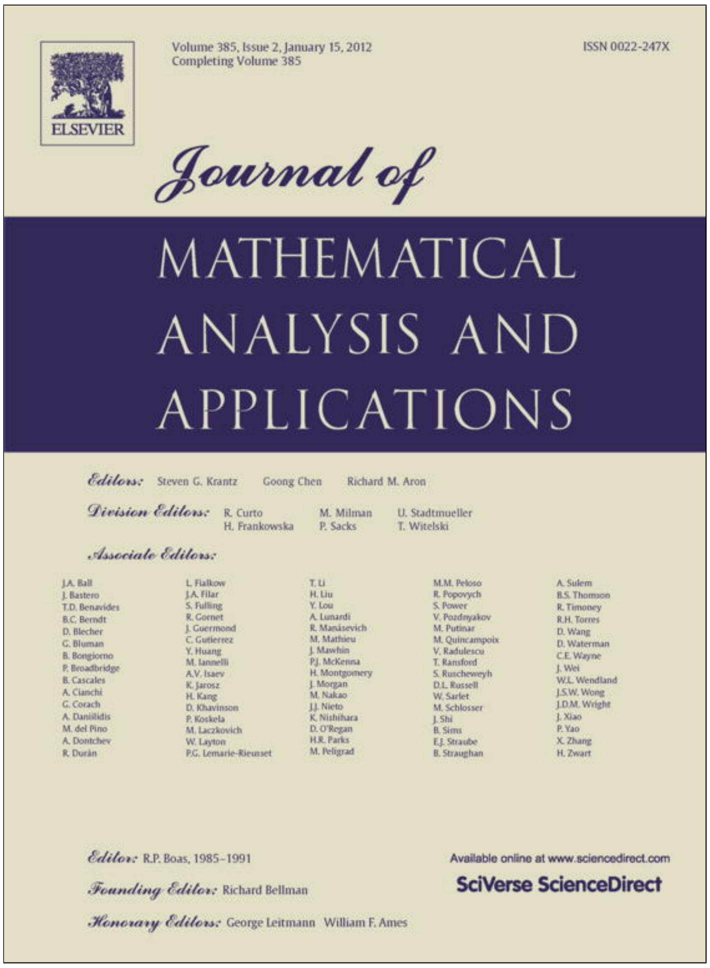
Volume 385, Issue 2, January 15, 2012
Completing Volume 385
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
Division Editors: R. Curto H. Frankowska M. Milman P. Sacks U. Stadtmueller T. Witelski
Associate Editors:
Editor: R.P. Boas, 1985–1991
Founding Editor: Richard Bellman
Honorary Editors: George Leitmann William F. Ames
Available online at www.sciencedirect.com
SciVerse ScienceDirect

J. Math. Anal. Appl. 385 (2012) 743–756

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Energy decay for the modified Kawahara equation posed in a bounded domain

F.D. Araruna [a,1], R.A. Capistrano-Filho [a,2], G.G. Doronin [b,*,3]

[a] Departamento de Matemática, Universidade Federal da Paraíba, 58051-900, João Pessoa, PB, Brazil
[b] Departamento de Matemática, Universidade Estadual de Maringá, 87020-900, Maringá, PR, Brazil

ARTICLE INFO

Article history:
Received 11 April 2011
Available online 8 July 2011
Submitted by M. Nakao

Keywords:
Kawahara's equation
Critical set
Energy decay

ABSTRACT

Studied here is the eventual dissipation of solutions to initial–boundary value problems for the modified Kawahara equation with and without a localized damping term included. It is shown that solutions of undamped problems posed on a bounded interval may not decay if the length of the interval is critical. In contrast, the energy associated to the locally damped problems is shown to be exponentially decreased independently of the interval length.

1. Introduction

The Kawahara equation [19]

$$u_t + u_x + u_{xxx} - u_{xxxxx} + uu_x = 0 \tag{1.1}$$

is a dispersive PDE describing numerous wave phenomena such as magneto-acoustic waves in a cold plasma [17], the propagation of long waves in a shallow liquid beneath an ice sheet [15], gravity waves on the surface of a heavy liquid [10], etc. In the literature this equation is also referred as the fifth-order KdV equation [5], or singularly perturbed KdV equation [24]. Jointly with (1.1) it is worthwhile to study the so-called modified Kawahara equation, i.e. the same fifth-order dispersive equation with a square nonlinearity in the convective term, namely

$$u_t + u_x + u_{xxx} - u_{xxxxx} + u^2u_x = 0. \tag{1.2}$$

There are some valuable efforts in the last years that focus on the analytical and numerical methods for solving (1.1) and (1.2). These methods include the tanh-function method [1], extended tanh-function method [2], sine-cosine method [38], Jacobi elliptic functions method [14], direct algebraic method [23], decompositions methods [18], as well as the variational iterations and homotopy perturbations methods [16]. For more details see [6,31,33,37,39], among others. These approaches deal, as a rule, with soliton-like solutions obtained while one considers problems posed on a whole real line. For numerical simulations, however, there appears the question of cutting-off the spatial domain [3,4]. This motivates the detail qualitative analysis of problems for (1.1) and/or (1.2) in bounded regions [12].

* Corresponding author.
E-mail addresses: fagner@mat.ufpb.br (F.D. Araruna), rob_capistrano@yahoo.com.br (R.A. Capistrano-Filho), ggdoronin@uem.br (G.G. Doronin).
1 Partially supported by INCTMat, FAPESQ-PB, CNPq (Brazil) by grants 308150/2008-2 and 620108/2008-8.
2 Partially supported by CAPES.
3 Partially supported by Araucária Foundation.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.07.003

Our aim here is to analyze qualitative properties of solutions to the initial–boundary value problem for (1.2) posed on a bounded interval under the presence of a localized damping term, that is

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^2 u_x + a(x)u = 0 & \text{in } Q_T = (0, L) \times (0, T), \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0, T), \\ u(x,0) = u_0(x), & x \in (0, L), \end{cases} \tag{1.3}$$

where the real function $a = a(x)$ satisfies the condition

$$\begin{cases} a \in L^\infty(0, L) \quad \text{and} \quad a(x) \geqslant a_0 > 0 \quad \text{a.e. in } \omega, \\ \text{with a nonempty } \omega \subset (0, L). \end{cases} \tag{1.4}$$

The term $a(x)u$ designs a feedback damping mechanism; therefore, one can expect the global well-posedness of (1.3) for all $L > 0$, and the decay of solutions. The main purpose of this paper is to prove that this is indeed true. There are basically two features to be emphasized in this way:

- one should be convinced that a damping is effectively important, i.e. there are solutions to undamped model (at least to its linear version) that do not decay;
- one should be capable to estimate the nonlinear term in appropriate norms, i.e. there are suitable functional spaces that allow to apply corresponding methods.

First, we show that the presence of an extra damping term is essential already in a linear case: if $a(x) \equiv 0$, then a nontrivial solution to

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0, T), \\ u(x,0) = u_0(x), & x \in (0, L), \end{cases}$$

is constructed to be not decayed as $t \to \infty$ if the length of an interval is critical. Observe that due to the drift term u_x the same occurs for the KdV equation [27]. Indeed, if for instance $L = 2\pi n$, $n \in \mathbb{N}$, then the function $v(x) = 1 - \cos x$ solves

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(L,t) = 0, & t \in (0, T), \\ u(x,0) = v(x), & x \in (0, L), \end{cases}$$

and clearly $v(x) \nrightarrow 0$ as $t \to \infty$. Despite the valuable advances in [7–9,13], the question whether solutions of undamped problems associated to nonlinear KdV and Kawahara equations decay as $t \to \infty$ for all finite $L > 0$ is open.

To overcome these difficulties, a damping of the type $a(x)u$ was introduced in [22] to stabilize the KdV system. More precisely, considering the damping localized at a subset $\omega \subset (0, L)$ containing nonempty neighborhoods of the end-points of an interval, it was shown that solutions of both linear and nonlinear problems for the KdV equation decay, independently on $L > 0$. In [25] it was proved that the same holds without cumbersome restrictions on $\omega \subset (0, L)$. In [34,36] the damping like in (1.4) was used for (1.1) without the drift term u_x. If, however, the linear term u_x is dropped, both the KdV and Kawahara equations do not possess critical set restrictions [27,35], and the damping is not necessary. The decay of solutions in such case was also proved in [11,12] by different methods.

Once the damping term $a(x)u \not\equiv 0$ is added to (1.2), the nonlinearity $u^2 u_x$ provides the second difficulty which should be treated with accurateness. In this context the mixed problems for the generalized KdV equation

$$u_t + u_x + u_{xxx} + u^p u_x + a(x)u = 0, \tag{1.5}$$

were studied in [28] when $p \in [2, 4)$. For the critical exponent, $p = 4$, the global well-posedness and the exponential stability were studied in [21]. The reader is also referred to [20,29] and the references therein for an overall literature review. To handle the nonlinearity in (1.3), i.e. the case $p = 2$ in (1.5) with $a(x)$ defined in (1.4), we follow mainly [22] and [28] to prove the exponential decay of the energy associated to (1.3) without any smallness restrictions.

It should be of interest to detect whether there is a difference between the powers $p \geqslant 2$ in the generalized Kawahara equation

$$u_t + u_x + u_{xxx} + u^p u_x - u_{xxxxx} + a(x)u = 0$$

while the decay of its solutions is studied. For $p \in (2, 4]$ we show that certain smallness conditions upon the initial data provide the desired decay. The case $p = 2$ is more appropriate in this sense.

2. Local and global well-posedness

For $L > 0$ and $T > 0$ let Q_T be a bounded rectangle: $Q_T = \{(x,t) \in (0,L) \times (0,T) \subset \mathbb{R}^2\}$. We study in this section the following nonlinear initial–boundary value problem:

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^2 u_x = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases} \tag{2.1}$$

Our aim is to put forward the well-posedness theory useful for posterior stability analysis.

2.1. Linear system

For the sake of completeness, we provide below the well-posedness results for the linear problem

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases} \tag{2.2}$$

Lemma 2.1. *Let $u_0 \in L^2(0,L)$. Then (2.2) possesses a unique (mild) solution*

$$u \in C^0\big([0,T]; L^2(0,L)\big) \cap L^2\big(0,T; H^2(0,L)\big) \tag{2.3}$$

with

$$u_{xx}(0,t) \in L^2(0,T).$$

Moreover, there exists a constant $C = C(T,L) > 0$ such that

$$\|u\|_{C^0([0,T];L^2(0,L))} + \|u\|_{L^2(0,T;H^2(0,L))} \leqslant C\|u_0\| \tag{2.4}$$

and

$$\big\|u_{xx}(0,t)\big\|_{L^2(0,T)} \leqslant \|u_0\|. \tag{2.5}$$

Proof. Let $A : D(A) \subset L^2(0,L) \to L^2(0,L)$ be the linear closed operator defined as

$$Av = -v' - v''' + v'''''$$

with the domain

$$D(A) = \big\{v \in H^5(0,L)\colon\ v(0) = v(L) = v'(0) = v'(L) = v''(L) = 0\big\}.$$

Then, for any $v \in D(A)$ it holds

$$(Av, v) = -\frac{1}{2}\int_0^L \big[(v')^2\big]'\,dx + \frac{1}{2}\int_0^L \big[(v'')^2\big]'\,dx = -\frac{1}{2}\big(v''\big)^2(0) \leqslant 0$$

which means that A is dissipative. Moreover, so is its adjoint

$$A^* : D\big(A^*\big) \subset L^2(0,L) \to L^2(0,L)$$

defined as

$$A^* v = v' + v''' - v'''''$$

with the domain

$$D\big(A^*\big) = \big\{v \in H^5(0,L)\colon\ v(0) = v(L) = v'(0) = v'(L) = v''(0) = 0\big\}.$$

Indeed, for $v \in D(A^*)$, we find $(v, A^* v) = -(v'')^2(0)/2 \leqslant 0$, as desired.

Notice that $\mathcal{D}(0,L)$ is dense in $L^2(0,L)$ and $\mathcal{D}(0,L) \subset D(A) \subset L^2(0,L)$ which implies that $D(A)$ is dense in $L^2(0,L)$. Therefore, due to classical semigroups results, A generates the semigroup of contractions, strongly continuous in $L^2(0,L)$

(see, for instance, Theorem 4.3 in [26]). Let $\{S(t)\}_{t\geqslant 0}$ be this semigroup. Then there exists a unique solution $u(\cdot,t)=S(t)u_0$ to (2.2) satisfying $u\in C^0([0,T];L^2(0,L))$ and

$$\|u\|_{C^0([0,T];L^2(0,L))}\leqslant\|u_0\|. \tag{2.6}$$

To prove that $u\in L^2(0,T;H^2(0,L))$, we first consider $u_0\in D(A)$. Multiply $(2.2)_1$ by xu and integrate over Q_T to obtain

$$\frac{1}{2}\int_0^L x\big|u(x,T)\big|^2\,dx+\frac{3}{2}\int_0^T\int_0^L\big|u_x(x,t)\big|^2\,dx\,dt+\frac{5}{2}\int_0^T\int_0^L\big|u_{xx}(x,t)\big|^2\,dx\,dt=\frac{1}{2}\int_0^L x\big|u_0(x)\big|^2\,dx+\frac{1}{2}\int_0^T\int_0^L\big|u(x,t)\big|^2\,dx\,dt,$$

which implies

$$\int_0^T\int_0^L\big\{\big|u(x,t)\big|^2+\big|u_x(x,t)\big|^2+\big|u_{xx}(x,t)\big|^2\big\}\,dx\,dt\leqslant L\int_0^L\big|u_0(x)\big|^2\,dx+2\int_0^T\int_0^L\big|u(x,t)\big|^2\,dx\,dt. \tag{2.7}$$

Combining (2.6) and (2.7), we get $u\in L^2(0,T;H^2(0,L))$ and

$$\|u\|_{L^2(0,T;H^2(0,L))}\leqslant(L+2T)^{1/2}\|u_0\|. \tag{2.8}$$

On the other hand, multiplying $(2.2)_1$ by u and integrating over Q_T gives

$$\int_0^L\big|u(x,T)\big|^2\,dx+\int_0^T\big|u_{xx}(0,t)\big|^2\,dt=\int_0^L\big|u_0(x)\big|^2\,dx, \tag{2.9}$$

i.e. $u_{xx}(0,T)\in L^2(0,T)$ and

$$\big\|u_{xx}(0,t)\big\|_{L^2(0,T)}\leqslant\|u_0\|. \tag{2.10}$$

Since $D(A)$ is dense in $L^2(0,L)$ it follows that (2.7)–(2.10) are valid for any $u_0\in L^2(0,L)$. This completes the proof. □

2.2. Nonlinear system. Local solutions

In this subsection we discuss the existence and uniqueness of a local solution to (2.1).

Lemma 2.2. *Let $T_0>0$ and $u_0\in L^2(0,L)$ be given. Then there exists $T\in(0,T_0]$ such that* (2.1) *possesses a unique solution*

$$u(x,t)\in C^0\big([0,T];L^2(0,L)\big)\cap L^2\big(0,T;H^2(0,L)\big). \tag{2.11}$$

Proof. Write a solution to (2.1) as

$$u=u_1+u_2, \tag{2.12}$$

where u_1 and u_2 solve, respectively, the following problems:

$$\begin{cases} u_{1t}+u_{1x}+u_{1xxx}-u_{1xxxxx}=0 & \text{in } Q_T,\\ u_1(0,t)=u_1(L,t)=u_{1x}(0,t)=u_{1x}(L,t)=u_{1xx}(L,t)=0, & t\in(0,T),\\ u_1(x,0)=u_0(x), & x\in(0,L), \end{cases} \tag{2.13}$$

and

$$\begin{cases} u_{2t}+u_{2x}+u_{2xxx}-u_{2xxxxx}=f & \text{in } Q_T,\\ u_2(0,t)=u_2(L,t)=u_{2x}(0,t)=u_{2x}(L,t)=u_{2xx}(L,t)=0, & t\in(0,T),\\ u_2(x,0)=0, & x\in(0,L), \end{cases} \tag{2.14}$$

where $f=-u^2u_x$.

For $u_0\in L^2(0,L)$, Lemma 2.1 assures that (2.13) admits the unique solution

$$u_1(x,t)\in C^0\big([0,T_0];L^2(0,L)\big)\cap L^2\big(0,T_0;H^2(0,L)\big). \tag{2.15}$$

Moreover, the map $u_0\mapsto u_1$ is linear and continuous.

To solve (2.14) observe that its linear part corresponds the semigroup $\{S(t)\}_{t\geqslant 0}$ considered in Lemma 2.1, and that $f=-u^2u_x$ is locally Lipschitz from $H^2(0,L)$ to $L^2(0,L)$. Indeed, since $\sup_{x\in[0,L]}|u^2|\leqslant\|u\|^2_{H^1(0,L)}$, one finds

$$\begin{aligned}\|u^2u_x-z^2z_x\| &\leqslant \|u^2(u_x-z_x)\|+\|z_x(u^2-z^2)\|\leqslant\|u^2\|_{L^\infty(0,L)}\|u_x-z_x\|+\|z_x\|_{L^\infty(0,L)}\|u-z\|\|u+z\|_{L^\infty(0,L)}\\ &\leqslant\big(\|u\|^2_{H^1(0,L)}+\|z\|_{H^2(0,L)}\|u+z\|_{H^1(0,L)}\big)\|u-z\|_{H^2(0,L)}.\end{aligned}\tag{2.16}$$

Therefore, there exists $T>0$ such that the implicitly defined function

$$u_2(\cdot,t)=\int_0^t S(t-s)f(\cdot,s)\,ds,\quad t\in[0,T]\subset[0,T_0),$$

solves (2.14) and satisfies (2.11). The proof is completed. □

2.3. Nonlinear system. Global solutions

In order to study the long-time asymptotic for nonlinear model (2.1), its global well-posedness has been established in this subsection. The main result here is

Lemma 2.3. *Let u be solution to* (2.1) *assured by Lemma* 2.2. *Then there exists $C=C(T,L)>0$ such that*

$$\|u\|^2_{L^2(0,T;H^2(0,L))}\leqslant C\|u_0\|^2\big(1+\|u_0\|^4\big)\tag{2.17}$$

and

$$u_t\in L^{4/3}\big(0,T;H^{-3}(0,L)\big).\tag{2.18}$$

Proof. As usual, the proof consists in several a priori estimates. First, multiply $(2.1)_1$ by u and integrate over $(0,L)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\big\|u(t)\big\|^2+\frac{1}{2}\big|u_{xx}(0,t)\big|^2=0$$

which means

$$\|u\|_{L^\infty(0,T;L^2(0,L))}\leqslant\|u_0\|.\tag{2.19}$$

Second, multiply $(2.1)_1$ by xu and integrate over Q_T to get

$$\int_0^T\int_0^L|u_x|^2\,dx\,dt+\frac{1}{3}\int_0^L x\big|u(x,T)\big|^2\,dx+\frac{5}{2}\int_0^T\int_0^L|u_{xx}|^2\,dx\,dt=\frac{1}{3}\int_0^T\int_0^L|u|^2\,dx\,dt+\frac{1}{3}\int_0^L x|u_0|^2\,dx+\frac{1}{6}\int_0^T\int_0^L|u|^4\,dx\,dt.$$

The use of (2.19) gives

$$\|u\|^2_{L^2(0,T;H^2(0,L))}\leqslant\left(\frac{4T+L}{3}\right)\|u_0\|^2+\frac{1}{6}\int_0^T\int_0^L u^4\,dx\,dt.\tag{2.20}$$

The Gagliardo–Nirenberg inequality jointly with (2.19) implies

$$\begin{aligned}\int_0^T\int_0^L u^4\,dx\,dt &\leqslant C\int_0^T\|u\|^3\,\|u_x\|\,dt\leqslant C\int_0^T\|u\|^6\,dt+3\int_0^T\|u_x\|^2\,dt\leqslant C\|u\|^6_{L^6(0,T;L^2(0,L))}+3\|u\|^2_{L^2(0,T;H^1(0,L))}\\ &\leqslant C(T)\|u\|^6_{L^\infty(0,T;L^2(0,L))}+3\|u\|^2_{L^2(0,T;H^2(0,L))}\leqslant C(T)\|u_0\|^6+3\|u\|^2_{L^2(0,T;H^2(0,L))}.\end{aligned}\tag{2.21}$$

Substituting (2.19) and (2.21) into (2.20), we obtain (2.17).

To get bounds for u_t we have to treat the nonlinear term in more details. First, note that (2.21) assures that

$$u^3 \text{ is bounded in } L^{4/3}(Q_T).$$

By the other hand, since $L^{4/3}(0,T)\hookrightarrow H^{-2}(0,T)$, we conclude that

$$u^2u_x=\frac{1}{3}\frac{\partial}{\partial x}\big(u^3\big)\text{ is bounded in }L^{4/3}\big(0,T;H^{-3}(0,L)\big).\tag{2.22}$$

This allows to estimate u_t. Indeed, write $(2.1)_1$ as

$$u_t = -u_x - u_{xxx} + u_{xxxxx} - u^2 u_x \quad \text{in } \mathcal{D}'\big(0, T; H^{-3}(0, L)\big).$$

Since (2.17), (2.19) and (2.22), it holds that

$$u_t \text{ is bounded in } L^{4/3}\big(0, T; H^{-3}(0, L)\big) \tag{2.23}$$

which completes the proof. □

Remark 2.1. Considering the generalized Kawahara equation

$$u_t + u_x + u_{xxx} - u_{xxxxx} + u^p u_x = 0 \quad \text{with } p \geqslant 2, \tag{2.24}$$

one can see that global solutions for $p \in (2, 4]$ can be obtained (at least by the method above) provided the initial data are sufficiently small. More precisely, if $p \in (2, 4]$ and $\|u_0\| \ll 1$, then

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leqslant c_1 \left(\frac{\|u_0\|^2}{1 - c_2\|u_0\|^2} \right) \tag{2.25}$$

where $c_k = c_k(T, L)$ $(k = 1, 2)$ are positive constants. In fact, for $2 < p \leqslant 4$, (2.19) remains true and the Gagliardo–Nirenberg inequality may be applied as follows:

$$\int_0^T \int_0^L u^{p+2}\,dx\,dt \leqslant C \int_0^T \|u\|^p \|u_x\|^2\,dt \leqslant C\|u_0\|^p \int_0^T \|u_x\|^2\,dt \leqslant C\|u_0\|^p \|u\|^2_{L^2(0,T;H^2(0,L))}. \tag{2.26}$$

Substituting (2.26) into appropriately modified (2.20), we obtain (2.25).

2.4. Example of a critical set

Due to dispersive properties, just obtained global solutions are expected to be dissipative, i.e. to approach zero as $t \to \infty$. However, the following example shows that not every solution of the Kawahara equation stabilizes. We prove here the existence of a nonempty critical set for the linearized system (2.2). More precisely, we construct a nontrivial steady-state solution to the initial–boundary value problem (2.2) with a non-zero initial datum $u_0(x) \not\equiv 0$ and homogeneous boundary conditions upon the endpoints of the interval with a critical length. Moreover, the critical set is shown to be at least countable, in the same manner as it is for the KdV equation (see [27]).

Let $L > 0$ be a real number and take

$$a = \sqrt{\frac{\sqrt{5} - 1}{2}}, \qquad b = \sqrt{\frac{\sqrt{5} + 1}{2}},$$
$$C_2 = 1 - e^{-aL}, \qquad C_3 = e^{aL} - 1, \qquad A = C_2 + C_3, \qquad B = C_3 - C_2,$$
$$C_1 = -\left(1 + \frac{a^2}{b^2}\right)A, \qquad C_4 = \frac{a^2}{b^2}A, \qquad C_5 = \frac{a}{b}B.$$

Define

$$\mathcal{N} = \left\{ L > 0\colon e^{ibL} = \left(\frac{C_4 + iC_5}{|C_4 + iC_5|} \right)^2 \right\} \subset \mathbb{R}^+$$

and

$$u(x) = C_1 + C_2 e^{ax} + C_3 e^{-ax} + C_4 \cos(bx) + C_5 \sin(bx) \not\equiv 0, \quad x \in (0, L).$$

If $L \in \mathcal{N}$, then $u = u(x)$ solves

$$-u''''' + u''' + u' = 0,$$

and satisfies

$$u(0) = u'(0) = u''(0) = u(L) = u'(L) = u''(L) = 0.$$

Observe that the countable critical set $\mathcal{N} \subset \mathbb{R}^+$ is due to a presence of the linear drift term u_x in $(2.2)_1$. It is shown in [35] that $\mathcal{N} = \emptyset$ for the model with u_x neglected. We note also that a complete description of critical lengths for (2.2) is an open problem.

3. Stability

Due to example provided above and in order to control the system described by (2.1), one is suggested to add some extra dissipative mechanism into the model. In this section, we consider the system appended with so-called "localized damping" term, i.e. the extra term corresponding, roughly speaking, to some kind of friction presented in a part of the spatial domain. We are looking for an exponential decay of the energy associated to the following problem:

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + u^2 u_x + a(x)u = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L), \end{cases} \tag{3.1}$$

where $a : (0,L) \to \mathbb{R}$ is a nonnegative function satisfying

$$\begin{cases} a \in L^\infty(0,L) \quad \text{and} \quad a(x) \geqslant a_0 > 0 \quad \text{a.e. in } \omega, \\ \text{with a nonempty } \omega \subset (0,L). \end{cases} \tag{3.2}$$

Considering (3.1) as a perturbation of (2.1) with $a(x) \equiv 0$, one can see that (3.1) is globally well-posed for $u_0 \in L^2(0,L)$, and (2.17) and (2.18) hold.

Note that multiplying $(3.1)_1$ by u and integrating over $(0,L)$ yields

$$\frac{d}{dt}\int_0^L \left|u(x,t)\right|^2 dx + \frac{1}{2}\left|u_{xx}(0,t)\right|^2 + \int_0^L a(x)\left|u(x,t)\right|^2 dx = 0. \tag{3.3}$$

This indicates that the energy $E(t) = \frac{1}{2}\|u\|^2(t)$ associated with (3.1) is not increasing, and the term $a(x)u$ designs a localized damping mechanism.

3.1. Linear case

We follow [34] and [40] to prove L^2-exponential decay of solutions to the linear problem

$$\begin{cases} u_t + u_x + u_{xxx} - u_{xxxxx} + a(x)u = 0 & \text{in } Q_T, \\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (0,L). \end{cases} \tag{3.4}$$

Despite the linear case provides no difficulties, we present it in detail in order to be applied in modified and improved form to nonlinear system (3.1). The stability result in this subsection is given by

Lemma 3.1. *Let $a(x)$ satisfy* (3.2). *Then for all $L > 0$ there exist constants $c > 0$ and $\mu > 0$ such that solutions of* (3.4) *obey*

$$\|u\|^2(t) \leqslant c\|u_0\|^2 e^{-\mu t}, \quad \forall t \geqslant 0. \tag{3.5}$$

Proof. Multiply $(3.4)_1$ by ux and integrate over Q_T to obtain

$$\begin{aligned} &\frac{1}{2}\int_0^L x\left|u(x,T)\right|^2 dx + \frac{3}{2}\int_0^T\int_0^L \left|u_x(x,t)\right|^2 dx\,dt + \frac{5}{2}\int_0^T\int_0^L \left|u_{xx}(x,t)\right|^2 dx\,dt + \int_0^T\int_0^L xa(x)\left|u(x,t)\right|^2 dx\,dt \\ &\quad = \frac{1}{2}\int_0^T\int_0^L \left|u(x,t)\right|^2 dx\,dt + \frac{1}{2}\int_0^L x\left|u_0(x)\right|^2 dx. \end{aligned}$$

Then

$$\|u\|^2_{L^2(0,T;H^2(0,L))} \leqslant (2T+L)\|u_0\|^2. \tag{3.6}$$

On the other hand, multiplying $(3.4)_1$ by $(T-t)u$ and integrating over Q_T gives

$$T\|u_0\|^2 = \int_0^T\int_0^L \left|u(x,t)\right|^2 dx\,dt + \int_0^T (T-t)\left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L (T-t)a(x)\left|u(x,t)\right|^2 dx\,dt \tag{3.7}$$

which can be written as

$$\|u_0\|^2 \leqslant \frac{1}{T}\int_0^T\int_0^L \left|u(x,t)\right|^2 dx\,dt + \int_0^T \left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt. \tag{3.8}$$

Our aim now is to prove that

$$\int_0^T\int_0^L \left|u(x,t)\right|^2 dx\,dt \leqslant C\left\{\int_0^T \left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt\right\} \tag{3.9}$$

for some positive constant $C = C(R,T)$ independent on u. We employ the contradiction argument following [40]. Suppose (3.9) is not true. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$,

$$u_n \in L^\infty\big(0,T;L^2(0,L)\big)\cap L^2\big(0,T;H^2(0,L)\big),$$

which solves (3.4) and

$$\lim_{n\to\infty}\frac{\int_0^T\int_0^L |u_n(x,t)|^2\,dx\,dt}{\int_0^T |u_{n,xx}(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)|u_n(x,t)|^2\,dx\,dt} = +\infty. \tag{3.10}$$

Let $\lambda_n = \|u_n\|_{L^2(Q_T)}$ and $v_n(x,t) = u_n(x,t)/\lambda_n$. Then v_n solves

$$\begin{cases} v_{n,t} + v_{n,x} + v_{n,xxx} - v_{n,xxxxx} + a(x)v_n = 0 & \text{in } Q_T,\\ v_n(0,t) = v_n(L,t) = v_{n,x}(0,t) = v_{n,x}(L,t) = v_{n,xx}(L,t) = 0, & t\in(0,T),\\ v_n(x,0) = v_{0,n} = \dfrac{u_n(x,0)}{\lambda_n}, & x\in(0,L). \end{cases} \tag{3.11}$$

On the other hand,

$$\|v_n\|_{L^2(Q_T)} = 1 \tag{3.12}$$

and, due to (3.10), it holds

$$\lim_{n\to\infty}\left\{\int_0^T \left|v_{n,xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|v_n(x,t)\right|^2 dx\,dt\right\} = 0. \tag{3.13}$$

Using (3.12) and (3.13), (3.8) means that $\{v_n(\cdot,0)\}$ is bounded in $L^2(0,L)$. Therefore, (3.6) implies

$$\|v_n\|_{L^2(0,T;H^2(0,L))} \leqslant C, \quad \forall n\in\mathbb{N}, \tag{3.14}$$

for some constant $C>0$. Similarly, $(3.11)_1$ and (3.14) assure that $\{(v_n)_t\}$ is bounded in $L^2(0,T;H^{-3}(0,L))$. In this way, due to Aubin–Lions' compactness theorem (see [32, Corollary 4]), one can extract a subsequence $\{v_n\}$ (still denoted as before) such that

$$v_n \to v \quad \text{strongly in } L^2(Q_T), \tag{3.15}$$

$$v_n \to v \quad \text{weakly in } L^2\big(0,T;H^2(0,L)\big), \tag{3.16}$$

$$v_{n,t} \to v_t \quad \text{weakly in } L^2\big(0,T;H^{-3}(0,L)\big). \tag{3.17}$$

By (3.12),

$$\|v\|_{L^2(Q_T)} = 1. \tag{3.18}$$

Moreover, since $a(x)$ satisfies (3.2) we get

$$0 = \liminf_{n\to\infty}\left\{\int_0^T \left|v_{n,xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|v_n(x,t)\right|^2 dx\,dt\right\} \geqslant \left\{\int_0^T \left|v_{xx}(0,t)\right|^2 dt + \int_0^T\int_0^L a(x)\left|v(x,t)\right|^2 dx\,dt\right\} \tag{3.19}$$

which implies $a(x)v \equiv 0$ in Q_T. In particular, $v \equiv 0$ in $\omega\times(0,T)$. However, $v(x,t)$ satisfies

$$v_t + v_x + v_{xxx} - v_{xxxxx} = 0$$

and, by Hölmgren's uniqueness theorem, $v \equiv 0$ in Q_T. This contradicts (3.18) and, therefore, (3.9) is verified.

To prove the energy decay we insert (3.9) into (3.8) to get

$$E(0) = \frac{1}{2}\|u_0\|^2 \leqslant C\left\{\frac{1}{2}\int_0^T \left|u_{xx}(0,t)\right|^2 dt + \int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt\right\} \tag{3.20}$$

for some constant $C = C(T) > 0$. On the other hand, (3.3) implies

$$E(T) = E(0) - \frac{1}{2}\int_0^T \left|u_{xx}(0,t)\right|^2 dt - \int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx, \tag{3.21}$$

which jointly with (3.20) insures

$$\begin{aligned}(1+C)E(T) &= (1+C)\left(E(0) - \frac{1}{2}\int_0^T \left|u_{xx}(0,t)\right|^2 dt - \int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt\right)\\ &\leqslant CE(0) - \left(\frac{1}{2}\int_0^T \left|u_{xx}(0,t)\right|^2 dt + \int_0^T\int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt\right) \leqslant CE(0).\end{aligned}$$

Consequently, $E(T) \leqslant \gamma E(0)$ with $0 < \gamma < 1$. Therefore, $\|u(\cdot,kT)\|^2 \leqslant \gamma^k\|u_0\|^2$ for all $k \geqslant 0$. Finally, since $\|u(\cdot,t)\| \leqslant \|u(\cdot,kT)\|$ for $kT \leqslant t \leqslant (k+1)T$, we get

$$E(t) \leqslant c\|u_0\|e^{-\mu t}, \quad \forall t \geqslant 0,$$

where $c = 1/\gamma$ and $\mu = \log\gamma/T$. □

3.2. Compactness-uniqueness method for nonlinear system

Considered here are preliminaries for nonlinear treatment; the last one is the crucial unique continuation property. Notice first that $v = u_t$ solves

$$\begin{cases} v_t + v_x + v_{xxx} - v_{xxxxx} + \left(u^2 v\right)_x + a(x)v = 0 & \text{in } Q_T,\\ v(0,t) = v(L,t) = v_x(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, & t \in (0,T),\\ v(x,0) = v_0(x), & x \in (0,L), \end{cases} \tag{3.22}$$

where u is a solution to (3.1) with $u_0 \in L^2(0,L)$ and

$$v_0(x) = u_t(x,0) = -u_{0,x} - u_{0,xxx} + u_{0,xxxxx} - u_0^2 u_{0,x} - a(x)u_0 \in H^{-5}(0,L). \tag{3.23}$$

Lemma 3.2. *There exists a constant $C = C(T,\|u_0\|) > 0$ such that*

$$\|v_0\|^2_{L^2(0,L)} \leqslant C\left\{\int_0^T v_{xx}^2(0,t)\,dt + \int_0^T\int_0^L a(x)v^2(x,t)\,dx\,dt + \|v_0\|^2_{H^{-5}(0,L)} + 1\right\}. \tag{3.24}$$

Proof. Multiplying $(3.22)_1$ by $(T-t)v$ and integrating over Q_T, we obtain

$$T\|v_0\|^2_{L^2(0,L)} = \int_0^T\int_0^L v^2\,dx\,dt + \int_0^T (T-t)v_{xx}^2(0,t)\,dt + 2\int_0^T\int_0^L a(x)v^2\,dx\,dt + \frac{1}{2}\int_0^T\int_0^L (T-t)\left(\frac{d}{dx}u^2\right)v^2\,dx\,dt, \tag{3.25}$$

this implies

$$\|v_0\|^2_{L^2(0,L)} \leqslant \frac{1}{T}\int_0^T\int_0^L v^2\,dx\,dt + 2\int_0^T\int_0^L a(x)v^2\,dx\,dt + \int_0^T v_{xx}^2(0,t)\,dt + \int_0^T\int_0^L |uu_x|v^2\,dx\,dt. \tag{3.26}$$

Moreover,

$$\int_0^T\int_0^L |uu_x|v^2\,dx\,dt \leqslant \int_0^T \|uu_x\|_{L^2(0,L)}\|v\|^2_{L^4(0,L)}\,dt \leqslant C\|u\|^2_{L^\infty(0,T;L^2(0,L))}\|u\|^2_{L^2(0,T;H^2(0,L))} + \frac{1}{2}\|v\|^4_{L^4(Q_T)}. \tag{3.27}$$

Substituting (3.27) in (3.26) and using (2.19), (2.20), we have

$$\|v_0\|^2_{L^2(0,L)} \leqslant C + \frac{1}{2}\|v\|^4_{L^4(Q_T)} + \int_0^T v^2_{xx}(0,t)\,dt + 2\int_0^T\int_0^L a(x)v^2\,dx\,dt, \tag{3.28}$$

where $C > 0$ is a constant depending on $\|u_0\|$. Therefore, to prove (3.24) it suffices to show that for all $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$\|v\|^2_{L^4(0,T;L^4(0,L))} \leqslant C\left\{\int_0^T v^2_{xx}(0,t)\,dt + 2\int_0^T\int_0^L a(x)v^2(x,t)\,dx\,dt + \|v_0\|^2_{H^{-5}(0,L)}\right\}. \tag{3.29}$$

Suppose this is false, i.e. there exists a sequence v_n of solutions to (3.1) satisfying

$$\lim_{n\to\infty}\frac{\|v_n\|^4_{L^4(Q_T)}}{\int_0^T |v_{n,xx}(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)v_n^2(x,t)\,dx\,dt + \|v_{0,n}\|^2_{H^{-5}(0,L)}} = \infty. \tag{3.30}$$

Let $\lambda_n = \|v_n\|_{L^4(0,T;L^4(0,L))}$ and define $w_n(x,t) = \frac{v_n(x,t)}{\lambda_n}$. Then w_n satisfies

$$\begin{cases} w_{n,t} + w_{n,x} + w_{n,xxx} - w_{n,xxxxx} + \left(u^2(x,t)w_n\right)_x + a(x)w_n = 0 & \text{in } Q_T,\\ w_n(0,t) = w_n(L,t) = w_{n,x}(0,t) = w_{n,x}(L,t) = w_{n,xx}(L,t) = 0, & t \in (0,T),\\ w_n(x,0) = \dfrac{v_n(x,0)}{\lambda_n}, & x \in (0,L), \end{cases} \tag{3.31}$$

and

$$\|w_n\|_{L^4(Q_T)} = 1. \tag{3.32}$$

Moreover, taking the limit as $n \to \infty$, it holds

$$\int_0^T \left|w_{n,xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)w_n^2(x,t)\,dx\,dt + \left\|w_n(\cdot,0)\right\|^2_{H^{-5}(0,L)} \to 0. \tag{3.33}$$

Since (3.28), (3.32) and (3.33), it follows that $\{w_n(\cdot,0)\}$ is bounded in $L^2(0,L)$. Therefore, using the same arguments as for (2.17) and (3.28), we obtain

$$\|w_n\|_{L^2(0,T;H^2(0,L))} \leqslant C, \tag{3.34}$$

for some constant $C > 0$. By the other hand

$$\left\|\left(u^2w_n\right)_x\right\|_{L^2(0,T;L^1(0,L))} \leqslant C\|w_n\|_{L^\infty(0,T;L^2(0,L))}\|u\|^2_{L^2(0,T;H^2(0,L))} + C\|u\|^2_{L^\infty(0,T;L^2(0,L))}\|w_n\|_{L^2(0,T;H^2(0,L))}, \tag{3.35}$$

where C is some other positive constant. Thus, it follows from (2.17), (2.19) and (3.34) that one can find a constant $C > 0$ such that

$$\left\|\left(u^2w_n\right)_x\right\|_{L^2(0,T;L^1(0,L))} \leqslant C. \tag{3.36}$$

Observe also that $(3.31)_1$, (3.34) and (3.36) insure

$$\{w_{n,t}\} \text{ is bounded in } L^2\left(0,T;H^{-3}(0,L)\right). \tag{3.37}$$

We claim now that there exists a constant $s > 0$ such that $\{w_n\}$ is bounded in $L^4(0,T;H^s(0,L))$ and the embedding $H^s(0,L) \hookrightarrow L^4(0,L)$ is compact. Indeed, because of $\{w_n\}$ is bounded in $L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))$, we deduce by interpolation that $\{w_n\}$ is bounded in

$$\left[L^q\big(0,T;L^2(0,L)\big),L^2\big(0,T;H^2(0,L)\big)\right]_\theta=L^p\big(0,T;\left[L^2(0,L);H^2(0,L)\right]_\theta\big),$$

where $\frac{1}{p}=\frac{1-\theta}{q}+\frac{\theta}{2}$ and $0<\theta<1$. Therefore, setting $p=4$, $q=\infty$ and $\theta=1/2$, we find $s=1/2$, i.e.

$$\left[L^2(0,T),H^2(0,L)\right]_{1/2}=H^{1/2}(0,L)$$

and the embedding $H^{\frac{1}{2}}(0,L)\hookrightarrow L^4(0,L)$ is compact.

Finally, using the above claim, (3.37) and the classical compactness results (see [32], for instance) we get a subsequence $\{w_n\}$ such that

$$w_n\to w\quad\text{strongly in } L^4(Q_T) \tag{3.38}$$

and, by (3.32), it holds

$$\|w\|_{L^4(Q_T)}=1. \tag{3.39}$$

Note that

$$\begin{aligned}0&=\lim_{n\to\infty}\inf\left\{\int_0^T\left|w_{n,xx}(0,t)\right|^2dt+2\int_0^T\int_0^L a(x)w_n^2(x,t)\,dx\,dt+\left\|w_n(\cdot,0)\right\|^2_{H^{-5}(0,L)}\right\}\\&\geqslant\int_0^T\left|w_{xx}(0,t)\right|^2dt+2\int_0^T\int_0^L a(x)w^2(x,t)\,dx\,dt+\left\|w(\cdot,0)\right\|^2_{H^{-5}(0,L)}\end{aligned} \tag{3.40}$$

which implies, in particular, that $w(\cdot,0)=0$. Therefore, the limit w, which solves the system

$$\begin{cases}w_t+w_x+w_{xxx}+\big(u^2w\big)_x-w_{xxxxx}+a(x)w=0 & \text{in } Q_T,\\ w(0,t)=w(L,t)=w_x(0,t)=w_x(L,t)=w_{xx}(L,t)=0, & t\in(0,T),\\ w(x,0)=0, & x\in(0,L),\end{cases}$$

is a null function, i.e. $w\equiv 0$. This contradicts (3.39) and, necessarily, (3.29) is valid. The proof of Lemma 3.2 is completed. □

As a corollary, the unique continuation property is done as follows.

Lemma 3.3. *Let u be a solution to* (3.1) *with $a(x)$ and ω defined in* (3.2)*. If*

$$\begin{cases}u_{xx}(0,\cdot)=0, & \forall t>0,\\ u\equiv 0 & \text{in } \omega\times(0,T),\end{cases}$$

then $u\equiv 0$ in Q_T.

Proof. Let $u_0\in L^2(0,L)$. Bearing in mind (3.2), (3.23) and Lemma 3.2, we deduce that $v_0\in L^2(0,L)$. Therefore, by Lemma 2.2, a solution v of (3.22) lyes in

$$u_t=v\in L^\infty\big(0,T;L^2(0,L)\big)\cap L^2\big(0,T;H^2(0,L)\big). \tag{3.41}$$

Since (2.11), (3.2) and (3.41), it follows from $(3.1)_1$ that $u_{xxxxx}\in L^2(Q_T)$. Therefore,

$$u\in L^2\big(0,T;H^5(0,L)\big)\cap H^1\big(0,T;H^2(0,L)\big)$$

which is sufficiently to the unique continuation principle from [30] be applied. This gives $u\equiv 0$ in Q_T which completes the proof. □

3.3. Nonlinear system. Main result

The main result of the paper is:

Theorem 3.1. *Let u be a global solution to* (3.1) *ensured by Lemma* 2.3 *with $\|u_0\|\leqslant R$. Then for all $L>0$ there exist positive constants $c=c(R)$ and $\mu=\mu(R)$ such that*

$$E(t):=\frac{1}{2}\|u\|^2(t)\leqslant c\|u_0\|^2e^{-\mu t},\quad\forall t\geqslant 0. \tag{3.42}$$

Proof. Integrate (3.3) over $(0,t)$ with $t \in (0,T)$ to obtain

$$E(t) = E(0) - \frac{1}{2}\int_0^t \left|u_{xx}(0,t)\right|^2 - \int_0^t \int_0^L a(x)\left|u(x,t)\right|^2 dx.$$

Hence,

$$E(t) \leqslant E(0) = \frac{1}{2}\|u_0\|^2, \quad \forall t \geqslant 0. \tag{3.43}$$

We are aimed to prove that for all $T > 0$ there exists a constant $C = C(T) > 0$ such that

$$\|u_0\|^2 \leqslant C\left\{ \int_0^T \left|u_{xx}(0,t)\right|^2 dt + \int_0^T \int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt \right\}. \tag{3.44}$$

In fact, multiplying $(3.1)_1$ by $(T-t)u$ and integrating over Q_T, we get

$$T\|u_0\|^2 = \int_0^T \int_0^L \left|u(x,t)\right|^2 dx\,dt + \int_0^T (T-t)\left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T \int_0^L (T-t)a(x)\left|u(x,t)\right|^2 dx\,dt, \tag{3.45}$$

which implies

$$\|u_0\|^2 \leqslant \frac{1}{T}\int_0^T \int_0^L \left|u(x,t)\right|^2 dx\,dt + \int_0^T \left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T \int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt. \tag{3.46}$$

Therefore, to prove (3.44) it suffices to show that for all $T > 0$ there exists a constant $C > 0$ such that

$$\int_0^T \int_0^L \left|u(x,t)\right|^2 dx\,dt \leqslant C\left\{ \int_0^T \left|u_{xx}(0,t)\right|^2 dt + 2\int_0^T \int_0^L a(x)\left|u(x,t)\right|^2 dx\,dt \right\} \tag{3.47}$$

provided $\|u_0\| \leqslant R$, for some $R > 0$. We use the contradiction argument again. Suppose (3.47) fails. Then one finds the sequence

$$\{u_n\} \in C^0\big([0,T]; L^2(0,L)\big) \cap L^2\big(0,T; H^2(0,L)\big)$$

satisfying

$$\big\|u_n(\cdot,0)\big\| \leqslant R, \tag{3.48}$$

which solves (3.1) and

$$\lim_{n\to\infty} \frac{\int_0^T \int_0^L |u_n(x,t)|^2 dx\,dt}{\int_0^T |u_{n,xx}(0,t)|^2 dt + 2\int_0^T \int_0^L a(x)|u_n(x,t)|^2 dx\,dt} = \infty. \tag{3.49}$$

As earlier, take $\lambda_n = \|u_n\|_{L^2(Q_T)}$ and define $v_n(x,t) = u_n(x,t)/\lambda_n$. For any $n \in \mathbb{N}$ the function v_n solves

$$\begin{cases} v_{n,t} + v_{n,x} + v_{n,xxx} + \lambda_n^2 v_n^2 v_{n,x} - v_{n,xxxxx} + a(x)v_n = 0 & \text{in } Q_T, \\ v_n(0,t) = v_n(0,L) = v_{n,x}(0,t) = v_{n,x}(L,t) = v_{n,xx}(L,t) = 0, & t \in (0,T), \\ v_n(x,0) = v_{0,n} = \dfrac{u_n(x,0)}{\lambda_n}, & x \in (0,L). \end{cases} \tag{3.50}$$

Moreover,

$$\|v_n\|_{L^2(Q_T)} = 1 \tag{3.51}$$

and

$$\int_0^T \left|v_{n,xx}(0,t)\right|^2 dt + 2\int_0^T \int_0^L a(x)\left|v_n(x,t)\right|^2 dx\,dt \to 0, \quad \text{as } n \to \infty. \tag{3.52}$$

Using (3.51) and (3.52) in (3.46), one concludes that $\{v_n(\cdot,0)\}$ is bounded in $L^2(0,L)$. Then (2.17) gives

$$\|v_n\|_{L^2(0,T;H^2(0,L))} \leqslant c, \quad \forall n \in \mathbb{N}, \tag{3.53}$$

and in the same way as in (2.23) we deduce

$$\{v_{n,t}\} \text{ is bounded in } L^{4/3}\big(0,T;H^{-3}(0,L)\big). \tag{3.54}$$

Since $H^2(0,L) \hookrightarrow L^2(0,L)$ compactly, due to (3.53), (3.54) and by Aubin–Lions' theorem we see that $\{v_n\}$ is relatively compact in $L^2(Q_T)$. Therefore, a subsequence $\{v_n\}$ can be extracted to obey

$$v_n \to v \quad \text{strongly in } L^2(Q_T) \tag{3.55}$$

and, by (3.51), it holds

$$\|v\|_{L^2(Q_T)} = 1. \tag{3.56}$$

Moreover,

$$\begin{aligned} 0 &= \liminf_{n\to\infty} \left\{ \int_0^T \left|v_{n,xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|v_n(x,t)\right|^2 dx\,dt \right\} \\ &\geqslant \int_0^T \left|v_{xx}(0,t)\right|^2 dt + 2\int_0^T\int_0^L a(x)\left|v(x,t)\right|^2 dx\,dt, \end{aligned} \tag{3.57}$$

which implies

$$\begin{cases} v_{xx}(0,t) = 0, & t \in (0,T), \\ v \equiv 0 & \text{in } \omega \times (0,T). \end{cases} \tag{3.58}$$

Combining (3.43) and (3.48), we have also a subsequence of $\{\lambda_n\}$, still denoted by $\{\lambda_n\}$ and $\lambda \geqslant 0$ such that

$$\lambda_n \to \lambda.$$

Two situations may occur: either $\lambda = 0$ or $\lambda > 0$. If $\lambda = 0$, then v solves linear problem (3.4) and satisfies (3.58). Therefore, by Hölmgren's uniqueness theorem, $v \equiv 0$ in Q_T which contradicts (3.56). If $\lambda > 0$, then v is a solution to nonlinear problem

$$\begin{cases} v_t + v_x + v_{xxx} - v_{xxxxx} + \lambda^2 v^2 v_x + a(x)v = 0 & \text{in } Q_T, \\ v(0,t) = v(L,t) = v_x(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, & t \in (0,T), \\ v(x,0) = v_0(x), & x \in (0,L), \end{cases}$$

satisfying (3.58). By Lemma 3.3 we get $v \equiv 0$ in Q_T which is a contradiction, as well.

Thus, (3.44) holds and the same arguments as in Lemma 3.1 lead to the desired result. □

Remark 3.1. Considering the generalized Kawahara equation (2.24) one concludes that the case $p = 2$ is more appropriate in the following sense: if $p = 2$, the exponent energy decay holds for the arbitrary initial data from $L^2(0,L)$. If $p \in (2,4]$, the decay may be proved in the same way as above, provided the global solutions exist. It was just mentioned, however, in Remark 2.1 that the existence of such solutions is available only for small initial data (at least by the suggested method).

References

[1] N. Berloff, L. Howard, Solitary and periodic solutions of nonlinear nonintegrable equations, Stud. Appl. Math. 99 (1) (1997) 1–24.

[2] A. Biswas, Solitary wave solution for the generalized Kawahara equation, Appl. Math. Lett. 22 (2009) 208–210.

[3] J.L. Bona, S.M. Sun, B.-Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain, Comm. Partial Differential Equations 28 (2003) 1391–1436.

[4] J.L. Bona, S.M. Sun, B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg–de Vries equation posed on a finite domain, II, J. Differential Equations 247 (9) (2009) 2558–2596.

[5] J.P. Boyd, Weakly non-local solitons for capillary-gravity waves: fifth degree Korteweg–de Vries equation, Phys. D 48 (1991) 129–146.

[6] T. Bridges, G. Derks, Linear instability of solitary wave solutions of the Kawahara equation and its generalizations, SIAM J. Math. Anal. 33 (6) (2002) 1356–1378.

[7] E. Cerpa, Exact controllability of a nonlinear Korteweg–de Vries equation on a critical spatial domain, SIAM J. Control Optim. 46 (2007) 877–899.

[8] E. Cerpa, E. Crépeau, Boundary controllability for the nonlinear Korteweg–de Vries equation on any critical domain, Ann. Inst. H. Poincaré Anal. Non Lineaire 26 (2009) 457–475.

[9] J.M. Coron, E. Crépeau, Exact boundary controllability of a nonlinear KdV equation with a critical length, J. Eur. Math. Soc. (JEMS) 6 (2004) 367–398.
[10] S.B. Cui, D.G. Deng, S.P. Tao, Global existence of solutions for the Cauchy problem of the Kawahara equation with L_2 initial data, Acta Math. Sin. (Engl. Ser.) 22 (2006) 1457–1466.
[11] G.G. Doronin, N.A. Larkin, Kawahara equation in a bounded domain, Discrete Contin. Dyn. Syst. 10 (4) (2008) 783–799.
[12] A.V. Faminskii, N.A. Larkin, Initial–boundary value problems for quasilinear dispersive equations posed on a bounded interval, Electron. J. Differential Equations 2010 (1) (2010) 1–20.
[13] O. Goubet, J. Shen, On the dual Petrov–Galerkin formulation of the KdV equation, Adv. Differential Equations 12 (2) (2007) 221–239.
[14] J.K. Hunter, J. Scheurle, Existence of perturbed solitary wave solutions to a model equation for water waves, Phys. D 32 (1998) 253–268.
[15] T. Iguchi, A long wave approximation for capillary-gravity waves and the Kawahara equations, Bull. Inst. Math. Acad. Sin. (N.S.) 2 (2) (2007) 179–220.
[16] L. Jin, Application of variational iteration method and homotopy perturbation method to the modified Kawahara equation, Math. Comput. Modelling 49 (2009) 573–578.
[17] T. Kakutani, Axially symmetric stagnation-point flow of an electrically conducting fluid under transverse magnetic field, J. Phys. Soc. Japan 15 (1960) 688–695.
[18] D. Kaya, K. Al-Khaled, A numerical comparison of a Kawahara equation, Phys. Lett. A 363 (5–6) (2007) 433–439.
[19] T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan 33 (1972) 260–264.
[20] V. Komornik, D.L. Russell, B.Y. Zhang, Stabilization de l'équation de Korteweg–de Vries, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991) 841–843.
[21] F. Linares, A.F. Pazoto, On the exponential decay of the critical generalized Korteweg–de Vries equation with localized damping, Proc. Amer. Math. Soc. 135 (2007) 1515–1522.
[22] G.P. Menzala, C.F. Vasconcellos, E. Zuazua, Stabilization of the Korteweg–de Vries equation with localized damping, Quart. Appl. Math. 1 (2002) 111–129.
[23] N. Polat, D. Kaya, H.I. Tutalar, An analytic and numerical solution to a modified Kawahara equation and a convergence analysis of the method, Appl. Math. Comput. 179 (2006) 466–472.
[24] Y. Pomeau, A. Ramani, B. Grammaticos, Structural stability of the Korteweg–de Vries solitons under a singular perturbation, Phys. D 31 (1988) 127–134.
[25] A.F. Pazoto, Unique continuation and decay for the Korteweg–de Vries equations with localized damping, ESAIM Control Optim. Calc. Var. 11 (2005) 473–486.
[26] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[27] L. Rosier, Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain, ESAIM Control Optim. Calc. Var. 2 (1997) 33–55.
[28] L. Rosier, B.Y. Zhang, Global stabilization of the generalized Korteweg–de Vries equation posed on a finite domain, SIAM J. Control Optim. 45 (3) (2006) 927–956.
[29] D.L. Russell, B.Y. Zhang, Controllability and stabilizability of the third order linear dispersion equation on a periodic domain, SIAM J. Control Optim. 31 (1993) 659–676.
[30] J.-C. Saut, B. Scheurer, Unique continuation for some evolutions equations, J. Differential Equations 66 (1987) 118–139.
[31] T. Shuangping, C. Shuangbin, Existence and uniqueness of solutions to nonlinear Kawahara equations, Chinese Ann. Math. Ser. A 23 (2) (2002) 221–228.
[32] S. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. CXLXVI IV (1987) 65–96.
[33] Sirendaoreji, New exact travelling wave solutions for the Kawahara and modified Kawahara equations, Chaos Solitons Fractals 19 (1) (2004) 147–150.
[34] C.F. Vasconcellos, P.N. Silva, Stabilization of the linear Kawahara equation with localized damping, Asymptot. Anal. 58 (2008) 229–252.
[35] C.F. Vasconcellos, P.N. Silva, Erratum, Asymptot. Anal. 66 (2010) 119–124.
[36] C.F. Vasconcellos, P.N. Silva, Stabilization of the Kawahara equation with localized damping, ESAIM Control Optim. Calc. Var. 17 (1) (2011) 102–116.
[37] A.M. Wazwaz, New solitary wave solutions to the modified Kawahara equation, Phys. Lett. A 8 (4–5) (2007) 588–592.
[38] E. Yusufoglu, A. Bekir, M. Alp, Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using Sine–Cosine method, Chaos Solitons Fractals 37 (2008) 1193–1197.
[39] D. Zhang, Doubly periodic solutions of the modified Kawahara equation, Chaos Solitons Fractals 25 (2005) 1155–1160.
[40] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Comm. Partial Differential Equations 152 (1990) 205–235.