{2}\alpha\pi i}|\tau|^{-\alpha}\chi_{(0,\infty)} + e^{\frac{1}{2}\alpha\pi i}|\tau|^{-\alpha}\chi_{(-\infty,0)}. \tag{2-6}$$

For $f \in C_0^\infty(\mathbb{R}^+)$, define $\mathcal{I}_\alpha f$ as

$$\mathcal{I}_\alpha f = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} * f.$$

Thus, for $\operatorname{Re}\alpha > 0$, we have

$$\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds. \tag{2-7}$$

The following properties easily hold:

$$\mathcal{I}_0 f = f, \quad \mathcal{I}_1 f(t) = \int_0^t f(s)\,ds, \quad \mathcal{I}_{-1} f = f' \quad \text{and} \quad \mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}.$$

Moreover, the lemmas below can be found in [Holmer 2006], and we will omit the proofs.

Lemma 2.7 [Holmer 2006, Lemma 2.1]. *If $f \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C_0^\infty(\mathbb{R}^+)$, for all $\alpha \in \mathbb{C}$.*

Lemma 2.8 [Holmer 2006, Lemma 5.3]. *If $0 \le \operatorname{Re}\alpha < \infty$ and $s \in \mathbb{R}$, then $\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \le c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)}$, where $c = c(\alpha)$.*

Lemma 2.9 [Holmer 2006, Lemma 5.4]. *If $0 \le \operatorname{Re}\alpha < \infty$, $s \in \mathbb{R}$ and $\mu \in C_0^\infty(\mathbb{R})$, then $\|\mu \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \le c\|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}$, where $c = c(\mu, \alpha)$.*

2D. *Oscillatory integral.* In this subsection, we will define the oscillatory integral which is the key to defining, in the next section, the Duhamel boundary forcing operator. Let

$$B(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-i\xi^4}\,d\xi. \tag{2-8}$$

We first calculate $B(0)$. A change of variable ($\eta = \xi^4$), gives us the following:

$$B(0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi^4}\,d\xi = \frac{1}{4\pi} \int_0^{+\infty} e^{-i\eta} \eta^{-3/4}\,d\eta.$$

Now, a change of contour yields

$$B(0) = \frac{(-i)^{1-3/4}}{4\pi} \int_0^{+\infty} e^{-t} t^{(1/4)-1}\,dt = \frac{(-i)^{1/4}}{4\pi} \Gamma\Big(\frac{1}{4}\Big) = -\frac{i^{7/4}}{\pi} \Gamma\Big(\frac{5}{4}\Big).$$

Let us obtain the Mellin transform of $B(x)$.

Lemma 2.10. *For $\operatorname{Re}\lambda > 0$ we have*

$$\int_0^\infty x^{\lambda-1} B(x)\,dx = \frac{\Gamma(\lambda)\Gamma\big(\frac{1}{4} - \frac{\lambda}{4}\big)}{8\pi} \big(e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)}\big). \tag{2-9}$$

Proof. By analytic argument, we can assume that λ is a real number in the set $\left(0, \frac{3}{8}\right)$.
Consider

$$B_1(x) = \frac{1}{2\pi}\int_0^{+\infty} e^{ix\xi}e^{-i\xi^4}\,d\xi$$

and

$$B_2(x) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{ix\xi}e^{-i\xi^4}\,d\xi = \frac{1}{2\pi}\int_0^{\infty} e^{-ix\xi}e^{-i\xi^4}\,d\xi,$$

then we have $B(x) = B_1(x) + B_2(x)$. Define

$$B_{1,\epsilon}(x) = \frac{1}{2\pi}\int_0^{+\infty} e^{ix\xi}e^{-i\xi^4}e^{-\epsilon\xi}\,d\xi.$$

By using the dominated convergence theorem and Fubini's theorem we have

$$\begin{aligned}\text{(2-10)}\quad \int_0^\infty x^{\lambda-1}B_1(x)\,dx &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\int_0^\infty e^{-\delta x}x^{\lambda-1}B_{1,\epsilon}(x)\,dx\\ &= \lim_{\epsilon\to 0}\lim_{\delta\to 0}\frac{1}{2\pi}\int_0^{+\infty} e^{-i\xi^4}e^{-\epsilon\xi}\int_0^\infty e^{ix\xi}e^{-\delta x}x^{\lambda-1}\,dx\,d\xi.\end{aligned}$$

Using a change of contour, we get that

$$\text{(2-11)}\qquad \int_0^{+\infty} e^{ix\xi}e^{-\delta x}x^{\lambda-1}\,dx = \xi^{-\lambda}e^{i\lambda\frac{\pi}{2}}\Gamma\left(\lambda, -\frac{\delta}{\xi}\right),$$

where $\Gamma(\lambda, z) = \int_0^{+\infty} r^{\lambda-1}e^{irz}e^{-r}\,dr$. Again, thanks to the dominated convergence theorem it follows that

$$\text{(2-12)}\qquad \lim_{\delta\to 0}\int_0^{+\infty} e^{ix\xi}e^{-\delta x}x^{\lambda-1}\,dx = \xi^{-\lambda}e^{i\lambda\frac{\pi}{2}}\Gamma(\lambda).$$

Once more applying the dominated convergence theorem and changing the contour we conclude that

$$\begin{aligned}\text{(2-13)}\quad \int_0^{+\infty} x^{\lambda-1}B_1(x)\,dx &= \frac{\Gamma(\lambda)}{2\pi}e^{i\lambda\frac{\pi}{2}}\lim_{\epsilon\to 0}\int_0^{+\infty} e^{-i\xi^4}e^{-\epsilon\xi}\xi^{-\lambda}\,d\xi\\ &= \frac{\Gamma(\lambda)}{2\pi}e^{i\lambda\frac{\pi}{2}}\frac{1}{4}\lim_{\epsilon\to 0}\int_0^{+\infty} e^{-i\eta}e^{-\epsilon\eta^{1/4}}(\eta)^{-(\lambda+3)/4}\,d\eta\\ &= \frac{\Gamma(\lambda)}{2\pi}e^{i\lambda\frac{\pi}{2}}\frac{1}{4}e^{-\frac{\pi i}{2}((1-\lambda)/4)}\Gamma\left(\frac{1}{4}-\frac{\lambda}{4}\right)\\ &= \frac{\Gamma(\lambda)\Gamma\left(\frac{1}{4}-\frac{\lambda}{4}\right)}{8\pi}e^{-i\frac{\pi}{8}(1-5\lambda)}.\end{aligned}$$

In a similar way, by using the identity

$$\int_0^{+\infty} e^{-ix\xi} e^{-\delta x} x^{\lambda-1}\,dx = \xi^{-\lambda} e^{-i\lambda\frac{\pi}{2}} \Gamma\Big(\lambda, \frac{\delta}{\xi}\Big), \tag{2-14}$$

we obtain

$$\begin{aligned}\int_0^{+\infty} x^{\lambda-1} B_2(x)\,dx &= \frac{\Gamma(\lambda)}{2\pi} e^{-i\lambda\frac{\pi}{2}} \frac{1}{4} e^{-\frac{\pi i}{2}(\frac{1-\lambda}{4})} \Gamma\Big(\frac{1}{4}-\frac{\lambda}{4}\Big)\\ &= \frac{\Gamma(\lambda)\Gamma\big(\frac{1}{4}-\frac{\lambda}{4}\big)}{8\pi} e^{-i\frac{\pi}{8}(1+3\lambda)}.\end{aligned} \tag{2-15}$$

Finally, as we can split by $B(x) = B_1(x) + B_2(x)$, equation (2-9) holds. □

3. Duhamel boundary forcing operator

In this section, we study the Duhamel boundary forcing operator, which was introduced by Colliander and Kenig [2002], in order to construct the solution to (1-2) forced by boundary conditions. We refer to [Cavalcante 2017; Cavalcante and Corcho 2019; Holmer 2005] for further exposition about this topic.

3A. ***Duhamel boundary forcing operator class.*** Let us introduce the Duhamel boundary forcing operator associated to the linearized biharmonic Schrödinger equation. Consider

$$M = \frac{1}{B(0)\Gamma\big(\frac{3}{4}\big)}. \tag{3-1}$$

For $f \in C_0^\infty(\mathbb{R}^+)$, define the boundary forcing operator $\mathcal{L}^0$ (of order 0) as

$$\mathcal{L}^0 f(t,x) := M \int_0^t e^{i(t-t')\partial_x^4} \delta_0(x) \mathcal{I}_{-3/4} f(t')\,dt', \tag{3-2}$$

where $e^{it\partial_x^4}$ denotes the group associated to (1-6) given by

$$e^{it\partial_x^4}\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^4} \hat{\psi}(\xi)\,d\xi.$$

Note that the property of convolution operator $(\partial_x(f*g) = (\partial_x f)*g = f*(\partial_x g))$ and the integration by parts in t' of (3-2) yield that

$$i\mathcal{L}^0(\partial_t f)(t,x) = iM\delta_0(x)\mathcal{I}_{-3/4} f(t) + \partial_x^4 \mathcal{L}^0 f(t,x). \tag{3-3}$$

By a change of variable and using (2-8), we get that

$$\mathcal{L}^0 f(t,x) = M\int_0^t e^{i(t-t')\partial_x^4}\delta_0(x)\mathcal{I}_{-3/4}f(t')\,dt' = M\int_0^t B\left(\frac{x}{(t-t')^{1/4}}\right)\frac{\mathcal{I}_{-3/4}f(t')}{(t-t')^{1/4}}\,dt'. \tag{3-4}$$

We are now in a position to make it precise when the boundary forcing term is continuous or discontinuous. More precisely, the following lemma holds.

Lemma 3.1. *Let $f \in C_{0,c}^\infty(\mathbb{R}^+)$.*

(a) *For fixed $0 \le t \le 1$, we have that $\partial_x^k\mathcal{L}^0 f(t,x)$, $k = 0, 1, 2$, is continuous in $x \in \mathbb{R}$ and has the decay property in terms of the spatial variable as follows*:

$$|\partial_x^k\mathcal{L}^0 f(t,x)| \lesssim_N \|f\|_{H^{N+k}}\langle x\rangle^{-N}, \quad N \ge 0. \tag{3-5}$$

(b) *For fixed $0 \le t \le 1$, we have that $\partial_x^3\mathcal{L}^0 f(t,x)$ is continuous in x for $x \ne 0$ and it is discontinuous at $x = 0$ satisfying*

$$\lim_{x\to 0^-}\partial_x^3\mathcal{L}^0 f(t,x) = -i\frac{M}{2}\mathcal{I}_{-3/4}f(t), \qquad \lim_{x\to 0^+}\partial_x^3\mathcal{L}^0 f(t,x) = i\frac{M}{2}\mathcal{I}_{-3/4}f(t).$$

Also, $\partial_x^3\mathcal{L}^0 f(t,x)$ has the decay property in terms of the spatial variable

$$|\partial_x^3\mathcal{L}^0 f(t,x)| \lesssim_N \|f\|_{H^{N+3}}\langle x\rangle^{-N}, \quad N \ge 0. \tag{3-6}$$

Proof. In fact, the continuity of $\partial_x^k\mathcal{L}^0 f(t,x)$ follows from (3-4), for $k = 0, 1, 2$, and the proof of (3-5) exactly follows the idea introduced by Holmer [2005, Lemma 12]. Moreover, (3-5) and (3-3) yield that $\partial_x^4\mathcal{L}^0 f(t,x)$ is discontinuous only at $x = 0$ of size $M\mathcal{I}_{-3/4}f(t)$ (where M is defined as (3-1)), and the decay bound (3-6) holds. □

Remark. Lemma 3.1 ensures that $\mathcal{L}^0 f(t,0) = f(t)$.

We are now in position to generalize the boundary forcing operator (3-2). For $\operatorname{Re}\lambda > -4$ and given $g \in C_0^\infty(\mathbb{R}^+)$, we define

$$\mathcal{L}^\lambda g(t,x) = \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{L}^0(\mathcal{I}_{-\lambda/4}g)(t,\cdot)\right](x), \tag{3-7}$$

where $*$ denotes the convolution operator and $x_-^{\lambda-1}/\Gamma(\lambda) = (-x)_+^{\lambda-1}/\Gamma(\lambda)$. In particular, for $\operatorname{Re}\lambda > 0$, we have

$$\mathcal{L}^\lambda g(t,x) = \frac{1}{\Gamma(\lambda)}\int_x^\infty (y-x)^{\lambda-1}\mathcal{L}^0(\mathcal{I}_{-\lambda/4}g)(t,y)\,dy. \tag{3-8}$$

A property of the convolution operator ($\partial_x^4(f*g)=(\partial_x^4 f)*g=f*(\partial_x^4 g)$) and (3-3) give us

$$
\begin{aligned}
\text{(3-9)}\quad \mathcal{L}^\lambda g(t,x) &= \left[\frac{x_-^{(\lambda+4)-1}}{\Gamma(\lambda+4)} * \partial_x^4 \mathcal{L}^0(\mathcal{I}_{-\lambda/4} g)(t,\cdot)\right](x) \\
&= iM\frac{x_-^{(\lambda+4)-1}}{\Gamma(\lambda+4)}\mathcal{I}_{-3/4-\lambda/4} g(t) \\
&\qquad + i\int_x^\infty \frac{(y-x)^{(\lambda+4)-1}}{\Gamma(\lambda+4)}\mathcal{L}^0(\partial_t \mathcal{I}_{-\lambda/4} g)(t,y)\,dy,
\end{aligned}
$$

for $\operatorname{Re}\lambda > -4$, where M is defined as in (3-1). From (3-3) and (3-7), we have

$$(i\partial_t - \partial_x^4)\mathcal{L}^\lambda g(t,x) = iM\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-3/4-\lambda/4} g(t),$$

in the distributional sense.

To finish this subsection, we will give two lemmas concerning the spatial continuity and decay properties of the $\mathcal{L}^\lambda g(t,x)$ and the explicit values for $\mathcal{L}^\lambda f(t,0)$, respectively.

Lemma 3.2. *Let $g \in C_0^\infty(\mathbb{R}^+)$ and M be as in* (3-1). *Then, we have*

$$\text{(3-10)}\qquad \mathcal{L}^{-k} g = \partial_x^k \mathcal{L}^0 \mathcal{I}_{k/4} g, \qquad k=0,1,2,3.$$

Moreover, $\mathcal{L}^{-3} g(t,x)$ is continuous in $x \in \mathbb{R}\setminus\{0\}$ and has a step discontinuity at $x=0$. For real λ satisfying $\lambda > -3$, $\mathcal{L}^\lambda g(t,x)$ is continuous in $x\in\mathbb{R}$. For $-3 \le \lambda \le 1$ and $0 \le t \le 1$, $\mathcal{L}^\lambda g(t,x)$ satisfies the following decay bounds:

$$
\begin{aligned}
|\mathcal{L}^\lambda g(t,x)| &\le c_{\lambda,g}\langle x\rangle^{\lambda-1} && \textit{for all } x\ge 0,\\
|\mathcal{L}^\lambda g(t,x)| &\le c_{m,\lambda,g}\langle x\rangle^{-m} && \textit{for all } x\ge 0 \textit{ and } m\ge 0.
\end{aligned}
$$

Proof. We give a sketch of the proof. The detailed argument can be found in [Holmer 2006]. By using (3-9), we have that (3-10) follows. Moreover, Lemma 3.1 together with (3-10) guarantee the continuity (except for $x=0$ when $\lambda=-3$) and discontinuity at $x=0$ of $\mathcal{L}^\lambda g$ for $\lambda \ge -3$ and $\lambda=-3$, respectively. The proof of decay bounds can be obtained by using (3-9), (3-3) and Lemma 3.1. □

Lemma 3.3. *For* $\operatorname{Re}\lambda > -4$ *and $f \in C_0^\infty(\mathbb{R}^+)$, we have the following value of $\mathcal{L}^\lambda f(t,0)$*:

$$\text{(3-11)}\qquad \mathcal{L}^\lambda f(t,0) = \frac{M}{8} f(t)\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)}}{\sin\left(\frac{1}{4}(1-\lambda)\pi\right)}\right).$$

Proof. By using (3-9) we get

$$\mathcal{L}^{\lambda} f(t,0) = i \int_0^{\infty} \frac{y^{(\lambda+4)-1}}{\Gamma(\lambda+4)} \mathcal{L}^0(\partial_t \mathcal{I}_{-\lambda/4} f)(t,y)\,dy.$$

This show that $\mathcal{L}^{\lambda} f(t,0)$ is analytic, in λ, for $\operatorname{Re}\lambda > -4$.

By analytic argument, it suffices to consider the case when λ is a positive real number and (3-4), where M is defined as in (3-1). In fact, in order to use (2-9), we take $\lambda \in \bigl(0, \frac{3}{8}\bigr)$ in (3-8). Thus, in the calculations, we use the representation (3-8) for $\lambda > 0$. Fubini's theorem, the change of variable, (2-10) and (2-7), yield that

$$\begin{aligned}
\mathcal{L}^{\lambda} f(t,0) &= \frac{M}{\Gamma(\lambda)} \int_0^{\infty} y^{\lambda-1} \int_0^t B\biggl(\frac{y}{(t-t')^{1/4}}\biggr) \frac{\mathcal{I}_{(-\lambda-3)/4} f(t')}{(t-t')^{1/4}}\,dt'\,dy \\
&= \frac{M}{\Gamma(\lambda)} \int_0^t (t-t')^{\frac{\lambda+3}{4}-1} \mathcal{I}_{(-\lambda-3)/4} f(t') \int_0^{\infty} y^{\lambda-1} B(y)\,dy\,dt' \\
&= \frac{M}{\Gamma(\lambda)} \Gamma\biggl(\frac{\lambda}{4}+\frac{3}{4}\biggr) f(t) \frac{\Gamma(\lambda)\Gamma\bigl(\frac{1}{4}-\frac{\lambda}{4}\bigr)}{8\pi} \bigl(e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)}\bigr) \\
&= \frac{M}{8} f(t) \biggl(\frac{e^{-i\frac{\pi}{8}(1+3\lambda)} + e^{-i\frac{\pi}{8}(1-5\lambda)}}{\sin\bigl(\frac{1}{4}(1-\lambda)\pi\bigr)}\biggr),
\end{aligned}$$

where in the last equality we used the fact that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Thus, the proof is complete. □

3B. ***Construction of the solution.*** Let us describe how we can construct the solution for the linear fourth order Schrödinger equation

$$i\partial_t u - \partial_x^4 u = 0. \tag{3-12}$$

3B1. *Linear version.* First, we define the unitary group associated to (3-12) as

$$e^{it\partial_x^4}\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-it\xi^4} \hat{\phi}(\xi)\,d\xi,$$

which allows

$$\begin{cases} (i\partial_t - \partial_x^4) e^{it\partial_x^4}\phi(x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ e^{it\partial_x^4}\phi(x)|_{t=0} = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{3-13}$$

Recall $\mathcal{L}^{\lambda}$ in (3-9) for the right half-line problem. Let

$$\begin{aligned}
u(t,x) &= \mathcal{L}^{\lambda_1}\gamma_1(t,x) + \mathcal{L}^{\lambda_2}\gamma_2(t,x), \\
\partial_x u(t,x) &= \mathcal{L}^{\lambda_1-1}\mathcal{I}_{-1/4}\gamma_1(t,x) + \mathcal{L}^{\lambda_2-1}\mathcal{I}_{-1/4}\gamma_2(t,x),
\end{aligned}$$

where γ_j $(j = 1, 2)$ will be chosen later in terms of the given boundary data f and g.

Let a_j and b_j be constants depending on λ_j, $j=1,2$, given by

$$
\begin{aligned}
a_j &= \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda_j)}+e^{-i\frac{\pi}{8}(1-5\lambda_j)}}{\sin\left(\frac{1}{4}(1-\lambda_j)\pi\right)}\right),\\
b_j &= \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-2+3\lambda)}+e^{-i\frac{\pi}{8}(6-5\lambda)}}{\sin\left(\frac{1}{4}(2-\lambda_j)\pi\right)}\right).
\end{aligned}
\tag{3-14}
$$

By Lemmas 3.2 and 3.3, we get

$$f(t)=u(t,0)=a_1\gamma_1(t)+a_2\gamma_2(t), \tag{3-15}$$

$$g(t)=\partial_x u(t,0)=b_1\mathcal{I}_{-1/4}\,\gamma_1(t)+b_2\mathcal{I}_{-1/4}\,\gamma_2(t). \tag{3-16}$$

Thanks to (3-15) and (3-16), we can write a matrix in the form

$$\begin{bmatrix} f(t) \\ \mathcal{I}_{1/4}\,g(t)\end{bmatrix}=A\begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix},$$

where

$$A(\lambda_1,\lambda_2)=\begin{bmatrix} a_1 & a_2\\ b_1 & b_2\end{bmatrix}.$$

Choosing appropriate λ_j, $j=1,2$, such that A is invertible, we have that u solves

$$
\begin{cases}
(i\partial_t-\partial_x^4)u(t,x)=iM\frac{x_-^{\lambda_1-1}}{\Gamma(\lambda_1)}\mathcal{I}_{-3/4-\lambda/4}\,\gamma_1(t) & \\
\qquad\qquad +iM\frac{x_-^{\lambda_2-1}}{\Gamma(\lambda)}\mathcal{I}_{-3/4-\lambda/4}\,\gamma_2(t), & (t,x)\in\mathbb{R}^+\times\mathbb{R},\\
u(0,x)=0, & x\in\mathbb{R},\\
u(t,0)=f(t),\quad \partial_x u(t,0)=g(t), & t\in\mathbb{R}^+.
\end{cases}
\tag{3-17}
$$

By restriction of the function u on the set $\mathbb{R}^+\times\mathbb{R}^+$, we can construct a solution for the linear fourth order dispersive equation (3-12) posed on the right half-line.

3B2. *Nonlinear version.* Now, we define the classical Duhamel inhomogeneous solution operator $\mathcal{D}$ by

$$\mathcal{D}w(t,x)=-i\int_0^t e^{i(t-t')\partial_x^4}w(t',x)\,dt'.$$

It follows that

$$
\begin{cases}
(i\partial_t-\partial_x^4)\mathcal{D}w(t,x)=w(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R},\\
\mathcal{D}w(x,0)=0, & x\in\mathbb{R}.
\end{cases}
$$

Let

$$u(t,x)=\mathcal{L}^{\lambda_1}\gamma_1(t,x)+\mathcal{L}^{\lambda_2}\gamma_2(t,x)+e^{it\partial_x^4}\phi(x)+\mathcal{D}w.$$

Similar to what was done in Section 3B, taking γ_1 and γ_2 appropriately, depending on f, g, $e^{it\partial_x^4}\phi(x)$ and $\mathcal{D}w$, we see that u solves

$$\begin{cases} (i\partial_t - \partial_x^4)u(t,x) = w(t,x), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^+,\\ u(0,x)=\phi(x), & x\in\mathbb{R}^+,\\ u(t,0)=f(t),\quad \partial_x u(t,0)=g(t), & t\in\mathbb{R}^+. \end{cases} \tag{3-18}$$

The discussion about the structure of the system (3-18) can be found in Section 5

4. Energy estimates

The main purpose of this section is to prove the energy estimate of the solutions of the fourth order nonlinear Schrödinger equation in the Bourgain spaces $X^{s,b}$.

Lemma 4.1. *Let $s\in\mathbb{R}$ and $b\in\mathbb{R}$. If $\phi\in H^s(\mathbb{R})$, then the following estimates hold*:

$$\|\psi(t)e^{it\partial_x^4}\phi(x)\|_{C_t(\mathbb{R};\,H_x^s(\mathbb{R}))} \lesssim_\psi \|\phi\|_{H^s(\mathbb{R})}, \tag{4-1}$$

$$\|\psi(t)\partial_x^j e^{it\partial_x^4}\phi(x)\|_{C_x(\mathbb{R};\,H_t^{(2s+3-2j)/8}(\mathbb{R}))} \lesssim_{\psi,s,j} \|\phi\|_{H^s(\mathbb{R})}, \qquad j\in\{0,1\}; \tag{4-2}$$

$$\|\psi(t)e^{it\partial_x^4}\phi(x)\|_{X^{s,b}} \lesssim_{\psi,b} \|\phi\|_{H^s(\mathbb{R})}. \tag{4-3}$$

Estimates (4-1), (4-2) *and* (4-3) *are so-called space traces*, *derivative time traces and Bourgain spaces estimates*, *respectively.*

Proof. The proofs of (4-1) and (4-3) are standard and the proof of (4-2) follows from the smoothness of ψ and the local smoothing estimate (1-8), thus we will omit the details. □

Lemma 4.2. *Let $0<b<\frac{1}{2}$ and $j=0,1$, we have the following inequalities*

$$\|\psi(t)\mathcal{D}w(t,x)\|_{C(\mathbb{R}_t;\,H^s(\mathbb{R}_x))} \lesssim \|w\|_{X^{s,-b}}, \tag{4-4}$$

for $s\in\mathbb{R}$;

$$\|\psi(t)\partial_x^j\mathcal{D}w(t,x)\|_{C(\mathbb{R}_x;\,H^{(2s+3-2j)/8}(\mathbb{R}_t))} \lesssim \|w\|_{X^{s,-b}}, \tag{4-5}$$

for $-\frac{3}{2}+j<s<\frac{1}{2}+j$;

$$\|\psi(t)\partial_x^j\mathcal{D}w(t,x)\|_{X^{s,b}} \lesssim \|w\|_{X^{s,-b}}, \tag{4-6}$$

for $s\in\mathbb{R}$.

Estimates (4-4), (4-5) *and* (4-6) *are so-called space traces*, *derivative time traces and Bourgain spaces estimates*, *respectively.*

Proof. The idea to prove this lemma follows a variation of the proof due to [Kenig et al. 1991]. Here, we will give the sketch of the proof for sake of completeness.

Estimate (4-4): By using $2\chi_{(0,t)}(t') = \operatorname{sgn} t' + \operatorname{sgn}(t - t')$, $\widehat{\operatorname{sgn}}(\tau) = \text{p.v.}\frac{2}{i\tau}$ and $e^{i\tau\xi^4}\hat{f}(\tau) = \hat{f}(\tau+\xi^4)$ we have

$$\psi(t)\mathcal{D}w(t,x) = c\int e^{ix\xi}e^{-it\xi^4}\psi(t)\int \widetilde{w}(\tau',\xi)\frac{e^{it(\tau'+\xi^4)}-1}{(\tau'+\xi^4)}\,d\tau'\,d\xi. \tag{4-7}$$

We denote by $w = w_1 + w_2$, where

$$\widetilde{w}_1(\tau,\xi) = \eta_0(\tau+\xi^4)\widetilde{w}(\tau,\xi),$$
$$\widetilde{w}_2(\tau,\xi) = (1-\eta_0(\tau+\xi^4))\widetilde{w}(\tau,\xi).$$

Here, $\eta_0 : \mathbb{R} \to [0, 1]$ is a smooth bump function supported in $[-2, 2]$ and equal to 1 in $[-1, 1]$. For w_1, we use the Taylor expansion of e^x at $x = 0$. Then, we can rewrite (4-7) for w_1 as

$$\begin{aligned}\psi(t)\mathcal{D}w(t,x) &= c\int e^{ix\xi}e^{-it\xi^4}\psi(t)\int \widetilde{w}_1(\tau',\xi)\frac{e^{it(\tau'+\xi^4)}-1}{(\tau'+\xi^4)}\,d\tau'\,d\xi\\ &= c\sum_{k=1}^{\infty}\frac{i^{k-1}}{k!}\psi^k(t)\int e^{ix\xi}e^{-it\xi^4}\widehat{F}_1^k(\xi)\,d\xi\\ &= c\sum_{k=1}^{\infty}\frac{i^{k-1}}{k!}\psi^k(t)e^{it\partial_x^4}F_1^k(x),\end{aligned}$$

where $\psi^k(t) = t^k\psi(t)$ and

$$\widehat{F}_1^k(\xi) = \int \widetilde{w}_1(\tau,\xi)(\tau+\xi^4)^{k-1}\,d\tau. \tag{4-8}$$

Since

$$\|F_1^k\|_{H^s} = \left(\int \langle\xi\rangle^{2s}\left|\int \widetilde{w}_1(\tau,\xi)(\tau+\xi^4)^{k-1}\,d\tau\right|^2 d\xi\right)^{\frac{1}{2}} \lesssim \|w\|_{X^{s,-b}}, \tag{4-9}$$

we have from (4-1) that

$$\|\psi(t)\mathcal{D}w(t,x)\|_{C_tH^s} \lesssim \sum_{k=1}^{\infty}\frac{1}{k!}\|F_1^k\|_{H_x^s} \lesssim \|w\|_{X^{s,-b}}.$$

For w_2, a direct calculation gives

$$\mathcal{F}[\psi\mathcal{D}w](\tau,\xi) = c\int \widetilde{w}_2(\tau',\xi)\frac{\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau+\xi^4)}{(\tau'+\xi^4)}\,d\tau'. \tag{4-10}$$

Since $\|\psi\mathcal{D}w\|_{C_tH^s} \lesssim \|\langle\xi\rangle^s\mathcal{F}[\psi\mathcal{D}w](\tau,\xi)\|_{L_\xi^2L_\tau^1}$, it suffices to bound the term

$$\left(\int \langle\xi\rangle^{2s}\left|\int |\widetilde{w}_2(\tau',\xi)|\int \frac{|\widehat{\psi}(\tau-\tau')-\widehat{\psi}(\tau+\xi^4)|}{|\tau'+\xi^4|}\,d\tau\,d\tau'\right|^2 d\xi\right)^{\frac{1}{2}}, \tag{4-11}$$

due to (4-10). We use the L^1 integrability of $\widehat{\psi}$, to bound (4-11) by

$$c\left(\int \langle\xi\rangle^{2s}\left|\int_{|\tau'+\xi^4|>1} \frac{|\widetilde{w}_2(\tau',\xi)|}{|\tau'+\xi^4|}\,d\tau'\right|^2 d\xi\right)^{\frac{1}{2}} \lesssim \|w\|_{X^{s,-b}}.$$

Estimate (4-5): We only consider the case $j=0$, since the estimate for $j=1$ is a direct consequence of the case $j=0$. Initially, take $\theta(\tau)\in C^\infty(\mathbb{R})$ such that $\theta(\tau)=1$ for $|\tau|<\frac{1}{2}$ and supp $\theta\subset\left[-\frac{2}{3},\frac{2}{3}\right]$. A standard calculation gives

$$\begin{aligned}
\mathcal{F}_x&\left(\psi(t)\int_0^t e^{(t-t')\partial_x^4}w(x,t')\right)(\xi)\\
&= c\psi(t)\int_\tau \frac{e^{it\tau}-e^{-it\xi^4}}{\tau+\xi^4}\widetilde{w}(\xi,\tau)\,d\tau\\
&= c\psi(t)e^{it\xi^4}\int_\tau \frac{e^{-it(\tau+\xi^4)}-1}{\tau+\xi^4}\theta(\tau+\xi^4)\widetilde{w}(\xi,\tau)\,d\tau\\
&\qquad + c\psi(t)\int_\tau e^{it\tau}\frac{1-\theta(\tau+\xi^4)}{\tau+\xi^4}\widetilde{w}(\xi,\tau)\,d\tau\\
&\qquad\qquad - c\psi(t)e^{it\xi^4}\int_\tau \frac{1-\theta(\tau+\xi^4)}{\tau+\xi^4}\widetilde{w}(\xi,\tau)\,d\tau\\
&:= \mathcal{F}_x w_1+\mathcal{F}_x w_2-\mathcal{F}_x w_3.
\end{aligned}$$

By the power series expansion for $e^{-it(\tau+\xi^4)}$, we have

$$w_1(x,t)=\sum_{k=1}^{\infty}\frac{\psi_k(t)}{k!}e^{it\partial_x^4}\phi_k(x).$$

Here, $\psi_k(t)=i^k t^k\theta(t)$ and

$$\hat{\phi}_k(\xi)=\int_\tau (\tau+\xi^4)^{k-1}\theta(\tau+\xi^4)\widetilde{w}(\xi,\tau)\,d\tau.$$

By using (4-2), it suffices to show that $\|\phi_k\|_{H^s(\mathbb{R})}\le c\|u\|_{X^{s,-b}}$, for $b<\frac{1}{2}$. Using the definition of ϕ_k and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\|\phi_k\|^2_{H^s(\mathbb{R}_x)} &= c\int_\xi \langle\xi\rangle^{2s}\left(\int_{\{\tau:|\tau+\xi^4|\le\frac{2}{3}\}}\sum_{k=1}^{\infty}(\tau+\xi^4)^{k-1}\theta(\tau+\xi^4)\tilde{u}(\xi,\tau)\right)^2 d\xi\\
&\le c\int_\xi \langle\xi\rangle^{2s}\int_\tau \langle\tau+\xi^4\rangle^{2c}|\tilde{u}(\xi,\tau)|^2\,d\tau\,d\xi.
\end{aligned}$$

This completes the estimate of w_1. Now we treat w_2. By using the change of variable $\eta = \xi^4$ and the Cauchy–Schwarz inequality we obtain

$$\|w_2\|^2_{C(\mathbb{R}_x;\, H^{(2s+3)/8}(\mathbb{R}_t))} \le c \int_\tau \langle\tau\rangle^{(2s+3)/4} G(\tau) \int_\xi \langle\tau+\xi^4\rangle^{-2b}\langle\xi\rangle^{2s}|\widetilde{w}_2(\xi,\tau)|^2\, d\xi\, d\tau,$$

where $G(\tau) = c\int_\eta \langle\tau+\eta\rangle^{-2+2b}|\eta|^{-3/4}\langle\eta\rangle^{-s/2}\, d\eta$. To conclude the estimate of w_2, we need to prove the following estimate:

$$G(\tau) \le c\langle\tau\rangle^{-(2s+3)/4}. \tag{4-12}$$

We split it in two cases. In the first case, we consider $|\tau| < 1$. For this, we use $\langle\tau+\eta\rangle \sim \langle\eta\rangle$ to get

$$G(\tau) \le c\int \langle\eta\rangle^{-2+2b-(s/2)}|\eta|^{-3/4}\, d\eta.$$

The above integral is bounded in the case $s > -\frac{7}{2} + 4b$, since $-b > -\frac{1}{2}$. Also, this estimate is valid for $s > -\frac{3}{2}$.

Now, the second case $|\tau| \ge 1$ can be estimated by separating the integral into three regions $|\eta| \le 1,\ 2|\eta| \le |\tau|,\ |\tau| \le 2|\eta|$ and using that $-\frac{3}{2} < s \le \frac{1}{2}$, so (4-12) follows.

Finally, to bound w_3, let us rewrite w_3 like $w_3 = \psi(t)e^{it\partial_x^4}\phi(x)$, where

$$\hat{\phi}(\xi) = \int \frac{1-\theta(\tau+\xi^4)}{\tau+\xi^4}\widetilde{w}(\xi,\tau)\, d\tau.$$

Thanks to (4-2) and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|w_3\|^2_{C(\mathbb{R}_x;H^{(2s+3)/8}(\mathbb{R}_t))} &= c\|\psi(t)e^{it\partial_x^4}\phi(x)\|^2_{C(\mathbb{R}_x;H^{(2s+3)/8}(\mathbb{R}_t))} \le c\|\phi\|^2_{H^s(\mathbb{R})}\\ &\le c\int_\xi \langle\xi\rangle^{2s}\left(\int_\tau |\widetilde{w}(\xi,\tau)|^2\langle\tau+\xi^4\rangle^{-2b}\, d\tau \int \frac{d\tau}{\langle\tau+\xi^4\rangle^{2-2b}}\right) d\xi.\end{aligned}$$

Since $b < \frac{1}{2}$, we have

$$\int \frac{1}{\langle\tau+\xi^4\rangle^{2-2b}}\, d\tau \le c.$$

By using (4-12), estimate (4-5) for w_3 follows and, consequently, (4-5) holds true for $w = w_1 + w_2 + w_3$.

Estimate (4-6): Finally, again we split $w = w_1 + w_2$, similar to what was done in the proof of (4-4). For w_1, estimates (4-3) and (4-9) yield that

$$\|\psi(t)\mathcal{D}w_1(t,x)\|_{X^{s,b}} \lesssim \sum_{k=1}^{\infty}\frac{1}{k!}\|F_1^k\|_{H_x^s} \lesssim \|w\|_{X^{s,-b}},$$

where F_1^k is defined as in (4-8).

For w_2, note that

$$\begin{aligned}\psi\partial_x^j\mathcal{D}w(t,x)&=c\int e^{ix\xi}e^{-it\xi^4}(i\xi)^j\psi(t)\int\frac{\widetilde{w}(\tau',\xi)}{(\tau'+\xi^4)}\big(e^{it(\tau'+\xi^4)}-1\big)\,d\tau'\,d\xi\\&=c\int e^{ix\xi}e^{-it\xi^4}(i\xi)^j\psi(t)\int\frac{\widetilde{w}(\tau',\xi)}{(\tau'+\xi^4)}e^{it(\tau'+\xi^4)}\,d\tau'\,d\xi\\&\qquad-c\int e^{ix\xi}e^{-it\xi^4}(i\xi)^j\psi(t)\int\frac{\widetilde{w}(\tau',\xi)}{(\tau'+\xi^4)}\,d\tau'\,d\xi\\&=I-II.\end{aligned}$$

Let

$$\widehat{W}(\xi)=\int\frac{\widetilde{w}_2(\tau,\xi)}{(\tau+\xi^4)}\,d\tau.$$

Therefore, we use (4-3) in II to obtain

$$\|\psi e^{it\partial_x^4}W\|_{X^{s,b}}\lesssim\|W\|_{H^s}\lesssim\|w\|_{X^{s,-b}}$$

for $b<\frac{1}{2}$.

Now, it remains to show the following estimate:

$$\left(\int_{|\xi|>1}|\xi|^{2s}\int\langle\tau+\xi^4\rangle^{2b}\left|\int\frac{\widetilde{w}_2(\tau',\xi)}{i(\tau'+\xi^4)}\widehat{\psi}(\tau-\tau')\,d\tau'\right|^2d\tau\,d\xi\right)^{\frac{1}{2}}\lesssim\|w\|_{X^{s,-b}}. \tag{4-13}$$

This follows by using the same argument as we used to prove (4-5). In fact, the proof of (4-13) is easier than proof of (4-5), since the L^2 integral with respect to ξ is negligible and hence it is enough to consider the relation between $\tau+\xi^4$ and $\tau'+\xi^4$. Thus, as a consequence, we have

$$\|\psi\mathcal{D}w\|_{X^{s,b}}\lesssim\|w\|_{X^{s,-b}}.$$

Therefore, Lemma 4.2 is proved. □

Lemma 4.3. *Let $s\in\mathbb{R}$.*

(a) *For $\frac{1}{2}(2s-7)<\lambda<\frac{1}{2}(1+2s)$ and $\lambda<\frac{1}{2}$ the following inequality holds*:

$$\|\psi(t)\mathcal{L}^\lambda f(t,x)\|_{C(\mathbb{R}_t;\,H^s(\mathbb{R}_x))}\le c\|f\|_{H_0^{(2s+3)/8}(\mathbb{R}^+)}. \tag{4-14}$$

(b) *For $-4+j<\lambda<1+j$, $j=0,1$, we have*

$$\|\psi(t)\partial_x^j\mathcal{L}^\lambda f(t,x)\|_{C(\mathbb{R}_x;\,H_0^{(2s+3-2j)/8}(\mathbb{R}_t^+))}\le c\|f\|_{H_0^{(2s+3)/8}(\mathbb{R}^+)}. \tag{4-15}$$

(c) *If $s<4-4b$, $b<\frac{1}{2}$, $-5<\lambda<\frac{1}{2}$ and $s+4b-2<\lambda<s+\frac{1}{2}$ yields that*

$$\|\psi(t)\mathcal{L}^\lambda f(t,x)\|_{X^{s,b}}\le c\|f\|_{H_0^{(2s+3)/8}(\mathbb{R}^+)}. \tag{4-16}$$

Estimates (4-14), (4-15) *and* (4-16) *are so-called space traces*, *derivative time traces and Bourgain spaces estimates*, *respectively.*

Proof. Let us first prove (4-14). By density, we may assume that $f \in C_{0,c}^{\infty}(\mathbb{R}^+)$. Moreover, from definition of $\mathcal{L}^{\lambda}$, it suffices to consider $\mathcal{L}^{\lambda} f(t,x)$ (removing ψ) for $\operatorname{supp} f \subset [0,1]$, thanks to Lemma 2.4.

From (2-4), (3-2) and (3-7), we see that

$$\mathcal{F}_x(\mathcal{L}^{\lambda} f)(t,\xi) = Me^{-i\pi\lambda/2}(\xi - i0)^{-\lambda} \int_0^t e^{-i(t-t')\xi^4} \mathcal{I}_{-\lambda/4-3/4} f(t')\, dt'.$$

By using the following change of variable $\eta = \xi^4$, (2-5) and the definition of the Fourier transform we have that

$$\begin{aligned} \|\mathcal{L}^{\lambda} f(t,\cdot)\|_{H^s(\mathbb{R})}^2 &\le c \int_{\eta} |\eta|^{-\lambda/2-3/4} \langle\eta\rangle^{s/2} \left| \int_0^t e^{-i(t-t')\eta} \mathcal{I}_{-\lambda/4-3/4} f(t')\, dt' \right|^2 d\eta \\ &= c \int_{\eta} |\eta|^{-\lambda/2-3/4} \langle\eta\rangle^{s/2} \left| (\chi_{(-\infty,t)} \mathcal{I}_{-\lambda/4-3/4} f)\widehat{\ }(\eta) \right|^2 d\eta, \end{aligned}$$

for a fixed t. Note that, by Lemma 2.2, we can replace $|\eta|^{-\lambda/2-3/4}$ by $\langle\eta\rangle^{-\lambda/2-3/4}$, since

$$-1 < -\frac{\lambda}{2} - \frac{3}{4} \Leftrightarrow \lambda < \frac{1}{2}.$$

Moreover, Lemma 2.1 (under the condition $-1 < -\frac{\lambda}{2} - \frac{3}{4} + \frac{s}{2} < 1$ for removing $\chi_{(-\infty,t)}$) and Lemma 2.8 (under the condition $-5 < \lambda$) yield that

$$\begin{aligned} &\int_{\eta} |\eta|^{-\lambda/2-3/4} \langle\eta\rangle^{s/2} \left| (\chi_{(-\infty,t)} \mathcal{I}_{-\lambda/4-3/4} f)\widehat{\ }(\eta) \right|^2 d\eta \\ &\qquad\le c \int_{\eta} \langle\eta\rangle^{s/2-\lambda/2-3/4} \left| (\chi_{(-\infty,t)} \mathcal{I}_{-\lambda/4-3/4} f)\widehat{\ }(\eta) \right|^2 d\eta \\ &\qquad\le c \|\mathcal{I}_{-\lambda/4-3/4} f\|_{H^{s/4-\lambda/4-3/8}}^2 \le c \|f\|_{H_0^{(2s+3)/8}}^2, \end{aligned}$$

which proves (4-14) thanks to the definition of $H_0^s(\mathbb{R}^+)$-norm.

Now we prove (4-15). A direct calculation gives

$$\partial_x^j \mathcal{L}^{\lambda} f = \mathcal{L}^{\lambda-j}(\mathcal{I}_{-j/4} f).$$

With the previous equality in hand and Lemma 2.8, it suffices to show (4-15) for $j = 0$. Lemma 2.4 ensures us to ignore the cut-off function ψ. The change of variable $t \to t - t'$ gives

$$\begin{aligned} &(I - \partial_t^2)^{(2s+3)/16} \left(\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \int_{-\infty}^t e^{i(t-t')\partial_x^4} \delta(x) h(t') dt' \right) \\ &\qquad= \left(\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \int_{-\infty}^t e^{i(t-t')\partial_x^4} \delta(x) (I - \partial_{t'}^2)^{(2s+3)/16} h(t') dt' \right). \end{aligned}$$

So, we just need to prove that

$$\left\| \int_\xi e^{ix\xi}(\xi - i0)^{-\lambda} \int_{-\infty}^t e^{-i(t-t')\xi^4} (\mathcal{I}_{-\frac{\lambda}{4}-\frac{3}{4}} f)(t')\,dt'\,d\xi \right\|_{L_x^\infty L_t^2(\mathbb{R})} \le c\|f\|_{L_t^2(\mathbb{R}^+)}, \tag{4-17}$$

thanks to $\partial_t^\sigma(\mathcal{I}_\alpha f) = \mathcal{I}_\alpha(\partial_t^\sigma f)$. We use $\chi_{(-\infty,t)} = \frac{1}{2}\operatorname{sgn}(t-t') + \frac{1}{2}$ to obtain

$$\begin{aligned}
&\int_\xi e^{ix\xi}(\xi - i0)^{-\lambda} \int_{-\infty}^t e^{-i(t-t')\xi^4} (\mathcal{I}_{-\lambda/4-3/4} f)(t')\,dt'\,d\xi \\
&\quad = \frac{1}{2}\int_\xi e^{ix\xi}(\xi - i0)^{-\lambda} \int_{-\infty}^\infty \operatorname{sgn}(t-t') e^{-i(t-t')\xi^4} (\mathcal{I}_{-\lambda/4-3/4} f)(t')\,dt'\,d\xi \\
&\qquad\qquad + \frac{1}{2}\int_\xi e^{ix\xi}(\xi - i0)^{-\lambda} \int_{-\infty}^\infty e^{-i(t-t')\xi^4} (\mathcal{I}_{-\lambda/4-3/4} f)(t')\,dt'\,d\xi \\
&\quad := I(t,x) + II(t,x).
\end{aligned}$$

We will treat $I(t,x) := I$ and $II(t,x) := II$ separately. To estimate I, we can rewrite it as

$$I(t,x) = \frac{1}{2}\int_\xi e^{ix\xi}(\xi - i0)^{-\lambda}\big((e^{-i\cdot\xi^4}\operatorname{sgn}(\cdot)) * \mathcal{I}_{-\lambda/4-3/4} f\big)(t)\,d\xi.$$

A direct calculation gives

$$\mathcal{F}_t\big((e^{-i\cdot\xi^4}\operatorname{sgn}(\cdot)) * \mathcal{I}_{-\lambda/4-3/4} f\big)(\tau) = \frac{(\tau - i0)^{\frac{1}{4}(3+\lambda)}\hat{f}(\tau)}{i(\tau+\xi^4)}.$$

Fubini's theorem and the dominated converge theorem imply that

$$I(t,x) = \int_\tau e^{it\tau} \lim_{\epsilon\to 0}\int_{|\tau+\xi^4|>\epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{1}{4}(\lambda+3)}(\xi - i0)^{-\lambda}}{i(\tau+\xi^4)}\hat{f}(\tau)\,d\xi\,d\tau.$$

Thus, once we show that the function

$$g(\tau) := \lim_{\epsilon\to 0}\int_{|\tau+\xi^4|>\epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{1}{4}(\lambda+3)}(\xi - i0)^{-\lambda}}{(\tau+\xi^4)}\,d\xi$$

is bounded independently of τ variable, the Plancherel's theorem enables us to obtain (4-17). The change of variable $\xi \mapsto |\tau|^{\frac{1}{4}}\xi$ and the fact that

$$(|\tau|^{\frac{1}{4}}\xi - i0)^{-\lambda} = |\tau|^{-\frac{\lambda}{4}}(\xi_+^{-\lambda} + e^{i\pi\lambda}\xi_-^{-\lambda})$$

gives

$$
\begin{aligned}
g(\tau) = \chi_{\{\tau>0\}} \int_\xi e^{ix|\tau|^{\frac14}\xi} \frac{\xi_+^{-\lambda} + e^{i\pi\lambda}\xi_-^{-\lambda}}{1+\xi^4}\, d\xi \\
- e^{-\frac14(i\pi(\lambda+3))}\chi_{\{\tau<0\}} \int_\xi e^{ix|\tau|^{\frac14}\xi} \frac{\xi_+^{-\lambda} + e^{i\pi\lambda}\xi_-^{-\lambda}}{1-\xi^4}\, d\xi \\
:= g_1 - e^{-\frac14(i\pi(\lambda+3))} g_2.
\end{aligned}
$$

We only consider g_2, since g_1 is uniformly bounded in τ for $-3<\lambda<1$. Let us define the following cut-off function $\zeta \in C^\infty(\mathbb{R})$ such that

$$
\zeta := \begin{cases} 1 & \text{in } \left[\frac34, \frac43\right], \\ 0 & \text{outside } \left(\frac12, \frac32\right). \end{cases}
$$

Then, we obtain

$$
\begin{aligned}
g_2 &= \chi_{\{\tau<0\}} \int_\xi e^{ix|\tau|^{\frac14}\xi} \zeta(\xi)\frac{\xi_+^{-\lambda}}{1-\xi^4}\, d\xi + \chi_{\{\tau<0\}} \int_\xi e^{ix|\tau|^{\frac14}\xi} (1-\zeta(\xi))\frac{\xi_+^{-\lambda}+e^{i\pi\lambda}\xi_-^{-\lambda}}{1-\xi^4}\, d\xi \\
&= g_{21} + g_{22}.
\end{aligned}
$$

It is clear that g_{22} is bounded independently of τ when $\lambda > -3$, and hence it remains to deal with g_{21}. Consider the functions

$$
\widehat{\Theta}(\xi) = \frac{\zeta(\xi)\xi_+^{-\lambda}}{1+\xi+\xi^2+\xi^3} \quad \text{and} \quad \widehat{\Psi}(\xi) = \frac{1}{i(\xi-1)}.
$$

We remark that $\widehat{\Theta}$ is a Schwartz function, and hence $\Theta \in \mathcal{S}(\mathbb{R})$. Moreover, we immediately know that

$$
\Psi(x) = \frac12 e^{ix}\,\mathrm{sgn}(x),
$$

since $\mathcal{F}_x[\mathrm{sgn}(x)](\xi) = \text{v.p.}\frac{2}{i\xi}$. Then, g_{21} can be written as

$$
g_{21}(\tau) = -i\chi_{\{\tau<0\}} \int_\xi e^{ix|\tau|^{\frac14}\xi}\widehat{\Theta}(\xi)\widehat{\Psi}(\xi)\, d\xi = -2i\pi\,\chi_{\{\tau<0\}}(\Theta * \Psi)(|\tau|^{\frac14}x),
$$

which implies

$$
|g_{21}(\tau)| \lesssim \left| \int \Theta(y)\Psi(|\tau|^{\frac14}x - y)\, dy \right| \lesssim \int |\Theta(y)|\, dy \lesssim_\zeta 1.
$$

Now, we bound II. By using the definition of Fourier transform and (2-5) we have, after the change of variable $\eta = \xi^4$ and contour, that

$$\begin{aligned} II(t,x) &= \frac{1}{2}\int_\xi e^{ix\xi}e^{-it\xi^4}(\xi^4 - i0)^{\frac{1}{4}(\lambda+3)}\hat{f}(\xi^4)(\xi - i0)^{-\lambda}\,d\xi \\ &= \frac{1}{2}\int_0^{+\infty} e^{it\eta}e^{-ix\eta^{\frac{1}{4}}}(\eta - i0)^{\frac{1}{4}(\lambda+3)}(\eta^{\frac{1}{4}} - i0)^{-\lambda}\eta^{-\frac{3}{4}}\hat{f}(\eta)\,d\eta = cf(t), \end{aligned}$$

for some $c \in \mathbb{C}$, implying $\|II(\cdot,x)\|_{L^2_t} \lesssim \|f\|_{L^2_t}$. This completes the proof of (4-15).

Lastly, let us show (4-16). A direct calculation ensures that

$$\begin{aligned} &\mathcal{F}_x(\psi(t)\mathcal{L}^\lambda f)(t,\xi) \\ &= Me^{-\frac{1}{2}(i\pi\lambda)}e^{\frac{1}{10}(i\pi(\lambda+4))}(\xi - i0)^{-\lambda}\psi(t)e^{-it\xi^4}\int \frac{e^{it(\tau'+\xi^4)}-1}{i(\tau'+\xi^4)}(\tau' - i0)^{\frac{\lambda}{4}+\frac{3}{4}}\hat{f}(\tau')\,d\tau', \end{aligned}$$

which can be divided into the following quantities:

$$\begin{aligned} \hat{f}_1(t,\xi) &= Me^{-\frac{1}{2}i\pi\lambda}e^{\frac{1}{10}i\pi(\lambda+4)}(\xi - i0)^{-\lambda}\psi(t) \\ &\qquad\times \int \frac{e^{it\tau'} - e^{-it\xi^4}}{i(\tau'+\xi^4)}\theta(\tau'+\xi^4)(\tau' - i0)^{\frac{\lambda}{4}+\frac{3}{4}}\hat{f}(\tau')\,d\tau', \\ \hat{f}_2(t,\xi) &= Me^{-\frac{1}{2}i\pi\lambda}e^{\frac{1}{10}i\pi(\lambda+4)}(\xi - i0)^{-\lambda}\psi(t) \\ &\qquad\times \int \frac{e^{it\tau'}}{i(\tau'+\xi^4)}(1-\theta(\tau'+\xi^4))(\tau' - i0)^{\frac{\lambda}{4}+\frac{3}{4}}\hat{f}(\tau')\,d\tau' \\ \hat{f}_3(t,\xi) &= Me^{-\frac{1}{2}i\pi\lambda}e^{\frac{1}{10}i\pi(\lambda+4)}(\xi - i0)^{-\lambda}\psi(t) \\ &\qquad\times \int \frac{e^{-it\xi^4}}{i(\tau'+\xi^4)}(1-\theta(\tau'+\xi^4))(\tau' - i0)^{\frac{\lambda}{4}+\frac{3}{4}}\hat{f}(\tau')\,d\tau'. \end{aligned}$$

Here $\theta \in \mathcal{S}(\mathbb{R})$ is defined by

$$\theta(\tau) := \begin{cases} 1 & \text{for } |\tau| \le 1, \\ 0 & \text{for } |\tau| \ge 2. \end{cases} \tag{4-18}$$

It follows that $\psi(t)\mathcal{L}^\lambda f = f_1 + f_2 - f_3$.

For f_1, we use the same argument as was done for w_1, in the proof of inequality (4-6). By the Taylor series expansion for $e^{it(\tau'+\xi^4)}$ at $it(\tau'+\xi^4) = 0$, we write

$$\psi(t)\mathcal{L}^\lambda f_1(t,x) = c\sum_{k=1}^{\infty}\frac{i^{k-1}}{k!}\psi^k(t)e^{it\partial_x^4}F_1^k(x),$$

for some constant $c \in \mathbb{C}$, where $\psi^k(t) = t^k\psi(t)$ and

$$\widehat{F_1^k}(\xi) = (\xi - i0)^{-\lambda}\int \theta(\tau'+\xi^4)(\tau'+\xi^4)^{k-1}\tau'^{\frac{\lambda}{4}+\frac{3}{4}}\hat{f}(\tau')\,d\tau'.$$

By using (2-5), (4-6) and (4-18), it is enough to show that

$$\int_\xi \langle\xi\rangle^{2s}|\xi|^{-2\lambda}\left|\int_{|\tau'+\xi^4|\le 1}|\tau'+\xi^4|^{k-1}|\tau'|^{\frac14(\lambda+3)}|\hat f(\tau')|\,d\tau'\right|^2 d\xi \lesssim \|f\|^2_{H_0^{(2s+3)/8}}. \tag{4-19}$$

Let us split $|\xi|$ into two regions: $|\xi|\le 1$ and $|\xi|>1$. For the region $|\xi|\le 1$ and $|\tau'|\lesssim 1$ ($|\xi|\le 1$ and $|\tau'+\xi^4|\le 1$ imply $|\tau'|\lesssim 1$) we have that both $|\xi|^{-2\lambda}$ and $|\tau'|^{\frac12(\lambda+3)}$ are integrable, for $-5<\lambda<\frac12$, respectively. So, we obtain (4-19) by using the Cauchy–Schwarz inequality in τ'.

Assume that $|\xi|>1$, which in addition with $|\tau'+\xi^4|\le 1$ implies $|\tau'|\sim|\xi|^4>1$. Let $\hat f^*(\tau')=\langle\tau'\rangle^{\frac18(2s+3)}\hat f(\tau')$. Then the change of variable $\xi^4\mapsto\eta$ gives that the left-hand side of (4-19) is bounded by

$$\int_{|\xi|>1}|\xi|^3|\mathcal M\hat f^*(\xi^4)|^2\,d\xi \lesssim \int_{|\eta|>1}|\mathcal M\hat f^*(\eta)|^2\,d\eta \lesssim \|f^*\|^2_{L^2}=\|f\|^2_{H_0^{(2s+3)/8}},$$

where $\mathcal M\hat f^*$ is the Hardy–Littlewood maximal function of $\hat f^*$, and f_1 is controlled.

For f_2, from (2-5), the definition of inverse Fourier transform and Lemma 2.5, it follows that

$$\begin{aligned}\|f_2\|^2_{X^{s,b}} &\lesssim \int\!\!\int \langle\xi\rangle^{2s}|\xi|^{-2\lambda}\langle\tau+\xi^4\rangle^{2b}\frac{(1-\theta(\tau+\xi^4))^2}{|\tau+\xi^4|^2}|\tau|^{\frac12(\lambda+3)}|\hat f(\tau)|^2\,d\tau d\xi\\ &\lesssim \int |\tau|^{\frac12(\lambda+3)}\left(\int\frac{\langle\xi\rangle^{2s}|\xi|^{-2\lambda}}{\langle\tau+\xi^4\rangle^{2-2b}}\,d\xi\right)|\hat f(\tau)|^2\,d\tau.\end{aligned}$$

Thus, by the change of variable $\eta=\xi^4$ and Lemma 2.2, for $-5<\lambda$ (we may assume $\operatorname{supp} f\subset[0,1]$, thanks to Lemma 2.4), it suffices to show

$$I(\tau)=\int\frac{|\eta|^{-\frac34-\frac{\lambda}{2}}\langle\eta\rangle^{\frac{s}{2}}}{\langle\tau+\eta\rangle^{2-2b}}\,d\eta\lesssim\langle\tau\rangle^{\frac{2s}{4}-\frac{2\lambda}{4}-\frac34}.$$

Here, we split $|\tau|$ into two regions: $|\tau|\le 2$ and $|\tau|>2$. When $|\tau|\le 2$, we have $\langle\tau+\eta\rangle\sim\langle\eta\rangle$. For $s<4-4b$ and $s+4b-\frac72<\lambda<\frac12$, we get

$$I(\tau)\lesssim\int_{|\eta|\le 1}|\eta|^{\frac14(-3-2\lambda)}+\int\frac{d\eta}{\langle\eta\rangle^{2-2b-\frac{s}{2}+\frac34+\frac{\lambda}{2}}}\lesssim 1.$$

Now, working in the region $|\tau|>2$, we divide the integral region in η into $|\eta|<\frac{|\tau|}{2}$ and $|\eta|\ge\frac{|\tau|}{2}$. In the first region, for $b<\frac12$ and $\lambda<\min\left(\frac12,s+\frac12\right)$, we bound in the following way:

$$\langle\tau\rangle^{2b-2}\left(\int_{|\eta|\le 1}|\eta|^{-\frac34-\frac{2\lambda}{4}}\,d\eta+\int_{1<|\eta|\le\frac{|\tau|}{2}}|\eta|^{\frac14(-3-2\lambda+2s)}d\eta\right)\lesssim\langle\tau\rangle^{\frac14(-3-2\lambda+2s)}.$$

On the other hand, in the second region, we have that $|\tau+\eta| \geq \frac{1}{2}|\tau| > 1$. Then, for $s-2 < \lambda$ and $b < \frac{1}{2}$, it holds that

$$I(\tau) \lesssim \langle\tau\rangle^{\frac{1}{4}(-3-2\lambda+2s)} \int \frac{d\eta}{\langle\tau+\eta\rangle^{2-2b}} \lesssim \langle\tau\rangle^{\frac{1}{4}(-3-2\lambda+2s)} \int_{|s|>1} \frac{ds}{|s|^{2-2b}} \lesssim \langle\tau\rangle^{\frac{1}{4}(-3-2\lambda+2s)},$$

so

$$\|f_2\|_{X^{s,b}} \lesssim \|f\|_{H_0^{(2s+3)/8}}.$$

This completes the estimate for f_2.

Finally, let us show that f_3 can be controlled. Similarly as for f_1, it suffices to show

$$\int \langle\xi\rangle^{2s}|\xi|^{-2\lambda} \left| \int (1-\theta(\tau'+\xi^4))|\tau'+\xi^4|^{-1}|\tau'|^{\frac{1}{4}(\lambda+3)}|\hat{f}(\tau')|\,d\tau' \right|^2 d\xi \lesssim \|f\|^2_{H_0^{(2s+3)/8}}. \tag{4-20}$$

Again, we split the region $|\xi|$ as follows: $|\xi| \leq 1$ and $|\xi| > 1$. Considering $|\xi| \leq 1$, since $|\xi|^{-2\lambda}$ is integrable, for $\lambda < \frac{1}{2}$, and we may ignore the integration in ξ. Let us work in the region $|\tau'| \leq 1$. In this region $|\tau'|^{\frac{1}{2}(\lambda+3)}$ is integrable, for $\lambda > -5$, and hence we get (4-20).

On the region $|\tau'| > 1$, since $|\tau'+\xi^4| \sim |\tau'|$ and $|\tau'|^{\frac{1}{5}(-s+\lambda-3)}$ are L^2 integrable, for $\lambda < s+\frac{1}{2}$, we also get (4-20) by using the Cauchy–Schwarz inequality in τ'. Still looking on the region $|\tau'| > 1$, since $|\tau'+\xi^4| \sim |\tau'|$ we have that the left-hand side of (4-20) is bounded by

$$\begin{aligned} c\left(\int_{|\tau'|>1} |\tau'|^{\frac{\lambda-1}{4}}|\hat{f}|^2 d\tau'\right)^2 &\sim \left(\int_{|\tau'|>1} \frac{|\tau'|^{\frac{\lambda-1}{4}}}{\langle\tau'\rangle^{\frac{2s+3}{8}}}\langle\tau'\rangle^{\frac{2s+3}{8}}|\hat{f}|^2 d\tau'\right)^2 \\ &\lesssim \int \frac{\langle\tau'\rangle^{\frac{\lambda-1}{2}}}{\langle\tau'\rangle^{\frac{2s+3}{8}}} d\tau' \|f\|^2_{H^{(2s+3)/4}} \lesssim \|f\|^2_{H_0^{(2s+3)/8}}, \end{aligned} \tag{4-21}$$

where we have used that $\lambda < s+\frac{1}{2}$, and the result follows on $|\xi| \leq 1$. On the other hand, in the region $|\xi| > 1$ and $|\tau'| \leq 1$, since $|\tau'+\xi^4| \sim |\xi|^4 \sim \langle\xi\rangle^4$ and $\langle\xi\rangle^{2s-2\lambda-8}$ are integrable for $\lambda > -\frac{7}{2}+s$, we also get (4-20).

Consider the region $|\xi| > 1$ and $|\tau'| > 1$. There are two possibilities:

(I) $|\tau'| \leq \frac{1}{2}|\xi|^4$.

(II) $\frac{1}{2}|\xi|^4 < |\tau'|$.

In view of the proof of [Holmer 2006, Lemma 5.8(d)] (see also [Holmer 2005]), one can replace

$$\frac{1-\theta(\tau'+\xi^4)}{\tau'+\xi^4}$$

by $\beta(\tau'+\xi^4)$ for some $\beta\in\mathcal{S}(\mathbb{R})$. Hence, the left-hand side of (4-20) is dominated by

$$c\int_{|\xi|>1}|\xi|^{2s-2\lambda}\left|\int_{|\tau'|>1}|\tau'+\xi^4|^{-N}|\tau'|^{\frac{1}{4}(\lambda+3)}|\hat{f}(\tau')|\,d\tau'\right|^2 d\xi, \tag{4-22}$$

for $N\geq 0$. By the Cauchy–Schwarz inequality and choosing $N=N(s,\lambda)\gg 1$, we have (4-20) for both cases. Indeed, for the case I (in this case $|\tau'+\xi^4|\sim|\xi|^4$), (4-22) can be controlled by

$$c\|f\|^2_{H^{\frac{1}{8}(2s+3)}}\int_{|\xi|>1}|\xi|^{2s-2\lambda-4N}\int_{1<|\tau'|\leq\frac{1}{2}|\xi|^4}|\tau'|^{\frac{1}{4}(-2s-3+2\lambda+6-4N)}d\tau'd\xi\lesssim\|f\|^2_{H_0^{\frac{1}{8}(2s+3)}}.$$

For case II (in this case $|\tau'+\xi^4|\sim|\tau'|$), (4-22) is bounded by

$$c\int_{|\xi|>1}|\xi|^{2s-2\lambda}\left|\int_{\frac{1}{2}|\xi|^4<|\tau'|}|\tau'|^{\frac{1}{4}(2\lambda+6-2s-3-4N)}|\tau'|^{\frac{1}{8}(2s+3)}|\hat{f}(\tau')|\,d\tau'\right|^2 d\xi \lesssim\|f\|^2_{H_0^{\frac{1}{8}(2s+3)}},$$

thus

$$\|f_3\|_{X^{s,b}}\lesssim\|f\|_{H_0^{\frac{1}{8}(2s+3)}},$$

finishing the estimate for f_3.

Remembering that $\psi(t)\mathcal{L}^\lambda f=f_1+f_2-f_3$, and using the estimates of f_i, $i=1,2,3$, equation (4-16) follows and the proof is complete. □

To close this section, let us enunciate the trilinear estimates associated to the fourth order nonlinear Schrödinger equation (1-2). The proof of this estimate can be found in [Oh and Tzvetkov 2017] (see also [Capistrano-Filho and Cavalcante 2019]), thus we will omit it.

Proposition 4.4. *For $s\geq 0$, there exists $b=b(s)<\frac{1}{2}$ such that we have*

$$\|u_1u_2\bar{u}_3\|_{X^{s,-b}}\leq c\|u_1\|_{X^{s,b}}\|u_2\|_{X^{s,b}}\|u_3\|_{X^{s,b}}. \tag{4-23}$$

5. Proof of Theorem 1.1

Initially, we pick an extension $\tilde{u}_0\in H^s(\mathbb{R})$ of u_0 such that

$$\|\tilde{u}_0\|_{H^s(\mathbb{R})}\leq 2\|u_0\|_{H^s(\mathbb{R}^+)}.$$

Let $b=b(s)<\frac{1}{2}$ such that the estimates given in Proposition 4.4 are valid.

By using similar arguments to those in Section 3B, let

$$u(t,x)=\mathcal{L}^{\lambda_1}\gamma_1(t,x)+\mathcal{L}^{\lambda_2}\gamma_2(t,x)+F(t,x), \tag{5-1}$$

where γ_i $(i=1,2)$ will be chosen in terms of initial and boundary data u_0, f, g and $F(t,x)=e^{it\partial_x^4}\tilde{u}_0+\lambda\mathcal{D}(|u|^2u)$.

Remember that a_j and b_j are defined by

$$\begin{aligned} a_j &= \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(1+3\lambda_j)}+e^{-i\frac{\pi}{8}(1-5\lambda_j)}}{\sin\left(\frac{1}{4}(1-\lambda_j)\pi\right)}\right), \\ b_j &= \frac{M}{8}\left(\frac{e^{-i\frac{\pi}{8}(-2+3\lambda_j)}+e^{-i\frac{\pi}{8}(6-5\lambda_j)}}{\sin\left(\frac{1}{4}(2-\lambda_j)\pi\right)}\right). \end{aligned} \tag{5-2}$$

By Lemmas 3.2 and 3.3, we get

$$f(t)=u(t,0)=a_1\gamma_1(t)+a_2\gamma_2(t)+F(t,0) \tag{5-3}$$

$$g(t)=\partial_x u(t,0)=b_1\mathcal{I}_{-1/4}\,\gamma_1(t)+b_2\mathcal{I}_{-1/4}\,\gamma_2(t)+\partial_x F(t,0). \tag{5-4}$$

Putting together (5-3) and (5-4), we can write a matrix in the form

$$\begin{bmatrix} f(t)-F(t,0) \\ \mathcal{I}_{1/4}\,g(t)-\mathcal{I}_{1/4}\,\partial_x F(t,0)\end{bmatrix} = A\begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix},$$

where

$$A(\lambda_1,\lambda_2)=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2\end{bmatrix}.$$

By using a mathematical software, the determinant of matrix $A(\lambda_1,\lambda_2)$ is given by

$$\begin{aligned} \det A = 2(-1)^{\frac{15}{8}}e^{-\frac{1}{8}i(6+3\lambda_1+\lambda_2)\pi}(1+e^{i\lambda_1\pi})(-1+e^{\frac{1}{2}i\lambda_2\pi})\sec\left(\frac{(1+\lambda_1)\pi}{4}\right) \\ +4(-1)^{\frac{3}{8}}e^{-\frac{1}{8}i(\lambda_1+\lambda_2)\pi}(-1+e^{\frac{1}{2}i\lambda_1\pi})(1-ie^{\frac{1}{2}i\lambda_2\pi}). \end{aligned}$$

Note that the following graphics, with real and imaginary parts, of the determinant function $A(\lambda_1,\lambda_2)$, help us to see when the matrix A is invertible.

Thus, matrix $A(\lambda_1,\lambda_2)$ is invertible if we get

$$\lambda_2 \neq \frac{2}{\pi}\left(2\pi n - i\log\left\{\frac{-2(-1)^{\frac{1}{4}}e^{\frac{i\pi\lambda_1}{4}}+2(-1)^{\frac{1}{4}}e^{\frac{3i\pi\lambda_1}{4}}+\left(e^{i\pi\lambda_1}+1\right)\sec\left(\frac{(1+\lambda_1)\pi}{4}\right)}{-2(-1)^{\frac{3}{4}}e^{\frac{i\pi\lambda_1}{4}}+2(-1)^{\frac{3}{4}}e^{\frac{3i\pi\lambda_1}{4}}+\left(e^{i\pi\lambda_1}+1\right)\sec\left(\frac{(1+\lambda_1)\pi}{4}\right)}\right\}\right) \tag{5-5}$$

and

$$\lambda_j \neq 1-4n, \quad \lambda_j\neq 2-4n, \quad j=1,2, \tag{5-6}$$

for all $n\in\mathbb{Z}$.

Figure 1 helps us to see that there are an infinite set of parameters which satisfy the relations (5-5) and (5-6). In fact, for example, pick $\lambda_1\approx 0$ and $\lambda_2\approx\frac{1}{3}$. Thus, for $0\le s<\frac{1}{2}$, the choice of parameters λ_1 and λ_2 satisfying the conditions

$$-3<\lambda_j<\tfrac{1}{2}, \quad s+4b-2<\lambda_j<s+\tfrac{1}{2}, \quad j=1,2,$$

ensures that Lemma 4.3 holds. Thus, for fixed $s \in \big[0, \frac{1}{2}\big)$, we can choose λ_1 and λ_2 as before and define the forcing functions $\gamma_1(t)$ and $\gamma_2(t)$ for any λ_j, $j = 1, 2$, given by

$$\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = A^{-1} \begin{bmatrix} f(t) - F(t, 0) \\ \mathcal{I}_{1/4}\, g(t) - \mathcal{I}_{1/4}\, \partial_x F(t, 0) \end{bmatrix}, \tag{5-7}$$

which shows that formula (5-1) restricted on the set $(0, +\infty) \times (0, +\infty)$ satisfies

$$(i\partial_t - \partial_x^4)u = \lambda |u|^2 u,$$

in the sense of distributions.

Thus, we define the solution operator by

$$\Lambda u(t, x) = \psi(t)\mathcal{L}^{\lambda_1}\gamma_1(t, x) + \psi(t)\mathcal{L}^{\lambda_2}\gamma_2(t, x) + \psi(t)F(t, x), \tag{5-8}$$

where

$$\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = A^{-1} \begin{bmatrix} f(t) - F(t, 0) \\ \mathcal{I}_{1/4}\, g(t) - \mathcal{I}_{1/4}\, \partial_x F(t, 0) \end{bmatrix},$$

$F(t, x) = e^{it\partial_x^4}\tilde{u}_0 + \lambda \mathcal{D}(\psi_T |u|^2 u)$ and ψ is defined by (2-1).

Recall the solution space $Z^{s,b}$, defined in Section 2B, under the norm

$$\|v\|_{Z^{s,b}} = \sup_{t \in \mathbb{R}} \|v(t, \cdot)\|_{H^s} + \sum_{j=0}^{1} \sup_{x \in \mathbb{R}} \|\partial_x^j v(\cdot, x)\|_{H^{\frac{1}{8}(2s+3-2j)}} + \|v\|_{X^{s,b}}.$$

The estimates obtained in Section 2 together with estimates of Section 4 and (4-23) yield that

$$\|\Lambda u\|_{Z^{s,b}} \le c(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+)} + \|g\|_{H^{\frac{1}{8}(2s+1)(\mathbb{R}^+)}}) + C_1 T^{\epsilon} \|u\|_{Z^{s,b}}^3,$$

for ϵ adequately small. Similarly,

$$\|\Lambda u_1 - \Lambda u_2\|_{Z^{s,b}} \le C_2 T^{\epsilon} (\|u_1\|_{Z^{s,b}}^2 + \|u_2\|_{Z^{s,b}}^2) \|u_1 - u_2\|_{Z^{s,b}},$$

for $u_1(0, x) = u_2(0, x)$.

Consider in Z the ball defined by $B = \{u \in Z^{s,b};\ \|u\|_{Z^{s,b}} \le M\}$, where

$$M = 2c\big(\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+)} + \|g\|_{H^{\frac{1}{8}(2s+1)(\mathbb{R}^+)}}\big).$$

Lastly, choosing $T = T(M)$ sufficiently small, such that

$$\|\Lambda u\|_{Z^{s,b}} \le M \quad \text{and} \quad \|\Lambda u_1 - \Lambda u_2\|_{Z^{s,b}} \le \tfrac{1}{2} \|u_1 - u_2\|_{Z^{s,b}},$$

it follows that Λ is a contraction map on B, finishing the proof of Theorem 1.1. □

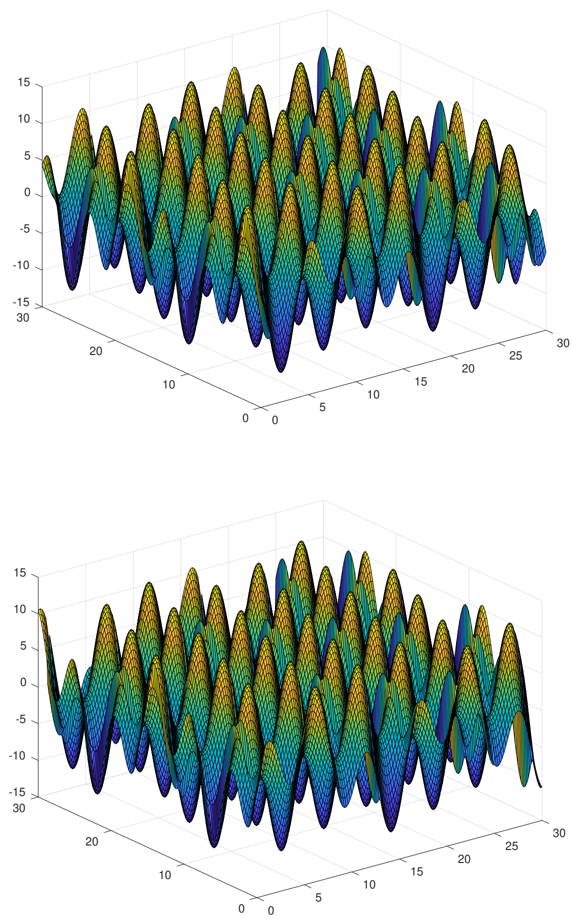

Figure 1. Real (top) and imaginary (bottom) parts of det A.

Remarks. Concerning our main result, Theorem 1.1, the following remarks are now in order:

1. An important point to treat in dispersive systems is the analysis of the scaling. For our case, that is, for the biharmonic Schrödinger equation on the half-line, we have the following: if $u(t,x)$ is solution for IBVP (1-2) on $[0,T)\times(0,\infty)$, then, for $\lambda>0$, the function $u_\lambda(t,x)=\lambda^2 u(\lambda^4 t,\lambda x)$ is solution for (1-2) on $[0,T/\lambda^4)\times(0,\infty)$ with initial-boundary conditions $u_\lambda(0,x)=\lambda^2 u_0(\lambda x):=u_{0\lambda}$, $u_\lambda(t,0)=\lambda^2 f(\lambda^4 t):=f_\lambda$ and $u_{x,\lambda}(t,0)=\lambda^3 g(\lambda^4 t):=g_\lambda$. A straightforward calculation gives

$$\begin{aligned}\|u_{0\lambda}\|_{H^s(\mathbb{R}^+)}+\|f_\lambda\|_{H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+)}+\|g_\lambda\|_{H^{\frac{1}{8}(2s+1)}(\mathbb{R}^+)}\\ \lesssim \lambda^{\frac{3}{2}}\langle\lambda\rangle^s\|u_0\|_{H^s(\mathbb{R}^+)}+\langle\lambda\rangle^{\frac{1}{2}(2s+3)}\|f\|_{H^{\frac{1}{8}(2s+3)}(\mathbb{R}^+)}+\lambda\langle\lambda\rangle^{\frac{1}{8}(2s+1)}\|g\|_{H^{\frac{1}{8}(2s+1)}(\mathbb{R}^+)}.\end{aligned}\tag{5-9}$$

2. In order to make the norms of our initial data u_0, f and g small, we rescale the data u_0 and g by choosing λ adequately small, by using (5-9). However, we can not rescale the function f since a positive power of λ does not appear in (5-9). To overcome this difficulty in our context, we introduce the cut-off function ψ_T, defined by (2-1), in the operator Λ (see (5-8)) to prove that Λ is thus a contraction, proving the main result of the article.

3. It is important to note that the scaling argument was successful in the cases of the quadratic NLS equation [Cavalcante 2017], KdV equation [Holmer 2006] and Kawahara equation [Cavalcante and Kwak 2019] posed on the half-line.

4. Finally, in view of (5-3), (5-4) and (5-7), it is necessary to check $\gamma_i(t)$, $i=1,2$ to be well-defined in $H_0^{(2s+3)/8}(\mathbb{R}^+)$. However, it follows from Lemmas 4.1, 4.2 and 4.3, Propositions 4.4 and Lemmas 2.1 and 2.3.

6. Further comments and open problems

In this section, our plan is to present four problems that can be treated with the approach used in this article.

6A. ***Biharmonic NLS on star graphs.*** The authors [Capistrano-Filho et al. 2019] considered the biharmonic Schrödinger equation on star graphs, given by N edges $(0,\infty)$ connected with a common vertex $(0,0,\dots,0)$ (see Figure 2), namely

$$\begin{cases} i\partial_t u_j - \partial_x^4 u_j + \lambda|u_j|^2 u_j = 0, & (t,x)\in(0,T)\times(0,\infty),\ j=1,2,\dots,N,\\ u_j(0,x)=u_{j0}(x), & x\in(0,\infty),\end{cases} \tag{6-1}$$

with initial conditions $(u_1(0,x),u_2(0,x),\dots,u_N(0,x))\in H^s(\mathbb{R}^+)$.

For a better understanding, we are interested in solving (6-1) with the following three classes boundary conditions:

$$\begin{cases} \partial_x^k u_1(t,0)=\partial_x^k u_2(t,0)=\cdots=u_N(t,0), & k=0,1,\ t\in(0,T),\\ \sum_{j=1}^n \partial_x^k u_j(t,0)=0, & k=2,3,\ t\in(0,T);\end{cases} \tag{6-2}$$

$$\begin{cases} \partial_x^k u_1(t,0)=\partial_x^k u_2(t,0)=\cdots=u_N(t,0), & k=2,3,\ t\in(0,T),\\ \sum_{j=1}^n \partial_x^k u_j(t,0)=0, & k=0,1,\ t\in(0,T);\end{cases} \tag{6-3}$$

$$\begin{cases} \partial_x^k u_1(t,0)=\partial_x^k u_2(t,0)=\cdots=u_N(t,0), & k=0,3,\ t\in(0,T),\\ \sum_{j=1}^n \partial_x^k u_j(t,0)=0, & k=1,2,\ t\in(0,T).\end{cases} \tag{6-4}$$

The motivation of these boundary conditions and how we can choose it, follows the ideas contained in [Cavalcante 2018], and are detailed in [Capistrano-Filho et al. 2019].

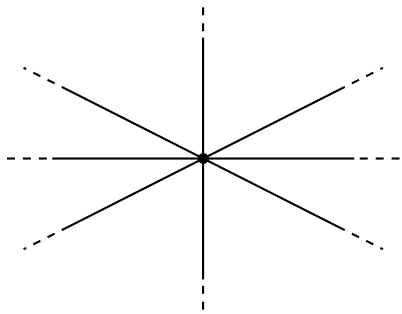

Figure 2. Star graphs connected with a common vertex $(0, 0, \ldots, 0)$.

6B. ***Control theory.*** We split this section in two parts: control theory for the biharmonic NLS on star graphs and on an unbounded domain, respectively.

6B1. *Control theory of biharmonic NLS on star graphs.* First, let us consider the controllability problem associated to (6-1) with three possibilities of boundary conditions, namely, (6-2), (6-3) and (6-4). Due of the recent development of graph theory for the Korteweg–de Vries equation, in the following paragraph we present a few comments about this study.

In three interesting papers Ammari and Crépeau [2018], Cavalcante [2018] and Mugnolo et al. [2018] dealt with the study of the KdV and Airy equations in graphs. In summary, in the first work, the authors proposed a model using the Korteweg–de Vries equation on a finite star-shaped network and proved the well-posedness of the system. Also, as the main result of the work, by using properties of the energy, they showed that the solutions of the system decays exponentially to zero (as $t \to \infty$) and they studied an exact boundary controllability problem. In the second work, Cavalcante showed local well-posedness for the Cauchy problem associated with Korteweg–de Vries equation on a metric star graph. More precisely, he used the Duhamel boundary forcing operator, in the context of half-line, introduced by Colliander and Kenig [2002] and Holmer [2006] to achieve his result. Finally, Mugnolo et al. obtained a characterization of all boundary conditions under which the Airy-type evolution equation $u_t = \alpha u_{xxx} + \beta u_x$, for $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$ on star graphs, generates contraction semigroups.

In this spirit, looking for the energy identity of the system (6-1), namely the L^2-energy, which satisfies an equality given by

$$\begin{aligned}(6\text{-}5)\quad & E(u_1(T,x), u_2(T,x), \ldots, u_N(T,x)) \\ &= -\sum_{i=1}^{N} \int_0^T \operatorname{Im}(\partial_x^3 u_j(t,0)\bar{u}_j(t,0))\,dt + \sum_{i=1}^{N} \int_0^T \operatorname{Im}(\partial_x^2 u_j(t,0)\partial_x \bar{u}_j(t,0))\,dt \\ &\qquad - E(u_1(0,x), u_2(0,x), \ldots, u_N(0,x)),\end{aligned}$$

where

$$E(u_1(t,x), u_2(t,x), \dots, u_N(t,x)) := \sum_{i=1}^{N} \int_0^{+\infty} |u(t,x)|^2\,dx,$$

the following natural questions arise.

Problem A: Which are the boundary conditions that we can impose in (6-2), (6-3) and (6-4) such that the energy is a nonincreasing function of the time variable t?

Problem B: If we can impose some boundary conditions such that the energy (6-5) is a nonincreasing function of the time variable t, is the system (6-1), with appropriate boundary conditions, asymptotically stable when the time tends to infinity?

Problem C: Can we find appropriate boundary controls such that the system (6-1) is controllable in some sense?

6B2. *Control theory of biharmonic NLS in unbounded domain.* In the context of control in unbounded domain Faminskii [2019] considered the initial-boundary value problem posed on infinite domain for the Korteweg–de Vries equation. Precisely, he elected a function f_0 on the right-hand side of the equation as an unknown function, regarded as a control. Thus he proved that this function must be chosen such that the corresponding solution should satisfy certain additional integral condition.

These techniques probably work well for the following biharmonic NLS system:

$$\begin{cases} i\partial_t u + \gamma\partial_x^4 u + \lambda|u|^2 u = f_0(t)v(x,t), & (t,x)\in(0,T)\times(0,\infty),\\ u(0,x) = u_0(x), & x\in(0,\infty),\\ u(t,0)=h(t),\ u_x(t,0)=g(t) & t\in(0,T), \end{cases} \tag{6-6}$$

for $\gamma, \lambda \in \mathbb{R}$, where v is a given function and f_0 is an unknown control function. Therefore, the issue here is:

Problem D: Is (6-6) controllable in the sense of Faminskii's work? Namely, can we find a pair $\{f_0, u\}$, satisfying appropriate additional integral conditions (for details see [Faminskii 2019])?

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