GENERAL BOUNDARY VALUE PROBLEMS OF THE KORTEWEG-DE VRIES EQUATION ON A BOUNDED DOMAIN

Roberto A. Capistrano-Filho

Departamento de Matemática
Universidade Federal de Pernambuco
Recife, Pernambuco 50740-545, Brazil

Shuming Sun

Department of Mathematics
Virginia Tech
Blacksburg, VA 24061, USA

Bing-Yu Zhang*

Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221-0025, USA
and
Yangtze Center of Mathematics
Sichuan University
Chengdu 610064, China

The paper is dedicated to Jiongmin Yong for his 60th birthday.

ABSTRACT. In this paper we consider the initial boundary value problem of the Korteweg-de Vries equation posed on a finite interval

$$u_t + u_x + u_{xxx} + uu_x = 0, \qquad u(x,0) = \phi(x), \qquad 0 < x < L, \ t > 0 \quad (0.1)$$

subject to the nonhomogeneous boundary conditions,

$$B_1 u = h_1(t), \qquad B_2 u = h_2(t), \qquad B_3 u = h_3(t) \qquad t > 0 \qquad (0.2)$$

where

$$B_i u = \sum_{j=0}^{2} \left(a_{ij} \partial_x^j u(0,t) + b_{ij} \partial_x^j u(L,t) \right), \qquad i = 1,2,3,$$

and a_{ij}, b_{ij} ($j = 0,1,2$ and $i = 1,2,3$) are real constants. Under some general assumptions imposed on the coefficients a_{ij}, b_{ij}, the IBVP (0.1)-(0.2) is shown to be locally well-posed in the space $H^s(0,L)$ for any $s \geq 0$ with $\phi \in H^s(0,L)$ and boundary values h_j, $j = 1,2,3$, belonging to some appropriate spaces with optimal regularity.

1. **Introduction.** In this paper we consider the initial-boundary value problem (IBVP) of the Korteweg-de Vries (KdV) equation posed on a finite interval $(0,L)$

$$u_t + u_x + u_{xxx} + uu_x = 0, \qquad u(x,0) = \phi(x), \qquad 0 < x < L, \ t > 0 \qquad (1.1)$$

with general non-homogeneous boundary conditions posed on the two ends of the interval $(0, L)$,

$$B_1 u = h_1(t), \qquad B_2 u = h_2(t), \qquad B_3 u = h_3(t) \qquad t > 0, \tag{1.2}$$

where

$$B_i u = \sum_{j=0}^{2} \left(a_{ij} \partial_x^j u(0, t) + b_{ij} \partial_x^j u(L, t) \right), \qquad i = 1, 2, 3,$$

and a_{ij}, b_{ij}, $j = 0, 1, 2$, $i = 1, 2, 3$, are real constants.

We are mainly concerned with the following question:

Under what assumptions on the coefficients a_{ij}, b_{ij} in (1.2) is the IBVP (1.1)-(1.2) well-posed in the classical Sobolev space $H^s(0, L)$?

As early as in 1979, Bubnov [12] studied the following IBVP of the KdV equation on the finite interval $(0, 1)$:

$$\begin{cases} u_t + u u_x + u_{xxx} = f, \quad u(x, 0) = 0, \quad x \in (0, 1), \ t \in (0, T), \\ \alpha_1 u_{xx}(0, t) + \alpha_2 u_x(0, t) + \alpha_3 u(0, t) = 0, \\ \beta_1 u_{xx}(1, t) + \beta_2 u_x(1, t) + \beta_3 u(1, t) = 0, \\ \chi_1 u_x(1, t) + \chi_2 u(1, t) = 0 \end{cases} \tag{1.3}$$

and obtained the result as described below.

Theorem A [12]. *Assume that*

$$\begin{cases} if \ \alpha_1 \beta_1 \chi_1 \neq 0, \ then \ F_1 > 0, \ F_2 > 0, \\ if \ \beta_1 \neq 0, \ \chi_1 \neq 0, \ \alpha_1 = 0, \ then \ \alpha_2 = 0, \ F_2 > 0, \ \alpha_3 \neq 0, \\ if \ \beta_1 = 0, \ \chi_1 \neq 0, \ \alpha_1 \neq 0, \ then \ F_1 > 0, \ F_3 \neq 0, \\ if \ \alpha_1 = \beta_1 = 0, \ \chi_1 \neq 0, \ then \ F_3 \neq 0, \ \alpha_2 = 0, \ \alpha_3 \neq 0, \\ if \ \beta_1 = 0, \ \alpha_1 \neq 0, \ \chi_1 = 0, \ then \ F_1 > 0, \ F_3 \neq 0, \\ if \ \alpha_1 = \beta_1 = \chi_1 = 0, \ then \ \alpha_2 = 0, \ \alpha_3 \neq 0, \ F_3 \neq 0, \end{cases} \tag{1.4}$$

where

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, \ \ F_2 = \frac{\beta_2 \chi_2}{\beta_1 \chi_1} - \frac{\beta_3}{\beta_1} - \frac{\chi_2^2}{2\chi_1^2}, \ \ F_3 = \beta_2 \chi_2 - \beta_1 \chi_1.$$

For any given

$$f \in H^1_{loc}(0, \infty; L^2(0, 1)) \ \ with \ \ f(x, 0) = 0,$$

there exists a $T > 0$ such that (1.3) admits a unique solution

$$u \in L^2(0, T; H^3(0, 1)) \ with \ u_t \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)).$$

The main tool used by Bubnov [12] to prove this theorem is the following Kato type smoothing property for the solution u of the linear system associated to the IBVP (1.3),

$$\begin{cases} u_t + u_{xxx} = f, \quad u(x, 0) = 0, \quad x \in (0, 1), \ t \in (0, T), \\ \alpha_1 u_{xx}(0, t) + \alpha_2 u_x(0, t) + \alpha_3 u(0, t) = 0, \\ \beta_1 u_{xx}(1, t) + \beta_2 u_x(1, t) + \beta_3 u(1, t) = 0, \\ \chi_1 u_x(1, t) + \chi_2 u(1, t) = 0. \end{cases} \tag{1.5}$$

Under the assumptions (1.4),

$$f \in L^2(0, T; L^2(0, 1)) \implies u \in L^2(0, T; H^1(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \tag{1.6}$$

and
$$\|u\|_{L^2(0,T;H^1(0,1))} + \|u\|_{L^\infty(0,T;L^2(0,1))} \le C\|f\|_{L^2(0,T;L^2(0,1))},$$
where $C > 0$ is a constant independent of f.

In the past thirty years since the work of Bubnov [12], various boundary-value problems of the KdV equation have been studied. In particular, the following two special classes of IBVPs of the KdV equation on the finite interval $(0,L)$,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \ u(x,0) = \phi(x), & x \in (0,L), \ t > 0, \\ u(0,t) = h_1(t), & u(L,t) = h_2(t), & u_x(L,t) = h_3(t) \end{cases} \tag{1.7}$$

and

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \ u(x,0) = \phi(x), & x \in (0,L), \ t > 0, \\ u(0,t) = h_1(t), & u(L,t) = h_2(t), & u_{xx}(L,t) = h_3(t), \end{cases} \tag{1.8}$$

as well as the IBVPs of the KdV equation posed on a quarter plane have been intensively studied in the past twenty years (cf. [5, 8, 14, 15, 16, 18, 19, 21, 31, 32, 33] and the references therein) following the rapid advances of the study of the pure initial value problem of the KdV equation posed on the whole line $\mathbb{R}$ or on the periodic domain $\mathbb{T}$ (see e.g. [1, 2, 9, 10, 11, 17, 18, 19, 20, 25, 26, 27, 28, 29, 30]).

The nonhomogeneous IBVP (1.7) was first shown by Faminskii in [18, 19] to be well-posed in the spaces $L^2(0,L)$ and $H^3(0,L)$ with boundary data

$$\vec{h} = (h_1, h_2, h_3)$$

belonging to the space

$$W^{\frac{1}{3},1}(0,T) \cap L^{6+\epsilon}(0,T) \cap H^{\frac{1}{6}}(0,T) \times W^{\frac{5}{6}+\epsilon,1}(0,T) \cap H^{\frac{1}{3}}(0,T) \times L^2(0,T),$$

and additionally

$$h_1' \in W^{\frac{1}{3},1}(0,T) \cap L^{6+\epsilon}(0,T) \cap H^{\frac{1}{6}}(0,T),$$

$$h_2' \in W^{\frac{5}{6}+\epsilon,1}(0,T) \cap H^{\frac{1}{3}}(0,T)$$

and

$$h_3' \in L^2(0,T),$$

respectively. Bona *et al.*, in [5], showed that IBVP (1.7) is well-posed in the space $H^s(0,L)$ for any $s \ge 0$ with boundary data

$$\vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

possessing optimal boundary regularity. Later on, in [21], Holmer showed that the IBVP (1.7) is locally well-posed in the space $H^s(0,L)$, for any $-\frac{3}{4} < s < \frac{1}{2}$, and Bona *et al.*, in [8], showed that the IBVP (1.7) is locally well-posed $H^s(0,L)$ for any $s > -1$.

The study of IBVP (1.8) began with the work of Colin and Ghidalia in late 1990's [14, 15, 16] and is now known to be well-posed in the space $H^s(0,L)$ for $s > -1$ with boundary data

$$\vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T)$$

possessing optimal boundary regularity [22, 32, 33].

As for the general IBVP (1.1)-(1.2), Kramer and Zhang, in [31], studied the following non-homogeneous boundary value problem,

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x \in (0,1), \ t \in (0,T), \\ \alpha_1 u_{xx}(0,t) + \alpha_2 u_x(0,t) + \alpha_3 u(0,t) = h_1(t), \\ \beta_1 u_{xx}(1,t) + \beta_2 u_x(1,t) + \beta_3 u(1,t) = h_2(t), \\ \chi_1 u_x(1,t) + \chi_2 u(1,t) = h_3(t). \end{cases} \tag{1.9}$$

They showed that the IBVP (1.9) is locally well-posed in the space $H^s(0,1)$, for any $s \geq 0$, under the assumption (1.4).

Theorem B [31]. *Let $s \geq 0$ with*

$$s \neq \frac{2j-1}{2}, j = 1,2,3...,$$

$T > 0$ be given and assume (1.4) holds. For any $r > 0$, there exists a $T^* \in (0,T]$ such that for any $s-$compatible [1] *$\phi \in H^s(0,1)$, $h_j \in H^{\frac{s+1}{3}}(0,T), j = 1,2,3$ with*

$$\|\phi\|_{H^s(0,1)} + \|h_1\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|_{H^{\frac{s+1}{3}}(0,T)} \leq r,$$

the IBVP (1.9) admits a unique solution

$$u \in C([0,T^*];H^s(0,1)) \cap L^2(0,T^*;H^{s+1}(0,1)).$$

Moreover, the solution u depends continuously on its initial data ϕ and the boundary values $h_j, j = 1,2,3$, in the respective spaces.

In this paper we continue to study the general IBVP (1.1)-(1.2) for its well-posedness in the space $H^s(0,L)$ and attempt to provide a (partial) answer for the following question asked earlier:

Under what assumptions on the coefficients a_{ij}, b_{ij} in (1.2) is the IBVP (1.1)-(1.2) well-posed in the classical Sobolev space $H^s(0,L)$?

We propose the following hypotheses on those coefficients a_{ij}, b_{ij}, $j = 0,1,2,3$, $i = 1,2,3$:

(A1) $a_{12} = a_{11} = 0$, $a_{10} \neq 0$, $b_{12} = b_{11} = b_{10} = 0$;
(A2) $a_{12} \neq 0$, $b_{12} = 0$;
(B1) $a_{22} = a_{21} = a_{20} = 0$, $b_{20} \neq 0, b_{22} = b_{21} = 0$;
(B2) $b_{22} \neq 0$, $a_{22} = 0$;
 (C) $a_{32} = a_{31} = 0$, $b_{31} \neq 0$, $b_{32} = 0$.

In addition, for any $s \geq 0$,

$$H_0^s(0,T] := \{h(t) \in H^s(0,T) : h^{(j)}(0) = 0\},$$

for $j = 0,1,...,[s]$[2]. Let us consider

$$\begin{cases} \mathcal{H}_1^s(0,T) := H_0^{\frac{s+1}{3}}(0,T] \times H_0^{\frac{s+1}{3}}(0,T] \times H_0^{\frac{s}{3}}(0,T], \\ \mathcal{H}_2^s(0,T) := H_0^{\frac{s+1}{3}}(0,T] \times H_0^{\frac{s}{3}}(0,T] \times H_0^{\frac{s-1}{3}}(0,T], \\ \mathcal{H}_3^s(0,T) := H_0^{\frac{s-1}{3}}(0,T] \times H_0^{\frac{s+1}{3}}(0,T] \times H_0^{\frac{s}{3}}(0,T], \\ \mathcal{H}_4^s(0,T) := H_0^{\frac{s-1}{3}}(0,T] \times H_0^{\frac{s-1}{3}}(0,T] \times H_0^{\frac{s}{3}}(0,T] \end{cases} \tag{1.10}$$

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[1]See [31] for exact definition, in this case, of $s-$compatibility.
[2] For any real number s, $[s]$ stands for its integer part.

and

$$\begin{cases} \mathcal{W}_1^s(0,T) := H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T), \\ \mathcal{W}_2^s(0,T) := H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T), \\ \mathcal{W}_3^s(0,T) := H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T), \\ \mathcal{W}_4^s(0,T) := H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T). \end{cases} \qquad (1.11)$$

We have the following well-posedness results for the IBVP (1.1)-(1.2).

Theorem 1.1. *Let $s \geq 0$ with $s \neq \frac{2j-1}{2}, j = 1, 2, 3..., $ and $T > 0$ be given. If one of the assumptions below are satisfied,*

(i) (A1), (B1) and (C) hold,
(ii) (A1), (B2) and (C) hold,
(iii) (A2), (B1) and (C) hold,
(iv) (A2), (B2) and (C) hold,

then, for any $r > 0$, there exists a $T^* \in (0,T]$ such that for any

$$(\phi, \vec{h}) \in H_0^s(0,L) \times \mathcal{H}_1^s(0,T)$$

satisfying

$$\|(\phi, \vec{h})\|_{L^2(0,L) \times \mathcal{H}_1^0(0,T)} \leq r$$

the IBVP (1.1)-(1.2) admits a solution

$$u \in C([0,T^*]; H^s(0,L)) \cap L^2(0,T^*; H^{s+1}(0,L))$$

possessing the hidden regularities (the sharp Kato smoothing properties)

$$\partial_x^l u \in L^\infty(0,L; H^{\frac{s+1-l}{3}}(0,T^*)) \text{ for } l = 0, 1, 2.$$

Moreover, the corresponding solution map is analytically continuous.

The following remarks are now in order.

(i) *The temporal regularity conditions imposed on the boundary values $\vec{h}$ are optimal (cf. [3, 6, 7]).*
(ii) *The assumptions imposed on the boundary conditions in Theorem 1.1 can be reformulated as follows:*
  (i) $((A1),(B1),(C)) \Leftrightarrow \mathcal{B}_1 v = \vec{h}$,
  (ii) $((A1),(C),(B2)) \Leftrightarrow \mathcal{B}_2 v = \vec{h}$,
  (iii) $((A2),(B1),(C)) \Leftrightarrow \mathcal{B}_3 v = \vec{h}$,
  (iv) $((A2),(C),(B2)) \Leftrightarrow \mathcal{B}_4 v = \vec{h}$.
  Here,

$$\mathcal{B}_1 v := (v(0,t),\ v(L,t),\ v_x(L,t)),$$

$$\mathcal{B}_2 v := (v(0,t),\ v_x(L,t)) + b_{30}v(L,t),\ v_{xx}(L,t) + a_{21}v_x(0,t) + b_{20}v(L,t)),$$

$$\mathcal{B}_3 v := (v_{xx}(0,t) + a_{10}v(0,t) + a_{11}v_x(0,t),\ v(L,t),\ v_x(L,t) + a_{30}v(0,t))$$

  and

$$\mathcal{B}_4 v := \left(v_{xx}(0,t) + \sum_{j=0}^{1} a_{1j}\partial_x^j v(0,t) + b_{10}v(L,t),\ v_x(L,t) + a_{30}v(0,t) + b_{30}v(L,t), \right.$$

$$\left. v_{xx}(L,t) + \sum_{j=0}^{1} a_{2j}\partial_x^j v(0,t) + b_{20}v(L,t) \right).$$

As a comparison, note that the assumptions of Theorem A are satisfied if and only if one of the following boundary conditions is imposed on the equation in (1.3):

(a) $u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0;$

(b)

$$u_{xx}(0,t) + au_x(0,t) + bu(0,t) = 0, \quad u_x(1,t) = 0, \quad u(1,t) = 0$$

with

$$a > b^2/2; \tag{1.12}$$

(c)

$$u(0,t) = 0, \quad u_{xx}(1,t) + au_x(1,t) + bu(1,t) = 0, \quad u_x(1,t) + cu(1,t) = 0,$$

with

$$ac > b - c^2/2; \tag{1.13}$$

(d) $u_{xx}(0,t) + a_1 u_x(0,t) + a_2 u(0,t) = 0, \quad u_{xx}(1,t) + b_1 u_x(1,t) + b_2 u(1,t) = 0,$

$$u_x(1,t) + cu(1,t) = 0,$$

with

$$a_2 > a_1^2/2, \quad b_1 c > b_2 - c^2/2. \tag{1.14}$$

Then, it follows from our results that the conditions (1.12), (1.13) and (1.14) for Theorem A can be removed completely.

(iii) In Theorem 1.1, we replace the $s-$compatibility of $(\phi, \vec{h})$ by assuming $(\phi, \vec{h}) \in H_0^s(0,L) \times H_0^{\frac{s+1}{3}}(0,T) \times H_0^{\frac{s+1}{3}}(0,T) \times H_0^{\frac{s}{3}}(0,T)$ for simplicity.

To prove our theorem, we rewrite the boundary operators $\mathcal{B}_k, \ k = 1,2,3,4$ as

$$\mathcal{B}_k = \mathcal{B}_{k,0} + \mathcal{B}_{k,1}$$

with

$$\mathcal{B}_{1,0}v := (v(0,t), v(L,t), v_x(L,t)), \quad \mathcal{B}_{2,0}v := (v(0,t), v_x(L,t), v_{xx}(L,t)),$$

$$\mathcal{B}_{3,0}v := (v_{xx}(0,t), v(L,t), v_x(L,t)), \quad \mathcal{B}_{4,0}v := (v_{xx}(0,t), v_x(L,t), v_{xx}(L,t))$$

and

$$\mathcal{B}_{1,1}v := (0, \ 0, \ 0),$$

$$\mathcal{B}_{2,1}v := (0, \ b_{30}v(L,t), \ a_{21}v_x(0,t) + b_{20}v(L,t)),$$

$$\mathcal{B}_{3,1}v := (a_{10}v(0,t) + a_{11}v_x(0,t), \ 0, \ a_{30}v(0,t)),$$

$$\mathcal{B}_{4,1}v := \left(\sum_{j=0}^{1} a_{1j}\partial_x^j v(0,t) + b_{10}v(L,t), \ a_{30}v(0,t) + b_{30}v(L,t) \ , \right.$$

$$\left. \sum_{j=0}^{1} a_{2j}\partial_x^j v(0,t) + b_{20}v(L,t) \right).$$

To prove our main result, we will first study the linear IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = f, \quad x \in (0,L), \quad t > 0 \\ u(x,0) = \phi(x), \\ \mathcal{B}_{k,0}u = \vec{h}, \end{cases} \tag{1.15}$$

for $k = 1, 2, 3, 4$ to establish all the linear estimates needed later for dealing with the nonlinear IBVP (1.1)-(1.2). Here $\delta_k = 0$ for $k = 1, 2, 3$ and $\delta_4 = 1$. Then we will consider the nonlinear map Γ defined by the following IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = -v_x - vv_x + \delta_k v, & x \in (0, L), \quad t > 0 \\ u(x, 0) = \phi(x), \\ \mathcal{B}_{k,0} u = \vec{h} - \mathcal{B}_{k,1} v \end{cases} \qquad (1.16)$$

for $k = 1, 2, 3, 4$ with

$$\Gamma(v) = u.$$

We will show that Γ is a contraction in an appropriate space whose fixed point will be the desired solution of the nonlinear IBVP (1.1)-(1.2). The key to show that Γ is a contraction in an appropriate space is the sharp Kato smoothing property of the solution of the IBVP (1.15) as described below, for example, for $s = 0$:

For given $\phi \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$ and $\vec{h} \in \mathcal{H}_k^0(0, T)$, the IBVP (1.15) admits a unique solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ with

$$\partial_x^l u \in L^\infty(0, L; H^{\frac{1-l}{3}}(0, T)) \text{ for } l = 0, 1, 2.$$

In order to demonstrate this smoothing properties for solutions of the IBVP (1.15), we need to study the following IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = 0, & x \in (0, L), \quad t > 0, \\ u(x, 0) = 0, \\ \mathcal{B}_{k,0} u = \vec{h} \end{cases} \qquad (1.17)$$

for $k = 1, 2, 3, 4$. The corresponding solution map

$$\vec{h} \to u$$

will be called the *boundary integral operator* denoted by $\mathcal{W}_{bdr}^{(k)}$. An explicit representation formula will be given for this boundary integral operator that will play an important role in showing the solution of the IBVP (1.17) possesses the smoothing properties. The needed smoothing properties for solutions of the IBVP (1.15) will then follow from the smoothing properties for solutions of the IBVP (1.17) and the well-known sharp Kato smoothing properties for solutions of the Cauchy problem

$$u_t + u_{xxx} + \delta_k u = 0, \quad u(x, 0) = \psi(x), \quad x, \ t \in \mathbb{R}.$$

The plan of the present paper is as follows.

— In Section 2 we will study the linear IBVP (1.15). The explicit representation formulas for the boundary integral operators $\mathcal{W}_{bdr}^{(k)}$, $k = 1, 2, 3, 4$, will be first presented. The various linear estimates for solutions of the IBVP (1.15) will be derived including the sharp Kato smoothing properties.

— Section 3 is devoted to the well-posedness of the nonlinear problem (1.1)-(1.2).

— Finally, in the Section 4, some concluding remarks will be provided together with a few open problems for further investigation.

2. **Linear problems.** This section is devoted to study the linear IBVP (1.15) which will be divided into two subsections. In subsection 2.1, we will present an explicit representation for the boundary integral operator $\mathcal{W}_{bdr}^{(k)}$ and then solution formulas for the solutions of the IBVP (1.15). Various linear estimates for solutions of the IBVP (1.15) will be derived in subsection 2.2.

2.1. **Boundary integral operators and their applications.** In this subsection, we first derive explicit representation formulas for the following four classes of non-homogeneous boundary-value problems

$$\begin{cases} v_t + v_{xxx} = 0, \quad v(x,0) = 0, & x \in (0,L), \ t \geq 0, \\ \mathcal{B}_{1,0}v = (h_{1,1}(t), \ h_{2,1}(t), \ h_{3,1}(t)), & t \geq 0, \end{cases} \tag{2.1}$$

$$\begin{cases} v_t + v_{xxx} = 0, \quad v(x,0) = 0, & x \in (0,L), \ t \geq 0, \\ \mathcal{B}_{2,0}v = (h_{1,2}(t), \ h_{2,2}(t), \ h_{3,2}(t)), & t \geq 0, \end{cases} \tag{2.2}$$

$$\begin{cases} v_t + v_{xxx} = 0, \quad v(x,0) = 0, & x \in (0,L), \ t \geq 0, \\ \mathcal{B}_{3,0}v = (h_{1,3}(t), \ h_{2,3}(t), \ h_{3,3}(t)), & t \geq 0 \end{cases} \tag{2.3}$$

and

$$\begin{cases} v_t + v_{xxx} + v = 0, \quad v(x,0) = 0, & x \in (0,L), \ t \geq 0, \\ \mathcal{B}_{4,0}v = (h_{1,4}(t), \ h_{2,4}(t), \ h_{3,4}(t)), & t \geq 0. \end{cases} \tag{2.4}$$

Without loss of generality, we assume that $L = 1$ in this subsection.

Consideration is first given to the IBVP (2.1). Applying the Laplace transform with respect to t, (2.1) is converted to

$$\begin{cases} s\hat{v} + \hat{v}_{xxx} = 0, \\ \hat{v}(0,s) = \hat{h}_{1,1}(s), \ \hat{v}(1,s) = \hat{h}_{2,1}(s), \ \hat{v}_x(1,s) = \hat{h}_{3,1}(s), \end{cases} \tag{2.5}$$

where

$$\hat{v}(x,s) = \int_0^{+\infty} e^{-st}v(x,t)dt$$

and

$$\hat{h}_{j,1}(s) = \int_0^{+\infty} e^{-st}h_{j,1}(t)dt, \quad j = 1,2,3.$$

The solution of (2.5) can be written in the form

$$\hat{v}(x,s) = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x},$$

where $\lambda_j(s), j = 1,2,3$ are solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and $c_j(s), j = 1,2,3$, solve the linear system

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} \\ \lambda_1 e^{\lambda_1} & \lambda_2 e^{\lambda_2} & \lambda_3 e^{\lambda_3} \end{pmatrix}}_{A^1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{h}_{1,1} \\ \hat{h}_{2,1} \\ \hat{h}_{3,1} \end{pmatrix}}_{\vec{\hat{h}}_1}.$$

By Cramer's rule,

$$c_j = \frac{\Delta_j^1(s)}{\Delta^1(s)}, \ j = 1, 2, 3,$$

with Δ^1 the determinant of A^1 and Δ_j^1 the determinant of the matrix A^1 with the column j replaced by $\vec{\hat{h}}_1$. Taking the inverse Laplace transform of $\hat{v}$ and following the same arguments as that in [5] yield the representation

$$v(x, t) = \sum_{m=1}^{3} v_m^1(x, t)$$

with

$$v_m^1(x, t) = \sum_{j=1}^{3} v_{j,m}^1(x, t)$$

and

$$v_{j,m}^1(x, t) = v_{j,m}^{+,1}(x, t) + v_{j,m}^{-,1}(x, t)$$

where

$$v_{j,m}^{+,1}(x, t) = \frac{1}{2\pi i} \int_0^{+i\infty} e^{st} \frac{\Delta_{j,m}^1(s)}{\Delta^1(s)} \hat{h}_{m,1}(s) e^{\lambda_j(s)x} ds$$

and

$$v_{j,m}^{-,1}(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^0 e^{st} \frac{\Delta_{j,m}^1(s)}{\Delta^1(s)} \hat{h}_{m,1}(s) e^{\lambda_j(s)x} ds,$$

for $j, m = 1, 2, 3$. Here $\Delta_{j,m}^1(s)$ is obtained from $\Delta_j^1(s)$ by letting $\hat{h}_{m,1}(s) = 1$ and $\hat{h}_{k,1}(s) = 0$ for $k \neq m$, $k, m = 1, 2, 3$. More precisely,

$$\Delta^1 = (\lambda_3 - \lambda_2)e^{-\lambda_1} + (\lambda_1 - \lambda_3)e^{-\lambda_2} + (\lambda_2 - \lambda_1)e^{-\lambda_3};$$

$$\Delta_{1,1}^1 = (\lambda_3 - \lambda_2)e^{-\lambda_1}, \ \Delta_{2,1}^1 = (\lambda_1 - \lambda_3)e^{-\lambda_2}, \ \Delta_{3,1}^1 = (\lambda_2 - \lambda_1)e^{-\lambda_3};$$

$$\Delta_{1,2}^1 = \lambda_2 e^{\lambda_2} - \lambda_3 e^{\lambda_3}, \ \Delta_{2,2}^1 = \lambda_3 e^{\lambda_3} - \lambda_1 e^{\lambda_1}, \ \Delta_{3,2}^1 = \lambda_1 e^{\lambda_1} - \lambda_2 e^{\lambda_2};$$

$$\Delta_{1,3}^1 = e^{\lambda_3} - e^{\lambda_2}, \ \Delta_{2,3}^1 = e^{\lambda_1} - e^{\lambda_3}, \ \Delta_{3,3}^1 = e^{\lambda_2} - e^{\lambda_1}.$$

Making the substitution $s = i\rho^3$, with $0 \leq \rho < \infty$, in the characteristic equation

$$s + \lambda^3 = 0,$$

the three roots $\lambda_j = \lambda_j(i\rho^3) := \lambda_j^+(\rho)$, $j = 1, 2, 3$, are

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{\sqrt{3} - i}{2}\rho, \quad \lambda_3^+(\rho) = -\frac{\sqrt{3} + i}{2}\rho.$$

Thus $v_{j,m}^{+,1}(x, t)$ has the form

$$v_{j,m}^{+,1}(x, t) = \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} \frac{\Delta_{j,m}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \hat{h}_{m,1}^+(\rho) e^{\lambda_j^+(\rho)x} 3\rho^2 d\rho$$

and

$$v_{j,m}^{-,1}(x, t) = \overline{v_{j,m}^{+,1}(x, t)},$$

where $\hat{h}_{m,1}^+(\rho) = \hat{h}_{m,1}(i\rho^3)$, $\Delta^{+,1}(\rho)$ and $\Delta_{j,m}^{+,1}(\rho)$ are obtained from $\Delta^1(s)$ and $\Delta_{j,m}^1(s)$ by replacing s with $i\rho^3$ and $\lambda_j^+(\rho) = \lambda_j(i\rho^3)$.

For given $m, j = 1, 2, 3$, let $W_{j,m}^1$ be an operator on $H_0^s(\mathbb{R}^+)$ defined as follows: for any $h \in H_0^s(\mathbb{R}^+)$,

$$[W_{j,m}^1 h](x, t) := [U_{j,m}^1 h](x, t) + \overline{[U_{j,m}^1 h](x, t)} \tag{2.6}$$

with

$$[U^1_{j,m}h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} 3\rho^2 [Q^{+,1}_{j,m}h](\rho) d\rho \qquad (2.7)$$

for $j = 1, 3, \ m = 1, 2, 3$ and

$$[U^1_{2,m}h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} 3\rho^2 [Q^{+,1}_{2,m}h](\rho) d\rho \qquad (2.8)$$

for $m = 1, 2, 3$. Here

$$[Q^{+,1}_{j,m}h](\rho) := \frac{\Delta^{+,1}_{j,m}(\rho)}{\Delta^{+,1}(\rho)} \hat{h}^+(\rho), \qquad [Q^{+,1}_{2,m}h](\rho) := \frac{\Delta^{+,1}_{2,m}(\rho)}{\Delta^{+,1}(\rho)} e^{\lambda_2^+(\rho)} \hat{h}^+(\rho) \qquad (2.9)$$

for $j = 1, 3$ and $m = 1, 2, 3$, $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$. Then the solution of the IBVP (2.1) has the following representation.

Lemma 2.1. *Given $\vec{h}_1 = (h_{1,1}, h_{2,1}, h_{3,1})$, the solution v of the IBVP (2.1) can be written in the form*

$$v(x,t) = [W^1_{bdr}\vec{h}_1](x,t) := \sum_{j,m=1}^{3} [W^1_{j,m}h_{m,1}](x,t).$$

Next we consider the IBVP (2.2). A similar argument shows the solution of the IBVP (2.2) has the following representation.

Lemma 2.2. *The solution v of the IBVP (2.2) can be written in the form*

$$v(x,t) = [W^2_{bdr}\vec{h}_2](x,t) := \sum_{j,m=1}^{3} [W^2_{j,m}h_{m,2}](x,t),$$

where

$$[W^2_{j,m}h](x,t) := [U^2_{j,m}h](x,t) + \overline{[U^2_{j,m}h](x,t)} \qquad (2.10)$$

with

$$[U^2_{j,m}h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} 3\rho^2 [Q^{+,2}_{j,m}h](\rho) d\rho \qquad (2.11)$$

for $j = 1, 3, \ m = 1, 2, 3$ and

$$[U^2_{2,m}h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} 3\rho^2 [Q^{+,2}_{2,m}h](\rho) d\rho \qquad (2.12)$$

for $m = 1, 2, 3$. Here

$$[Q^{+,2}_{j,m}h](\rho) := \frac{\Delta^{+,2}_{j,m}(\rho)}{\Delta^{+,2}(\rho)} \hat{h}^+(\rho), \qquad [Q^{+,2}_{2,m}h](\rho) := \frac{\Delta^{+,2}_{2,m}(\rho)}{\Delta^{+,2}(\rho)} e^{\lambda_2^+(\rho)} \hat{h}^+(\rho) \qquad (2.13)$$

for $j = 1, 3$ and $m = 1, 2, 3$. Here $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$, $\Delta^{+,2}(\rho)$ and $\Delta^{+,2}_{j,m}(\rho)$ are obtained from $\Delta^2(s)$ and $\Delta^2_{j,m}(s)$ by replacing s with $i\rho^3$ and $\lambda_j^+(\rho) = \lambda_j(i\rho^3)$ where

$$\Delta^2 = \lambda_2 \lambda_3 (\lambda_3 - \lambda_2) e^{-\lambda_1} + \lambda_1 \lambda_3 (\lambda_1 - \lambda_3) e^{-\lambda_2} + \lambda_2 \lambda_1 (\lambda_2 - \lambda_1) e^{-\lambda_3};$$

$$\Delta^2_{1,1} = e^{-\lambda_1} \lambda_2 \lambda_3 (\lambda_3 - \lambda_2), \ \Delta^2_{2,1} = e^{-\lambda_2} \lambda_1 \lambda_3 (\lambda_3 - \lambda_1), \ \Delta^2_{3,1} = e^{-\lambda_3} \lambda_1 \lambda_2 (\lambda_2 - \lambda_1);$$

$$\Delta^2_{1,2} = \lambda_2^2 e^{\lambda_2} - \lambda_3^2 e^{\lambda_3}, \ \Delta^2_{2,2} = \lambda_3^2 e^{\lambda_3} - \lambda_1^2 e^{\lambda_1}, \ \Delta^2_{3,2} = \lambda_1^2 e^{\lambda_1} - \lambda_2^2 e^{\lambda_2};$$

$$\Delta^2_{1,3} = \lambda_3 e^{\lambda_3} - \lambda_2 e^{\lambda_2}, \ \Delta^2_{2,3} = \lambda_1 e^{\lambda_1} - \lambda_3 e^{\lambda_3}, \ \Delta^2_{3,3} = \lambda_2 e^{\lambda_2} - \lambda_1 e^{\lambda_1}.$$

For solutions of (2.3), we have the following lemma.

Lemma 2.3. *The solution v of the IBVP (2.3) can be written in the form*

$$v(x,t) = [W_{bdr}^3 \vec{h}_3](x,t) := \sum_{j,m=1}^3 [W_{j,m}^3 h_{m,3}](x,t),$$

where

$$[W_{j,m}^3 h](x,t) := [U_{j,m}^3 h](x,t) + \overline{[U_{j,m}^3 h](x,t)} \tag{2.14}$$

with

$$[U_{j,m}^3 h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} 3\rho^2 [Q_{j,m}^{+,3} h](\rho) d\rho \tag{2.15}$$

for $j = 1, 3$, $m = 1, 2, 3$ and

$$[U_{2,m}^3 h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} 3\rho^2 [Q_{2,m}^{+,3} h](\rho) d\rho \tag{2.16}$$

for $m = 1, 2, 3$. Here

$$[Q_{j,m}^{+,3} h](\rho) := \frac{\Delta_{j,m}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \hat{h}^+(\rho), \qquad [Q_{2,m}^{+,3} h](\rho) := \frac{\Delta_{2,m}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} e^{\lambda_2^+(\rho)} \hat{h}^+(\rho) \tag{2.17}$$

for $j = 1, 3$ and $m = 1, 2, 3$. Here $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$, $\Delta^{+,1}(\rho)$ and $\Delta_{j,m}^{+,3}(\rho)$ are obtained from $\Delta^3(s)$ and $\Delta_{j,m}^3(s)$ by replacing s with $i\rho^3$ and $\lambda_j^+(\rho) = \lambda_j(i\rho^3)$ where

$$\Delta^3 = \lambda_1^2(\lambda_3 - \lambda_2)e^{-\lambda_1} + \lambda_2^2(\lambda_1 - \lambda_3)e^{-\lambda_2} + \lambda_3^2(\lambda_2 - \lambda_1)e^{-\lambda_3};$$

$$\Delta_{1,1}^3 = e^{-\lambda_1}(\lambda_3 - \lambda_2), \ \Delta_{2,1}^3 = e^{-\lambda_2}(\lambda_1 - \lambda_3), \ \Delta_{3,1}^3 = e^{-\lambda_3}(\lambda_2 - \lambda_1);$$

$$\Delta_{1,2}^3 = \lambda^2 \lambda_3(\lambda_3 e^{\lambda_2} - \lambda_2 e^{\lambda_3}), \ \Delta_{2,2}^3 = \lambda_1 \lambda_3(\lambda_1 e^{\lambda_3} - \lambda_3 e^{\lambda_1}),$$

$$\Delta_{3,2}^3 = \lambda_1 \lambda_2(\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2});$$

$$\Delta_{1,3}^3 = \lambda_2^2 e^{\lambda_3} - \lambda_3^2 e^{\lambda_2}, \ \Delta_{2,3}^3 = \lambda_3^2 e^{\lambda_1} - \lambda_1^2 e^{\lambda_3}, \ \Delta_{3,3}^3 = \lambda_1^2 e^{\lambda_2} - \lambda_2^2 e^{\lambda_1}.$$

For solutions of (2.4), we have

Lemma 2.4. *The solution v of the IBVP (2.4) can be written in the form*

$$v(x,t) = [W_{bdr}^4 \vec{h}_4](x,t) := \sum_{j,m=1}^3 [W_{j,m}^4 h_{m,4}](x,t),$$

where

$$[W_{j,m}^4 h](x,t) := [U_{j,m}^4 h](x,t) + \overline{[U_{j,m}^4 h](x,t)} \tag{2.18}$$

with

$$[U_{j,m}^4 h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} 3\rho^2 [Q_{j,m}^{+,4} h](\rho) d\rho \tag{2.19}$$

for $j = 1, 3$, $m = 1, 2, 3$ and

$$[U_{2,m}^4 h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} 3\rho^2 [Q_{2,m}^{+,4} h](\rho) d\rho \tag{2.20}$$

for $m = 1, 2, 3$. Here

$$[Q_{j,m}^{+,4} h](\rho) := \frac{\Delta_{j,m}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \hat{h}^+(\rho), \qquad [Q_{2,m}^{+,4} h](\rho) := \frac{\Delta_{2,m}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} e^{\lambda_2^+(\rho)} \hat{h}^+(\rho) \tag{2.21}$$

for $j = 1, 3$ and $m = 1, 2, 3$. Here $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$, $\Delta^{+,4}(\rho)$ and $\Delta_{j,m}^{+,4}(\rho)$ are obtained from $\Delta^4(s)$ and $\Delta_{j,m}^4(s)$ by replacing s with $i\rho^3$ and $\lambda_j^+(\rho) = \lambda_j(i\rho^3)$ where

$$\Delta^4 = \lambda_1 \lambda_2 \lambda_3 \left(\lambda_1(\lambda_3 - \lambda_2)e^{-\lambda_1} + \lambda_2(\lambda_1 - \lambda_3)e^{-\lambda_2} + \lambda_3(\lambda_2 - \lambda_1)e^{-\lambda_3} \right);$$

$$\Delta_{1,1}^4 = e^{-\lambda_1}\lambda_2\lambda_3(\lambda_3 - \lambda_2), \ \ \Delta_{2,1}^4 = e^{-\lambda_2}\lambda_1\lambda_3(\lambda_1 - \lambda_3), \ \ \Delta_{3,1}^4 = e^{-\lambda_3}\lambda_1\lambda_2(\lambda_2 - \lambda_1);$$

$$\Delta_{1,2}^4 = \lambda_2^2\lambda_3^2(e^{\lambda_2} - e^{\lambda_3}), \ \ \Delta_{2,2}^4 = \lambda_1^2\lambda_3^2(e^{\lambda_3} - e^{\lambda_1}), \ \ \Delta_{3,2}^4 = \lambda_1^2\lambda_2^2(e^{\lambda_1} - e^{\lambda_2});$$

$$\Delta_{1,3}^4 = \lambda_2\lambda_3(\lambda_2 e^{\lambda_3} - \lambda_3 e^{\lambda_2}), \ \ \Delta_{2,3}^4 = \lambda_1\lambda_3(\lambda_3 e^{\lambda_1} - \lambda_1 e^{\lambda_3}),$$

$$\Delta_{3,3}^4 = \lambda_1\lambda_2(\lambda_1 e^{\lambda_2} - \lambda_2 e^{\lambda_1}),$$

and $\lambda_j(s), j = 1, 2, 3$ are solutions of the characteristic equation

$$s + 1 + \lambda^3 = 0.$$

Remark 2.5. From $s\hat{v} + \hat{v}_{xxx} = 0$ with boundary conditions $\mathcal{B}_j\hat{v} = 0$ for $j = 1, 2, 3$ or $s\hat{v} + \hat{v} + \hat{v}_{xxx} = 0$ with boundary conditions $\mathcal{B}_4\hat{v} = 0$, it can be easily shown that there are no nontrivial solutions $\hat{v}$ for any s with $Re\,s \geq 0$. Therefore, $\Delta^j(s) \neq 0, j = 1, 2, 3, 4$ for any s with $Re\,s \geq 0$.

The following lemma is helpful in deriving various linear estimates for solutions of the IBVP (1.15) in the next subsection.

Lemma 2.6. *For $m = 1, 2, 3$, $k = 1, 2, 3, 4$ and $j = 1, 3$, set*

$$\hat{h}_{j,m,k}^*(\rho) := 3\rho^2[Q_{j,m}^{+,k} h_{m,k}](\rho) = 3\rho^2 \frac{\Delta_{j,m}^{+,k}(\rho)}{\Delta^{+,k}(\rho)}\hat{h}_{m,k}^+(\rho)$$

and

$$\hat{h}_{2,m,k}^*(\rho) := 3\rho^2[Q_{2,m}^{+,k} h_{m,k}](\rho) = 3\rho^2 \frac{\Delta_{2,m}^{+,k}(\rho)}{\Delta^{+,k}(\rho)}e^{\lambda_2^+(\rho)}\hat{h}_{m,k}^+(\rho)$$

and view $h_{j,m,k}^*$ as the inverse Fourier transform of $\hat{h}_{j,m,k}^*$. Then for any $s \in \mathbb{R}$,

$$\begin{cases} h_{1,1} \in H_0^{(s+1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,1,1}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{2,1} \in H_0^{(s+1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,2,1}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{3,1} \in H_0^{s/3}(\mathbb{R}^+) & \Rightarrow h_{j,3,1}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3. \end{cases} \qquad (2.22)$$

$$\begin{cases} h_{1,2} \in H_0^{(s+1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,1,2}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{2,2} \in H_0^{s/3}(\mathbb{R}^+) & \Rightarrow h_{j,2,2}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{3,2} \in H_0^{(s-1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,3,2}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3. \end{cases} \qquad (2.23)$$

$$\begin{cases} h_{1,3} \in H_0^{(s-1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,1,3}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{2,3} \in H_0^{(s+1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,2,3}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{3,3} \in H_0^{s/3}(\mathbb{R}^+) & \Rightarrow h_{j,3,3}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3. \end{cases} \qquad (2.24)$$

$$\begin{cases} h_{1,4} \in H_0^{(s-1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,1,4}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{2,4} \in H_0^{s/3}(\mathbb{R}^+) & \Rightarrow h_{j,2,4}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3, \\[2mm] h_{3,4} \in H_0^{(s-1)/3}(\mathbb{R}^+) & \Rightarrow h_{j,3,4}^* \in H^s(\mathbb{R}), \quad j = 1, 2, 3. \end{cases} \qquad (2.25)$$

Proof. Recall that for $k = 1, 2, 3$, we have

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{\sqrt{3}-i}{2}\rho, \quad \lambda_3^+(\rho) = -\frac{\sqrt{3}+i}{2}\rho$$

for $\rho \geq 0$, and for $k = 4$,

$$\lambda_1^+(\rho) \sim i\rho, \quad \lambda_2^+(\rho) \sim \frac{\sqrt{3}-i}{2}\rho, \quad \lambda_3^+(\rho) \sim -\frac{\sqrt{3}+i}{2}\rho$$

as $\rho \to +\infty$. Thus, the following asymptotic estimates of $\frac{\Delta_{n,m}^{+,k}(\rho)}{\Delta^{+,k}(\rho)}$, for $m, n = 1, 2, 3$, $k = 1, 2, 3, 4$, as $\rho \to +\infty$, hold:

$\frac{\Delta_{1,1}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,1}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim e^{-\sqrt{3}\rho}$	$\frac{\Delta_{3,1}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim 1$
$\frac{\Delta_{1,2}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim 1$	$\frac{\Delta_{2,2}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,2}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim 1$
$\frac{\Delta_{1,3}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,3}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,3}^{+,1}(\rho)}{\Delta^{+,1}(\rho)} \sim \rho^{-1}$

$\frac{\Delta_{1,1}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,1}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim e^{-\sqrt{3}\rho}$	$\frac{\Delta_{3,1}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim 1$
$\frac{\Delta_{1,2}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,2}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,2}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-1}$
$\frac{\Delta_{1,3}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-2}$	$\frac{\Delta_{2,3}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,3}^{+,2}(\rho)}{\Delta^{+,2}(\rho)} \sim \rho^{-2}$

$\frac{\Delta_{1,1}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,1}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-2}e^{-\sqrt{3}\rho}$	$\frac{\Delta_{3,1}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-2}$
$\frac{\Delta_{1,2}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,2}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,2}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim 1$
$\frac{\Delta_{1,3}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,3}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,3}^{+,3}(\rho)}{\Delta^{+,3}(\rho)} \sim \rho^{-1}$

$\frac{\Delta_{1,1}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{2,1}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}e^{-\sqrt{3}\rho}$	$\frac{\Delta_{3,1}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}$
$\frac{\Delta_{1,2}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-1}$	$\frac{\Delta_{2,2}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-1}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,2}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-1}$
$\frac{\Delta_{1,3}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}$	$\frac{\Delta_{2,3}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho}$	$\frac{\Delta_{3,3}^{+,4}(\rho)}{\Delta^{+,4}(\rho)} \sim \rho^{-2}$

Then (2.22)-(2.25) follow consequently. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We consider next the linear IBVP with homogeneous boundary conditions

$$\begin{cases} z_t + z_{xxx} + \delta_k z = f(x,t), & x \in (0, L), \ t > 0, \\ z(x, 0) = \phi(x), \\ \mathcal{B}_{k,0}z = 0 \end{cases} \tag{2.26}$$

for $k = 1, 2, 3, 4$. By the standard semigroup theory, for any $\phi \in L^2(0, L)$, $f \in L_{loc}^1(\mathbb{R}^+; L^2(0, L))$, the IBVP (2.26) admits a unique solution $z \in C(\mathbb{R}^+; L^2(0, L))$ which can be written as

$$z(x, t) = W_{0,k}(t)\phi + \int_0^t W_{0,k}(t - \tau)f(\cdot, \tau)d\tau$$

where $W_{0,k}(t)$ is the C_0-semigroup associated with the IBVP (2.26) with $f := 0$. Recall the solution of the Cauchy problem of the linear KdV equation,

$$\begin{cases} w_t + w_{xxx} + \delta_k w = 0, & x \in \mathbb{R}, \ t \geq 0, \\ w(x,0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{2.27}$$

has the explicit representation

$$v(x,t) = [W_{\mathbb{R},k}(t)]\psi(x) = c \int_{\mathbb{R}} e^{i\xi^3 t - \delta_k t} e^{ix\xi} \hat{\psi}(\xi) d\xi. \tag{2.28}$$

Here $\hat{\psi}$ denotes the Fourier transform of ϕ. In terms of the C_0-group $W_{\mathbb{R},k}(t)$ and the boundary integral operator $\mathcal{W}_{bdr}^{(k)}$, we can have a more explicit representation of solutions of the IBVP (2.26).

Let $s \geq 0$ be given and $B_s : H^s(0,L) \to H^s(\mathbb{R})$ be the standard extension operator from $H^s(0,L)$ to $H^s(\mathbb{R})$. For any $\phi \in H^s(0,L)$ and $f \in L^1_{loc}(\mathbb{R}^+; H^s(0,L))$ let

$$\phi^* = B_s \phi$$

and

$$f^* = B_s f.$$

Lemma 2.7. *For given $\phi \in L^2(0,L)$ and $f \in L^1_{loc}(\mathbb{R}^+; L^2(0,L))$, let*

$$q_k(x,t) = W_{\mathbb{R},k}(t)\phi^* + \int_0^t W_{\mathbb{R},k}(t-\tau)f^*(\tau)d\tau$$

and

$$\vec{h}_k := \mathcal{B}_{k,0}q, \quad k = 1,2,3,4.$$

Then the solution of the IBVP (2.26) can be written as

$$z(x,t) = W_{\mathbb{R},k}(t)\phi^* + \int_0^t W_{\mathbb{R},k}(t-\tau)f^*(\tau)d\tau - \mathcal{W}_{bdr}^{(k)}\vec{h}_k.$$

2.2. **Linear estimates.** In this subsection we consider the following IBVP of the linear equations:

$$\begin{cases} v_t + v_{xxx} + \delta_k v = f, & v(x,0) = \phi(x), & x \in (0,L), \ t \geq 0, \\ \mathcal{B}_{k,0}v = \vec{h}(t), & t \geq 0 \end{cases} \tag{2.29}$$

and present various linear estimates for its solutions. For given $s \geq 0$ and $T > 0$, let us consider:

$$\mathcal{Z}_{s,T} := C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$$

and

$$X_{s,T}^k := H_0^s(0,L) \times \mathcal{H}_k^s(0,T), \ \text{for } k = 1,2,3,4.$$

Recall that when $f = 0$ and $\phi = 0$, the solution v of the IBVP (2.29) can be written in the form

$$v(x,t) = [W_{bdr}^{(k)}\vec{h}](x,t) := \sum_{j,m=1}^{3} [W_{j,m}^{(k)}h_m](x,t),$$

where

$$[W_{j,m}^{(k)}h](x,t) := [U_{j,m}^{(k)}h](x,t) + \overline{[U_{j,m}^{(k)}h](x,t)}$$

with

$$[U_{j,m}^{(k)}h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \hat{h}_{j,m,k}^*(\rho)d\rho$$

for $k = 1, 2, 3, 4$, $j = 1, 3$, $m = 1, 2, 3$ and

$$[U_{2,m}^{(k)} h](x,t) := \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{-\lambda_2^+(\rho)(1-x)} \hat{h}_{2,m,k}^*(\rho) d\rho$$

for $k = 1, 2, 3, 4$ and $m = 1, 2, 3$. Here

$$\hat{h}_{j,m,k}^*(\rho) = 3\rho^2 \frac{\Delta_{j,m}^{+,k}(\rho)}{\Delta^{+,k}(\rho)} \hat{h}^+(\rho), \qquad \hat{h}_{2,m,k}^*(\rho) = 3\rho^2 \frac{\Delta_{2,m}^{+,k}(\rho)}{\Delta^{+,k}(\rho)} e^{\lambda_2^+(\rho)} \hat{h}^+(\rho)$$

for $k = 1, 2, 3, 4$, $j = 1, 3$ and $m = 1, 2, 3$.

Proposition 2.8. *Let $0 \le s \le 3$ with $s \ne \frac{2j-1}{2}, j = 1, 2, 3$, and $T > 0$ be given. There exists a constant $C > 0$ such that for any $\vec{h} \in \mathcal{H}_k^s(0,T)$,*

$$z_k = \mathcal{W}_{bdr}^{(k)} \vec{h}$$

satisfies

$$\|z_k\|_{\mathcal{Z}_{s,T}} + \sum_{j=0}^{2} \|\partial_x^j z_k\|_{L^\infty(0,L;H^{\frac{s+1-j}{3}}(0,T))} \le C\|\vec{h}\|_{\mathcal{H}_k^s(0,T)}$$

for $k = 1, 2, 3, 4$.

Proof. We only consider the case that $\vec{h} = (h_1, 0, 0)$ and $k = 4$; the proofs for the other cases are similar. Note that, the solution z_4 can be written as

$$z_4(x,t) = w_1(x,t) + w_2(x,t) + w_3(x,t)$$

with

$$w_j(x,t) := [W_{j,1}^{(4)} h_1](x,t) = [U_{j,1}^{(4)} h_1](x,t) + \overline{[U_{j,1}^{(4)} h_1](x,t)}, \quad j = 1, 2, 3.$$

Let us prove Proposition 2.8 for w_1. It suffices to only consider

$$w_1^+(x,t) := [U_{1,1}^{(4)} h_1](x,t) = \frac{1}{2\pi} \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_1^+(\rho)x} \hat{h}_{1,1,4}^*(\rho) d\rho.$$

Applying [5, Lemma 2.5] yields

$$\sup_{t \in [0,T]} \|w_1^+(\cdot,t)\|_{L^2(0,L)}^2 \le C \int_0^{+\infty} \left| \hat{h}_{1,1,4}^*(\rho) \right|^2 d\rho$$

$$\le C\|h_1\|_{H^{-\frac{1}{3}}(\mathbb{R}^+)}^2$$

and

$$\sup_{t \in [0,T]} \|\partial_x^3 w_1^+(\cdot,t)\|_{L^2(0,L)}^2 \le C \int_0^{+\infty} \left| \lambda_1^+(\rho) \right|^6 \left| \hat{h}_{1,1,4}^*(\rho) \right|^2 d\rho$$

$$\le C\|h_1\|_{H^{\frac{2}{3}}(\mathbb{R}^+)}^2.$$

By interpolation,

$$\sup_{t \in [0,T]} \|w_1^+(\cdot,t)\|_{H^s(0,L)}^2 \le C\|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}^2$$

for $0 \le s \le 3$. Furthermore, for $l = 0, 1, 2$, let $\mu = \rho^3$, $\rho \ge 0$, then

$$\partial_x^l w_1(x,t) = \frac{1}{2\pi} \int_0^{+\infty} \left(\lambda_1^+(\rho) \right)^l e^{i\rho^3 t} e^{\lambda_1^+(\rho)x} \hat{h}_{1,1,4}^*(\rho) d\rho$$

$$= \frac{1}{2\pi} \int_0^{+\infty} \left(\lambda_1^+(\mu^{\frac{1}{3}}) \right)^l e^{i\mu t} e^{\lambda_1^+(\mu^{\frac{1}{3}})x} \hat{h}_{1,1,4}^*(\mu^{\frac{1}{3}}) \mu^{-\frac{2}{3}} d\mu.$$

Applying Plancherel theorem, in time t, yields that, for all $x \in (0, L)$,

$$\|\partial_x^l w_1(x, \cdot)\|^2_{H^{\frac{s+1-l}{3}}(0,T)} \le C \int_0^{+\infty} \mu^{\frac{2(s+1-l)}{3}} \left| \left(\lambda_1^+(\mu^{\frac{1}{3}})\right)^l e^{\lambda_1^+(\mu^{\frac{1}{3}})x} \hat{h}_{1,1,4}^*(\mu^{\frac{1}{3}})\mu^{-\frac{2}{3}} \right|^2 d\mu$$

$$\le C \int_0^{+\infty} \left| (\lambda_1^+(\rho))^l) \hat{h}_{1,1,4}^*(\rho) \right|^2 \rho^{2s-2l} d\rho$$

$$\le C \int_0^{+\infty} \rho^{2s} \left| \hat{h}_{1,1,4}^*(\rho) \right|^2 d\rho$$

$$\le C \|h_1\|^2_{H^{\frac{s-1}{3}}(\mathbb{R}^+)},$$

for $l = 0, 1, 2$. Consequently, for $0 \le s \le 3$ and $l = 0, 1, 2$, we have

$$\sup_{x \in (0,L)} \|\partial_x^l w_1(x, \cdot)\|_{H^{\frac{s+1-l}{3}}(0,T)} \le C \|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)},$$

which ends the proof of Proposition 2.8 for w_1. The proof for w_j, $j = 2, 3$, are similar, and therefore will be omitted. □

Next we consider the following initial boundary-value problem:

$$\begin{cases} v_t + v_{xxx} + \delta_k v = f, & v(x,0) = \phi(x), \quad x \in (0, L), \ t \ge 0, \\ B_{k,0} v = 0, & t \ge 0, \end{cases} \quad (2.30)$$

for $k = 1, 2, 3, 4$. Recall that for any $s \in \mathbb{R}$, $\psi \in H^s(\mathbb{R})$ and $g \in L^1_{loc}(\mathbb{R}^+; H^s(\mathbb{R}))$, the Cauchy problem of the following linear KdV equation posed on $\mathbb{R}$,

$$\begin{cases} w_t + w_{xxx} + \delta_k w = g, & x \in \mathbb{R}, \ t \ge 0, \\ w(x,0) = \psi(x), & x \in \mathbb{R} \end{cases} \quad (2.31)$$

admits a unique solution $v \in C(\mathbb{R}^+; H^s(\mathbb{R}))$ and possesses the well-known sharp Kato smoothing properties.

Lemma 2.9. *Let $T > 0$ be given. For any $\psi \in L^2(\mathbb{R})$ and $g \in L^1(0, T; L^2(\mathbb{R}))$, the system* (2.31) *admits a unique solution $w \in \mathcal{Z}_{0,T}$ with*

$$\partial_x^l w \in L_x^\infty(\mathbb{R}; H^{\frac{1-l}{3}}(0,T)) \qquad for \ l = 0, 1, 2$$

and

$$\|w\|_{\mathcal{Z}_{0,T}} + \sum_{l=0}^2 \|\partial_x^l w(x, \cdot)\|_{L_x^\infty(\mathbb{R}; H^{\frac{1-l}{3}}(0,T))} \le C \left(\|\psi\|_{H^s(\mathbb{R})} + \|g\|_{L^1(0,T;L^2(\mathbb{R}))} \right)$$

where $C > 0$ is a constant depending only on T.

Corollary 2.10. *Let $T > 0$ be given. For any $\phi \in L^2(0, L)$ and $g \in L^1(0, T; L^2(\mathbb{R}))$, let $\psi \in L^2(\mathbb{R})$ be zero extension of ϕ from $(0, L)$ to $\mathbb{R}$. If*

$$\vec{q}_k := \mathcal{B}_{k,0} w, \quad k = 1, 2, 3, 4,$$

then

$$\vec{q}_k \in \mathcal{H}_k^0(0, T).$$

Moreover, for $k = 1, 2, 3, 4$,

$$\|\vec{q}_k\|_{\mathcal{H}_k^0(0,T)} \le C \left(\|\phi\|_{L^2(0,L)} + \|g\|_{L^1(0,T;L^2(\mathbb{R}))} \right).$$

The following two propositions follow from Proposition 2.8 and Lemma 2.9.

Proposition 2.11. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $(\phi, \vec{h}) \in X_{0,T}^k$ and $f \in L^1(0,T; L^2(0,L))$, the IBVP (2.29) admits a unique solution $v \in Z_{0,T}$ satisfying*

$$\|v\|_{Z_{0,T}} \leq C \left(\|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right).$$

Proposition 2.12. *Let $T > 0$ be given. For any $\phi \in L^2(0,L)$, $\vec{h} \in \mathcal{H}_{0,T}^k$ and $f \in L^1(0,T; L^2(0,L))$ the solution v of the system (2.29) satisfies*

$$\sup_{x \in (0,L)} \|\partial_x^r v(x, \cdot)\|_{H^{\frac{1-r}{3}}(0,T)} \leq C \left(\|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right) \qquad (2.32)$$

for $r = 0, 1, 2$.

3. **Nonlinear problems.** In this section, we will consider the IBVP of the nonlinear KdV equation on $(0,L)$ with the general boundary conditions

$$\begin{cases} u_t + u_{xxx} + u_x + uu_x = 0, & x \in (0,L), \ t > 0 \\ u(x,0) = \phi(x), & x \in (0,L), \\ \mathcal{B}_k u = \vec{h}(t), & t \geq 0, \end{cases} \qquad (3.1)$$

where the boundary operators $\mathcal{B}_k$, $k = 1, 2, 3, 4$, are introduced in the introduction.
    For given $s \geq 0$ and $T > 0$, let

$$Y_{s,T} := \left\{ w \in Z_{s,T}; \sum_{l=0}^{2} \|\partial_x^l w(x, \cdot)\|_{L_x^\infty(\mathbb{R}; H^{\frac{s+1-l}{3}}(0,T))} < +\infty \right\}$$

and

$$\|w\|_{Y_{s,T}} := \left(\|w\|_{Z_{s,T}}^2 + \sum_{l=0}^{2} \|\partial_x^l w(x, \cdot)\|_{L_x^\infty(\mathbb{R}; H^{\frac{s+1-l}{3}}(0,T))}^2 \right)^{\frac{1}{2}}.$$

The next lemma is helpful in establishing the well-posedness of (3.1) whose proof can be found in [5, 31].

Lemma 3.1. *There exists a $C > 0$ and $\mu > 0$ such that for any $T > 0$ and $u, \ v \in Y_{0,T}$,*

$$\int_0^T \|uv_x\|_{L^2(0,L)} \, d\tau \leq C(T^{\frac{1}{2}} + T^{\frac{1}{3}})\|u\|_{Y_{0,T}}\|v\|_{Y_{0,T}} \qquad (3.2)$$

and

$$\|\mathcal{B}_{k,1} v\|_{\mathcal{H}_k^0(0,T)} \leq CT^\mu \|v\|_{Y_{0,T}},$$

for $k = 1, 2, 3, 4$.

Consider the following linear IBVPs

$$\begin{cases} v_t + v_{xxx} + \delta_k v = f, & x \in (0,L), \ t > 0 \\ v(x,0) = \phi(x), & x \in (0,L), \\ \mathcal{B}_{k,0} v = \vec{h}, & t \geq 0, \end{cases} \qquad (3.3)$$

for $k = 1, 2, 3, 4$. The following lemma follows from the discussion in the Section 2, and therefore proof will be omitted.

Lemma 3.2. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $(\phi, \vec{h}) \in X_{0,T}^k$ and $f \in L^1(0,T; L^2(0,L))$, the IBVP (3.3) admits a unique solution $v \in Y_{0,T}$ satisfying*

$$\|v\|_{Y_{0,T}} \leq C \left(\|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right),$$

for $k = 1, 2, 3, 4$.

Next, we consider the following linearized IBVPs associated to (3.1)

$$\begin{cases} v_t + v_x + v_{xxx} + (a(x,t)v)_x = f, & x \in (0,L), \ t > 0 \\ v(x,0) = \phi(x), & x \in (0,L), \\ \mathcal{B}_k v = \vec{h}(t), & t \geq 0, \end{cases} \tag{3.4}$$

for $k = 1, 2, 3, 4$ and $a(x,t)$ is a given function.

Proposition 3.3. *Let $T > 0$ be given. Assume that $a \in Y_{0,T}$. Then for any $(\phi, \vec{h}) \in X_{0,T}^k$ and $f \in L^1(0,T; L^2(0,L))$, the IBVP (3.4) admits unique solution*

$$v \in Y_{0,T}.$$

Moreover, there exists a constant $C > 0$ depending only on T and $\|a\|_{Y_{0,T}}$ such that

$$\|v\|_{Y_{0,T}} \leq C \left(\|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right).$$

Proof. Let $r > 0$ and $0 < \theta \leq T$ be a constant to be determined. Set

$$S_{\theta,r} := \{ u \in Y_{0,\theta} : \|u\|_{Y_{0,\theta}} \leq r \},$$

which is a bounded closed convex subset of $Y_{0,\theta}$. For given $(\phi, \vec{h}) \in X_{0,T}^k$, $a \in Y_{0,T}$ and $f \in L^1(0,T; L^2(0,L))$, define a map Γ on $S_{\theta,r}$ by

$$v = \Gamma(u)$$

for any $u \in S_{\theta,r}$ where v is the unique solution of

$$\begin{cases} v_t + v_{xxx} + \delta_k v = -u_x - (a(x,t)u)_x + \delta_k u & x \in (0,L), \ t \geq 0, \\ v(x,0) = \phi(x), & x \in (0,L), \\ \mathcal{B}_{k,0} v = \vec{h}(t) - \mathcal{B}_{k,1} u, & t \geq 0. \end{cases} \tag{3.5}$$

By Lemma 3.2 (see also Propositions 2.11 and 2.12), for any $u, w \in S_{\theta,r}$,

$$\begin{aligned} \|\Gamma(u)\|_{Y_{0,\theta}} &\leq C_1 \left(\|(\phi, \vec{h})\|_{X_{k,T}^0} + \|f\|_{L^1(0,T;L^2(0,L))} \right) \\ &\quad + C_2 \|B_{k,1} v\|_{\mathcal{H}_k^0(0,\theta)} + C_3 \|(av)_x\|_{L^1(0,\theta;L^2(0,L))} \\ &\leq C_1 \left(\|(\phi, \vec{h})\|_{X_{k,T}^0} + \|f\|_{L^1(0,T;L^2(0,L))} \right) \\ &\quad + \left(C_2 \theta^\mu + \left[\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}} \right] \|a\|_{Y_{0,T}} \right) \|v\|_{Y_{0,\theta}} \end{aligned}$$

and

$$\|\Gamma(w) - \Gamma(u)\|_{Y_{0,\theta}} \leq \left(C_2 \theta^\mu + \left[\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}} \right] \|a\|_{Y_{0,T}} \right) \|w - u\|_{Y_{0,\theta}}.$$

Thus Γ is a contraction mapping from $S_{r,\theta}$ to $S_{r,\theta}$ if one chooses r and θ by

$$r = 2C_0 \left(\|(\phi, \vec{h})\|_{X_{0,T}^k} + \|f\|_{L^1(0,T;L^2(0,L))} \right)$$

and

$$\left(C_2 \theta^\mu + \left[\theta^{\frac{1}{2}} + \theta^{\frac{1}{3}} \right] \|a\|_{Y_{0,T}} \right) \leq \frac{1}{2}.$$

Its fixed point $v = \Gamma(u)$ is desired solution of (3.5) in the time interval $[0, \theta]$. Note that θ only depends on $\|a\|_{Y_{0,T}}$ and thus by standard extension argument, the solution v can be extended to the time interval $[0, T]$. The proof is completed. □

Now, we turn to consider the well-posedness problem of the nonlinear IBVP (3.1).

Theorem 3.4. *Let $s \geq 0$ with $s \neq \frac{2j-1}{2}, j = 1, 2, 3..., T > 0$ and $r > 0$ be given. There exists a $T^* \in (0, T]$ such that for any $(\phi, \vec{h}) \in X_k^0(0, T)$ with*

$$\|(\phi, \vec{h})\|_{X_k^s(0,T)} \leq r,$$

the IBVP (3.1) admits a unique solution $u \in Y_{s,T^*}$. Moreover, the corresponding solution map is real analytic.

Proof. We only prove the theorem in the case of $0 \leq s \leq 3$. When $s > 3$ it follows from a standard procedure developed in [3]. First we consider the case of $s = 0$. As in the proof of Proposition 3.3, let $r > 0$ and $0 < \theta \leq T$ be a constant to be determined. Set

$$S_{\theta,r} := \{u \in Y_{s,\theta} : \|u\|_{Y_{s,\theta}} \leq r\}.$$

For given $(\phi, \vec{h}) \in X_{k,T}^0$, define a map Γ on $S_{\theta,r}$ by

$$v = \Gamma(u) \quad for \ u \in Y_{0,\theta}$$

where v is the unique solution of

$$\begin{cases} v_t + v_{xxx} + \delta_k v = -u_x - uu_x + \delta_k u, & x \in (0, L), \ t \geq 0 \\ v(x,0) = \phi(x), & x \in (0, L), \\ \mathcal{B}_k \vec{v}(t) = \vec{h}(t), & t \geq 0. \end{cases} \quad (3.6)$$

By Proposition 3.3, for any $u, \ w \in S_{\theta,r}$,

$$\|\Gamma(u)\|_{Y_{0,\theta}} \leq C_0 \|(\phi, \vec{h})\|_{X_{0,T}^k} + C_1 \theta \|u\|_{Y_{0,\theta}} + C_2 \left(\theta^{1/3} + \theta^{1/2}\right) \|u\|_{Y_{0,\theta}}^2$$

and

$$\|\Gamma(u) - \Gamma(w)\|_{Y_{0,\theta}} \leq C_1 \theta \|u - w\|_{Y_{0,\theta}} + \frac{C_2}{2} \left(\theta^{1/3} + \theta^{1/2}\right) \|u + w\|_{Y_{0,\theta}} \|u - w\|_{Y_{0,\theta}}.$$

Choosing r and θ with

$$r = 2C_0 \|(\phi, \vec{h})\|_{X_{0,T}^k}, \qquad C_1 \theta + C_2 \left(\theta^{1/3} + \theta^{1/2}\right) r \leq \frac{1}{2},$$

Γ is a contraction whose critical point is the desired solution.

Next we consider the case of $s = 3$. Let $v = u_t$ we have v solves

$$\begin{cases} v_t + v_x + v_{xxx} + (a(x,t)v)_x = 0, & x \in (0, L), \ t > 0 \\ v(x,0) = \phi^*(x), & x \in (0, L), \\ \mathcal{B}_k v = \vec{h}'(t), & t \geq 0, \end{cases} \quad (3.7)$$

where $\phi^*(x) = -\phi'(x) - \phi'''(x)$ and $a(x,t) = \frac{1}{2}u(x,t)$. Applying Proposition 3.3 implies that $v = u_t \in Y_{0,T^*}$. Then it follows from the equation

$$u_t + u_x + uu_x + u_{xxx} = 0$$

that $u_{xxx} \in Y_{0,T^*}$ and $u \in Y_{3,T^*}$. The case of $0 < s < 3$ follows using Tartar's nonlinear interpolation theory [34] and the proof is achived. □

4. **Concluding remarks.** In this paper we have studied the nonhomogeneous boundary value problem of the KdV equation on the finite interval (0,L) with general boundary conditions,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & 0 < x < L, \ t > 0 \\ u(x,0) = \phi(x) \\ \mathcal{B}_k u = \vec{h} \end{cases} \tag{4.1}$$

and have shown that the IBVP (4.1) is locally well-posed in the space $H^s(0, L)$ for any $s \geq 0$ with $s \neq \frac{2j-1}{2}, j = 1, 2, 3...$, and $(\phi, \vec{h}) \in X^s_{k,T}$. Two important tools have played indispensable roles in approach; one is the explicit representation of the boundary integral operators $\mathcal{W}^{(k)}_{bdr}$ associated to the IBVP (4.1) and the other one is the sharp Kato smoothing property. We have obtained our results by first investigating the associated linear IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = f, & 0 < x < L, \ t > 0, \\ u(x,0) = \phi(x), \\ \mathcal{B}_{k,0} u = \vec{h}. \end{cases} \tag{4.2}$$

The local well-posedness of the nonlinear IBVP (4.1) follows via *contraction mapping principle*.

While the results reported in this paper have significantly improved the earlier works on general boundary value problems of the KdV equation on a finite interval, there are still many questions to be addressed regarding the IBVP (4.1). Here we list a few of them which are most interesting to us.

(1) *Is the IBVP (4.1) globally well-posed in the space $H^s(0, L)$ for some $s \geq 0$ or equivalently, does any solution of the IBVP (4.1) blow up in the some space $H^s(0, L)$ in finite time?*

   It is not clear if the IBVP (4.1) is globally well-posed or not even in the case of $\vec{h} := 0$. It follows from our results that a solution u of the IBVP (4.1) blows up in the space $H^s(0, L)$ for some $s \geq 0$ at a finite time $T > 0$ if and only if

$$\lim_{t \to T^-} \|u(\cdot, t)\|_{L^2(0,L)} = +\infty.$$

Consequently, it suffices to establish a global a priori $L^2(0, L)$ estimate

$$\sup_{0 \leq t < \infty} \|u(\cdot, t)\|_{L^2(0,L)} < +\infty, \tag{4.3}$$

for solutions of the IBVP (4.1) in order to obtain the global well-posedness of the IBVP (4.1) in the space $H^s(0, L)$ for any $s \geq 0$. However, estimate (4.3) is known to be held only in one case

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f, & 0 < x < L, \ t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = h_1(t), \ u(L,t) = h_2(t), \ u_x(L,t) = h_3(t). \end{cases}$$

(2) *Is the IBVP well-posed in the space $H^s(0, L)$ for some $s \leq -1$?*

   We have shown that the IBVP (4.1) is locally well-posed in the space $H^s(0, L)$ for any $s \geq 0$. Our results can also be extended to the case of $-1 < s \leq 0$ using the same approach developed in [8]. For the pure initial

value problem (IVP) of the KdV equation posed on the whole line $\mathbb{R}$ or on torus $\mathbb{T}$,

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x, \ t \in \mathbb{R} \tag{4.4}$$

and

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \tag{4.5}$$

it is well-known that the IVP (4.4) is well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -\frac{3}{4}$ and is (conditionally) ill-posed in the space $H^s(\mathbb{R})$ for any $s < -\frac{3}{4}$ in the sense the corresponding solution map cannot be uniformly continuous. As for the IVP (4.5), it is well-posed in the space $H^s(\mathbb{T})$ for any $s \geq -1$. The solution map corresponding to the IVP (4.5) is real analytic when $s > -\frac{1}{2}$, but only continuous (not even locally uniformly continuous) when $-1 \leq s < -\frac{1}{2}$. Whether the IVP (4.4) is well-posed in the space $H^s(\mathbb{R})$ for any $s < -\frac{3}{4}$ or the IVP (4.5) is well-posed in the space $H^s(\mathbb{T})$ for any $s < -1$ is still an open question. On the other hand, by contrast, the IVP of the KdV-Burgers equation

$$u_t + uu_x + u_{xxx} - u_{xx} = 0, \quad u(x,0) = \phi(x), \quad x \in \mathbb{R}, \ t > 0$$

is known to be well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -1$, but is known to be ill-posed for any $s < -1$. We conjecture that the IBVP (4.1) is ill-posed in the space $H^s(0, L)$ for any $s < -1$.

(3) While the approach developed in this paper can be used to study the non-homogeneous boundary value problems of the KdV equation on $(0, L)$ with quite general boundary conditions, there are still some boundary value problems of the KdV equation that our approach does not work. Among them the following two boundary value problems of the KdV equation on $(0, L)$ stand out:

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \quad x \in (0, L) \\ u(x,0) = \phi(x), \\ u(0,t) = u(L,t), \ u_x(0,t) = u_x(L,t), \ u_{xx}(0,t) = u_{xx}(L,t) \end{cases} \tag{4.6}$$

and

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, \quad x \in (0, L), \\ u(x,0) = \phi(x), \\ u(0,t) = 0, \ u(L,t) = 0, \ u_x(0,t) = u_x(L,t). \end{cases} \tag{4.7}$$

A common feature for these two boundary value problems is that the L^2-norm of their solutions are conserved:

$$\int_0^L u^2(x,t)dx = \int_0^L \phi^2(x)dx \qquad \text{for any } t \in \mathbb{R}.$$

The IBVP (4.6) is equivalent to the IVP (4.5) which was shown by Kato [23, 24] to be well-posed in the space $H^s(\mathbb{T})$ when $s > \frac{3}{2}$ as early as in the late 1970s. Its well-posedness in the space $H^s(\mathbb{T})$ when $s \leq \frac{3}{2}$, however, was established 24 years later in the celebrated work of Bourgain [9, 10] in 1993. For the IBVP (4.7), its associated linear problem

$$\begin{cases} u_t + u_{xxx} = 0, \quad x \in (0, L), \\ u(x,0) = \phi(x), u(0,t) = 0, \\ u(L,t) = 0, \ u_x(0,t) = u_x(L,t) \end{cases} \tag{4.8}$$

has been shown by Cerpa (see, for instance, [13]) to be well-posed in the space $H^s(0, L)$ forward and backward in time. However, whether the nonlinear IBVP (4.7) is well-posed in the space $H^s(0, L)$ for some s is still unknown.

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E-mail address: capistranofilho@dmat.ufpe.br
E-mail address: sun@math.vt.edu
E-mail address: zhangb@ucmail.uc.edu