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Rapid exponential stabilization of a Boussinesq system of KdV–KdV Type

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This paper studies the exponential stabilization of a Boussinesq system describing the two-way propagation of small amplitude gravity waves on the surface of an ideal fluid, the so-called *Boussinesq system of the Korteweg–de Vries type*. We use a Gramian-based method introduced by Urquiza to design our feedback control. By means of spectral analysis and Fourier expansion, we show that the solutions of the linearized system decay uniformly to zero when the feedback control is applied. The decay rate can be chosen as large as we want. The main novelty of our work is that we can exponentially stabilize this system of two coupled equations using only one scalar input.

Keywords: KdV–KdV system; Gramian-based method; stabilization; feedback control.

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1. Introduction

1.1. *Setting of the problem*

Boussinesq introduced in [3] several nonlinear partial differential equations to explain certain physical observations concerning the water waves, e.g. the emergence and stability of solitons. Unfortunately, several systems derived by Boussinesq proved to be ill-posed, so that there was a need to propose other systems with better mathematical properties. In that direction, the four-parameter family of Boussinesq systems

$$\begin{cases} \eta_t + v_x + (\eta v)_x + a v_{xxx} - b\eta_{xxt} = 0, \\ v_t + \eta_x + v v_x + c\eta_{xxx} - d v_{xxt} = 0 \end{cases} \tag{1.1}$$

was introduced by Bona *et al.* in [1] to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two-dimensional. In (1.1), η is the elevation of the fluid surface from the equilibrium position and v is the horizontal velocity in the flow. The parameters a, b, c, d are required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}\left(1 - \theta^2\right) \geq 0, \tag{1.2}$$

where $\theta \in [0, 1]$ and thus $a + b + c + d = \frac{1}{3}$. As it has been proved in [1], the initial value problem for the *linear system* associated with (1.1) is well posed on $\mathbb{R}$ if and only if the parameters a, b, c, d fall in one of the following cases:

$$\text{(C1)} \quad b, d \geq 0, \quad a \leq 0, \quad c \leq 0;$$

$$\text{(C2)} \quad b, d \geq 0, \quad a = c > 0.$$

The well-posedness of the system (1.1) on the line ($x \in \mathbb{R}$) was investigated in [2]. Considering (C_2) with $b = d = 0$, then necessarily $a = c = 1/6$. Using the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives a coupled system of two Korteweg–de Vries (KdV) equations equivalent to (1.1) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + w_x + w_{xxx} + (\eta w)_x = 0, & \text{in } (0, L) \times (0, +\infty), \\ w_t + \eta_x + \eta_{xxx} + w w_x = 0, & \text{in } (0, L) \times (0, +\infty), \\ \eta(x, 0) = \eta_0(x), \quad w(x, 0) = w_0(x), & \text{in } (0, L), \end{cases} \tag{1.3}$$

which is the so-called *Boussinesq system of KdV–KdV type.*

The goal of this paper is to investigate the boundary stabilization for the linear Boussinesq system of KdV–KdV type

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0, & \text{in } (0, L) \times (0, +\infty), \\ w_t + \eta_x + \eta_{xxx} = 0, & \text{in } (0, L) \times (0, +\infty), \\ \eta(x, 0) = \eta_0(x), \quad w(x, 0) = w_0(x), & \text{in } (0, L), \end{cases} \tag{1.4}$$

with boundary conditions

$$\begin{cases} \eta(0,t) = 0, \quad \eta(L,t) = 0, \quad \eta_x(0,t) = f(t), \quad \text{in } (0,+\infty), \\ w(0,t) = 0, \quad w(L,t) = 0, \quad w_x(L,t) = 0, \quad \text{in } (0,+\infty), \end{cases} \tag{1.5}$$

where $f(t)$ is the boundary control. We are mainly concerned with the following problem.

***Stabilization Problem*:** *Can one find a linear feedback control law*

$$f(t) = F[(\eta(\cdot,t), w(\cdot,t)], \quad t \in (0,\infty),$$

such that the closed-loop system (1.4) *with boundary condition* (1.5) *is exponentially stable*?

1.2. *Previous results*

Abstract methods have been developed to obtain the rapid stabilization of linear partial differential equations. Among them, we cite the works [9, 15, 16] based on the Gramian approach. In this paper, we are interested in applying this method to design the feedback control law for the system (1.4) and (1.5). The method presented here was successfully applied by Cerpa and Crépeau in [6] to study the rapid stabilization of the KdV equation. Although this method is typically used in a single equation, this approach has not yet widespread applied to coupled systems.

Stability properties of systems (1.3) or (1.4) on a bounded domain have been studied by several authors. The pioneering work, for the system under consideration in this work, is due to Rosier and Pazoto in [13]. They showed the asymptotic behavior for the solutions of the system (1.4) satisfying the boundary conditions

$$\begin{cases} w(0,t) = w_{xx}(0,t) = 0, & \text{on } (0,T), \\ w_x(0,t) = \alpha_0 \eta_x(0,t), & \text{on } (0,T), \\ w(L,t) = \alpha_2 \eta(L,t), & \text{on } (0,T), \\ w_x(L,t) = -\alpha_1 \eta_x(L,t), & \text{on } (0,T), \\ w_{xx}(L,t) = -\alpha_2 \eta_{xx}(L,t), & \text{on } (0,T). \end{cases} \tag{1.6}$$

In (1.6), α_0, α_1 and α_2 denote some nonnegative real constants. Under the above boundary conditions, they observed that the derivative of the energy associated to the system (1.3) satisfies

$$\frac{dE}{dt} = -\alpha_2 |\eta(L,t)|^2 - \alpha_1 |\eta_x(L,t)|^2 - \alpha_0 |\eta_x(0,t)|^2,$$

where

$$E(t) = \frac{1}{2} \int_0^L (\eta^2 + w^2) dx.$$

This indicates that the boundary conditions play the role of a damping mechanism, at least for the linearized system. In [13] the authors provide the following result.

Theorem A (Pazoto and Rosier [13]). *Assume that $\alpha_0 \geq 0, \alpha_1 > 0$, and that $\alpha_2 = 1$. Then there exist two constants $C_0, \mu_0 > 0$ such that for any $(\eta_0, w_0) \in L^2(0,L) \times L^2(0,L)$, the solution of* (1.4) *with boundary condition* (1.6) *satisfies*

$$\|(\eta(t), w(t))\|_{L^2(0,L)\times L^2(0,L)} \leq C_0 \mathrm{e}^{-\mu_0 t}\|(\eta_0, w_0)\|_{L^2(0,L)\times L^2(0,L)}, \quad \forall\, t \geq 0.$$

Recently, Capistrano–Filho and Gallego [4] investigated the system (1.4) with two controls in the boundary conditions

$$\begin{cases} \eta(0,t) = 0, \quad \eta(L,t) = 0, \quad \eta_x(0,t) = f(t), & \text{in } (0,+\infty), \\ w(0,t) = 0, \quad w(L,t) = 0, \quad w_x(L,t) = g(t), & \text{in } (0,+\infty), \end{cases} \tag{1.7}$$

and deal with the local rapid exponential stabilization by using the backstepping method. They designed boundary feedback controls

$$f(t) = F_1(\eta(\cdot,t), \omega(\cdot,t)) \quad \text{and} \quad g(t) = F_2(\eta(\cdot,t), \omega(\cdot,t)),$$

that lead to the stabilization of the system. The authors proved that the solution of the closed-loop system decays exponentially to zero in the $L^2(0,L)$–norm and the decay rate can be tuned to be as large as desired.

Theorem B (Capistrano–Filho and Gallego [4]). *Let $L \in (0,+\infty)\backslash\mathcal{N}$ where*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2 + kl + l^2};\ k, l \in \mathbb{N}^* \right\}. \tag{1.8}$$

For every $\lambda > 0$, there exist a continuous linear feedback control law

$$F := (F_1, F_2) \colon L^2(0,L) \times L^2(0,L) \to \mathbb{R} \times \mathbb{R},$$

and positive constant $C > 0$. Then, for every $(\eta_0, w_0) \in L^2(0,L) \times L^2(0,L)$, the solution (η, w) of (1.4) *with boundary conditions* (1.7) *belongs to space $C([0,T]; (L^2(0,L) \times L^2(0,L)))$ and satisfies*

$$\|(\eta(t), w(t))\|_{L^2(0,L)\times L^2(0,L)} \leq Ce^{-\frac{\lambda}{2}t}\|(\eta_0, w_0)\|_{L^2(0,L)\times L^2(0,L)}, \quad \forall\, t \geq 0.$$

It is important to emphasize that our goal in this paper is to stabilize the system (1.4) using only one feedback control, improving thus the result in [4]. Concerning controllability, the paper [5] studied different configurations for the position of the control, in particular they proved the following.

Theorem C (Capistrano–Filho *et al.* [5]). *Let $T > 0$ and $L \in (0,+\infty)\backslash\mathcal{N}$. For all states $(\eta_0, w_0), (\eta_1, w_1) \in [L^2(0,L)]^2$ one can find a control $f \in L^2(0,T)$ such that the solution*

$$(\eta, w) \in C([0,T], [L^2(0,L)]^2) \cap L^2(0,T, [H^1(0,L)]^2)$$

of (1.4) *and* (1.5) *satisfies $\eta(T,x) = \eta_1(x)$ and $w(T,x) = w_1(x)$, $x \in (0,L)$.*

As in the case of the KdV equation [14, Lemma 3.5], when $L \in \mathcal{N}$, the linear system (1.4) and (1.5) is not controllable. To prove Theorem C, the authors used the classical duality approach based upon the Hilbert Uniqueness Method (HUM) due to Lions [11], which reduces the exact controllability of the system to some observability inequality to be proved for the adjoint system. Then, to establish the required observability inequality, was used the compactness-uniqueness argument due to Lions [12] and some multipliers, which reduces the analysis to study a spectral problem. The spectral problem is finally solved by using a method introduced by Rosier in [14], based on the use of complex analysis, namely, the Paley–Wiener theorem. As we will prove later, in this paper we obtain a controllability result as in Theorem C but in different state space. This is required by our stabilization method.

1.3. *Main result and outline of the work*

In order to present the main result of the paper, let us now define the spaces that we will work on. To do that, we first need a spectral analysis (see Sec. 3 for details) for the operator $A : D(A) \subset L^2(0,L)^2 \to L^2(0,L)^2$ given by

$$A(\eta, w) = (-w' - w''', -\eta' - \eta''')$$

and

$$D(A) = \{(\eta, w) \in [H^3(0,L) \cap H_0^1(0,L)]^2 : \eta'(0) = w'(L) = 0\}.$$

As can be seen in [5], the operator A has a compact resolvent and it can be diagonalized in an orthonormal basis, i.e. the spectrum $\sigma(A)$ of A consists only of eigenvalues and the eigenfunctions form an orthonormal basis of X_0. Thus, due to the results presented in [5], there exists an orthonormal basis $\{(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}\}$ in $[L^2_{\mathbb{C}}(0,L)]^2$, endowed with the natural scalar product

$$((\theta, u), (\varphi, \omega)) = \int_0^L (\theta(x)\overline{\varphi(x)} + u(x)\overline{\omega(x)})dx,$$

composed of eigenfunctions of A satisfying

$$A(\theta_n^+, u_n^+) = i\lambda_n(\theta_n^+, u_n^+)$$

and

$$A(\theta_n^-, u_n^-) = (-i\lambda_n)(\theta_n^-, u_n^-),$$

where $\pm\lambda_n$ are the eigenvalues. Consider then this orthonormal basis in $[L^2_{\mathbb{C}}(0,L)]^2$

$$\{(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}\},$$

and let $Z = \operatorname{span}\{(\theta_n^+, u_n^+) \cup (\theta_n^-, u_n^-)\}$. For any $s \in \mathbb{R}$, consider the norm

$$\left\| \sum_{n\in\mathbb{Z}} (c^{n,+}(\theta_n^+, u_n^+) + c^{n,-}(\theta_n^-, u_n^-)) \right\|_s$$

$$:= \left(\sum_{n\in\mathbb{Z}} (1 + |\lambda_n|)^{\frac{2}{3}s} (|c^{n,+}|^2 + |c^{n,-}|^2) \right)^{\frac{1}{2}}$$

which is used in the following definition.

Definition 1.1. Let $s \in \mathbb{R}$. The spaces H_s will be defined as the completion of Z with respect of the norm $\|\cdot\|_s$. In each space H_s, one has the orthonormal basis

$$\{(1+|\lambda_n|)^{-\frac{s}{3}}\theta_n^+, (1+|\lambda_n|)^{-\frac{s}{3}}u_n^+\}_{n\in\mathbb{Z}} \cup \{(1+|\lambda_n|)^{-\frac{s}{3}}\theta_n^-, (1+|\lambda_n|)^{-\frac{s}{3}}u_n^-\}_{n\in\mathbb{Z}}.$$

Our result deals with the *stabilization problem* already mentioned showing the following theorem in the H_1-level.

Theorem 1.1. *Let $L \in (0,+\infty)\backslash\mathcal{N}$ and $\omega > 0$. Then, there exist a continuous linear map*

$$F_\omega : H_1 \to \mathbb{R}$$

and a positive constant C, such that for every $(\eta_0, w_0) \in H_1$, the solution (η, w) of the closed-loop system (1.4)–(1.5), *with $f(t) = F_w(\eta(t), w(t))$ satisfies*

$$\|(\eta(t), w(t))\|_{H_1} \le Ce^{-2\omega t}\|(\eta_0, w_0)\|_{H_1}, \quad \forall\, t \ge 0.$$

Theorem 1.1 is shown using the result proved by Urquiza in [15, Theorem 2.1]. It is important to emphasize that our control acts only on one equation and through a boundary condition, instead of four or two controls as in Theorems A and B, respectively.

The content of this paper is divided as follows. Section 2 is devoted to presenting the Urquiza approach, which requires four hypotheses to be satisfied called (H1), (H2), (H3) and (H4). In Sec. 3, we deal with some preliminary results including the proof of (H1). We will note that (H2) is easily verified. Next, we prove the hypotheses (H3) and (H4) in Sec. 4. Section 5 is dedicated to the construction of the feedback and to finish the proof of Theorem 1.1. Some final comments are provided in Sec. 6.

2. Urquiza Approach

In this section, we present the Urquiza method [15] to prove rapid exponential stabilization of the following system:

$$\begin{cases} \eta_t + w_x + w_{xxx} = 0, & \text{in } (0,L)\times(0,+\infty), \\ w_t + \eta_x + \eta_{xxx} = 0, & \text{in } (0,L)\times(0,+\infty), \\ \eta(x,0) = \eta_0(x), \quad w(x,0) = w_0(x), & \text{in } (0,L), \end{cases} \tag{2.1}$$

with boundary conditions

$$\begin{cases} \eta(0,t) = 0, \quad \eta(L,t) = 0, \quad \eta_x(0,t) = f(t), & \text{in } (0,+\infty), \\ w(0,t) = 0, \quad w(L,t) = 0, \quad w_x(L,t) = 0, & \text{in } (0,+\infty). \end{cases} \tag{2.2}$$

We will extensively use the space operator already mentioned given by

$$A(\eta, w) = (-w' - w''', -\eta' - \eta'''), \tag{2.3}$$

with domain

$$D(A) = \{(\eta, w) \in [H^3(0,L) \cap H_0^1(0,L)]^2 : \eta'(0) = w'(L) = 0\}, \tag{2.4}$$

where

$$D(A) \subset X_0 := [L^2(0,L)]^2.$$

2.1. *Gramian method*

Let us first write our system in the abstract framework. Set A the operator defined by (2.3)–(2.4) and B given by

$$\begin{aligned} B : \mathbb{R} &\to D(A^*)' \\ s &\mapsto Bs, \end{aligned} \tag{2.5}$$

where $s \in \mathbb{R}$ and Bs is a functional given by

$$\begin{aligned} Bs : D(A^*) &\to \mathbb{R} \\ (u,v) &\mapsto Bs(u,v) := -sv_x(0). \end{aligned} \tag{2.6}$$

We will see in Proposition 3.1 that $D(A) = D(A^*)$, which are obviously closed and dense in X_0. Thus, we have that B^* is

$$\begin{aligned} B^* : D(A) &\to \mathbb{R} \\ (u,v) &\mapsto B^*(u,v) = -v_x(0). \end{aligned} \tag{2.7}$$

Note that system (2.1) and (2.2) takes the abstract form

$$\begin{cases} \dot{y}(x,t) = Ay(x,t) + Bv(t), & \text{in } [D(A^*)]', \\ y(x,0) = y^0(x). \end{cases} \tag{2.8}$$

Here $y^0 = (\eta_0, w_0) \in X_0$ is the initial condition and the control is $v(t) = -f(t)$.

In order to get the rapid exponential stabilization, we use the Urquiza approach [15]. Let us explain the method on the abstract control system (2.8) with state $y(t)$ in a Hilbert space Y and control $s(t)$ in a Hilbert space U. Here, the initial condition $y^0 \in Y$, A is a skew-adjoint operator in Y whose domain is dense in Y, and B is an unbounded operator from U to Y.

The method to prove rapid stabilization consists on building a feedback control using the following four hypothesis for the operators A and B:

- (H1) The skew-adjoint operator A is an infinitesimal generator of a strongly continuous group in the state space Y.
- (H2) The operator $B : U \to D(A^*)'$ is linear and continuous.
- (H3) (*Regularity property*) For every $T > 0$ there exists $C_T > 0$ such that
$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \leq C_T \|y\|_Y^2, \quad \forall\, y \in D(A^*).$$
- (H4) (*Observability property*) There exist $T > 0$ and $c_T > 0$ such that
$$\int_0^T \|B^* e^{-tA^*} y\|_U^2 dt \geq c_T \|y\|_Y^2, \quad \forall\, y \in D(A^*).$$

With these hypotheses in hand, the next result holds. Its proof mainly relies on general results about the algebraic Riccati equation associated with the linear quadratic regulator problem (see [7]).

Theorem 2.1 (see [15, Theorem 2.1]). *Consider operators A and B under assumptions $(H1)$–$(H4)$. For any $\omega > 0$, we have*

(i) *The symmetric positive operator Λ_ω defined by*

$$(\Lambda_\omega x, z)_Y = \int_0^\infty (B^* e^{-\tau(A+\omega I)^*} x, B^* e^{-\tau(A+\omega I)^*} z)_U d\tau, \quad \forall\, x, z \in Y,$$

is coercive and is an isomorphism on Y.

(ii) *Let $F_\omega := -B^*\Lambda_\omega^{-1}$. The operator $A + BF_\omega$ with $D(A + BF_\omega) = \Lambda_\omega(D(A^*))$ is the infinitesimal generator of a strongly continuous semigroup on Y.*

(iii) *The closed-loop system (system (2.8) with the feedback law $v = F_\omega(y)$) is exponentially stable with a decay equals to 2ω, that is,*

$$\exists\, C > 0, \quad \forall\, y \in Y, \quad \|e^{t(A+BF_\omega)} y\|_Y \le C e^{-2\omega t} \|y\|_Y.$$

In order to apply this method, we have to verify the four hypotheses. This will be done in the next sections. It is worth mentioning that the observability property (H4) is equivalent to the controllability of system (2.8) in the appropriate spaces.

3. Preliminaries

To apply Theorem 2.1 to our linear Boussinesq control system is necessary to check the hypotheses (H1)–(H4). First, we will prove that operator A defined by (2.3) satisfies (H1). Note that by the definition of the operator B, see (2.6), it is easy to see that (H2) also follows true. Moreover, in this section we establish the asymptotic behavior of eigenfunctions.

3.1. *Hypothesis* (H1)

We first comment that in order to verify hypothesis (H1) it is enough to prove that A is a skew-adjoint operator in $H_1 \subset X_0 = L^2(0, L)^2$. In fact, this will imply that A has a semigroup property on H_1. Remember that H_1 plays the role of Y in the general statement of Urquiza's method.

Proposition 3.1. *A is a skew-adjoint H_1 and thus generates a group of isometries $(e^{tA})_{t\in\mathbb{R}}$ in H_1.*

Proof. First, it is clear that $D(A)$ is dense in H_1. We have to prove that $A^* = -A$ in H_1. Note that we have $-A \subset A^*$ (i.e. $(\theta, u) \in D(A^*)$ and $A^*(\theta, u) = -A(\theta, u)$ for all $(\theta, u) \in D(A)$). Indeed, for any $(\eta, v), (\theta, u) \in D(A)$, we have the following series representation by the orthonormal basis $\{(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}\}$ given by Definition 1.1

$$(\eta, v) = \sum_{n\in\mathbb{Z}} (c_n^+(\theta_n^+, u_n^+) + c_n^-(\theta_n^-, u_n^-))$$

and

$$(\theta, u) = \sum_{n\in\mathbb{Z}} (d_n^+(\theta_n^+, u_n^+) + d_n^-(\theta_n^-, u_n^-)).$$

In this case, we have that

$$A(\eta, u) = \sum_{n\in\mathbb{Z}} (i\lambda_n c_n^+(\theta_n^+, u_n^+) - i\lambda_n c_n^-(\theta_n^-, u_n^-))$$

and

$$A(\theta, u) = \sum_{n\in\mathbb{Z}} (i\lambda_n d_n^+(\theta_n^+, u_n^+) - i\lambda_n d_n^-(\theta_n^-, u_n^-)),$$

respectively. Therefore, it yields that

$$\begin{aligned}((\theta, u), A(\eta, v))_{H_1} &= \sum_{n\in\mathbb{Z}} (1 + |\lambda_n|^2)^{\frac{2}{3}} (c_n^+ (\overline{i\lambda d_n^+}) + c_n^- (\overline{-i\lambda_n d_n^-})) \\ &= \sum_{n\in\mathbb{Z}} (1 + |\lambda_n|^2)^{\frac{2}{3}} (-i\lambda_n c_n^+ \overline{d_n^+} + i\lambda_n c_n^- \overline{d_n^-}) \\ &= -(A(\theta, u), (\eta, v))_{H_1}.\end{aligned}$$

Now, let us prove now that $A^* \subset -A$. Pick any $(\theta, u) \in D(A^*)$. Then, we have for some constant $C > 0$

$$|((\theta, u), A(\eta, v))_{X_0}| \leq C\|(\eta, v)\|_{X_0} \quad \forall\, (\eta, v) \in D(A),$$

i.e.

$$\left| \int_0^L [\theta(v_x + v_{xxx}) + u(\eta_x + \eta_{xxx})]dx \right| \leq C \left(\int_0^L [\eta^2 + v^2]dx \right)^{\frac{1}{2}}, \tag{3.1}$$

for all $(\eta, v) \in D(A)$. Picking $v = 0$ and $\eta \in C_c^\infty(0, L)$, we infer from (3.1) that $u_x + u_{xxx} \in L^2(0, L)$, and hence that $u \in H^3(0, L)$. Similarly, we obtain that $\theta \in H^3(0, L)$. Integrating by parts in the left-hand side of (3.1), we obtain that

$$\begin{aligned}&|\theta(L)v_{xx}(L) - \theta(0)v_{xx}(0) + \theta_x(0)v_x(0) + u(L)\eta_{xx}(L) - u(0)\eta_{xx}(0) - u_x(L)\eta_x(L)| \\ &\quad \leq C \left(\int_0^L [\eta^2 + v^2]dx \right), \quad \forall\, (\eta, v) \in D(A).\end{aligned}$$

It easily follows that

$$\theta(0) = \theta(L) = \theta_x(0) = u(0) = u(L) = u_x(L) = 0,$$

so that $(\theta, u) \in D(A) = D(-A)$. Thus $D(A^*) = D(-A)$ and $A^* = -A$, which ends the proof of this proposition. □

3.2. *Behavior of the traces*

As already mentioned (see [5] for details), there exists an orthonormal basis

$$\{(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}\}$$

in $[L^2_{\mathbb{C}}(0,L)]^2$, composed of eigenfunctions of A satisfying

$$A(\theta_n^+, u_n^+) = i\lambda_n(\theta_n^+, u_n^+)$$

and

$$A(\theta_n^-, u_n^-) = (-i\lambda_n)(\theta_n^-, u_n^-),$$

where the real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ are the eigenvalues. Moreover, they have the following asymptotic form:

$$\lambda_n = \begin{cases} \left(\dfrac{\pi + 12\pi(k_1+n)}{6L}\right)^3 + O(n), & \text{as } n \to +\infty \\ -\left(\dfrac{7\pi + 12\pi(k_2-n)}{6L}\right)^3 + O(n), & \text{as } n \to -\infty \end{cases} \tag{3.2}$$

for some numbers $k_1, k_2 \in \mathbb{Z}$. Next result provides the behavior of boundary traces associated with the orthonormal basis $\{(\theta_n^+, u_n^+)_{n\in\mathbb{Z}} \cup (\theta_n^-, u_n^-)_{n\in\mathbb{Z}}\}$. The proof is given in Appendix A.

Proposition 3.2. *There exist positive constants $C_1^\pm$ and $C_2^\pm$, such that*

$$\lim_{|n|\to\infty} \frac{|\theta^\pm_{n,x}(L))|}{|n|} = C_1^\pm \quad \textit{and} \quad \lim_{|n|\to\infty} \frac{|u^\pm_{n,x}(0)|}{|n|} = C_2^\pm. \tag{3.3}$$

4. Proof of Hypothesis (H3) and (H4)

In this section, we are interested in proving the hypothesis (H3) and (H4). We start presenting some auxiliary results that will be used to prove the regularity condition and observability inequality, respectively.

4.1. *Auxiliary lemmas*

To find the regularity needed and to prove the observability inequality we use the following classical Ingham inequality, see e.g. [8, 10] for details.

Lemma 4.1. *Let $T > 0$ and $\{\beta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of pairwise distinct real numbers such that*

$$\lim_{|n|\to\infty} (\beta_{n+1} - \beta_n) = +\infty.$$

Then, the series

$$h(t) = \sum_{n\in\mathbb{Z}} \gamma_n e^{i\beta_n t} \quad \textit{converges in } L^2(0,T),$$

for any sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ *satisfying* $\sum_{n\in\mathbb{Z}} \gamma_n^2 < \infty$. *Moreover, there exist two strictly positive constants* C_1 *and* C_2 *such that*

$$C_1 \sum_{n\in\mathbb{Z}} \gamma_n^2 \le \int_0^T |h(t)|^2 dt \le C_2 \sum_{n\in\mathbb{Z}} \gamma_n^2.$$

Now, observe that following the spectral analysis for the operator A given in [5, Appendix: Proof of Theorem 3.11], we obtain that

$$\theta_n^{\pm}(x) := \mp\frac{i}{\sqrt{2}} v_n(L-x) \quad \text{and} \quad u_n^{\pm}(x) = \frac{1}{\sqrt{2}} v_n(x), \tag{4.1}$$

where $\{v_n\}_{n\in\mathbb{Z}}$ are the eigenvectors of the operator $\mathcal{B}$ defined as

$$\mathcal{B}y = -y'''(L-x) - y'(L-x),$$

with domain $D(\mathcal{B}) = \{y \in H^3(0,L) \cap H_0^1(0,L) : y'(L) = 0\}$, which is closely related with the operator A. So, operator $\mathcal{B}$ has the following properties that can be seen in [5, Appendix: Proof of Theorem 3.11].

Lemma 4.2. *The operator* $\mathcal{B}$ *is self-adjoint in* $L^2(0,L)$. *Moreover, the following claims hold*:

(i) *If* $L \in (0,\infty)\backslash\mathcal{N}$, *then*

$$\mathcal{B}^{-1} : L^2(0,L) \to H^3(0,L)$$

is well-defined continuous operator. Here, $\mathcal{N}$ *is defined by* (1.8);

(ii) *There is an orthonormal basis* $\{v_n\}_{n\in\mathbb{Z}}$ *in* $L^2(0,L)$ *composed of eigenvectors of* $\mathcal{B}$: $v_n \in D(\mathcal{B})$ *and* $\mathcal{B}v_n = \lambda_n v_n$ *for all* $n \in \mathbb{N}$ *for some* $\lambda_n \in \mathbb{R}$.

Finally, the next result is a direct consequence of the spectral analysis for the operator A and ensures the well-posedness for the homogeneous system associated to the system (2.1) and (2.2).

Lemma 4.3. *For any* $(\eta_0, w_0) = \sum_{n\in\mathbb{Z}} (z_0^{n,+}(\theta_n^+, u_n^+) + z_0^{n,-}(\theta_n^-, u_n^-)) \in H_s$, *there exists a unique solution of*

$$\begin{cases} (\eta_t, w_t) = A(\eta, w), \\ (\eta(0), w(0)) = (\eta_0, w_0), \end{cases}$$

belonging of $C(\mathbb{R}, H_s)$ *and given by*

$$(\eta(x,t), w(x,t)) = \sum_{n\in\mathbb{Z}} (e^{i\lambda_n t} z_0^{n,+}(\theta_n^+, u_n^+) + e^{-i\lambda_n t} z_0^{n,-}(\theta_n^-, u_n^-)).$$

Additionally, as $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$, *we have*

$$\|(\eta(t), w(t))\|_s = \|(\eta_0, w_0)\|_s.$$

4.2. *Proof of* (H3)

Let us take

$$(\eta_0, w_0) = \sum_{n\in\mathbb{Z}} (z_0^{n,+}(\theta_n^+, u_n^+) + z_0^{n,-}(\theta_n^-, u_n^-)) \in H_1.$$

Note that by Lemma 4.3, we have

$$(\eta(x,t), w(x,t)) = \sum_{n\in\mathbb{Z}} (e^{i\lambda_n t} z_0^{n,+}(\theta_n^+, u_n^+) + e^{-i\lambda_n t} z_0^{n,-}(\theta_n^-, u_n^-)),$$

it implies that

$$w_x(0,t) = \sum_{n\in\mathbb{Z}} (e^{i\lambda_n t} z_0^{n,+} u_{n,x}^+(0) + e^{-i\lambda_n t} z_0^{n,-} u_{n,x}^-(0)).$$

Thanks to (4.1), we deduce that

$$\begin{cases} \theta_{n,x}^+(L) := -\dfrac{i}{\sqrt{2}} v_{n,x}(0), \quad \overline{\theta_{n,x}^-(L)} := -\dfrac{i}{\sqrt{2}} \overline{v_{n,x}(0)}, \\ u_{n,x}^+(0) = \dfrac{1}{\sqrt{2}} v_{n,x}(0), \quad \overline{u_{n,x}^-(0)} = \dfrac{1}{\sqrt{2}} \overline{v_{n,x}(0)}. \end{cases} \tag{4.2}$$

Thus, there exists a positive constant C, such that

$$\begin{aligned} |w_x(0,t)|^2 &\leq C \sum_{n\in\mathbb{Z}} (|z_0^{n,+}|^2 |u_{n,x}^+(0)|^2 + |z_0^{n,-}|^2 |u_{n,x}^-(0)|^2) \\ &\leq C \sum_{n\in\mathbb{Z}} |v_{n,x}^+(0)|^2 (|z_0^{n,+}|^2 + |z_0^{n,-}|^2|). \end{aligned}$$

Hence,

$$|w_x(0,t)|^2 \leq C \sum_{n\in\mathbb{Z}} \frac{|v_{n,x}(0)|^2}{(1+|\lambda_n|)^{\frac{2}{3}}} [(1+|\lambda_n|)^{\frac{2}{3}} (|z_0^{n,+}|^2 + |z_0^{n,-}|^2)].$$

Using the asymptotic behavior (3.2), there exists a positive constant C_1 such that

$$\frac{|v_{n,x}(0)|^2}{(1+|\lambda_n|)^{\frac{2}{3}}} \leq C_1, \quad \forall\, n \in \mathbb{Z}.$$

Therefore,

$$|w_x(0,t)|^2 \leq C_1 \sum_{n\in\mathbb{Z}} (1+|\lambda_n|)^{\frac{2}{3}} (|z_0^{n,+}|^2 + |z_0^{n,-}|^2) = C_1 \|(\eta_0, w_0)\|_1^2, \tag{4.3}$$

and so condition $(H3)$ holds in H_1-norm.

4.3. *Proof of* (H4)

Note that (3.2) implies that sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ satisfies the following gap condition:

$$\lim_{|n|\to\infty}(\lambda_{n+1}-\lambda_n)=+\infty.$$

Then, from Lemma 4.1 and relations (4.1) and (4.3), we obtain that

$$\begin{aligned} C_2\left(\sum_{n\in\mathbb{Z}}|z_0^{n,+}u_{n,x}^+(0)|^2+\sum_{n\in\mathbb{Z}}|z_0^{n,-}u_{n,x}^-(0)|^2\right) &\le \int_0^T |w_x(0,t)|^2dt \\ &\le C_1\|(\eta_0,w_0)\|_1^2, \end{aligned} \tag{4.4}$$

for some positive constants C_1 and C_2. By using (4.2) and (4.4), we have that

$$\begin{aligned} \int_0^T |w_x(0,t)|^2dt &\ge C_3\sum_{n\in\mathbb{Z}}|v_{n,x}(0)|^2(|z_0^{n,+}|^2+|z_0^{n,-}|^2) \\ &= C_3\sum_{n\in\mathbb{Z}}\frac{|v_{n,x}(0)|^2}{(1+|\lambda_n|)^{\frac{2}{3}}}[(1+|\lambda_n|)^{\frac{2}{3}}(|z_0^{n,+}|^2+|z_0^{n,-}|^2)], \end{aligned} \tag{4.5}$$

holds for some $C_3>0$. Observe that we can estimate the right-hand side of (4.5) in terms of any H_s–norm for $s\ge 1$. To finalize the proof of the hypothesis (H4) for the H_1–norm, we cannot lose any coefficient $z_0^{n,\pm}$. Thus, we claim the following.

Claim 1. $v_{n,x}(0)\neq 0$ for all $n\in\mathbb{Z}$.

Indeed, suppose by contradiction that there exists $n_0\in\mathbb{Z}$ such that $v_{n_0,x}(0)=0$. This implies that

$$(\theta_{n_0,x}^+(L),u_{n_0,x}^+(0))=0 \quad \text{and} \quad (\theta_{n_0,x}^-(L),u_{n_0,x}^-(0))=0. \tag{4.6}$$

In particular, considering $u(x)=\theta_{n_0}^+(x)+u_{n_0}^+(x)$, there exist $\lambda_{n_0}\in\mathbb{C}$ such that

$$\begin{cases} u'''+u'+\lambda_{n_0}u=0, \\ u(0)=u(L)=u'(0)=u'(L)=0. \end{cases}$$

From [14, Lemma 3.5], it follows that $L\in\mathcal{N}$, which is a contradiction, and Claim 1 holds.

Lastly, due to the asymptotic behavior (3.2) and Claim 1, there exists a positive constant $C_4>0$ such that

$$\frac{|v_{n,x}(0)|^2}{(1+|\lambda_n|)^{\frac{2}{3}}}\ge C_4,$$

for all $n\in\mathbb{N}$, Thus, follows by (4.6) that

$$\int_0^T |w_x(0,t)|^2dt\ge C_4\|(\eta_0,w_0)\|_1^2. \tag{4.7}$$

Therefore, relation (H4) is satisfied. As we already mentioned, this property gives us an additional result. The exact controllability of (2.1) and (2.2) with control space $L^2(0,T)$ and state space H_1.

5. Rapid Stabilization: Control Design

In this section, we design the feedback law using the Urquiza approach to show the rapid exponential stabilization for solutions of the system (2.1) and (2.2). Recall that this system takes an abstract form (2.8) and the operators A and B are given by (2.3)–(2.4) and (2.5)–(2.6), respectively. With this in hand, we are in position to prove our main result.

5.1. *Proof of Theorem* 1.1

For any (p_0, q_0), (r_0, s_0) in H_1 and $\omega > 0$, consider the bilinear form defined by

$$a_\omega((p_0, q_0), (r_0, s_0)) := \int_0^\infty e^{-2\omega\tau} q_x(0,\tau) s_x(0,\tau)\, d\tau. \tag{5.1}$$

Here (p, q) and (r, s) are solutions of

$$\begin{cases} p_\tau + q_x + q_{xxx} = 0, \\ q_\tau + p_x + p_{xxx} = 0, \\ p(0,\tau) = p(L,\tau) = p_x(0,\tau) = 0, \\ q(0,\tau) = q(L,\tau) = q_x(L,\tau) = 0, \\ p(x,0) = p_0(x), \quad q(x,0) = q_0(x) \end{cases}$$

and

$$\begin{cases} r_\tau + s_x + s_{xxx} = 0, \\ s_\tau + r_x + r_{xxx} = 0, \\ r(0,\tau) = r(L,\tau) = r_x(0,\tau) = 0, \\ s(0,\tau) = s(L,\tau) = s_x(L,\tau) = 0, \\ r(x,0) = r_0(x), \quad s(x,0) = s_0(x), \end{cases}$$

respectively. Finally, consider the following operator $\Lambda_\omega : H_1 \to H_{-1}$ satisfying the relation

$$\langle \Lambda_\omega(p_0, q_0), (r_0, s_0)\rangle_{H_{-1},H_1} = a_\omega((p_0, q_0), (r_0, s_0)), \tag{5.2}$$

for all $(p_0, q_0), (r_0, s_0) \in H_1$ and $\omega > 0$. Note that, from (2.7) we have that

$$\begin{aligned} &\langle \Lambda_\omega(p_0, q_0), (r_0, s_0)\rangle_{H_{-1},H_1} \\ &\quad = \int_0^\infty e^{-2\omega\tau} q_x(0,\tau) s_x(0,\tau)\, d\tau, \\ &\quad = \int_0^\infty e^{-2\omega\tau} B^*(p(x,\tau), q(x,\tau)) B^*(r(x,\tau), s(x,\tau))\, d\tau. \end{aligned}$$

Thanks to Theorem 2.1, the operator Λ_ω is coercive and an isomorphism. On the other hand, set the functional

$$F_\omega : H_1 \to \mathbb{R}$$
$$(z_1, z_2) \mapsto F_\omega((z_1, z_2)) := q_0'(0),$$

where (p_0, q_0) is the solution of the following Lax–Milgram problem

$$a_\omega((p_0, q_0), (r_0, s_0)) = \langle (z_1, z_2), (r_0, s_0)\rangle_{H_{-1}, H_1}, \quad \forall\, (r_0, s_0) \in H_1. \tag{5.3}$$

From (5.2), we deduce that $(z_1, z_2) = \Lambda_\omega(p_0, q_0)$ in H_{-1}. Moreover, observe that

$$F_\omega(z_1, z_2) = q_0'(0) = -B^*(p_0, q_0) = -B^*\Lambda_\omega^{-1}(z_1, z_2), \quad \forall\, (z_1, z_2) \in H_1.$$

Thus, we are in the hypothesis of Theorem 2.1, which one can be applied and guarantees the rapid exponential stabilization to the solutions of the system (2.1) and (2.2). It means that for any $\omega > 0$, there exists a continuous linear feedback control

$$f(t) = F_\omega(\eta(t), w(t))$$

with $F_\omega = -B^*\Lambda_\omega^{-1}$ where Λ_ω is given by (5.1) and (5.2) and a positive constant C, such that for every initial conditions $(\eta_0, w_0) \in H_1$, the solution (η, w) of the closed-loop system (2.1) and (2.2), satisfies

$$\|(\eta(t), w(t))\|_{H_1} \leq Ce^{-2\omega t}\|(\eta_0, w_0)\|_{H_1},$$

with a decay equals to 2ω.

6. Further Comments

We have applied the Gramian approach to build some boundary feedback law to prove the rapid stabilization for a coupled KdV–KdV type system. Considering one control acting on the Neumann boundary condition at the right-hand side of the interval where the system evolves we are able to prove that the closed-loop system is locally exponentially stable with a decay rate that can be chosen to be as large as we want. In what follows, we present some final remarks.

- Theorem A guarantees the stabilization of the KdV–KdV system with four controls and Theorem B ensures the rapid stabilization with two controls. However, Theorem 1.1 gives us a best result for the linear system, that is, we are able to make the solutions of the linear system go to zero with only one control acting at the boundary. It is important to point out here that the drawback is that we are not able to treat the nonlinear case. This is due to the lack of any Kato smoothing effect, as in the case of a single KdV [6], which leaves the rapid stabilization for the full system (1.3) with boundary conditions (1.5) completely open to study. The same lack of smoothing effect brings us to work in the state space H_1 in order to have traces (and then controls) in the space $L^2(0, T)$. This regularity is unlikely to be sharp and we think that a better result in fractional spaces should be expected.

- Note that we can also prove that there exists a continuous linear feedback control

$$g(t) = F_\omega(\eta(t), w(t))$$

such that the closed-loop system (1.4) with boundary conditions

$$\begin{cases} \eta(0,t) = 0, \quad \eta(L,t) = 0, \quad \eta_x(0,t) = 0, & \text{in } (0,+\infty), \\ w(0,t) = 0, \quad w(L,t) = 0, \quad w_x(L,t) = g(t), & \text{in } (0,+\infty), \end{cases}$$

satisfies

$$\|(\eta(t), w(t))\|_{H_1} \leq Ce^{-2\omega t}\|(\eta_0, w_0)\|_{H_1},$$

with a decay equals to 2ω. To prove this consider the operator B given by

$$\begin{aligned} B : \mathbb{R} &\to D(A^*)' \\ s &\mapsto Bs := L_s, \end{aligned}$$

where $s \in \mathbb{R}$ and L_s is a functional given by

$$\begin{aligned} L_s : D(A^*) &\to \mathbb{R} \\ (u,v) &\mapsto L_s(u,v) := su_x(L). \end{aligned}$$

With these information in hand and the following observability inequality

$$\int_0^T |\eta_x(L,t)|^2 dt \geq C\|(\eta_0, w_0)\|_1^2, \quad C > 0,$$

the result follows using the same idea as done in the proof of Theorem 1.1.

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Appendix A. Proof of Proposition 3.2

Following the ideas of [5, Appendix A. Proof of Theorem 3.11], we observe that v_n takes the form

$$v_n(x) = \sum_{j=1}^{3} a_j[e^{r_j x} - ie^{r_j(L-x)}], \tag{A.1}$$

with $a_j = a_j(n) \in \mathbb{C}$, for $j = 1, 2, 3$, where

$$\sum_{j=1}^{3} a_j(e^{r_j L} - i) = 0,$$
$$\sum_{j=1}^{3} a_j(1 - ie^{r_j L}) = 0, \tag{A.2}$$
$$\sum_{j=1}^{3} r_j a_j(e^{r_j L} + i) = 0$$

and r_j, $j = 1, 2, 3$, are pairwise distinct such that

$$r_1 = r_1(n) \sim -i\lambda_n^{1/3}, \quad r_2 = r_2(n) \sim -ip\lambda_n^{1/3}, \quad r_3 = r_3(n) \sim -ip^2\lambda_n^{1/3}, \tag{A.3}$$

for $p = e^{i\frac{2\pi}{3}}$. Note that the equations in (A.2) imply that

$$\sum_{j=1}^{3} a_j = \sum_{j=1}^{3} a_j e^{r_j L} = 0,$$

that is,

$$a_3 = -a_1 - a_2$$

and

$$a_1(e^{r_1 L} - e^{r_3 L}) + a_2(e^{r_2 L} - e^{r_3 L}) = 0.$$

Moreover, if we assume $\lambda_n \to \infty$, we have

$$r_1 = -i\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim -i\lambda_n^{1/3}$$
$$r_2 = -ip\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\lambda_n^{1/3}$$

and

$$r_3 = -ip^2\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim \left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\lambda_n^{1/3}.$$

Additionally, if $\lambda_n \to -\infty$, we have

$$r_1 = -i\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim i|\lambda_n^{1/3}|,$$
$$r_2 = -ip\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim -\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)|\lambda_n^{1/3}|$$

and

$$r_3 = -ip^2\lambda_n^{1/3} + O(\lambda_n^{-1/3}) \sim \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)|\lambda_n^{1/3}|.$$

Note that the previous relations implies

$$|e^{r_1L}| \to 1, \quad |e^{r_2L}| \to +\infty, \quad |e^{r_3L}| \to 0, \quad \text{as } n \to \infty \tag{A.4}$$

and

$$|e^{r_1L}| \to 1, \quad |e^{r_2L}| \to 0, \quad |e^{r_3L}| \to +\infty, \quad \text{as } n \to -\infty. \tag{A.5}$$

The convergences (A.4) and (A.5) ensure that

$$\lambda_n \to \pm\infty, \quad \text{as } n \to \pm\infty.$$

With these relations in hand, the following claim can be verified.

Claim 2. *The behaviors* (3.3) *hold whenever there exist positive constant C_1 and C_2 such that*

$$\lim_{|n|\to\infty} \frac{|v_n'(0)|}{|n|} = C_1 \quad \text{and} \quad \lim_{|n|\to\infty} \frac{|v_n'(L)|}{|n|} = C_2. \tag{A.6}$$

In fact, to obtain the limit (A.6), we have to analyze the asymptotic behavior of the $a_j(n)$ terms. First, note that $\|v_n\|_{L^2(0,L)} = 1$, since $\{v_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis thanks to Lemma 4.2, thus

$$\begin{aligned} 1 &= \int_0^L \left| \sum_{j=1}^3 a_j \left[e^{r_jx} - ie^{r_j(L-x)} \right] \right|^2 dx \\ &= \int_0^L \left(\sum_{j=1}^3 A_j^2(x) + 2 \sum_{i,j=1, i\neq j}^3 A_i(x)A_j(x) \right) dx, \end{aligned} \tag{A.7}$$

where $A_j(x) = a_j[e^{r_jx} - ie^{r_j(L-x)}]$. Then,

$$\begin{aligned} \int_0^L A_j^2(x)dx &= \int_0^L a_j^2(e^{2r_jx} - 2ie^{r_jx}e^{r_j(L-x)} - e^{2r_j(L-x)})dx \\ &= a_j^2 \left[\frac{e^{2r_jx}}{2r_j} - 2ie^{r_jL}x + \frac{e^{2r_j(L-x)}}{2r_j} \right]_0^L \end{aligned}$$

hence

$$\int_0^L A_j^2(x)dx = -2ia_j^2 e^{r_jL} L. \tag{A.8}$$

On the other hand,

$$\begin{aligned} &\int_0^L A_i(x)A_j(x)dx \\ &\quad = \int_0^L a_ia_j(e^{r_ix} - ie^{r_i(L-x)})(e^{r_jx} - ie^{r_j(L-x)})dx \\ &\quad = a_ia_j \left[\frac{e^{(r_i+r_j)x}}{r_i+r_j} - \frac{ie^{(r_i-r_j)x}e^{r_jL}}{r_i-r_j} + \frac{ie^{-(r_i-r_j)x}e^{r_iL}}{r_i-r_j} + \frac{e^{(r_i+r_j)(L-x)}}{r_i+r_j} \right]_0^L, \end{aligned}$$

therefore

$$\int_0^L A_i(x)A_j(x)dx = 2ia_ia_j\frac{e^{r_jL}-e^{r_iL}}{r_i-r_j}, \quad \forall\, i \neq j. \tag{A.9}$$

Putting together (A.8) and (A.9) in (A.7), we get

$$\begin{aligned}&-2iL(a_1^2e^{r_1L}+a_2^2e^{r_2L}+a_3^2e^{r_3L})\\ &\quad+4i\left(a_1a_2\frac{e^{r_2L}-e^{r_1L}}{r_1-r_2}+a_1a_3\frac{e^{r_3L}-e^{r_1L}}{r_1-r_3}+a_2a_3\frac{e^{r_3L}-e^{r_2L}}{r_2-r_3}\right)=1.\end{aligned} \tag{A.10}$$

Thanks to the second and first relation in (A.2), respectively, we have

$$a_2 = -\Gamma a_1, \quad a_3 = -a_1(1-\Gamma),$$

where $\Gamma = (e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_3L})^{-1}$. Now, using (A.10), we obtain

$$\begin{aligned}&-2iLa_1^2(e^{r_1L}+\Gamma^2e^{r_2L}+(1-\Gamma)^2e^{r_3L})\\ &\quad+4ia_1^2\left(-\Gamma\frac{e^{r_2L}-e^{r_1L}}{r_1-r_2}-(1-\Gamma)\frac{e^{r_3L}-e^{r_1L}}{r_1-r_3}+\Gamma(1-\Gamma)\frac{e^{r_3L}-e^{r_2L}}{r_2-r_3}\right)=1.\end{aligned} \tag{A.11}$$

Moreover, note that

$$1-\Gamma = \frac{e^{r_2L}-e^{r_1L}}{e^{r_2L}-e^{r_3L}}$$

and

$$\Gamma(1-\Gamma) = \frac{(e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_1L})}{(e^{r_2L}-e^{r_3L})^2}.$$

Therefore, we have

$$\begin{aligned}\Gamma(1-\Gamma)\frac{e^{r_3L}-e^{r_2L}}{r_2-r_3} &= -\frac{(e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_1L})}{(r_2-r_3)(e^{r_2L}-e^{r_3L})},\\ \Gamma\frac{e^{r_2L}-e^{r_1L}}{r_1-r_2} &= \frac{(e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_1L})}{(r_1-r_2)(e^{r_2L}-e^{r_3L})}\end{aligned}$$

and

$$(1-\Gamma)\frac{e^{r_3L}-e^{r_1L}}{r_1-r_3} = -\frac{(e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_1L})}{(r_1-r_3)(e^{r_2L}-e^{r_3L})}.$$

Using the previous equalities in (A.11), it follows that

$$\begin{aligned}&-2iLa_1^2(e^{r_1L}+\Gamma^2e^{r_2L}+(1-\Gamma)^2e^{r_3L})\\ &\quad+4ia_1^2\frac{(e^{r_1L}-e^{r_3L})(e^{r_2L}-e^{r_1L})}{(e^{r_2L}-e^{r_3L})}\left(\frac{1}{r_1-r_3}-\frac{1}{r_2-r_3}-\frac{1}{r_1-r_2}\right)=1\end{aligned}$$

or, equivalently,

$$-2ia_1^2[L(e^{r_1L}+\Gamma^2e^{r_2L}+(1-\Gamma)^2e^{r_3L})-2\Gamma(e^{r_2L}-e^{r_1L})\phi(r_1,r_2,r_3)]=1,$$

where

$$\phi(r_1,r_2,r_3)=\frac{1}{r_1-r_3}-\frac{1}{r_2-r_3}-\frac{1}{r_1-r_2}.$$

Noting that

$$|a_1(n)|^2=\frac{1}{2|L(e^{r_1L}+\Gamma^2e^{r_2L}+(1-\Gamma)^2e^{r_3L})-2\Gamma(e^{r_2L}-e^{r_1L})\phi(r_1,r_2,r_3)|},$$

it follows that

$$\begin{aligned}&\frac{1}{2(L|e^{r_1L}|+L|\Gamma^2e^{r_2L}|+L|(1-\Gamma)^2e^{r_3L}|+2|\Gamma||e^{r_2L}-e^{r_1L}||\phi(r_1,r_2,r_3)|)}\\&\quad\le|a_1(n)|^2\\&\quad\le\frac{1}{2|L|e^{r_1L}|-L|\Gamma^2e^{r_2L}|-L|(1-\Gamma)^2e^{r_3L}|-2|\Gamma||e^{r_2L}-e^{r_1L}||\phi(r_1,r_2,r_3)||}.\end{aligned}\tag{A.12}$$

Additionally, from (A.3), we have

$$\begin{aligned}r_1-r_2&=-i(1-p)\lambda_n^{1/3}+O(\lambda_n^{-1/3}),\\r_2-r_3&=-ip(1-p)\lambda_n^{1/3}+O(\lambda_n^{-1/3})\end{aligned}$$

and

$$r_1-r_3=-i(1-p^2)\lambda_n^{1/3}+O(\lambda_n^{-1/3}),$$

which allow us to conclude that $\phi(r_1,r_2,r_3)\to0$, as $|n|\to\infty$.

Observe that

$$\begin{aligned}|\Gamma|&=\left|\frac{(e^{r_1L}-e^{r_3L})}{e^{r_2L}(1-e^{r_3L}e^{-r_2L})}\right|\le\frac{|e^{r_1L}|+|e^{r_3L}|}{|e^{r_2L}||1-e^{r_3L}e^{-r_2L}|}\\|\Gamma^2e^{r_2L}|&=\left|\frac{(e^{r_1L}-e^{r_3L})^2}{e^{r_2L}(1-e^{r_3L}e^{-r_2L})^2}\right|\le\frac{(|e^{r_1L}|+|e^{r_3L}|)^2}{|e^{r_2L}||1-e^{r_3L}e^{-r_2L}|^2}\end{aligned}$$

and

$$|(1-\Gamma)^2e^{r_3L}|=\left|\left(1-\frac{e^{r_1L}-e^{r_3L}}{e^{r_2L}-e^{r_3L}}\right)^2e^{r_3L}\right|\le\frac{(|e^{r_2L}|+|e^{r_1L}|)^2}{|e^{r_3L}||e^{r_2L}e^{-r_3L}-1|^2}.$$

Due to the asymptotic behavior of $\{\lambda_n\}_{n\in\mathbb{N}}$ (3.2) and (A.4), we get

$$|\Gamma|\to0,\quad|\Gamma^2e^{r_2L}|\to0\quad\text{and}\quad|(1-\Gamma)^2e^{r_3L}|\to0,\quad\text{as }n\to\infty.\tag{A.13}$$

Note that $|\Gamma e^{r_2L}|\to1$, which implies that

$$|\Gamma||e^{r_2L}-e^{r_1L}||\phi(r_1,r_2,r_3)|\to0\quad\text{as }n\to\infty.$$

On the other hand, thanks to (A.5), it follows that

$$\begin{aligned}&|\Gamma| \to 1, \quad |\Gamma||e^{r_2L} - e^{r_1L}| \to 1, \quad |\Gamma^2 e^{r_2L}| \to 0 \quad \text{and}\\ &|(1-\Gamma)^2 e^{r_3L}| \to 0, \quad \text{as } n \to -\infty.\end{aligned} \tag{A.14}$$

Thus, using (A.13), (A.14) and passing to the limit in (A.12), we deduce that

$$\lim_{|n|\to\infty} |a_1(n)| = \frac{1}{\sqrt{2L}}. \tag{A.15}$$

Let us prove (A.6). From (A.1) we have that

$$v_n'(x) = \sum_{j=1}^{3} a_j r_j [e^{r_j x} + i e^{r_j (L-x)}],$$

then

$$v_n'(0) = \sum_{j=1}^{3} a_j r_j [1 + i e^{r_j L}]$$

and

$$v_n'(L) = \sum_{j=1}^{3} a_j r_j [e^{r_j L} + i].$$

Thanks to (A.2), it follows that

$$\begin{aligned}
v_n'(0) &= a_1 r_1 (1 + i e^{r_1 L}) + a_2 r_2 (1 + i e^{r_2 L}) - (a_1 + a_2) r_3 (1 + i e^{r_3 L})\\
&= a_1 r_1 (1 + i e^{r_1 L}) + a_2 (r_2 - r_3) + a_2 r_2 i (e^{r_2 L} - e^{r_3 L})\\
&\quad + a_2 i e^{r_3 L} (r_2 - r_3) - a_1 r_3 (1 + i e^{r_3 L})\\
&= a_1 r_1 (1 + i e^{r_1 L}) + a_2 (r_2 - r_3) - a_1 r_2 i (e^{r_1 L} - e^{r_3 L})\\
&\quad + a_2 i e^{r_3 L} (r_2 - r_3) - a_1 r_3 (1 + i e^{r_3 L})\\
&= a_1 [r_1 (1 + i e^{r_1 L}) - r_3 - r_2 i e^{r_1 L}]\\
&\quad + a_2 (r_2 - r_3)(1 + i e^{r_3 L}) + a_1 i e^{r_3 L} (r_2 - r_3)\\
&= a_1 [r_1 (1 + i e^{r_1 L}) - r_3 - r_2 i e^{r_1 L}]\\
&\quad - \frac{a_1 (e^{r_1 L} - e^{r_3 L})}{(e^{r_2 L} - e^{r_3 L})} (r_2 - r_3)(1 + i e^{r_3 L}) + a_1 i e^{r_3 L} (r_2 - r_3).
\end{aligned}$$

We analyze the case when $n \to +\infty$, the case when $n \to -\infty$ can be shown analogously. Noting that

$$\frac{e^{r_1 L} - e^{r_3 L}}{e^{r_2 L} - e^{r_3 L}} = O(e^{-\frac{\sqrt{3}}{2}\lambda^{1/3} L}),$$

$$r_2 - r_3 = O(\lambda^{1/3})$$

and

$$1 + ie^{r_3 L} = O(1 + e^{-\frac{\sqrt{3}}{2}\lambda^{1/3}L}),$$

we have that

$$\frac{a_1(e^{r_1 L} - e^{r_3 L})}{(e^{r_2 L} - e^{r_3 L})}(r_2 - r_3)(1 + ie^{r_3 L}) + a_1 ie^{r_3 L}(r_2 - r_3) = O(e^{-\frac{\sqrt{3}}{2}\lambda^{1/3}L}).$$

Note that $(1 - p^4) = (1 - p)$, due the fact that $p = e^{i\frac{2\pi}{3}}$. Thus,

$$\begin{aligned} v_n'(0) &= a_1((r_1 - r_3) + ie^{r_1 L}(r_1 - r_2) + O(\lambda_n^{1/3}e^{-\frac{\sqrt{3}}{2}\lambda_n^{1/3}L})) \\ &= a_1((-i\lambda_n^{1/3})(1 - p^2) + ie^{r_1 L}(-i\lambda_n^{1/3})(1 - p) + O(1)) \\ &= a_1((-i\lambda_n^{1/3})(1 - p^2) + ie^{r_1 L}(-i\lambda_n^{1/3})(1 - p^4) + O(1)) \\ &= a_1(-i\lambda_n^{1/3})(1 - p^2)[1 + ie^{r_1 L}(1 + p^2) + O(1)] \\ &= a_1(-i\lambda_n^{1/3})(1 - p^2)(1 + ie^{r_1 L}p^2 + O(1)). \end{aligned}$$

Since $e^{r_1 L} \sim e^{-i\lambda_n^{1/3}L} \sim e^{-i\pi/6} \sim ip^2$, there exists $K_1^+ \in \mathbb{C}\backslash\{0\}$, such that

$$v_n'(0) \sim K_1^+ a_1(n)\lambda_n^{1/3}.$$

Similarly, there exists $K_2^+ \in \mathbb{C}\backslash\{0\}$, such that

$$v_n'(L) \sim K_2^+ a_1(n)\lambda_n^{1/3}.$$

Analogously, when $n \to -\infty$, we get

$$v_n'(0) \sim K_1^- a_1(n)\lambda_n^{1/3}$$

and

$$v_n'(L) \sim K_2^- a_1(n)\lambda_n^{1/3},$$

for some complex constants nonzero K_1^- and K_2^-, respectively. Finally, (3.2) and (A.15) ensure that (A.6) follows and, consequently, Proposition 3.2 is achieved.

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