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Neumann boundary controllability of the Gear–Grimshaw system with critical size restrictions on the spatial domain

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Abstract. In this paper, we study the boundary controllability of the Gear–Grimshaw system posed on a finite domain $(0, L)$, with Neumann boundary conditions:

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1 vv_x + a_2(uv)_x = 0, & \text{in } (0,L)\times(0,T),\\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2 buu_x + a_1 b(uv)_x = 0, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases}$$

We first prove that the corresponding linearized system around the origin is exactly controllable in $\left(L^2(0,L)\right)^2$ when $h_2(t) = g_2(t) = 0$. In this case, the exact controllability property is derived for any $L > 0$ with control functions $h_0, g_0 \in H^{-\frac{1}{3}}(0,T)$ and $h_1, g_1 \in L^2(0,T)$. If we change the position of the controls and consider $h_0(t) = h_2(t) = 0$ (resp. $g_0(t) = g_2(t) = 0$), we obtain the result with control functions $g_0, g_2 \in H^{-\frac{1}{3}}(0,T)$ and $h_1, g_1 \in L^2(0,T)$ if and only if the length L of the spatial domain $(0,L)$ does not belong to a countable set. In all cases, the regularity of the controls are sharp in time. If only one control act in the boundary condition, $h_0(t) = g_0(t) = h_2(t) = g_2(t) = 0$ and $g_1(t) = 0$ (resp. $h_1(t) = 0$), the linearized system is proved to be exactly controllable for small values of the length L and large time of control T. Finally, the nonlinear system is shown to be locally exactly controllable via the contraction mapping principle, if the associated linearized systems are exactly controllable.

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1. Introduction

1.1. Setting of the problem

The goal of this paper is to investigate the boundary controllability properties of the nonlinear dispersive system

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1 vv_x + a_2(uv)_x = 0, & \text{in } (0,L)\times(0,T),\\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2 buu_x + a_1 b(uv)_x = 0, & \text{in } (0,L)\times(0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{1}$$

with the following boundary conditions

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t). \end{cases} \tag{2}$$

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In (1), a_1, a_2, a, b, c and r are real constants, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions of the two variables x and t and subscripts indicate partial differentiation. The boundary functions h_i and g_i, for $i = 0, 1, 2$, are considered as control inputs acting on the boundary conditions. Our purpose is to see whether we can force the solutions of the system to have certain properties by choosing appropriate control inputs. More precisely, we are mainly concerned with the following exact control problem:

Given $T > 0$ and $u^0, v^0, u^1, v^1 \in L^2(0,L)$, can one find appropriate control inputs h_i, g_i, for $i = 0, 1, 2$, such that the corresponding solution (u, v) of (1), (2) satisfies

$$(u(x,T), v(x,T)) = \left(u^1(x), v^1(x)\right)? \tag{3}$$

In order to provide the tools to handle with this problem, we assume that the coefficients a, b, c and r satisfy

$$b, c \text{ and } r \text{ are positive and } 1 - a^2 b > 0. \tag{4}$$

System (1) was derived by Gear and Grimshaw [8] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion (we also refer to [1,14] for an extensive discussion on the physical relevance of the system). This somewhat complicated model has the structure of a pair of Korteweg–de Vries (KdV) equations coupled through both dispersive and nonlinear effects and has been object of intensive research in recent year. It is a special case of a broad class of nonlinear evolution equations for which the well-posedness theory associated with the pure initial-value problem posed on the whole real line $\mathbb{R}$, or on a finite interval with periodic boundary conditions, has been intensively investigated. By contrast, the mathematical theory pertaining to the study of the boundary value problem is considerably less advanced, specially in what concerns the study of the controllability properties. As far as we know, the controllability results for system (1) were first obtained in [11], when the model is posed on a periodic domain and $r = 0$. In this case, a diagonalization of the main terms allows to decouple the corresponding linear system into two scalar KdV equations and use the previous results available in the literature. Later on, assuming that (4) holds, Micu et al. [12] proved the local exact boundary controllability property for the nonlinear system, posed on a bounded interval, considering the following boundary conditions:

$$\begin{cases} u(0,t) = 0, \ u(L,t) = f_1(t), \ u_x(L,t) = f_2(t), \\ v(0,t) = 0, \ v(L,t) = k_1(t), \ v_x(L,t) = k_2(t). \end{cases} \tag{5}$$

The analysis developed in [12] was inspired by the results obtained by Rosier [10] for the scalar KdV equation. It combines the analysis of the linearized system and the Banach's fixed-point theorem. Following the classical duality approach [7,9], the exact controllability of system linearized system is equivalent to an observability for the adjoint system. Then, the problem is reduced to prove a nonstandard unique continuation property of the eigenfunctions of the corresponding differential operator. Their main result reads as follows:

Theorem A. (Micu et al. [12]) *Let $L > 0$ and $T > 0$. Then there exists a constant $\delta > 0$, such that, for any initial and final data $u^0, v^0, u^1, v^1 \in L^2(0,L)$ verifying*

$$|| \left(u^0, v^0\right) ||_{(L^2(0,L))^2} \leq \delta \quad \text{and} \quad || \left(u^1, v^1\right) ||_{(L^2(0,L))^2} \leq \delta,$$

there exist four control functions $f_1, k_1 \in H_0^1(0,T)$ and $f_2, k_2 \in L^2(0,T)$, such that the solution

$$(u, v) \in C([0,T]; \left(L^2(0,L)\right)^2) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,T; (H^{-2}(0,L))^2)$$

of (1)–(5) verifies (3).

Later on, the same problem was addressed by Cerpa and Pazoto [5] when only two controls act on the Neumann boundary conditions, i.e., assuming that $f_1 = k_1 = 0$. In this case, the analysis of the linearized system is much more complicated; therefore, the authors used a direct approach based on the multiplier technique that gives the observability inequality for small values of the length L and large time of control T. The fixed-point argument as well as the existence and regularity results needed in order to consider the nonlinear system runs exactly in the same way as in [12].

The program of this work was carried out for a particular choice of boundary conditions and aims to establish as a fact that such a model predicts the interesting controllability properties initially observed for the KdV equation. Therefore, to introduce the reader to the theory developed for KdV with the boundary conditions of types (2) and (5), we present below a summary of the results achieved in [10] and [3], respectively.

Rosier [10], studied the following boundary control problem for the KdV equation posed on the finite domain $(0, L)$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = 0,\ u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u^0(x) & \text{in } (0, L), \end{cases} \tag{6}$$

where the boundary value function $g(t)$ is considered as a control input. First, the author studies the associated linear system

$$\begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0,\ u(L, t) = 0,\ u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u^0(x) & \text{in } (0, L) \end{cases} \tag{7}$$

and discovered the so-called *critical length* phenomena, i.e., whether the system (7) is exactly controllable depends on the length L of the spatial domain $(0, L)$. More precisely, the following result was proved:

Theorem B. (Rosier [10]) *The linear system (7) is exactly controllable in the space $L^2(0, L)$ if and only if the length L of the spatial domain $(0, L)$ does not belong to the set*

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \, : k,\, l \in \mathbb{N}^* \right\}. \tag{8}$$

Then, by using a fixed-point argument, the controllability result was extended to the nonlinear system when $L \notin \mathcal{N}$.

Theorem C. (Rosier [10]) *Let $T > 0$ be given. If $L \notin \mathcal{N}$, there exists $\delta > 0$, such that, for any $u^0, u^{\mathrm{T}} \in L^2(0, L)$ with*

$$||u^0||_{L^2(0,L)} + ||u^{\mathrm{T}}||_{L^2(0,L)} \leq \delta,$$

one can find a control input $g \in L^2(0, T)$, such that the nonlinear system (6) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, T) = u^{\mathrm{T}}(x).$$

More recently, in [3], Caicedo et al. investigated the boundary control problem of the KdV equation with new boundary conditions, namely the Neumann boundary conditions:

$$\begin{cases} u_t + (1 + \beta)u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = 0,\ u_x(L, t) = h(t),\ u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u^0(x) & \text{in } (0, L). \end{cases} \tag{9}$$

In (9), β is a given real constant and g a control input. For any $\beta \neq -1$, the authors obtained the following set of *critical lengths*

$$\mathcal{R}_\beta := \left\{ \frac{2\pi}{\sqrt{3(1 + \beta)}} \sqrt{k^2 + kl + l^2} \, : k,\, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{\beta + 1}} : k \in \mathbb{N}^* \right\}, \tag{10}$$

and proved that the following result holds:

Theorem D. (Caicedo et al. [3])

(i) *If $\beta \neq -1$, the linear system (9) is exactly controllable in the space $L^2(0,L)$ if and only if the length L of the spatial domain $(0,L)$ does not belong to the set $\mathcal{R}_\beta$.*

(ii) *If $\beta = -1$, then the system (9) is not exact controllable in the space $L^2(0,L)$ for any $L > 0$.*

In addition, for the nonlinear system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = 0,\ u_x(L,t) = h(t),\ u_{xx}(L,t) = 0 & \text{in } (0,T),\\ u(x,0) = u_0(x) & \text{in } (0,L), \end{cases} \tag{11}$$

the result below was proved by using a fixed-point argument:

Theorem E. (Caicedo et al. [3]) *Let $T > 0$, $\beta \neq -1$ and $L \notin \mathcal{R}_\beta$ be given. There exists $\delta > 0$, such that, for any $u^0, u^{\mathrm{T}} \in L^2(0,L)$ with*

$$||u^0 - \beta||_{L^2(0,L)} + ||u^{\mathrm{T}} - \beta||_{L^2(0,L)} \leq \delta,$$

one can find a control input $h \in L^2(0,T)$, such that the system (11) admits unique solution

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u(x,T) = u^{\mathrm{T}}(x).$$

Both theorems, Theorems B and D, were proved following the classical duality approach [7,9] which reduces the problem to prove an observability inequality for the solutions of the corresponding adjoint system. Then, the controllability is obtained with the aid of a compactness argument that leads the issue to a nonstandard unique continuation principle for the eigenfunctions of the differential operator associated with the model. The critical lengths in (8) and (10) are such that there are eigenfunctions of the linear scalar problem for which the observability inequality associated with the adjoint system fails.[1] However, in [3], the authors encountered some difficulties that require special attention. For instance, the adjoint system of the linear system (9) is given by

$$\begin{cases} \psi_t + (1+\beta)\psi_x + \psi_{xxx} = 0 & \text{in } (0,L)\times(0,T),\\ (1+\beta)\psi(0,t) + \psi_{xx}(0,t) = 0 & \text{in } (0,T),\\ (1+\beta)\psi(L,t) + \psi_{xx}(L,t) = 0 & \text{in } (0,T),\\ \psi_x(0,t) = 0 & \text{in } (0,T),\\ \psi(x,T) = \psi^{\mathrm{T}}(x) & \text{in } (0,L). \end{cases} \tag{12}$$

The exact controllability of system (9) is equivalent to the following observability inequality for the adjoint system (12):

$$||\psi^{\mathrm{T}}||_{L^2(0,L)} \leq C||\psi_x(L,\cdot)||_{L^2(0,T)},$$

for some $C > 0$. Nonetheless, the usual multiplier method and compactness arguments used to deal with the system (12) only lead to

$$||\psi^{\mathrm{T}}||^2_{L^2(0,L)} \leq C_1||\psi_x(L,\cdot)||^2_{L^2(0,T)} + C_2||\psi(L,\cdot)||^2_{L^2(0,T)}, \tag{13}$$

where C_1 and C_2 are positive constants. In order to absorb the extra term present in (13), Caicedo et al. derived a technical result, which reveals some hidden regularity (sharp trace regularities) for solutions of the adjoint system (12):

[1] In the case of $L \in \mathcal{N}$ (resp. $L \in \mathcal{R}_\beta$), Rosier (resp. Caicedo et al. [3]) proved in [10] that the associated linear system (7) is not controllable; there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $\mathcal{M} = \mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi \in \mathcal{M}$, $g \in L^2(0,T)$ and $u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$ satisfying (7) and $u(\cdot,0) = 0$, one has $u(\cdot,T) \neq \psi$. A spatial domain $(0,L)$ is called critical for the system (7) (resp. (9)) if its domain length $L \in \mathcal{N}$ (resp. $L \in \mathcal{R}_\beta$).

Theorem F. (Caicedo et al. [3]) *For any* $\psi^{\mathrm{T}} \in L^2(0,L)$, *the solution*
$$\psi \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$
of the problem (12) *possesses the following sharp trace properties*
$$\sup_{x\in(0,L)} ||\partial_x^r \psi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r ||\psi^{\mathrm{T}}||_{L^2(0,L)}, \tag{14}$$
for $r = 0,1,2$, *where* C_r *are positive constants.*

Estimate (14) is then combined with compactness argument to remove the extra term in (13). We remark that the sharp Kato smoothing properties obtained by Kenig, Ponce and Vega [13] for the solutions of the KdV equation posed on the line, played an important role in the proof of the previous result. The same strategy has been successfully applied by Cerpa et al. [6] for the study of a similar boundary controllability problem.

1.2. Main result

We are now in a position to return considerations to the control properties of the system (1). First, we prove that the corresponding linear system with the following boundary conditions
$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = 0, \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = 0, \end{cases}$$
is exactly controllable in $\left(L^2(0,L)\right)^2$ with controls h_0, $g_0 \in H^{-\frac{1}{3}}(0,T)$ and h_1, $g_1 \in L^2(0,T)$. In this case, no restriction on the length L of the spatial domain is required. However, if we change the position of the controls, a critical size restriction can appear. This is the case when we consider the following boundary conditions
$$\begin{cases} u_{xx}(0,t) = 0,\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = 0, \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t). \end{cases}$$
In this case, the exact controllability result in $\left(L^2(0,L)\right)^2$ is derived with controls g_0, $g_2 \in H^{-\frac{1}{3}}(0,T)$ and h_1, $g_1 \in L^2(0,T)$ if and only if the length L does not belong of the following set
$$\mathcal{F}_r := \left\{ 2\pi k \sqrt{\frac{1-a^2b}{r}} : k \in \mathbb{N}^* \right\} \cup \left\{ \pi \sqrt{\frac{(1-a^2b)\alpha(k,l,m,n,s)}{3r}} : k,l,m,n,s \in \mathbb{N}^* \right\}, \tag{15}$$
where
$$\begin{aligned} \alpha := \alpha(k,l,m,n,s) = 5k^2 + 8l^2 + 9m^2 + 8n^2 + 5s^2 + 8kl + 6km \\ +\ 4kn + 2ks + 12ml + 8ln + 3ls + 12mn + 6ms + 8ns. \end{aligned}$$
As in [3], the hidden regularity for the corresponding adjoint system (1) was required. Here, the result is given in Proposition 2.4, which is the key point to prove the controllability result.

Finally, for small values of the length L and large time of control T we derive a exact controllability result in $\left(L^2(0,L)\right)^2$ by assuming that the controls $g_1(t) = 0$ (resp. $h_1(t) = 0$) and $g_0(t) = g_2(t) = 0$. In this case, the analysis of the linearized system is much more complicated; therefore, we use a direct approach based on the multipliers technique, as in [5]. In all cases, the result obtained for the linear system allows to prove the local controllability property of the nonlinear system (1) by means of a fixed-point argument.

The analysis describe above are summarized in the main result of the paper, Theorem 1.1. However, in order to make the reading easier, throughout the paper we use the following notation for the boundary functions:

$$\vec{h}_1=(0,h_1,0),\ \vec{g}_1=(g_0,g_1,g_2) \text{ and } \vec{h}_2=(h_0,h_1,h_2),\ \vec{g}_2=(0,g_1,0),$$
$$\vec{h}_3=(h_0,h_1,0),\ \vec{g}_3=(g_0,g_1,0) \text{ and } \vec{h}_4=(0,h_1,h_2),\ \vec{g}_4=(0,g_1,g_2),$$
$$\vec{h}_5=(0,h_1,0),\ \vec{g}_5=(0,0,0) \text{ and } \vec{h}_6=(0,0,0),\ \vec{g}_6=(0,g_1,0).$$

We also introduce the space $\mathcal{X}:=\left(L^2(0,L)\right)^2$ endowed with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle := \int_0^L u(x)\varphi(x)\mathrm{d}x+\frac{b}{c}\int_0^L v(x)\psi(x)\mathrm{d}x, \qquad \forall (u,v),(\varphi,\psi)\in\mathcal{X},$$

and the spaces

$$\mathcal{H}_T:=H^{-\frac{1}{3}}(0,T)\times L^2(0,T)\times H^{-\frac{1}{3}}(0,T) \text{ and } \mathcal{Z}_T:=C([0,T];\left(L^2(0,L)\right)^2)\cap L^2(0,T,(H^1(0,L))^2)$$

endowed with their natural inner products.

Thus, our main result reads as follows:

Theorem 1.1. *Let $T>0$. Then, there exists $\delta>0$, such that, for any $\left(u^0,v^0\right),\left(u^1,v^1\right)\in\mathcal{X}:=\left(L^2(0,L)\right)^2$ verifying*

$$\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(u^1,v^1\right)\|_{\mathcal{X}}\le\delta,$$

the following holds:

(i) *If $L\in(0,\infty)\setminus\mathcal{F}_r$, one can find $\vec{h}_i,\vec{g}_i\in\mathcal{H}_T$, for $i=1,2$, such that the system (1), (2) admits a unique solution $(u,v)\in\mathcal{Z}_T$ satisfying (3).*

(ii) *For any $L>0$, one can find $\vec{h}_i,\vec{g}_j\in\mathcal{H}_T$, for $j=3,4$, such that the system (1), (2) admits a unique solution $(u,v)\in\mathcal{Z}_T$, satisfying (3).*

(iii) *Let $T>0$ and $L>0$ satisfying*

$$1>\frac{\beta C_T}{T}\left[L+\frac{r}{c}\right],$$

where C_T is the constant in (35) and β is the constant given by the embedding $H^{\frac{1}{3}}(0,T)\subset L^2(0,T)$. Then, one can find $\vec{h}_k,\vec{g}_k\in\mathcal{H}_T$, for $k=5,6$, such that the system (1), (2) admits a unique solution $(u,v)\in\mathcal{Z}_T$, satisfying (3).

Before close this section, we observe that the exact controllability result given in Theorem A holds without any restriction of the Length L. However, we believe that, with another configuration of the controls, it is possible to prove the existence of a critical set for the system (1).

The article is organized as follows:

- In Sect. 2, we show that the system (1), (2) is locally well posed in $\mathcal{Z}_T$, whenever $\left(u^0,v^0\right)\in(L^2(0,L))^2$, $h_0,\ g_0\in H^{-\frac{1}{3}}(\mathbb{R}^+)$, $h_1,\ g_1\in L^2(\mathbb{R}^+)$ and $h_2,\ g_2\in H^{-\frac{1}{3}}(\mathbb{R}^+)$. Various linear estimates, including hidden regularities, are presented for solutions of the corresponding linear system. As we pointed before, such estimates will play important roles in studying the controllability properties.
- In Sect. 3, the boundary control system (1) is investigated for its controllability. We investigate first the linearized system and its corresponding adjoint system for their controllability and observability. In particular, the hidden regularities for the solutions of the adjoint system presented in Sect. 2 are used to prove observability inequalities associated with the control problem.
- The proof of our main result, Theorem 1.1, is presented in Sect. 4. Finally, the paper ends with an appendix, where the proof of a technical lemma used in the paper is furnished.

2. Well-posedness

2.1. Linear system

In this section, we establish the well-posedness of the linear system associated with (1), (2):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{16}$$

We begin by considering the following linear nonhomogeneous boundary value problem

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = f, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = s, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = g_2(t), & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L), \end{cases} \tag{17}$$

with the notation introduced in Sect. 1. Then, the next proposition shows that the problem (17) is well posed in the space $\mathcal{X}$.

Proposition 2.1. *Let $T>0$ be given. Then, for any $\left(u^0,v^0\right)$ in $\mathcal{X}$, f,s in $L^1(0,T;L^2(0,L))$ and $\overrightarrow{h},\overrightarrow{g}\in\mathcal{H}_T$, problem (17) admits a unique solution $(u,v)\in\mathcal{Z}_T$, with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)), \quad k=0,1,2. \tag{18}$$

Moreover, there exist $C>0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T} + \sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)} \le C\Big\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}$$
$$+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_T} + \|(f,s)\|_{L^1(0,T;(L^2(0,L))^2)}\Big\}.$$

Proof. We diagonalize the main term in (16) and consider the change of variable

$$\begin{cases} u = 2a\widetilde{u} + 2a\widetilde{v},\\ v = \left(\left(\frac{1}{c}-1\right)+\lambda\right)\widetilde{u} + \left(\left(\frac{1}{c}-1\right)-\lambda\right)\widetilde{v}, \end{cases}$$

where $\lambda = \sqrt{\left(\frac{1}{c}-1\right)^2 + \frac{4a^2b}{c}}$. Thus, we can transform the linear system (17) into

$$\begin{cases} \widetilde{u}_t + \alpha_-\widetilde{u}_{xxx} = \widetilde{f},\\ \widetilde{v}_t + \alpha_+\widetilde{v}_{xxx} = \widetilde{s},\\ \widetilde{u}_{xx}(0,t) = \widetilde{h}_0(t),\ \widetilde{u}_x(L,t) = \widetilde{h}_1(t),\ \widetilde{u}_{xx}(L,t) = \bar{h}_2(t),\\ \widetilde{v}_{xx}(0,t) = \widetilde{g}_0(t),\ \widetilde{v}_x(L,t) = \widetilde{g}_1(t),\ \widetilde{v}_{xx}(L,t) = \widetilde{g}_2(t),\\ \widetilde{u}(x,0) = \widetilde{u}^0(x), \quad \widetilde{v}(x,0) = \widetilde{v}^0(x), \end{cases} \tag{19}$$

where $\alpha_\pm = -\frac{1}{2}\left(\left(\frac{1}{c}-1\right)\pm\lambda\right)$ and

$$\begin{cases} \widetilde{f} = -\frac{1}{2}\left(\frac{\alpha_+}{a\lambda}f + \frac{1}{\lambda}s\right), \quad \widetilde{u}_0 = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}u^0 - \frac{1}{\lambda}v^0\right), \quad \widetilde{h}_i = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}h_i - \frac{1}{\lambda}g_i\right), \quad i=0,1,2,\\ \widetilde{s} = -\frac{1}{2}\left(\frac{\alpha_-}{a\lambda}f - \frac{1}{\lambda}s\right), \quad \widetilde{v}_0 = \frac{1}{2}\left(\frac{\alpha_+}{a\lambda}u^0 - \frac{1}{\lambda}v^0\right), \quad \widetilde{g}_i = \frac{1}{2}\left(\frac{\alpha_+}{a\lambda}h_i - \frac{1}{\lambda}g_i\right), \quad i=0,1,2. \end{cases}$$

Note that condition (4) guarantees that $\alpha_\pm$ are nonzero. Therefore, system (19) can be decoupled into two single KdV equations as follows:

$$\begin{cases} \widetilde{u}_t + \alpha_- \widetilde{u}_{xxx} = \widetilde{f}, \\ \widetilde{u}_{xx}(0,t) = \widetilde{h}_0(t),\ \widetilde{u}_x(L,t) = \widetilde{h}_1(t),\ \widetilde{u}_{xx}(L,t) = \widetilde{h}_2(t), \\ \widetilde{u}(0,x) = \widetilde{u}^0(x) \end{cases} \tag{20}$$

and

$$\begin{cases} \widetilde{v}_t + \alpha_+ \widetilde{v}_{xxx} = \widetilde{s}, \\ \widetilde{v}_{xx}(0,t) = \widetilde{g}_0(t),\ \widetilde{v}_x(L,t) = \widetilde{g}_1(t),\ \widetilde{v}_{xx}(L,t) = \widetilde{g}_2(t), \\ \widetilde{v}(x,0) = \widetilde{v}^0(x). \end{cases} \tag{21}$$

Here, we consider the solutions written in the form $\{W^{\pm}_{bdr}(t)\}_{t\geq 0}$ that will be called the boundary integral operator. For this purpose, we use a lemma, which can be found in [4, Lemma 2.4] (see also [3, Lemma 2.1]), for solutions of (20) (or (21)). For the sake of completeness, we will present the proof in Appendix A.
□

Lemma 2.2. *The solution u of the IBVP (20) (or (21)), when $\widetilde{f}=0$, $\widetilde{s}=0$ and null initial data, can be written in the form*

$$u(x,t) = [W^+_{bdr}\vec{\widetilde{h}}](x,t) := [W^+_{bdr}\vec{h}](x,t) := \sum_{j,m=1}^{3} [W^+_{j,m} h_m](x,t),$$

where

$$[W^+_{j,m}h](x,t) \equiv [U_{j,m}h](x,t) + \overline{[U_{j,m}h](x,t)} \tag{22}$$

with

$$[U_{j,m}h](x,t) \equiv \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} 3\rho^2 [Q^+_{j,m}h](\rho)\mathrm{d}\rho \tag{23}$$

for $j=1,3,\ m=1,2,3$ and

$$[U_{2,m}h](x,t) \equiv \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} 3\rho^2 [Q^+_{2,m}h](\rho)\mathrm{d}\rho \tag{24}$$

for $m=1,2,3$. Here

$$[Q^+_{j,m}h](\rho) := \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}\hat{h}^+(\rho), \qquad [Q^+_{2,m}h](\rho) = \frac{\Delta^+_{2,m}(\rho)}{\Delta^+(\rho)} e^{\lambda_2^+(\rho)}\hat{h}^+(\rho) \tag{25}$$

for $j=1,3$ and $m=1,2,3$. Here $\hat{h}^+(\rho)=\hat{h}(i\rho^3)$, $\Delta^+(\rho)$ and $\Delta^+_{j,m}(\rho)$ are obtained from $\Delta(s)$ and $\Delta_{j,m}(s)$ by replacing s with $i\rho^3$ and $\lambda_j^+(\rho)=\lambda_j(i\rho^3)$ where

$$\begin{aligned} \Delta &= \lambda_1\lambda_2\lambda_3\left(\lambda_1(\lambda_3-\lambda_2)e^{-\lambda_1} + \lambda_2(\lambda_1-\lambda_3)e^{-\lambda_2} + \lambda_3(\lambda_2-\lambda_1)e^{-\lambda_3}\right); \\ \Delta_{1,1} &= e^{-\lambda_1}\lambda_2\lambda_3(\lambda_3-\lambda_2),\ \Delta_{2,1} = e^{-\lambda_2}\lambda_1\lambda_3(\lambda_1-\lambda_3),\ \Delta_{3,1} = e^{-\lambda_3}\lambda_1\lambda_2(\lambda_2-\lambda_1); \\ \Delta_{1,2} &= \lambda_2^2\lambda_3^2(e^{\lambda_2}-e^{\lambda_3}),\ \Delta_{2,2} = \lambda_1^2\lambda_3^2(e^{\lambda_3}-e^{\lambda_1}),\ \Delta_{3,2} = \lambda_1^2\lambda_2^2(e^{\lambda_1}-e^{\lambda_2}); \\ \Delta_{1,3} &= \lambda_2\lambda_3(\lambda_2 e^{\lambda_3} - \lambda_3 e^{\lambda_2}),\ \Delta_{2,3} = \lambda_1\lambda_3(\lambda_3 e^{\lambda_1} - \lambda_1 e^{\lambda_3}),\ \Delta_{3,3} = \lambda_1\lambda_2(\lambda_1 e^{\lambda_2} - \lambda_2 e^{\lambda_1}). \end{aligned}$$

Since

$$(\widetilde{u}^0, \widetilde{v}^0) \in \mathcal{X}, \quad (\widetilde{f}, \widetilde{s}) \in L^1(0,T;\left(L^2(0,L)\right)^2) \text{ and } \vec{\widetilde{h}}, \vec{\widetilde{g}} \in \mathcal{H}_T,$$

by [3, Propositions 2.2 and 2.5], we obtain the existence of $(\widetilde{u},\widetilde{v})\in\mathcal{Z}_T$, solution of the system (19), such that

$$\partial_x^k\widetilde{u},\partial_x^k\widetilde{v}\in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)),\quad k=0,1,2,$$

and

$$\|(\widetilde{u},\widetilde{v})\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\|(\partial_x^k\widetilde{u},\partial_x^k\widetilde{v})\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}\le C\left\{\|(\widetilde{u}^0,\widetilde{v}^0)\|_{\mathcal{X}}+\|(\overrightarrow{\widetilde{h}},\overrightarrow{\widetilde{g}})\|_{\mathcal{H}_T}\right.$$
$$\left.+\|(\widetilde{f},\widetilde{s})\|_{L^1(0,T;(L^2(0,L))^2)}\right\},$$

for some constant $C>0$. Furthermore, we can write $\widetilde{u}$ and $\widetilde{v}$ in its integral form as follows

$$\widetilde{u}(t)=W_0^-(t)\widetilde{u}^0+W_{bdr}^-(t)\overrightarrow{\widetilde{h}}+\int_0^t W_0^-(t-\tau)\widetilde{f}(\tau)\mathrm{d}\tau,$$

$$\widetilde{v}(t)=W_0^+(t)\widetilde{v}^0+W_{bdr}^+(t)\overrightarrow{\widetilde{g}}+\int_0^t W_0^+(t-\tau)\widetilde{s}(\tau)\mathrm{d}\tau,$$

where $\{W_0^\pm(t)\}_{t\ge0}$ are the C_0 semigroup in the space $L^2(0,L)$ generated by the linear operators

$$A^\pm=-\alpha_\pm g''',$$

with domain

$$D(A^\pm)=\{g\in H^3(0,L):g''(0)=g'(L)=g''(L)=0\},$$

and $\{W_{bdr}^\pm(t)\}_{t\ge0}$ are the operator given in Lemma 2.2 (see also [3, Lemma 2.1] for more details). Then, by change of variable we can easily verify that

$$\begin{cases} u(t)=W_0^-(t)u^0+W_{bdr}^-(t)\overrightarrow{h}+\displaystyle\int_0^t W_0^-(t-\tau)f(\tau)\mathrm{d}\tau,\\ v(t)=W_0^+(t)v^0+W_{bdr}^+(t)\overrightarrow{g}+\displaystyle\int_0^t W_0^+(t-\tau)s(\tau)\mathrm{d}\tau\end{cases}$$

and the result follows.

The global well-posedness of the system (16) is obtained using a fixed-point argument.

Proposition 2.3. *Let $T>0$ be given. Then, for any $\left(u^0,v^0\right)\in\mathcal{X}$ and $\overrightarrow{h},\overrightarrow{g}\in\mathcal{H}_T$, problem (16) admits a unique solution $(u,v)\in\mathcal{Z}_T$ with*

$$\partial_x^k u,\partial_x^k v\in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)),\quad k=0,1,2.$$

Moreover, there exist $C>0$, such that

$$\|(u,v)\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}\le C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_T}\right.$$
$$\left.+\|(f,s)\|_{L^1(0,T;(L^2(0,L))^2)}\right\}.$$

Proof. Let $\mathcal{F}_T:=\left\{(u,v)\in\mathcal{Z}_T:(u,v)\in L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2),k=0,1,2\right\}$ equipped with the norm

$$\|(u,v)\|_{\mathcal{F}_T}=\|(u,v)\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}.$$

Let $0<\beta\leq T$ to be determined later. For each $u,v\in\mathcal{F}_\beta$, consider the problem

$$\begin{cases} \omega_t+\omega_{xxx}+a\eta_{xxx}=0, & \text{in } (0,L)\times(0,\beta),\\ \eta_t+\frac{ab}{c}\omega_{xxx}+\frac{1}{c}\eta_{xxx}=-\frac{r}{c}v_x, & \text{in } (0,L)\times(0,\beta),\\ \omega_{xx}(0,t)=h_0(t),\ \omega_x(L,t)=h_1(t),\ \omega_{xx}(L,t)=h_2(t), & \text{in } (0,\beta),\\ \eta_{xx}(0,t)=g_0(t),\ \eta_x(L,t)=g_1(t),\ \eta_{xx}(L,t)=g_2(t), & \text{in } (0,\beta),\\ \omega(x,0)=u^0(x), \quad v(x,0)=v^0(x), & \text{in } (0,L). \end{cases} \tag{26}$$

According to Proposition 2.1, we can define the operator

$$\Gamma:\mathcal{F}_\beta\to\mathcal{F}_\beta, \quad \text{given by} \quad \Gamma(u,v)=(\omega,\eta),$$

where (ω,η) is the solution of (26). Moreover,

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta}\leq C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_\beta}+\|(0,v_x)\|_{L^1(0,\beta;(L^2(0,L))^2)}\right\}, \tag{27}$$

where the positive constant C depends only on T. Since

$$\|(0,v_x)\|_{L^1(0,\beta;L^2(0,L))}\leq\beta^{\frac{1}{2}}\|(u,v)\|_{\mathcal{F}_\beta},$$

we obtain a positive constant $C>0$, such that

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta}\leq C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_\beta}\right\}+C\beta^{\frac{1}{2}}\|(u,v)\|_{\mathcal{F}_\beta}. \tag{28}$$

Let $(u,v)\in B_r(0):=\left\{(u,v)\in\mathcal{F}_\beta:\|(u,v)\|_{\mathcal{F}_\beta}\leq r\right\}$, with $r=2C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_\beta}\right\}$. Choosing $\beta>0$, satisfying

$$C\beta^{\frac{1}{2}}\leq\frac{1}{2}, \tag{29}$$

from (28) we obtain

$$\|\Gamma(u,v)\|_{\mathcal{F}_\beta}\leq r.$$

The above estimate allows us to conclude that

$$\Gamma:B_r(0)\subset\mathcal{F}_\beta\to B_r(0).$$

On the other hand, note that $\Gamma(u_1,v_1)-\Gamma(u_2,v_2)$ solves the following system

$$\begin{cases} \omega_t+\omega_{xxx}+a\eta_{xxx}=0, & \text{in } (0,L)\times(0,\beta),\\ \eta_t+\frac{ab}{c}\omega_{xxx}+\frac{1}{c}\eta_{xxx}=-\frac{r}{c}(v_{1x}-v_{2x}), & \text{in } (0,L)\times(0,\beta),\\ \omega_{xx}(0,t)=\omega_x(L,t)=\omega_{xx}(L,t)=0, & \text{in } (0,\beta),\\ \eta_{xx}(0,t)=\eta_x(L,t)=\eta_{xx}(L,t)=0, & \text{in } (0,\beta),\\ \omega(x,0)=0, \quad v(x,0)=0, & \text{in } (0,L). \end{cases}$$

Again, from Proposition 2.1 and (29), we have

$$\begin{aligned}\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_\beta}&\leq C\|(0,v_{1x}-v_{2x})\|_{L^1(0,\beta;(L^2(0,L))^2)}\leq C\beta^{\frac{1}{2}}\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{F}_\beta}\\ &\leq\frac{1}{2}\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{F}_\beta}.\end{aligned}$$

Hence, $\Gamma:B_r(0)\to B_r(0)$ is a contraction and, by Banach fixed-point theorem, we obtain a unique $(u,v)\in B_r(0)$, such that

$$\Gamma(u,v)=(u,v)\in\mathcal{F}_\beta,$$

and (27) holds, for all $t\in(0,\beta)$. Since the choice of β is independent of $\left(u^0,v^0\right)$, the standard continuation extension argument yields that the solution (u,v) belongs to $\mathcal{F}_T$. The proof is complete. □

2.1.1. Adjoint system. Consider the following homogeneous initial-value problem associated with (1), (2):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ v_t + \frac{r}{c}v_x + \frac{ab}{c}u_{xxx} + \frac{1}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ u_{xx}(0,t) = u_x(L,t) = u_{xx}(L,t) = 0, & \text{in } (0,T),\\ v_{xx}(0,t) = v_x(L,t) = v_{xx}(L,t) = 0, & \text{in } (0,T),\\ u(x,0) = u^0(x), \quad v(x,0) = v^0(x), & \text{in } (0,L). \end{cases} \tag{30}$$

In order to introduce the backward system associated with (30), we multiply the first equation of (30) by φ, the second one by ψ and integrate over $(0,L)\times(0,T)$. Assuming that the functions u, v, φ and ψ are regular enough to justify all the computations, we obtain, after integration by parts, the following identity:

$$\begin{aligned} &\int_0^L (u(x,T)\varphi(x,T) + v(x,T)\psi(x,T))\,\mathrm{d}x - \int_0^L \left(u^0(x)\varphi(x,0) + v^0(x)\psi(x,0)\right)\mathrm{d}x \\ &= \int_0^T\int_0^L u(x,t)\left(\varphi(x,t) + \varphi_{xxx}(x,t) + \frac{ab}{c}\psi_{xxx}(x,t)\right)\mathrm{d}x\mathrm{d}t \\ &\quad + \int_0^T\int_0^L v(x,t)\left(\psi(x,t) + \frac{r}{c}\psi(x,t) + a\varphi_{xxx}(x,t) + \frac{1}{c}\psi_{xxx}(x,t)\right)\mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^T u_{xx}(L,t)\left(\varphi(L,t) + \frac{ab}{c}\psi(L,t)\right)\mathrm{d}t + \int_0^T u_{xx}(0,t)\left(\varphi(0,t) + \frac{ab}{c}\psi(0,t)\right)\mathrm{d}t \\ &\quad + \int_0^T u_x(L,t)\left(\varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t)\right)\mathrm{d}t - \int_0^T u_x(0,t)\left(\varphi_x(0,t) + \frac{ab}{c}\psi_x(0,t)\right)\mathrm{d}t \\ &\quad - \int_0^T u(L,t)\left(\varphi_{xx}(L,t) + \frac{ab}{c}\psi_{xx}(L,t)\right)\mathrm{d}t + \int_0^T u(0,t)\left(\varphi_{xx}(0,t) + \frac{ab}{c}\psi_{xx}(0,t)\right)\mathrm{d}t \\ &\quad - \int_0^T v_{xx}(L,t)\left(a\varphi(L,t) + \frac{1}{c}\psi(L,t)\right)\mathrm{d}t + \int_0^T v_{xx}(0,t)\left(a\varphi(0,t) + \frac{1}{c}\psi(0,t)\right)\mathrm{d}t \\ &\quad + \int_0^T v_x(L,t)\left(a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right)\mathrm{d}t - \int_0^T v_x(0,t)\left(a\varphi_x(0,t) + \frac{1}{c}\psi_x(0,t)\right)\mathrm{d}t \\ &\quad - \int_0^T v(L,t)\left(a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t) + \frac{r}{c}\psi(L,t)\right)\mathrm{d}t \\ &\quad + \int_0^T v(0,t)\left(a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t) + \frac{1}{c}\psi(0,t)\right)\mathrm{d}t. \end{aligned}$$

Having the previous equality in hands, we consider backward system as follows

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T) \end{cases} \tag{31}$$

satisfying the boundary conditions,

$$\begin{cases} a\varphi_x(0,t) + \frac{1}{c}\psi_x(0,t) = 0, & \text{in } (0,T), \\ \varphi_x(0,t) + \frac{ab}{c}\psi_x(0,t) = 0, & \text{in } (0,T), \\ \varphi_{xx}(L,t) + \frac{ab}{c}\psi_{xx}(L,t) = 0, & \text{in } (0,T), \\ \varphi_{xx}(0,t) + \frac{ab}{c}\psi_{xx}(0,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t) + \frac{r}{c}\psi(L,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t) + \frac{r}{c}\psi(0,t) = 0, & \text{in } (0,T) \end{cases} \tag{32}$$

and the final conditions

$$\varphi(x,T) = \varphi^1(x), \qquad \psi(x,T) = \psi^1(x), \qquad \text{in } (0,L). \tag{33}$$

Since the coefficients satisfy $1-a^2b>0$, we can deduce from the first and second equations of (32) that the above boundary conditions can be written as

$$\begin{cases} \varphi_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,T), \\ \varphi_{xx}(L,t) + \frac{ab}{c}\psi_{xx}(L,t) = 0, & \text{in } (0,T), \\ \varphi_{xx}(0,t) + \frac{ab}{c}\psi_{xx}(0,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(L,t) + \frac{1}{c}\psi_{xx}(L,t) + \frac{r}{c}\psi(L,t) = 0, & \text{in } (0,T), \\ a\varphi_{xx}(0,t) + \frac{1}{c}\psi_{xx}(0,t) + \frac{r}{c}\psi(0,t) = 0, & \text{in } (0,T). \end{cases} \tag{34}$$

The following proposition is the key to prove the controllability of the linear system (16). The result ensures the hidden regularity for the solution of the adjoint system (31)–(34).

Proposition 2.4. *For any $(\varphi^1,\psi^1)\in\mathcal{X}$, the system (31)–(34) admits a unique solution $(\varphi,\psi)\in\mathcal{Z}_T$, such that it possess the following sharp trace properties*

$$\begin{cases} \sup\limits_{0<x<L} \|\partial_x^k \varphi(x,\cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \le C_T \|\varphi^1\|_{L^2(0,L)}, \\ \sup\limits_{0<x<L} \|\partial_x^k \psi(x,\cdot)\|_{H^{\frac{1-k}{3}}(0,T)} \le C_T \|\psi^1\|_{L^2(0,L)}, \end{cases} \tag{35}$$

for $k=0,1,2$, where C_T is a positive constant.

Proof. Proceeding as the proof of Proposition 2.3, we obtain the result. Indeed, first we consider the change of variable $t\to T-t$ and $x\to L-x$, then for any (φ,ψ) in $\mathcal{Z}_T$, we consider the system

$$\begin{cases} u_t + u_{xxx} + \frac{ab}{c}v_{xxx} = 0, & \text{in } (0,L)\times(0,T), \\ v_t + au_{xxx} + \frac{1}{c}v_{xxx} = -\frac{r}{c}v_x, & \text{in } (0,L)\times(0,T), \\ \varphi(x,0) = \varphi^0(x),\ \psi(x,0) = \psi^0(x), & \text{in } (0,L), \end{cases}$$

with boundary conditions

$$\begin{cases} u_x(L,t) = v_x(L,t) = 0, & \text{in } (0,T), \\ u_{xx}(L,t) = -\frac{ab}{c}\psi_{xx}(L,t), & \text{in } (0,T), \\ u_{xx}(0,t) = -\frac{ab}{c}\psi_{xx}(0,t), & \text{in } (0,T), \\ v_{xx}(L,t) = -ac\varphi_{xx}(L,t) - r\psi(L,t), & \text{in } (0,T), \\ v_{xx}(0,t) = -ac\varphi_{xx}(0,t) - r\psi(0,t), & \text{in } (0,T). \end{cases}$$

By using a fixed-point argument, the result is archived. □

The adjoint system possesses a relevant estimate as described below.

Proposition 2.5. *Any solution* (φ, ψ) *of the adjoint system* (31)–(34) *satisfies*

$$\begin{aligned}\|(\varphi^1, \psi^1)\|^2_{\mathcal{X}} \leq \frac{1}{T}\|(\varphi, \psi)\|_{L^2(0,T;\mathcal{X})} + \frac{1}{2}\|\varphi_x(L, \cdot)\|^2_{L^2(0,T)} + \frac{b}{2c}\|\psi_x(L, \cdot)\|^2_{L^2(0,T)} + \frac{br}{c^2}\|\psi(L, \cdot)\|^2_{L^2(0,T)} \\ + \frac{1}{2}\left\|\varphi_x(L, \cdot) + \frac{ab}{c}\psi_x(L, \cdot)\right\|^2_{L^2(0,T)} + \frac{b}{2c}\left\|a\varphi_x(L, \cdot) + \frac{1}{c}\psi_x(L, \cdot)\right\|^2_{L^2(0,T)},\end{aligned} \tag{36}$$

with initial data $(\varphi^1, \psi^1) \in \mathcal{X}$.

Proof. Multiplying the first equation of (31) by $-t\varphi$, the second one by $-\frac{b}{c}t\psi$ and integrating by parts over $(0, T) \times (0, L)$, we obtain

$$\begin{aligned}\frac{T}{2}\int_0^L \varphi^2(x, T)\mathrm{d}x = \frac{1}{2}\int_0^T\int_0^L \varphi^2(x, t)\mathrm{d}x\mathrm{d}t + \frac{ab}{c}\int_0^T\int_0^L t\varphi_{xxx}(x, t)\psi(x, t)\mathrm{d}x\mathrm{d}t \\ - \int_0^T t\left[\varphi_{xx}(x, t)\varphi(x, t) - \frac{1}{2}\varphi_x^2(x, t) + \frac{ab}{c}\psi_{xx}(x, t)\varphi(x, t) - \frac{ab}{c}\psi_x(x, t)\varphi_x(x, t)\right. \\ \left. + \frac{ab}{c}\psi(x, t)\varphi_{xx}(x, t)\right]_0^L \mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}\frac{Tb}{2c}\int_0^L \psi^2(x, T)\mathrm{d}x = \frac{b}{2c}\int_0^T\int_0^L \psi^2(x, t)\mathrm{d}x\mathrm{d}t - \frac{ab}{c}\int_0^T\int_0^L t\varphi_{xxx}(x, t)\psi(x, t)\mathrm{d}x\mathrm{d}t \\ - \int_0^T t\left[\frac{b}{c^2}\psi_{xx}(x, t)\psi(x, t) - \frac{b}{2c^2}\psi_x^2(x, t) + \frac{br}{2c^2}\psi^2(x, t)\right]_0^L \mathrm{d}t.\end{aligned}$$

Adding the above identities, it follows that

$$\begin{aligned}\frac{T}{2}\|(\varphi^1, \psi^1)\|^2_{\mathcal{X}} = \frac{1}{2}\|(\varphi, \psi)\|^2_{L^2(0,T;\mathcal{X})} - \int_0^T t\left[\frac{b}{c}\psi(x, t)\left(a\varphi_{xx}(x, t) + \frac{1}{c}\psi_{xx}(x, t) + \frac{r}{c}\psi(x, t)\right)\right]_0^L \mathrm{d}t \\ - \int_0^T t\left[\frac{b}{2c}\psi_x(x, t)\left(a\varphi_x(x, t) + \frac{1}{c}\psi_x(x, t)\right) - \frac{1}{2}\varphi_x(x, t)\left(\varphi_x(x, t) + \frac{ab}{c}\psi_x(x, t)\right)\right]_0^L \mathrm{d}t \\ + \int_0^T t\left[\varphi(x, t)\left(\varphi_{xx}(x, t) + \frac{ab}{c}\psi_{xx}(x, t)\right) - \frac{br}{2c^2}\psi^2(x, t)\right]_0^L \mathrm{d}t.\end{aligned}$$

Then, from (34), we obtain

$$\frac{T}{2}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \leq \frac{1}{2}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})} + \frac{bT}{2c}\int_0^T \psi_x(L,t)\left(a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t)\right)\mathrm{d}t$$
$$+\frac{T}{2}\int_0^T \varphi_x(L,t)\left(\varphi_x(L,t)+\frac{ab}{c}\psi_{x,t}(L,t)\right)\mathrm{d}t$$
$$+\frac{brT}{2c^2}\int_0^T \psi^2(L,t)\mathrm{d}t - \frac{brT}{2c^2}\int_0^T \psi^2(0,t)\mathrm{d}t.$$

Finally, (36) is obtained by applying Young inequality in the right-hand side of the above inequality. □

2.2. Nonlinear system

In this subsection, attention will be given to the full nonlinear system (1), (2). The proof of the lemma below is available in [2, Lemma 3.1], and therefore, we will omit it.

Lemma 2.6. *There exists a constant $C>0$, such that, for any $T>0$ and $(u,v)\in\mathcal{Z}_T$,*

$$\|uv_x\|_{L^1(0,T;L^2(0,L))} \leq C(T^{\frac{1}{2}}+T^{\frac{1}{3}})\|u\|_{\mathcal{Z}_T}\|v\|_{\mathcal{Z}_T}.$$

We first show that system (1), (2) is locally well posed in the space $\mathcal{Z}_T$.

Theorem 2.7. *For any $\left(u^0,v^0\right)\in\mathcal{X}$ and $\overrightarrow{h}=(h_0,h_1,h_2)$, $\overrightarrow{g}=(g_0,g_1,g_2)\in\mathcal{H}_T$, there exists $T^*>0$, depending on $\|\left(u^0,v^0\right)\|_{\mathcal{X}}$, such that the problem (1), (2) admits a unique solution $(u,v)\in\mathcal{Z}_{T^*}$ with*

$$\partial_x^k u, \partial_x^k v \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T^*)), \quad k=0,1,2.$$

Moreover, the corresponding solution map is Lipschitz continuous.

Proof. Let $\mathcal{F}_T=\left\{(u,v)\in\mathcal{Z}_T : (u,v)\in L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2), k=0,1,2\right\}$ equipped with the norm

$$\|(u,v)\|_{\mathcal{F}_T}=\|(u,v)\|_{\mathcal{Z}_T}+\sum_{k=0}^{2}\|(\partial_x^k u,\partial_x^k v)\|_{L_x^\infty(0,L;(H^{\frac{1-k}{3}}(0,T))^2)}.$$

Let $0<T^*\leq T$ to be determined later. For each $u,v\in\mathcal{F}_{T^*}$, consider the problem

$$\begin{cases} \omega_t+\omega_{xxx}+a\eta_{xxx}=f(u,v), & \text{in } (0,L)\times(0,T^*),\\ \eta_t+\frac{ab}{c}\omega_{xxx}+\frac{1}{c}\eta_{xxx}=s(u,v), & \text{in } (0,L)\times(0,T^*),\\ \omega_{xx}(0,t)=h_0(t),\ \omega_x(L,t)=h_1(t),\ \omega_{xx}(L,t)=h_2(t), & \text{in } (0,T^*),\\ \eta_{xx}(0,t)=g_0(t),\ \eta_x(L,t)=g_1(t),\ \eta_{xx}(L,t)=g_2(t), & \text{in } (0,T^*),\\ \omega(x,0)=u^0(x),\quad v(x,0)=v^0(x), & \text{in } (0,L), \end{cases} \tag{37}$$

where

$$f(u,v)=-a_1(vv_x)-a_2(uv)_x$$

and

$$s(u,v)=-\frac{r}{c}v_x-\frac{a_2 b}{c}(uu_x)-\frac{a_1 b}{c}(uv)_x.$$

Since $\|v_x\|_{L^1(0,T^*;L^2(0,L))} \leq \beta^{\frac{1}{2}}\|v\|_{\mathcal{Z}_{T^*}}$, from Lemma 2.6 we deduce that $f(u,v)$ and $s(u,v)$ belong to $L^1(0,T^*;L^2(0,L))$ and satisfies

$$\|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)} \leq C_1\left((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}}\right)\left(\|u\|^2_{\mathcal{Z}_{T^*}}+(\|u\|_{\mathcal{Z}_{T^*}}+1)\|v\|_{\mathcal{Z}_{T^*}}+\|v\|^2_{\mathcal{Z}_{T^*}}\right), \tag{38}$$

for some positive constant C_1. Then, according to Proposition 2.1, we can define the operator

$$\Gamma:\mathcal{F}_{T^*}\to\mathcal{F}_{T^*}, \quad \text{given by} \quad \Gamma(u,v)=(\omega,\eta),$$

where (ω,η) is the solution of (37). Moreover,

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_{T^*}}+\|(f,s)\|_{L^1(0,T^*;(L^2(0,L))^2)}\right\}, \tag{39}$$

where the positive constant C depends only on T^*. Combining (38) and (39), we obtain

$$\begin{aligned}\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq & C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_{T^*}}\right\}\\ &+CC_1\left((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}}\right)\left(\|u\|^2_{\mathcal{Z}_{T^*}}+(\|u\|_{\mathcal{Z}_{T^*}}+1)\|v\|_{\mathcal{Z}_{T^*}}+\|v\|^2_{\mathcal{Z}_{T^*}}\right).\end{aligned}$$

Let $(u,v)\in B_r(0):=\{(u,v)\in\mathcal{F}_{T^*}:\|(u,v)\|_{\mathcal{F}_{T^*}}\leq r\}$, where $r=2C\left\{\|\left(u^0,v^0\right)\|_{\mathcal{X}}+\|\left(\overrightarrow{h},\overrightarrow{g}\right)\|_{\mathcal{H}_T}\right\}$. From the estimate above, it follows that

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}} \leq \frac{r}{2}+CC_1\left((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}}\right)(3r+1)\,r. \tag{40}$$

Then, by choosing $T^*>0$, such that

$$CC_1\left((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}}\right)(3r+1)\leq\frac{1}{2}, \tag{41}$$

from (40), we have

$$\|\Gamma(u,v)\|_{\mathcal{F}_{T^*}}\leq r.$$

Thus, we conclude that

$$\Gamma: B_r(0)\subset\mathcal{F}_{T^*}\to B_r(0).$$

On the other hand, $\Gamma(u_1,v_1)-\Gamma(u_2,v_2)$ solve the system

$$\begin{cases}\omega_t+\omega_{xxx}+a\eta_{xxx}=f(u_1,v_1)-f(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \eta_t+\frac{ab}{c}\omega_{xxx}+\frac{1}{c}\eta_{xxx}=s(u_1,v_1)-s(u_2,v_2), & \text{in } (0,L)\times(0,T^*),\\ \omega_{xx}(0,t)=\omega_x(L,t)=\omega_{xx}(L,t)=0, & \text{in } (0,T^*),\\ \eta_{xx}(0,t)=\eta_x(L,t)=\eta_{xx}(L,t)=0, & \text{in } (0,T^*),\\ \omega(x,0)=0, \quad v(x,0)=0, & \text{in } (0,L),\end{cases}$$

where $f(u,v)$ and $s(u,v)$ were defined in (37). Note that

$$|f(u_1,v_1)-f(u_2,v_2)|\leq C_2|\left((v_2-v_1)v_{2,x}+v_1(v_2-v_1)_x+(u_2(v_2-v_1))_x+((u_2-u_1)v_1)_x\right)|$$

and

$$\begin{aligned}|s(u_1,v_1)-s(u_2,v_2)|\leq & C_2|\left((v_2-v_1)_x+(u_2-u_1)u_{2,x}+u_1(u_2-u_1)_x\right.\\ &\left.+(u_2(v_2-v_1))_x+((u_2-u_1)v_1)_x\right)|,\end{aligned}$$

for some positive constant C_2. Then, Proposition 2.1 and Lemma 2.6, give us the following estimate

$$\begin{aligned}\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}} &\leq C\|(f(u_1,v_1)-f(u_2,v_2),s(u_1,v_1)-s(u_2,v_2))\|_{L^1(0,T^*;(L^2(0,L))^2)}\\ &\leq C_3((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}})(8r+1)\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}},\end{aligned}$$

for some positive constant C_3. Choosing T^*, satisfying (41) and such that

$$C_3((T^*)^{\frac{1}{2}}+(T^*)^{\frac{1}{3}})(8r+1)\leq \frac{1}{2},$$

we obtain

$$\|\Gamma(u_1,v_1)-\Gamma(u_2,v_2)\|_{\mathcal{F}_{T^*}}\leq \frac{1}{2}\|(u_1-u_2,v_1-v_2)\|_{\mathcal{F}_{T^*}}.$$

Hence, $\Gamma : B_r(0)\to B_r(0)$ is a contraction and, by Banach fixed-point theorem, we obtain a unique $(u,v)\in B_r(0)$, such that $\Gamma(u,v)=(u,v)\in\mathcal{F}_{T^*}$, and therefore, the proof is complete. □

Remark 2.8. From the proof of Proposition 2.1, we deduce that solution of the system (1), (2) can be written as

$$\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}=W_0(t)\begin{pmatrix}u^0(x)\\ v^0(x)\end{pmatrix}+W_{bdr}(t)\begin{pmatrix}\overrightarrow{h}\\ \overrightarrow{g}\end{pmatrix}$$
$$-\int_0^t W_0(t-\tau)\begin{pmatrix}a_1(vv_x)(\tau)+a_2(uv)_x(\tau)\\ \frac{r}{c}v_x(\tau)+\frac{a_2b}{c}(uu_x)(\tau)+\frac{a_1b}{c}(uv)_x(\tau)\end{pmatrix}\mathrm{d}\tau,$$

with

$$W_0(t)=\begin{pmatrix}W_0^-(t) & 0\\ 0 & W_0^+(t)\end{pmatrix}\quad\text{and}\quad W_{bdr}(t)=\begin{pmatrix}W_{bdr}^-(t) & 0\\ 0 & W_{bdr}^+(t)\end{pmatrix},$$

where $\{W_0^{\pm}(t)\}_{t\geq 0}$ are the C_0 semigroup in the space $L^2(0,L)$ generated by the linear operators

$$A^{\pm}=-\alpha_{\pm}g''',$$

where

$$\alpha_{\pm}=-\frac{1}{2}\left(\left(\frac{1}{c}-1\right)\pm\sqrt{\left(\frac{1}{c}-1\right)^2+\frac{4a^2b}{c}}\right),$$

with domain

$$D(A^{\pm})=\{g\in H^3(0,L):g''(0)=g'(L)=g''(L)=0\},$$

and $\{W_{bdr}^{\pm}(x)\}_{t\geq 0}$ is the operator defined in Lemma 2.2.

3. Exact boundary controllability for the linear system

In this section, we study the existence of controls $\overrightarrow{h}:=(h_0,h_1,h_2)$ and $\overrightarrow{g}:=(g_0,g_1,g_2)\in\mathcal{H}_T$, such that the solution (u,v) of the system

$$\begin{cases} u_t+u_{xxx}+av_{xxx}=0 & \text{in } (0,L)\times(0,T),\\ v_t+\frac{r}{c}v_x+\frac{ab}{c}u_{xxx}+\frac{1}{c}v_{xxx}=0 & \text{in } (0,L)\times(0,T),\\ u(x,0)=u^0(x),\quad v(x,0)=v^0(x), & \text{in } (0,L),\end{cases}\tag{42}$$

satisfying the boundary conditions

$$\begin{cases} u_{xx}(0,t)=h_0(t),\ u_x(L,t)=h_1(t),\ u_{xx}(L,t)=h_2(t) & \text{in } (0,T),\\ v_{xx}(0,t)=g_0(t),\ v_x(L,t)=g_1(t),\ v_{xx}(L,t)=g_2(t) & \text{in } (0,T),\end{cases}\tag{43}$$

satisfies

$$u(\cdot,T)=u^1(\cdot),\qquad\text{and}\qquad v(\cdot,T)=v^1(\cdot).\tag{44}$$

More precisely, we have the following definition:

Definition 3.1. Let $T > 0$. System (42), (43) is exactly controllable in time T if for any initial and final data $\left(u^0, v^0\right)$ and $\left(u^1, v^1\right)$ in $\mathcal{X}$, there exist control functions $\overrightarrow{h} = (h_0, h_1, h_2)$ and $\overrightarrow{g} = (g_0, g_1, g_2)$ in $\mathcal{H}_T$, such that the solution of (42), (43) satisfies (44).

Remark 3.1. Without any loss of generality, we shall consider only the case $u^0 = v^0 = 0$. Indeed, let $\left(u^0, v^0\right)$, $\left(u^1, v^1\right)$ in $\mathcal{X}$ and $\overrightarrow{h}$, $\overrightarrow{g}$ in $\mathcal{H}_T$ be controls which lead the solution $(\widetilde{u}, \widetilde{v})$ of (42) from the zero initial data to the final state $\left(u^1, v^1\right) - (u(T), v(T))$, where (u, v) is the mild solution corresponding to (42), (43) with initial data $\left(u^0, v^0\right)$. It follows immediately that these controls also lead to the solution $(\widetilde{u}, \widetilde{v}) + (u, v)$ of (42), (43) from $\left(u^0, v^0\right)$ to the final state $\left(u^1, v^1\right)$.

In the following pages, we will analyze the exact controllability of the system (42), (43) for different combinations of four controls and one control.

3.1. Four controls

3.1.1. Case 1. Consider the following boundary conditions:

$$\begin{cases} u_{xx}(0,t) = h_0(t),\ u_x(L,t) = h_1(t),\ u_{xx}(L,t) = 0 & \text{in } (0,T), \\ v_{xx}(0,t) = g_0(t),\ v_x(L,t) = g_1(t),\ v_{xx}(L,t) = 0 & \text{in } (0,T). \end{cases} \tag{45}$$

We first give an equivalent condition for the exact controllability property.

Lemma 3.2. *For any $\left(u^1, v^1\right)$ in $\mathcal{X}$, there exist four controls $\overrightarrow{h} = (h_0, h_1, 0)$ and $\overrightarrow{g} = (g_0, g_1, 0)$ in $\mathcal{H}_T$, such that the solution (u, v) of (42)–(45) satisfies (44) if and only if*

$$\begin{aligned} \int_0^L \left(u^1(x)\varphi^1(x) + v^1(x)\psi^1(x)\right) \mathrm{d}x = & \int_0^T h_0(t) \left(\varphi(0,t) + \frac{ab}{c}\psi(0,t)\right) \mathrm{d}t \\ & + \int_0^T h_1(t) \left(\varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t)\right) \mathrm{d}t \\ & + \int_0^T g_0(t) \left(a\varphi(0,t) + \frac{1}{c}\psi(0,t)\right) \mathrm{d}t \\ & + \int_0^T g_1(t) \left(a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right) \mathrm{d}t, \end{aligned} \tag{46}$$

for any (φ^1, ψ^1) in $\mathcal{X}$, where (φ, ψ) is the solution of the backward system (31)–(34) with initial data (φ^1, ψ^1).

Proof. The relation (46) is obtained by multiplying the equations in (42) by the solution (φ, ψ) of (31)–(34), integrating by parts and using the boundary conditions (45). □

The following observability inequality plays a fundamental role for the study of the controllability properties.

Proposition 3.3. *For $T>0$ and $L>0$, there exists a constant $C:=C(T,L)>0$, such that*

$$\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \leq C\left\{\left\|(-\Delta_t)^{\frac{1}{6}}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}+\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\right.$$
$$\left.+\left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\right\}, \tag{47}$$

for any $(\varphi^1,\psi^1)\in\mathcal{X}$, where (φ,ψ) is a solution of (31)–(34) with initial data (φ^1,ψ^1), where $\Delta_t:=\partial_t^2$.

Proof. We argue by contradiction, as in [10, Proposition 3.3], and suppose that (47) does not hold. In this case, we obtain a sequence $\{(\varphi^1_n,\psi^1_n)\}_{n\in\mathbb{N}}$, satisfying

$$1=\|(\varphi^1_n,\psi^1_n)\|^2_{\mathcal{X}} \geq n\left\{\left\|(-\Delta_t)^{\frac{1}{6}}\left(\varphi_n(0,\cdot)+\frac{ab}{c}\psi_n(0,\cdot)\right)\right\|^2_{L^2(0,T)}+\left\|\varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}\right.$$
$$\left.+\left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi_n(0,\cdot)+\frac{1}{c}\psi_n(0,\cdot)\right)\right\|^2_{L^2(0,L)}+\left\|a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}\right\}. \tag{48}$$

Consequently, (48) imply that

$$\begin{cases}\varphi_n(0,\cdot)+\frac{ab}{c}\psi_n(0,\cdot)\to 0 & \text{in} \quad H^{\frac{1}{3}}(0,T),\\ a\varphi_n(0,\cdot)+\frac{1}{c}\psi_n(0,\cdot)\to 0 & \text{in} \quad H^{\frac{1}{3}}(0,T),\\ \varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T),\\ a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T),\end{cases} \tag{49}$$

as $n\to\infty$. Since $1-a^2b>0$, (49) guarantees that the following converge hold

$$\begin{cases}\varphi_n(0,\cdot)\to 0, \quad \psi_n(0,\cdot)\to 0 & \text{in} \quad H^{\frac{1}{3}}(0,T),\\ \varphi_{n,x}(L,\cdot)\to 0, \quad \psi_{n,x}(L,\cdot)\to 0 & \text{in} \quad L^2(0,T),\end{cases} \tag{50}$$

as $n\to\infty$. The next steps are devoted to pass the strong limit in the left-hand side of (48). First, observe that from Proposition 2.4 we deduce that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;(H^1(0,L))^2)$. Then, (31) implies that $\{(\varphi_{t,n},\psi_{t,n})\}_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;(H^{-2}(0,L))^2)$ and the compact embedding

$$H^1(0,L)\hookrightarrow L^2(0,L)\hookrightarrow H^{-2}(0,L)$$

allows us to conclude that $\{(\varphi_n,\psi_n)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;\mathcal{X})$. Consequently, we obtain a subsequence, still denoted by the same index n, satisfying

$$(\varphi_n,\psi_n)\to(\varphi,\psi) \text{ in } L^2(0,T;\mathcal{X}), \text{ as } n\to\infty. \tag{51}$$

On the other hand, (35) and (48) imply that the sequences

$$\{\varphi_n(0,\cdot)\}_{n\in\mathbb{N}} \text{ and } \{\psi_n(0,\cdot)\}_{n\in\mathbb{N}} \text{ are bounded in } H^{\frac{1}{3}}(0,T).$$

Then, the following compact embedding

$$H^{\frac{1}{3}}(0,T)\hookrightarrow L^2(0,T) \tag{52}$$

guarantees that the above sequences are relatively compact in $L^2(0,T)$, that is, we obtain a subsequence, still denoted by the same index n, satisfying

$$\begin{cases}\varphi_n(0,\cdot)\to\varphi(0,\cdot) & \text{in} \quad L^2(0,T),\\ \psi_n(0,\cdot)\to\psi(0,\cdot) & \text{in} \quad L^2(0,T),\end{cases} \tag{53}$$

as $n\to\infty$. Then, from (50) and (53) we deduce that

$$\varphi(0,\cdot)=\psi(0,\cdot)=0.$$

Moreover, (35), (48) and (52) imply that $\{\varphi_n(L,t)\}_{n\in\mathbb{N}}$ and $\{\psi_n(L,t)\}_{n\in\mathbb{N}}$ are relatively compact in $L^2(0,T)$. Hence, we obtain a subsequence, still denoted by the same index, satisfying

$$\begin{cases}\varphi_n(L,\cdot)\to\varphi(L,\cdot) \text{ in } L^2(0,T),\\ \psi_n(L,\cdot)\to\psi(L,\cdot) \text{ in } L^2(0,T),\end{cases} \tag{54}$$

as $n\to\infty$. In addition, according to Proposition 2.5, we have

$$\begin{aligned}\|(\varphi_n^1,\psi_n^1)\|^2_{\mathcal{X}} \leq & \frac{1}{T}\|(\varphi_n,\psi_n)\|_{L^2(0,T;\mathcal{X})}+\frac{1}{2}\|\varphi_{n,x}(L,\cdot)\|^2_{L^2(0,T)}\\ &+\frac{b}{2c}\|\psi_{n,x}(L,\cdot)\|^2_{L^2(0,T)}+\frac{br}{c^2}\|\psi_n(L,\cdot)\|^2_{L^2(0,T)}\\ &+\frac{1}{2}\left\|\varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}+\frac{b}{2c}\left\|a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\right\|^2_{L^2(0,T)}.\end{aligned}$$

Then, from (49), (50), (51) and (54) we conclude that $\{(\varphi_n^1,\psi_n^1)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}$, and therefore, we get

$$(\varphi_n^1,\psi_n^1)\to(\varphi^1,\psi^1) \text{ in } \mathcal{X}, \text{ as } n\to\infty. \tag{55}$$

Thus, Proposition 2.4 together with (55) imply that

$$\begin{cases}\varphi_{n,x}(L,\cdot)\to\varphi_x(L,\cdot) \text{ in } L^2(0,T),\\ \psi_{n,x}(L,\cdot)\to\psi_x(L,\cdot) \text{ in } L^2(0,T)\end{cases} \tag{56}$$

and

$$\begin{cases}\varphi_{n,xx}(L,\cdot)+\frac{ab}{c}\psi_{n,xx}(L,\cdot)\to\varphi_{xx}(L,\cdot)+\frac{ab}{c}\psi_{xx}(L,\cdot) & \text{in } L^2(0,T),\\ \varphi_{n,xx}(0,\cdot)+\frac{ab}{c}\psi_{n,xx}(0,\cdot)\to\varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot) & \text{in } L^2(0,T),\\ a\varphi_{n,xx}(L,\cdot)+\frac{1}{c}\psi_{n,xx}(L,\cdot)+\frac{r}{c}\psi_n(L,\cdot)\to a\varphi_{xx}(L,\cdot)+\frac{1}{c}\psi_{xx}(L,\cdot)+\frac{r}{c}\psi(L,\cdot) & \text{in } L^2(0,T),\\ a\varphi_{n,xx}(0,\cdot)+\frac{1}{c}\psi_{n,xx}(0,\cdot)+\frac{r}{c}\psi_n(0,\cdot)\to a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)+\frac{r}{c}\psi(0,\cdot) & \text{in } L^2(0,T),\end{cases}$$

as $n\to\infty$. Since (φ_n,ψ_n) is a solution of the adjoint system, we obtain that

$$\begin{cases}\varphi_{xx}(L,\cdot)+\frac{ab}{c}\psi_{xx}(L,\cdot)=0,\\ \varphi_{xx}(0,\cdot)+\frac{ab}{c}\psi_{xx}(0,\cdot)=0,\\ a\varphi_{xx}(L,\cdot)+\frac{1}{c}\psi_{xx}(L,\cdot)+\frac{r}{c}\psi(L,\cdot)=0,\\ a\varphi_{xx}(0,\cdot)+\frac{1}{c}\psi_{xx}(0,\cdot)+\frac{r}{c}\psi(L,\cdot)=0.\end{cases}$$

On the other hand, from (50) and (56), we have

$$\varphi_x(L,\cdot)=\psi_x(L,\cdot)=0.$$

Finally, we obtain that (φ,ψ) is a solution of

$$\begin{cases}\varphi_t+\varphi_{xxx}+a\frac{ab}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T),\\ \psi_t+\frac{r}{c}\psi_x+a\varphi_{xxx}+\frac{1}{c}\psi_{xxx}=0, & \text{in } (0,L)\times(0,T),\\ a\varphi_{xx}(L,t)+\frac{1}{c}\psi_{xx}(L,t)+\frac{r}{c}\psi(L,t)=0, & \text{in } (0,T),\\ a\varphi_{xx}(0,t)+\frac{1}{c}\psi_{xx}(0,t)+\frac{r}{c}\psi(0,t)=0, & \text{in } (0,T),\\ \varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)=0, & \text{in } (0,T),\\ \varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)=0, & \text{in } (0,T),\\ \varphi_x(0,t)=\psi_x(0,t)=0, & \text{in } (0,T),\\ \varphi(x,T)=\varphi^1(x), \qquad \psi(x,T)=\psi^1(x), & \text{in } (0,L),\end{cases} \tag{57}$$

satisfying the additional boundary conditions

$$\varphi(0,t)=\psi(0,t)=\varphi_x(L,t)=\psi_x(L,t)=0 \ \text{ in } \ (0,T) \tag{58}$$

and

$$\|(\varphi^1,\psi^1)\|_{\mathcal{X}}=1. \tag{59}$$

Observe that (59) implies that the solutions of (57), (58) cannot be identically zero. However, by Lemma 3.4, one can conclude that $(\varphi,\psi)=(0,0)$, which drive us to a contradiction. □

Lemma 3.4. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^1,\psi^1)\in\mathcal{X}$, such that the solution of* (57) *satisfies* (58)*. Then, $N_T=\{0\}$.*

Proof. The proof uses the same arguments as those given in [10].

If $N_T\neq\{0\}$, the map $(\varphi^1,\psi^1)\in N_T\to A(N_T)\subset\mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue. Hence, there exist $\lambda\in\mathbb{C}$ and $\varphi_0,\psi_0\in H^3(0,L)\setminus\{0\}$, such that

$$\begin{cases}\lambda\varphi_0+\varphi_0'''+\frac{ab}{c}\psi_0'''=0, & \text{in } (0,L),\\ \lambda\psi_0+\frac{r}{c}\psi_0'+a\varphi_0'''+\frac{1}{c}\psi_0'''=0, & \text{in } (0,L),\\ \varphi_0'(x)=\psi_0'(x)=0, & \text{in } \{0,L\},\\ a\varphi_0''(x)+\frac{1}{c}\psi_0''(x)+\frac{r}{c}\psi_0(x)=0, & \text{in } \{0,L\},\\ \varphi_0''(x)+\frac{ab}{c}\psi_0''(x)=0, & \text{in } \{0,L\},\\ \varphi_0(0)=\psi_0(0)=0.\end{cases}$$

The notation $\{0,L\}$, used above, mean that the expression is applied in 0 and L.

Since $1-a^2b>0$, the above system becomes

$$\begin{cases}\lambda\varphi_0+\varphi_0'''+\frac{ab}{c}\psi_0'''=0, & \text{in } (0,L),\\ \lambda\psi_0+\frac{r}{c}\psi_0'+a\varphi_0'''+\frac{1}{c}\psi_0'''=0, & \text{in } (0,L),\\ \varphi_0(0)=\varphi_0'(0)=\varphi_0''(0)=0,\\ \psi_0(0)=\psi_0'(0)=\psi_0''(0)=0.\end{cases} \tag{60}$$

By straightforward computations, we see that $(\varphi_0,\psi_0)=(0,0)$ is the unique solution of (60) for all $L>0$, which concludes the proof of Lemma 3.4 and Proposition 3.3. □

The following theorem gives a positive answer for the control problem:

Theorem 3.5. *Let $T>0$ and $L>0$. Then, the system* (42)–(45) *is exactly controllable in time T.*

Proof. Let us denote by Γ the linear and bounded map defined by

$$\begin{array}{rccl}\Gamma: & L^2(0,L)\times L^2(0,L) & \longrightarrow & L^2(0,L)\times L^2(0,L)\\ & (\varphi^1(\cdot),\psi^1(\cdot)) & \longmapsto & \Gamma(\varphi^1(\cdot),\psi^1(\cdot))=(u(\cdot,T),v(\cdot,T)),\end{array}$$

where (u,v) is the solution of (42)–(45), with

$$\begin{cases} h_0(t)=(-\Delta_t)^{\frac13}\left(\varphi(0,t)+\frac{ab}{c}\psi(0,t)\right), & h_1(t)=\varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t),\\ g_0(t)=(-\Delta_t)^{\frac13}\left(a\varphi(0,t)+\frac{1}{c}\psi(0,t)\right), & g_1(t)=a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t),\end{cases} \tag{61}$$

and (φ,ψ) the solution of the system (31)–(34) with $\Delta_t=\partial_t^2$ and initial data (φ^1,ψ^1). According to Lemma 3.2 and Proposition 3.3, we obtain

$$\begin{aligned}\left(\Gamma(\varphi^1,\psi^1),(\varphi^1,\psi^1)\right)_{(L^2(0,L))^2} &= \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac1c\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &\quad+\left((-\Delta_t)^{\frac13}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right),\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)_{L^2(0,T)}\\ &\quad+\left((-\Delta_t)^{\frac13}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right),a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)_{L^2(0,T)}\\ &=\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac1c\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &\quad+\left\|(-\Delta_t)^{\frac16}\left(a\varphi(0,\cdot)+\frac1c\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\ &\quad+\left\|(-\Delta_t)^{\frac16}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)}\\ &\geq C^{-1}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}}.\end{aligned}$$

Thus, by the Lax–Milgram theorem, Γ is invertible. Consequently, for given $\left(u^1,v^1\right)\in(L^2(0,L))^2$, we can define $(\varphi^1,\psi^1):=\Gamma^{-1}(u^1,v^1)$ to solve the system (31)–(34) and get $(\varphi,\psi)\in\mathcal{Z}_T$. Then, if $h_0(t)$, $h_1(t)$, $g_0(t)$ and $g_1(t)$ are given by (61), the corresponding solution (u,v) of the system (42)–(45), satisfies

$$(u(\cdot,0),v(\cdot,0))=(0,0)\quad\text{and}\quad(u(\cdot,T),v(\cdot,T))=(u^1(\cdot),=v^1(\cdot)).$$

□

Remark 3.6. An important question is whether the exact controllability holds, in time $T>0$, when we consider the boundary condition with another configuration, for example,

$$\begin{cases} u_{xx}(0,t)=0 \quad u_x(L,t)=h_1(t) \quad u_{xx}(L,t)=h_2(t), & \text{in } (0,T),\\ v_{xx}(0,t)=0, \quad v_x(L,t)=g_1(t), \quad v_{xx}(L,t)=g_2(t), & \text{in } (0,T).\end{cases} \tag{62}$$

Observe that, in this case it would be necessary to prove that the following observability inequality

$$\begin{aligned}\|(\varphi^1,\psi^1)\|^2_{\mathcal{X}}\leq C\Bigg\{&\left\|(-\Delta_t)^{\frac16}\left(\varphi(L,\cdot)+\frac{ab}{c}\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)}+\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\\ &+\left\|(-\Delta_t)^{\frac16}\left(a\varphi(L,\cdot)+\frac1c\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac1c\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\Bigg\},\end{aligned}$$

holds for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is solution of (31)–(34) with initial data (φ^1,ψ^1). It can be done using Proposition 2.4 together with the contradiction argument used in the proof of Proposition 3.3. Thus, the next result about the exact controllability of the system (42)–(62) also holds:

Theorem 3.7. *Let $T>0$ and $L>0$. Then, the system (42)–(62) is exactly controllable in time T.*

3.1.2. Case 2. We consider the following boundary conditions:

$$\begin{cases} u_{xx}(0,t)=0 & u_x(L,t)=h_1(t) \quad u_{xx}(L,t)=0, \quad \text{in } (0,T), \\ v_{xx}(0,t)=g_0(t), & v_x(L,t)=g_1(t), \quad v_{xx}(L,t)=g_2(t), \quad \text{in } (0,T). \end{cases} \tag{63}$$

First, as in subsection above, we give an equivalent condition for the exact controllability property. It can be done using the same idea of the proof of Lemma 3.2.

Lemma 3.8. *For any (u^1,v^1) in $\mathcal{X}$, there exist four controls $\overrightarrow{h}=(0,h_1,0)$ and $\overrightarrow{g}=(g_0,g_1,g_2)$ in $\mathcal{H}_T$, such that the solution (u,v) of (42)–(63) satisfies (44) if and only if*

$$\begin{aligned} \int_0^L (u^1(x)\varphi^1(x)+v^1(x)\psi^1(x))\mathrm{d}x = & \int_0^{\mathrm{T}} g_0(t)\left(a\varphi(0,t)+\frac{1}{c}\psi(0,t)\right)\mathrm{d}t \\ & + \int_0^t g_1(t)\left(a\varphi_x(L,t)+\frac{1}{c}\psi_x(L,t)\right)\mathrm{d}t \\ & - \int_0^{\mathrm{T}} g_2(t)\left(a\varphi(L,t)+\frac{1}{c}\psi(L,t)\right)\mathrm{d}t \\ & + \int_0^{\mathrm{T}} h_1(t)\left(\varphi_x(L,t)+\frac{ab}{c}\psi_x(L,t)\right)\mathrm{d}t, \end{aligned} \tag{64}$$

for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is the solution of the backward system (31)–(34).

To prove the exact controllability property, it suffices to prove the following observability inequality:

Proposition 3.9. *Let $T>0$ and $L\in(0,\infty)\setminus\mathcal{F}_r$, where $\mathcal{F}_r$ is given by (15). Then, there exists a constant $C(T,L)>0$, such that*

$$\begin{aligned} \|(\varphi^1,\psi^1)\|^2_{\mathcal{X}} \le C\Bigg\{ & \left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(0,\cdot)+\frac{1}{c}\psi(0,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)} \\ & + \left\|(-\Delta_t)^{\frac{1}{6}}\left(a\varphi(L,\cdot)+\frac{1}{c}\psi(L,\cdot)\right)\right\|^2_{L^2(0,T)} + \left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|^2_{L^2(0,T)}\Bigg\}, \end{aligned}$$

for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is solution of (31)–(34) with initial data (φ^1,ψ^1), where $\Delta_t:=\partial_t^2$.

Proof. We proceed as in the proof of Proposition 3.3 using the contradiction argument. Therefore, we will summarize it. Firstly, we show that the sequences $\{(\varphi_n^1,\psi_n^1)\}_{n\in\mathbb{N}}$,

$$\{a\varphi_n(0,\cdot)+\frac{1}{c}\psi_n(0,\cdot)\}_{n\in\mathbb{N}},$$

$$\{a\varphi_n(L,\cdot)+\frac{1}{c}\psi_n(L,\cdot)\}_{n\in\mathbb{N}},$$

$$\{a\varphi_{n,x}(L,\cdot)+\frac{1}{c}\psi_{n,x}(L,\cdot)\}_{n\in\mathbb{N}},$$

and

$$\{\varphi_{n,x}(L,\cdot)+\frac{ab}{c}\psi_{n,x}(L,\cdot)\}_{n\in\mathbb{N}},$$

are relatively compact in $\mathcal{X}$ and $L^2(0,T;\mathcal{X})$, respectively. Next, we proceed as in the proof of Proposition 3.3 to get that

$$a\varphi_n(0,\cdot)+\frac{1}{c}\psi_n(0,\cdot)\to 0,$$
$$a\varphi_n(L,\cdot)+\frac{1}{c}\psi_n(L,\cdot)\to 0,$$
$$\varphi_{n,x}(L,\cdot)\to 0,\ \psi_x(L,\cdot)\to 0,$$

as $n\to\infty$, and

$$||(\varphi,\psi)||_{(L^2(0,L))^2}=1.$$

Finally, combining the hidden regularity of the solutions of the adjoint system (35) and the compact embedding $H^{\frac{1}{3}}(0,T)\hookrightarrow L^2(0,T)$, we conclude that (φ,ψ) satisfies

$$\begin{cases}\varphi_t+\varphi_{xxx}+\frac{ab}{c}\psi_{xxx}=0 & \text{in } (0,L)\times(0,T),\\ \psi_t+\frac{r}{c}\psi_x+a\varphi_{xxx}+\frac{1}{c}\psi_{xxx}=0 & \text{in } (0,L)\times(0,T),\\ \varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)=0 & \text{in } (0,T),\\ \varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)=0 & \text{in } (0,T),\\ a\varphi_{xx}(L,t)+\frac{1}{c}\psi_{xx}(L,t)+\frac{r}{c}\psi(L,t)=0 & \text{in } (0,T),\\ a\varphi_{xx}(0,t)+\frac{1}{c}\psi_{xx}(0,t)+\frac{r}{c}\psi(0,t)=0 & \text{in } (0,T),\\ \varphi_x(0,t)=\psi_x(0,t)=0 & \text{in } (0,T),\\ \varphi(x,T)=\varphi^1(x),\ \psi(x,T)=\psi^1(x) & \text{in } (0,L)\end{cases} \tag{65}$$

and

$$\begin{cases}a\varphi(L,t)+\frac{1}{c}\psi(L,t)=0 & \text{in } (0,T),\\ a\varphi(0,t)+\frac{1}{c}\psi(0,t)=0 & \text{in } (0,T),\\ \varphi_x(L,t)=\psi_x(L,t)=0 & \text{in } (0,T),\\ \|(\varphi,\psi)\|_{\mathcal{X}}=1.\end{cases} \tag{66}$$

Notice that the solutions of (65), (66) cannot be identically zero. Therefore, from Lemma 3.10, one can conclude that $(\varphi,\psi)=(0,0)$, which drive us to a contradiction. □

Lemma 3.10. *For any $T>0$, let N_T denote the space of the initial states $(\varphi^1,\psi^1)\in\mathcal{X}$, such that the solution of* (65) *satisfies* (66)*. Then, for $L\in(0,\infty)\setminus\mathcal{F}_r$, $N_T=\{0\}$.*

Proof. By the same arguments given in [10], if $N_T\neq\{0\}$, the map $(\varphi^1,\psi^1)\in N_T\to A(N_T)\subset\mathbb{C}N_T$ has (at least) one eigenvalue. Hence, there exist $\lambda\in\mathbb{C}$ and $\varphi_0,\psi_0\in H^3(0,L)\setminus\{0\}$, such that

$$\begin{cases}\lambda\varphi_0+\varphi_0'''+\frac{ab}{c}\psi_0'''=0, & \text{in } (0,L),\\ \lambda\psi_0+\frac{r}{c}\psi_0'+a\varphi_0'''+\frac{1}{c}\psi_0'''=0, & \text{in } (0,L),\\ a\varphi_0(x)+\frac{1}{c}\psi_0(x)=0, & \text{in } \{0,L\},\\ \varphi_0'(x)=\psi_0'(x)=0, & \text{in } \{0,L\},\\ \varphi_0''(x)+\frac{ab}{c}\psi_0''(x)=0, & \text{in } \{0,L\},\\ a\varphi_0''(x)+\frac{1}{c}\psi_0''(x)+\frac{r}{c}\psi_0(x)=0, & \text{in } \{0,L\}.\end{cases} \tag{67}$$

□

To conclude the proof of the Lemma 3.10, we prove that this does not hold if $L\in(0,\infty)\setminus\mathcal{F}_r$. To simplify the notation, henceforth we denote $(\varphi_0,\psi_0):=(\varphi,\psi)$.

Lemma 3.11. *Let $L>0$. Consider the assertion*

$$(\mathcal{N}): \quad \exists \lambda \in \mathbb{C}, \exists \varphi, \psi \in H^3(0,L)\setminus(0,0), \text{ such that } \begin{cases} \lambda\varphi + \varphi''' + \frac{ab}{c}\psi''' = 0, & \text{in } (0,L), \\ \lambda\psi + \frac{r}{c}\psi' + a\varphi''' + \frac{1}{c}\psi''' = 0, & \text{in } (0,L), \\ a\varphi(x) + \frac{1}{c}\psi(x) = 0, & \text{in } \{0,L\}, \\ \varphi'(x) = \psi'(x) = 0, & \text{in } \{0,L\}, \\ \varphi''(x) + \frac{ab}{c}\psi''(x) = 0, & \text{in } \{0,L\}, \\ a\varphi''(x) + \frac{1}{c}\psi''(x) + \frac{r}{c}\psi(x) = 0, & \text{in } \{0,L\}. \end{cases}$$

Then, $(\mathcal{N})$ holds if and only if $L \in \mathcal{F}_r$.

Proof. We use an argument similar to the one used in [10, Lemma 3,5]. Let us introduce the notation $\hat{\varphi}(\xi) = \int_0^L \mathrm{e}^{-ix\xi}\varphi(x)\mathrm{d}x$ and $\hat{\psi}(\xi) = \int_0^L \mathrm{e}^{-ix\xi}\psi(x)\mathrm{d}x$. Then, multiplying the first and the second equations in $(\mathcal{N})$ by $\mathrm{e}^{-ix\xi}$ and integrating by part in $(0,L)$, it follows that

$$\begin{aligned} &\left((i\xi)^3+\lambda\right)\hat{\varphi}(\xi) + \frac{ab}{c}(i\xi)^3\hat{\psi}(\xi) \\ &\quad + \left[\left(\left(\varphi''(x)+\frac{ab}{c}\psi''(x)\right) + (i\xi)\left(\varphi'(x)+\frac{ab}{c}\psi'(x)\right) + (i\xi)^2\left(\varphi(x)+\frac{ab}{c}\psi(x)\right)\right)\mathrm{e}^{-ix\xi}\right]_0^L = 0 \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{c}(i\xi)^3 + \frac{r}{c}(i\xi) + \lambda\right)\hat{\psi}(\xi) + a(i\xi)^3\hat{\varphi}(\xi) + \left[\left(\left(a\varphi''(x) + \frac{1}{c}\psi''(x) + \frac{r}{c}\psi(\xi)\right)\right.\right. \\ &\quad \left.\left. + (i\xi)\left(a\varphi'(x) + \frac{1}{c}\psi'(x)\right) + (i\xi)^2\left(a\varphi(x) + \frac{1}{c}\psi(x)\right)\right)\mathrm{e}^{-ix\xi}\right]_0^L = 0. \end{aligned}$$

The boundary conditions allow us to conclude that

$$\begin{cases} [(i\xi)^3+\lambda]\hat{\varphi}(\xi) + \dfrac{ab}{c}(i\xi)^3\hat{\psi}(\xi) = (i\xi)^2\left(\varphi(0) + \dfrac{ab}{c}\psi(0) - \left(\varphi(L) + \dfrac{ab}{c}\psi(L)\right)\mathrm{e}^{-iL\xi}\right), \\ \\ \dfrac{1}{c}[(i\xi)^3 + r(i\xi) + c\lambda]\hat{\psi}(\xi) + a(i\xi)^3\hat{\varphi}(\xi) = 0. \end{cases} \tag{68}$$

Then, from the first equation in (68), we obtain

$$\hat{\varphi}(\xi) = \frac{(i\xi)^2\left(\alpha + \beta\mathrm{e}^{-iL\xi}\right)}{(i\xi)^3+\lambda} - \frac{ab(i\xi)^3\hat{\psi}(\xi)}{c\left((i\xi)^3+\lambda\right)}, \tag{69}$$

where $\alpha = \varphi(0) + \frac{ab}{c}\psi(0)$ and $\beta = -\varphi(L) - \frac{ab}{c}\psi(L)$. Replacing the above expression in the second equation in (68) it follows that

$$\frac{1}{c}\left[(i\xi)^3 + r(i\xi) + c\lambda - \frac{a^2b(i\xi)^6}{(i\xi)^3+\lambda}\right]\hat{\psi}(\xi) = -\frac{a(i\xi)^5\left(\alpha+\beta\mathrm{e}^{-iL\xi}\right)}{(i\xi)^3+\lambda}.$$

Thus,

$$\hat{\psi}(\xi) = -\frac{ac(i\xi)^5\left(\alpha+\beta\mathrm{e}^{-iL\xi}\right)}{(1-a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}. \tag{70}$$

Replacing (70) in (69), we obtain

$$\hat{\varphi}(\xi) = \frac{(i\xi)^2\left((i\xi)^3 + r(i\xi) + c\lambda\right)\left(\alpha+\beta\mathrm{e}^{-iL\xi}\right)}{(1-a^2b)(i\xi)^6 + r(i\xi)^4 + (c+1)\lambda(i\xi)^3 + r\lambda(i\xi) + c\lambda^2}.$$

Setting $\lambda = ip$, $p \in \mathbb{C}$, from the previous identities we can write $\hat{\psi}(\xi) = -i\,[acf(\xi)]$ and $\hat{\varphi}(\xi) = -ig(\xi)$, where

$$\begin{cases} f(\xi) = \dfrac{\xi^5\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}, \\ \\ g(\xi) = \dfrac{\xi^2\left(\xi^3 - r\xi - cp\right)\left(\alpha + \beta e^{-iL\xi}\right)}{P(\xi)}, \end{cases}$$

with

$$P(\xi) := (1 - a^2 b)\xi^6 - r\xi^4 - (c+1)p\xi^3 + rp\xi + cp^2.$$

Using Paley–Wiener theorem (see [15, Section 4, p. 161]) and the usual characterization of $H^2(\mathbb{R})$ functions by means of their Fourier transforms, we see that $(\mathcal{N})$ is equivalent to the existence of $p \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus (0,0)$, such that

(i) f and g are entire functions in $\mathbb{C}$,

(ii) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty$ and $\int_{\mathbb{R}} |g(\xi)|^2(1 + |\xi|^2)^2 d\xi < \infty$,

(iii) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \le c_1(1 + |\xi|)^k e^{L|Im\xi|}$ and $|g(\xi)| \le c_1(1 + |\xi|)^k e^{L|Im\xi|}$, for some positive constants c_1 and k.

Notice that if (i) holds true, then (ii) and (iii) are satisfied. Recall that f and g are entire functions if and only if the roots $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$ and ξ_5 of $P(\xi)$ are roots of $\xi^5\left(\alpha + \beta e^{-iL\xi}\right)$ and $\xi^2(\xi^3 - r\xi - cp)\left(\alpha + \beta e^{-iL\xi}\right)$.

Let us first assume that $\xi = 0$ is not a root of $P(\xi)$. Thus, it is sufficient to consider the case when $\alpha + \beta e^{-iL\xi}$ and $P(\xi)$ share the same roots. Observe that the roots of $\alpha + \beta e^{-iL\xi}$ are simple, unless $\alpha = \beta = 0$ (indeed, in this case $\varphi(0) + \frac{ab}{c}\psi(0) = 0$ and $\varphi(L) + \frac{ab}{c}\psi(L) = 0$ and using the system (67) we conclude that $(\varphi, \psi) = (0,0)$, which is a contradiction). Then, (i) holds provided that the roots of $P(\xi)$ are simple. Therefore, it follows that $(\mathcal{N})$ is equivalent to the existence of complex numbers p and ξ_0 and positive integers k, l, m, n and s, such that, if we set

$$\xi_1 = \xi_0 + \frac{2\pi}{L}k, \quad \xi_2 = \xi_1 + \frac{2\pi}{L}l, \quad \xi_3 = \xi_2 + \frac{2\pi}{L}m, \quad \xi_4 = \xi_3 + \frac{2\pi}{L}n \quad \text{and} \quad \xi_5 = \xi_4 + \frac{2\pi}{L}s, \tag{71}$$

$P(\xi)$ can be written as follows

$$P(\xi) = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_5).$$

In particular, we obtain the following relations:

$$\xi_0 + \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 = 0, \tag{72}$$

$$\begin{aligned} &\xi_0(\xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5) + \xi_1(\xi_2 + \xi_3 + \xi_4 + \xi_5) + \xi_2(\xi_3 + \xi_4 + \xi_5) \\ &\quad + \xi_3(\xi_4 + \xi_5) + \xi_4\xi_5 = -\frac{r}{1 - a^2 b} \end{aligned} \tag{73}$$

and

$$\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5 = \left(\frac{c}{1 - a^2 b}\right)p^2.$$

(71) and (72) imply that

$$\begin{aligned} &\xi_0 + \left(\xi_0 + \frac{2\pi}{L}k\right) + \left(\xi_0 + \frac{2\pi}{L}(k+l)\right) + \left(\xi_0 + \frac{2\pi}{L}(k+l+m)\right) + \left(\xi_0 + \frac{2\pi}{L}(k+l+m+n)\right) \\ &\quad + \left(\xi_0 + \frac{2\pi}{L}(k+l+m+n+s)\right) = 0. \end{aligned}$$

Straightforward computations lead to

$$\xi_0 = -\frac{\pi}{3L}(5k + 4l + 3m + 2n + s). \tag{74}$$

On the other hand, from (73), we obtain

$$\begin{aligned}
&\xi_0 \left(5\xi_0 + \frac{2\pi}{L}(5k + 4l + 3m + 2n + s)\right) + \left(\xi_0 + \frac{2\pi}{L}k\right)\left(4\xi_0 + \frac{2\pi}{L}(4k + 4l + 3m + 2n + s)\right)\\
&\quad + \left(\xi_0 + \frac{2\pi}{L}(k + l)\right)\left(3\xi_0 + \frac{2\pi}{L}(3k + 3l + 3m + 2n + s)\right)\\
&\quad + \left(\xi_0 + \frac{2\pi}{L}(k + l + m)\right)\left(2\xi_0 + \frac{2\pi}{L}(2k + 2l + 2m + 2n + s)\right)\\
&\quad + \left(\xi_0 + \frac{2\pi}{L}(k + l + m + n)\right)\left(\xi_0 + \frac{2\pi}{L}(k + l + m + n + s)\right) = -\frac{r}{1 - a^2 b}.
\end{aligned}$$

Thus, we have

$$15\xi_0^2 + \frac{2\pi}{L}(25k + 20 + 15m + 10n + 5s)\xi_0 + \frac{4\pi^2}{L^2}\eta = -\frac{r}{1 - a^2 b}, \tag{75}$$

where

$$\begin{aligned}
\eta = &\, k(10k + 10l + 9m + 7n + 4s) + l(6k + 6l + 6m + 5n + 3s)\\
&+ m(3k + 3l + 3m + 3n + 2s) + n(k + l + m + n + s).
\end{aligned}$$

Replacing (74) in (75), we obtain

$$\frac{3rL^2}{1 - a^2 b} = \pi^2\left(5(5k + 4l + 3m + 2n + s)^2 - 12\eta\right).$$

From the discussion above, we can conclude that

$$\begin{cases}
L = \pi\sqrt{\dfrac{(1 - a^2 b)\alpha(k, l, m, n, s)}{3r}},\\[2ex]
\xi_0 = -\dfrac{\pi}{3}(5k + 4l + 3m + 2n + s),\\[2ex]
p = \sqrt{\dfrac{(1 - a^2 b)\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5}{c}},
\end{cases} \tag{76}$$

where

$$\begin{aligned}
\alpha(k, l, m, n, s) :=&\, 5k^2 + 8l^2 + 9m^2 + 8n^2 + 5s^2 + 8kl + 6km + 4kn + 2ks + 12ml\\
&+ 8ln + 3ls + 12mn + 6ms + 8ns.
\end{aligned}$$

Now, we assume that $\xi_0 = 0$ is a root of $P(\xi)$. Then, it follows that $p = 0$ and

$$\begin{cases}
f(\xi) = \dfrac{\xi^5\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2 b)\xi^6 - r\xi^4} = \dfrac{\xi\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2 b)\xi^2 - r},\\[3ex]
g(\xi) = \dfrac{\xi^2\left(\xi^3 - r\xi\right)\left(\alpha + \beta e^{-iL\xi}\right)}{(1 - a^2 b)\xi^6 - r\xi^4} = \dfrac{\left(\xi^2 - r\right)\left(\alpha + \beta e^{-iL\xi}\right)}{\xi\left((1 - a^2 b)\xi^2 - r\right)}.
\end{cases}$$

In this case, $(\mathcal{N})$ holds if and only if f and g satisfy (i), (ii) and (iii). Thus, (i) holds provided that

$$\xi_0 = 0, \quad \xi_1 = \sqrt{\frac{r}{1 - a^2 b}} \quad \text{and} \quad \xi_2 = -\sqrt{\frac{r}{1 - a^2 b}}$$

are roots of $\alpha+\beta e^{-iL\xi}$. Therefore, we can write $\xi_1=\xi_0+\frac{2\pi}{L}k$, for $k\in\mathbb{Z}$. Consequently, it follows that

$$L=2\pi k\sqrt{\frac{1-a^2b}{r}}. \tag{77}$$

Finally, from (76) and (77), we deduce that $(\mathcal{N})$ holds if and only if $L\in\mathcal{F}_r$, where $\mathcal{F}_r$ is given by (15). This completes the proof of Lemma 3.11, Lemma 3.10 and, consequently, the proof of Proposition 3.9. □

The next result gives a positive answer for the control problem, and can be proved using the same ideas presented in Theorem 3.5, and thus, we will omit the proof.

Theorem 3.12. *Let $T>0$ and $L\in(0,\infty)\backslash\mathcal{F}_r$, where $\mathcal{F}_r$ is given by (15). Then, the system (42)–(63) is exactly controllable in time T.*

Remark 3.13. As in the previous subsection, the question here is whether system (42)–(78) is exactly controllable with another configuration of the boundary condition, for example,

$$\begin{cases} u_{xx}(0,t)=h_0(t), & u_x(L,t)=h_1(t), & u_{xx}(L,t)=h_2(t) & \text{in } (0,T),\\ v_{xx}(0,t)=0, & v_x(L,t)=g_1(t), & v_{xx}(L,t)=0 & \text{in } (0,T). \end{cases} \tag{78}$$

The answer for this question is positive if we prove that the following observability inequality

$$\begin{aligned}\|(\varphi^1,\psi^1)\|_{\mathcal{X}}^2\le C\Bigg\{&\left\|(-\Delta)^{\frac{1}{6}}\left(\varphi(0,\cdot)+\frac{ab}{c}\psi(0,\cdot)\right)\right\|_{L^2(0,T)}^2+\left\|\varphi_x(L,\cdot)+\frac{ab}{c}\psi_x(L,\cdot)\right\|_{L^2(0,T)}^2\\ &+\left\|(-\Delta)^{\frac{1}{6}}\left(\varphi(L,\cdot)+\frac{ab}{c}\psi(L,\cdot)\right)\right\|_{L^2(0,T)}+\left\|a\varphi_x(L,\cdot)+\frac{1}{c}\psi_x(L,\cdot)\right\|_{L^2(0,T)}^2\Bigg\},\end{aligned}$$

holds, for any (φ^1,ψ^1) in $\mathcal{X}$, where (φ,ψ) is solution of (31)–(34) with initial data (φ^1,ψ^1). Note that it can be proved using Proposition 2.4 together with the contradiction argument as in the proof of Proposition 3.9.

Thus, the exact controllability result is also true in this case.

Theorem 3.14. *Let $T>0$ and $L\in(0,\infty)\setminus\mathcal{F}_r$. Then, the system (42)–(78) is exactly controllable in time T.*

3.2. One control

In this subsection, we intend to prove the exact controllability of the system by using only one boundary control h_1 or g_1 and fixing $h_0=h_2=g_0=g_2=0$, namely

$$\begin{cases} u_{xx}(0,t)=0 & u_x(L,t)=h_1(t), & u_{xx}(L,t)=0, & \text{in } (0,T),\\ v_{xx}(0,t)=0, & v_x(L,t)=0, & v_{xx}(L,t)=0, & \text{in } (0,T). \end{cases} \tag{79}$$

or

$$\begin{cases} u_{xx}(0,t)=0 & u_x(L,t)=0 & u_{xx}(L,t)=0, & \text{in } (0,T),\\ v_{xx}(0,t)=0, & v_x(L,t)=g_1(t), & v_{xx}(L,t)=0, & \text{in } (0,T). \end{cases} \tag{80}$$

The result below give us an equivalent condition for the exact controllability and the proof is analogous to the proof of the Lemma 3.2.

Lemma 3.15. *For any* (u^1, v^1) *in* $\mathcal{X}$*, there exist one control* $\overrightarrow{h} = (0, h_1, 0)$ *and* $\overrightarrow{g} = (0,0,0)$ *(resp.* $\overrightarrow{h} = (0,0,0)$ *and* $\overrightarrow{g} = (0, g_1, 0)$*) in* $\mathcal{H}_T$*, such that the solution* (u, v) *of* (42)–(79) *(resp.* (42)–(80)*) satisfies* (44) *if and only if*

$$\int_0^L (u^1(x)\varphi^1(x) + v^1(x)\psi^1(x))\mathrm{d}x = \int_0^T h_1(t)\left[\varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t)\right]\mathrm{d}t$$

$$\left(\text{resp.} \quad \int_0^L (u^1(x)\varphi^1(x) + v^1(x)\psi^1(x))\mathrm{d}x = \int_0^T g_1(t)\left[a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t)\right]\mathrm{d}t\right)$$

for any (φ^1, ψ^1) *in* $\mathcal{X}$*, where* (φ, ψ) *is the solution of the backward system* (31)–(34).

Note that using the change of variable $x' = L - x$ and $t' = T - t$, the system (31)–(34) is equivalent to the following forward system

$$\begin{cases} \varphi_t + \varphi_{xxx} + \frac{ab}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ \psi_t + \frac{r}{c}\psi_x + a\varphi_{xxx} + \frac{1}{c}\psi_{xxx} = 0, & \text{in } (0,L)\times(0,T),\\ \varphi(x,0) = \varphi^0(x),\ \psi(x,0) = \psi^0(x), & \text{in } (0,L), \end{cases} \tag{81}$$

with boundary conditions

$$\begin{cases} \varphi_{xx}(x,t) + \frac{ab}{c}\psi_{xx}(x,t) = 0, & \text{in } \{0,L\}\times(0,T),\\ a\varphi_{xx}(x,t) + \frac{1}{c}\psi_{xx}(x,t) + \frac{r}{c}\psi(x,t) = 0, & \text{in } \{0,L\}\times(0,T),\\ \varphi_x(L,t) = \psi_x(L,t) = 0, & \text{in } (0,T). \end{cases} \tag{82}$$

It is well known (according to the previous sections) that the observability inequality

$$\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \le C\left\|\varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)} \tag{83}$$

or

$$\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \le C\left\|a\varphi_x(0,\cdot) + \frac{1}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)} \tag{84}$$

plays a fundamental role for the study of the controllability. To prove (83) (resp. (84)), we use a direct approach based on the multiplier technique that gives us the observability inequality for small values of the length L and large time of control T.

Proposition 3.16. *Let us suppose that* $T > 0$ *and* $L > 0$ *satisfy*

$$1 > \frac{\beta C_T}{T}\left[L + \frac{r}{c}\right], \tag{85}$$

where C_T *is the constant in* (35) *and* β *is the constant given by the embedding* $H^{\frac{1}{3}}(0,T) \subset L^2(0,T)$*. Then, there exists a constant* $C(T,L) > 0$*, such that for any* (φ^0,ψ^0) *in* $\mathcal{X}$ *the observability inequality* (83) *(resp.* (84)*) holds, where* (φ,ψ) *is solution of* (81)*,* (82) *with initial data* (φ^0,ψ^0)*.*

Proof. We multiply the first equation in (81) by $(T-t)\varphi$, the second one by $\frac{b}{c}(T-t)\psi$ and integrate over $(0,T)\times(0,L)$, to give us:

$$
\begin{aligned}
\frac{T}{2}\int_0^L \left(\varphi_0^2(x)+\frac{b}{c}\psi_0^2(x)\right)\mathrm{d}x &= \frac{1}{2}\int_0^T\int_0^L \left(\varphi^2(x,t)+\frac{b}{c}\psi^2(x,t)\right)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T (T-t)\left[\varphi(L,t)\left(\varphi_{xx}(L,t)+\frac{ab}{c}\psi_{xx}(L,t)\right)\right]\mathrm{d}t \\
&- \int_0^T (T-t)\left[\varphi(0,t)\left(\varphi_{xx}(0,t)+\frac{ab}{c}\psi_{xx}(0,t)\right)\right]\mathrm{d}t \\
&+ \int_0^T (T-t)\left[\frac{b}{c}\psi(L,t)\left(a\varphi_{xx}(L,t)+\frac{\psi_{xx}(L,t)}{c}+\frac{r}{2c}\psi(L,t)\right)\right]\mathrm{d}t \\
&+ \int_0^T (T-t)\left[-\frac{b}{c}\psi(0,t)\left(a\varphi_{xx}(0,t)+\frac{\psi_{xx}(0,t)}{c}+\frac{r}{2c}\psi(0,t)\right)\right]\mathrm{d}t \\
&+ \frac{1}{2}\int_0^T (T-t)\left[\varphi_x^2(0,t)+\frac{2ab}{c}\psi_x(0,t)\varphi_x(0,t)+\frac{b}{c^2}\psi_x^2(0,t)\right]\mathrm{d}t.
\end{aligned}
$$

From the boundary conditions (82), we have that

$$
\begin{aligned}
\|(\varphi^0,\psi^0)\|_{\mathcal{X}}^2 &\leq \frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{br}{c^2T}\|\psi(0,\cdot)\|^2_{L^2(0,T)}-\frac{br}{c^2}\int_0^T\frac{T-t}{T}\psi(L,t)^2\mathrm{d}t \\
&+\int_0^T\left[\varphi_x^2(0,t)+\frac{2ab}{c}\psi_x(0,t)\varphi_x(0,t)+\frac{b}{c^2}\psi_x^2(0,t)\right]\mathrm{d}t, \\
&\leq \frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{\beta br}{c^2T}\|\psi(0,\cdot)\|^2_{H^{\frac{1}{3}}(0,T)}+\frac{1}{a^2b}\left\|\varphi_x(0,\cdot)+\frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}, \\
&\left(\text{resp. } \|(\varphi^0,\psi^0)\|_{\mathcal{X}}^2\leq \frac{1}{T}\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})}+\frac{\beta br}{c^2T}\|\psi(0,\cdot)\|^2_{H^{\frac{1}{3}}(0,T)}\right. \\
&\left.+\frac{1}{a^2}\left\|a\varphi_x(0,\cdot)+\frac{1}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}\right)
\end{aligned}
$$

where β is the constant given by the compact embedding $H^{\frac{1}{3}}(0,T)\subset L^2(0,T)$. On the other hand, note that $L^\infty(0,L)\subset L^2(0,L)$; thus,

$$
\|\varphi(\cdot,t)\|^2_{L^2(0,L)}\leq L\|\varphi(\cdot,t)\|^2_{L^\infty(0,L)}, \quad \text{and} \quad \|\psi(\cdot,t)\|^2_{L^2(0,L)}\leq L\|\psi(\cdot,t)\|^2_{L^\infty(0,L)}, \tag{86}
$$

Hence,

$$
\begin{aligned}
\|(\varphi,\psi)\|^2_{L^2(0,T;\mathcal{X})} &= \int_0^T\left\{\|\varphi(\cdot,t)\|^2_{L^2(0,L)}+\frac{b}{c}\|\psi(\cdot,t)\|^2_{L^2(0,L)}\right\}\mathrm{d}t \\
&\leq L\int_0^T\left\{\|\varphi(\cdot,t)\|^2_{L^\infty(0,L)}+\frac{b}{c}\|\psi(\cdot,t)\|^2_{L^\infty(0,L)}\right\}\mathrm{d}t
\end{aligned}
$$

$$\leq L\beta\|\varphi\|^2_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))} + \frac{bL\beta}{c}\|\psi\|^2_{H^{\frac{1}{3}}(0,T;L^\infty(0,L))}.$$

Thanks to Proposition 2.4, we obtain

$$\begin{aligned}\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \leq& \frac{L\beta C_T}{T}\|\varphi^0\|^2_{L^2(0,L)} + \frac{bL\beta C_T}{cT}\|\psi^0\|^2_{L^2(0,L)} + \frac{\beta C_T br}{c^2T}\|\psi^0\|^2_{L^2(0,L)} \\ &+ \frac{1}{a^2b}\left\|\varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)} \\ \leq& \frac{L\beta C_T}{T}\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} + \frac{\beta C_T r}{cT}\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} + \frac{1}{a^2b}\left\|\varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}.\end{aligned}$$

Finally, it follows that

$$\|(\varphi^0,\psi^0)\|^2_{\mathcal{X}} \leq K\left\|\varphi_x(0,\cdot) + \frac{ab}{c}\psi_x(0,\cdot)\right\|^2_{L^2(0,T)}$$

under the condition

$$K = \frac{1}{a^2b}\left(1 - \frac{\beta C_T}{T}\left[L + \frac{r}{c}\right]\right)^{-1} > 0. \tag{87}$$

□

From the observability inequality (83), the following result holds.

Theorem 3.17. *Let $T > 0$ and $L > 0$ satisfying (85). Then, the system (42)–(79) (resp. (42)–(80)) is exactly controllable in time T.*

Proof. Consider the map

$$\begin{aligned}\Gamma : L^2(0,L)\times L^2(0,L) &\longrightarrow L^2(0,L)\times L^2(0,L)\\ (\varphi^1(\cdot),\psi^1(\cdot)) &\longmapsto \Gamma(\varphi^1(\cdot),\psi^1(\cdot)) = (u(\cdot,T),v(\cdot,T))\end{aligned}$$

where (u,v) is the solution of (42)–(63), with

$$\begin{cases} h_1(t) = \varphi_x(L,t) + \frac{ab}{c}\psi_x(L,t),\\ g_1(t) = a\varphi_x(L,t) + \frac{1}{c}\psi_x(L,t),\end{cases}$$

and (φ,ψ) is the solution of the system (31)–(34) with initial data (φ^1,ψ^1). By (83) (resp. (84)) and the Lax–Milgram theorem, the proof is achieved. □

4. The nonlinear control system

We are now in a position to prove our main result considering several configurations of the control in the boundary conditions. Let $T > 0$, from Theorems 3.5, 3.7, 3.12, 3.14 and 3.17, we can define the bounded linear operators

$$\Lambda_i : \mathcal{X}\times\mathcal{X} \longrightarrow \mathcal{H}_T\times\mathcal{H}_T \qquad (i = 1,2,3,4,5,6),$$

such that, for any $(u^0,v^0)\in\mathcal{X}$ and $(u^1,v^1)\in\mathcal{X}$,

$$\Lambda_i\left(\begin{pmatrix}u^0\\v^0\end{pmatrix},\begin{pmatrix}u^1\\v^1\end{pmatrix}\right) := \begin{pmatrix}\vec{h}_i\\ \vec{g}_i\end{pmatrix},$$

where $\vec{h}_i$ and $\vec{g}_i$ were defined in the Introduction.

Proof of Theorem 1.1. We treat the nonlinear problem (1), (2) using a classical fixed-point argument.

According to Remark 2.8, the solution can be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = W_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + W_{bdr}(t)\begin{pmatrix} \vec{h}_i \\ \vec{g}_i \end{pmatrix}$$
$$- \int_0^t W_0(t-\tau)\begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c}v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix} \mathrm{d}\tau,$$

for $i = 1,2,3,4,5,6$, where $\{W_0(t)\}_{t\geq 0}$ and $\{W_{bdr}(t)\}_{t\geq 0}$ are the operators defined in Proposition 2.1. We only analyze the case $i=1$, since the other cases are analogous we will omit them.

For $u, v \in \mathcal{Z}_T$, let us define

$$\begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} := \int_0^{\mathrm{T}} W_0(T-\tau)\begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_2 b}{c}(uv)_x(\tau) \end{pmatrix} \mathrm{d}\tau$$

and consider the map

$$\Gamma\begin{pmatrix} u \\ v \end{pmatrix} = W_0(t)\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} + W_{bdr}(x)\Lambda_1\left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right)$$
$$- \int_0^t W_0(t-\tau)\begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c}v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix} \mathrm{d}\tau.$$

If we choose

$$\begin{pmatrix} \vec{h}_1 \\ \vec{g}_1 \end{pmatrix} = \Lambda_1\left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right), \tag{88}$$

from Theorem 3.12, we get

$$\Gamma\begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$$

and

$$\Gamma\begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=T} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} - \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} = \begin{pmatrix} u^1 \\ v^1 \end{pmatrix}.$$

Now we prove that the map Γ is a contraction in an appropriate metric space, then its fixed point (u,v) is the solution of (1), (2) with $\vec{h}_1$ and $\vec{g}_1$ defined by (88), satisfying (3). In order to prove the existence of the fixed point, we apply the Banach fixed-point theorem to the restriction of Γ on the closed ball

$$B_r = \left\{(u,v) \in \mathcal{Z}_T : \|(u,v)\|_{\mathcal{Z}_T} \leq r\right\},$$

for some $r > 0$.

(i) Γ maps B_r into itself.

Using Proposition 2.3 there exists a constant $C_1 > 0$, such that

$$\left\|\Gamma\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\mathcal{Z}_T} \leq C_1\left\{\left\|\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}\right\|_{\mathcal{X}} + \left\|\Lambda_1\left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix}\right)\right\|_{\mathcal{H}_T}\right\}$$
$$+ C_1\left\{\int_0^t \left\|\begin{pmatrix} a_1(vv_x)(\tau) + a_2(uv)_x(\tau) \\ \frac{r}{c}v_x(\tau) + \frac{a_2 b}{c}(uu_x)(\tau) + \frac{a_1 b}{c}(uv)_x(\tau) \end{pmatrix}\right\|_{\mathcal{X}} \mathrm{d}\tau\right\}.$$

Moreover, since

$$\left\| \Lambda_1 \left(\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} \right) \right\|_{\mathcal{H}_T} \le C_2 \left\{ \left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{\mathcal{X}} + \left\| \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} \right\|_{\mathcal{X}} + \left\| \begin{pmatrix} v \\ \nu(T,u,v) \end{pmatrix} \right\|_{\mathcal{X}} \right\},$$

applying Lemma 2.6, we can deduce that

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\mathcal{Z}_T} \le C_3 \delta + C_4 (r+1) r,$$

where C_4 is a constant depending only T. Thus, choosing r and δ such that

$$r = 2C_3\delta$$

and

$$2C_3C_4\delta + C_4 \le \frac{1}{2},$$

the operator Γ maps B_r into itself for any $(u,v) \in \mathcal{Z}_T$.

(ii) Γ is contractive.

Proceeding as the proof of Theorem 2.7, we obtain

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} - \Gamma \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T} \le C_5 (r+1) r \left\| \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T},$$

for any $(u,v), (\widetilde{u},\widetilde{v}) \in B_r$ and a constant C_5 depending only on T. Thus, taking $\delta > 0$, such that

$$\gamma = 2C_3C_5\delta + C_5 < 1,$$

we obtain

$$\left\| \Gamma \begin{pmatrix} u \\ v \end{pmatrix} - \Gamma \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T} \le \gamma \left\| \begin{pmatrix} u - \widetilde{u} \\ v - \widetilde{v} \end{pmatrix} \right\|_{\mathcal{Z}_T}.$$

Therefore, the map Γ is a contraction. Thus, from (i), (ii) and the Banach fixed-point theorem, Γ has a fixed point in B_r and its fixed point is the desired solution. The proof of Theorem 1.1 is, thus, complete. □

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Appendix A. Proof of lemma 2.2

In this appendix, we prove the Lemma 2.2 used in the proof of Proposition 2.1. Without loss of generality we can consider the following linear nonhomogeneous boundary value problem,

$$\begin{cases} w_t + w_{xxx} = 0, \quad w(x,0) = 0 & x \in (0,L), t > 0, \\ w_{xx}(0,t) = h_1(t), \ w_x(L,t) = h_2(t), \ w_{xx}(L,t) = h_3(t) & t > 0. \end{cases} \tag{89}$$

Proof of Lemma 2.2. Applying the Laplace transform with respect to t, (89) is converted to

$$\begin{cases} s\hat{w} + \hat{w}_{xxx} = 0, \\ \hat{w}_{xx}(0,s) = \hat{h}_1(s),\ \hat{w}_x(L,s) = \hat{h}_2(s),\ \hat{w}_{xx}(L,s) = \hat{h}_3(s), \end{cases} \tag{90}$$

where

$$\hat{w}(x,s) = \int_0^{+\infty} \mathrm{e}^{-st} w(x,t)\mathrm{d}t$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} \mathrm{e}^{-st} h(t)\mathrm{d}t, \quad j = 1,2,3.$$

The solution $\hat{w}(x,s)$ can be written in the form

$$\hat{w}(x,s) = \sum_{j=1}^{3} c_j(s)\mathrm{e}^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1,2,3$, are the solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and $c_j(s)$, $j = 1,2,3$, solve the linear system

$$\underbrace{\begin{pmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1\mathrm{e}^{\lambda_1 L} & \lambda_2\mathrm{e}^{\lambda_2 L} & \lambda_3\mathrm{e}^{\lambda_3 L} \\ \lambda_1^2\mathrm{e}^{\lambda_1 L} & \lambda_2^2\mathrm{e}^{\lambda_2 L} & \lambda_3^2\mathrm{e}^{\lambda_3 L} \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix}}_{\vec{h}}. \tag{91}$$

Let $\Delta(s)$ be the determinant of the coefficient matriz A and $\Delta_j(s)$, $j = 1,2,3$, the determinant of the matrix A with the jth column replaced by $\vec{h}$. By Cramer's rule,

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1,2,3,$$

provided that $\Delta(s) \neq 0$.
Claim: $\Delta(s) \neq 0$, for any $Re(s) \geq 0$.

Indeed, if otherwise, suppose $\Delta(s) = 0$, for some s with $Re(s) \geq 0$. Then, there exists a nontrivial $f \in H^3(0,L)$ satisfying

$$\begin{cases} sf(x) + f'''(x) = 0, & x \in (0,L), \\ f''(0) = 0, \quad f'(L) = 0, & f''(L) = 0. \end{cases} \tag{92}$$

Consider now the conjugate of (92), that is, the following system

$$\begin{cases} s\overline{f(x)} + \overline{f'''(x)} = 0, & x \in (0,L), \\ \overline{f''(0)} = 0, \quad \overline{f'(L)} = 0, & \overline{f''(L)} = 0. \end{cases} \tag{93}$$

Multiplying both sides of (92) by $\overline{f}$ and integrating over $(0,L)$, we get

$$\int_0^L sf\overline{f}\mathrm{d}x + \int_0^L f'''\overline{f}\mathrm{d}x = 0. \tag{94}$$

Then, if we multiply both sides of (93) by f and integrate over $(0, L)$, it follows that

$$\int_0^L s\overline{f} f \mathrm{d}x + \int_0^L \overline{f'''} f \mathrm{d}x = 0. \tag{95}$$

Integrating by parts (94) and (95) and adding the two resulting identities together yields that

$$2Re(s)\int_0^L |f(x)|^2 \mathrm{d}x = -|f'(0)|^2.$$

Consequently, we must have $Re(s) < 0$, as $||f||_{L^2(0,L)} \neq 0$ by the assumption. This is a contradiction. Thus, we conclude that $\Delta(s) \neq 0$, for any $Re(s) \geq 0$.

Note that the solution $w(x,t)$ for (89) can be written in the form

$$w(x,t) = \sum_{m=1}^{3} w_m(x,t), \tag{96}$$

where $w_m(x,t)$ solves (89) with $h_j \equiv 0$ when $j \neq m$, $j, m = 1, 2, 3$. Using the inverse Laplace transform yields

$$w(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \mathrm{e}^{\mathrm{st}} \hat{w}(x,s)\mathrm{d}s = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\Delta_j(s)}{\Delta(s)} \mathrm{e}^{\lambda_j(s)x}\mathrm{d}s,$$

for $r > 0$. Combining this formula and (96) we can write the values of w_m as follows, for $m = 1, 2, 3$,

$$w_m(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\Delta_{j,m}(s)}{\Delta(s)} \mathrm{e}^{\lambda_j(s)x}\hat{h}_m(s)\mathrm{d}s \equiv [W_{m,j}(t)h_m](x).$$

In the last two formulas, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r, we can take $r = 0$ in these formulas. Moreover,

$$w_{j,m}(x,t) = w^+_{j,m}(x,t) + w^-_{j.m}(x,t)$$

where

$$w^+_{j,m}(x,t) = \frac{1}{2\pi i}\int_0^{+i\infty} \mathrm{e}^{\mathrm{st}} \frac{\Delta_{j,m}(s)}{\Delta(s)} \hat{h}_m(s)\mathrm{e}^{\lambda_j(s)x}\mathrm{d}s$$

and

$$w^-_{j,m}(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{0} \mathrm{e}^{\mathrm{st}} \frac{\Delta_{j,m}(s)}{\Delta(s)} \hat{h}_m(s)\mathrm{e}^{\lambda_j(s)x}\mathrm{d}s,$$

for $j, m = 1, 2, 3$. Here, $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m$, $k, m = 1, 2, 3$. Making the substitution $s = i\rho^3$ with $\rho \geq 0$ in the characteristic equation

$$s + \lambda^3 = 0,$$

the three roots are given in terms of ρ by

$$\lambda_1(\rho) = i\rho, \quad \lambda_2(\rho) = -i\rho\left(\frac{1 + i\sqrt{3}}{2}\right), \quad \lambda_3(\rho) = -i\rho\left(\frac{1 - i\sqrt{3}}{2}\right), \tag{97}$$

thus $w^+_{j,m}$ has the following form

$$w^+_{j,m}(x,t) = \frac{1}{2\pi i} \int_0^{+\infty} \mathrm{e}^{i\rho^3 t} \frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)} \hat{h}^+_m(\rho) \mathrm{e}^{\lambda^+_j(\rho)x} 3i\rho^2 \mathrm{d}\rho$$

and

$$w^-_{j,m}(x,t) = \overline{w^+_{j,m}(x,t)},$$

where $\hat{h}^+_m(\rho) = \hat{h}_m(i\rho^3)$, $\Delta^+(\rho) = \Delta(i\rho^3)$, $\Delta^+_{j,m}(\rho) = \Delta_{j,m}(i\rho^3)$ and $\lambda^+_j(\rho) = \lambda_j(i\rho^3)$.

Therefore, we have that the solution of the IBVP (89) has the representation in the form (22)–(25) as required. Thus, the proof is finished. □

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