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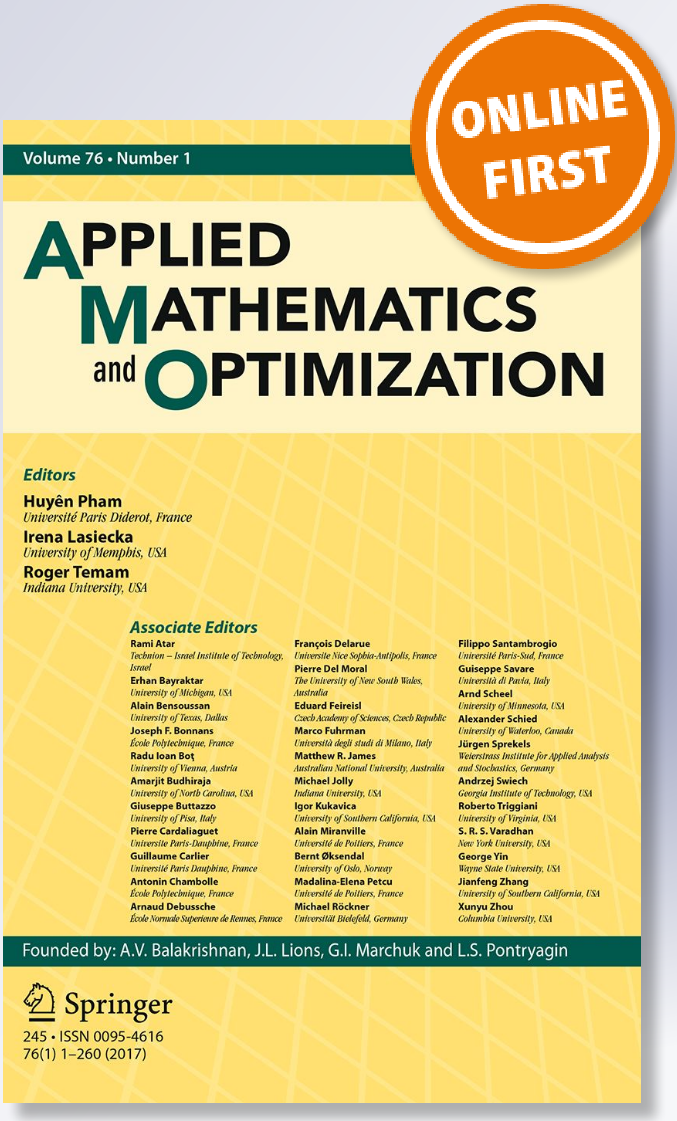

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Stabilization and Control for the Biharmonic Schrödinger Equation

Roberto A. Capistrano–Filho[1] · **Márcio Cavalcante**[2]

Abstract

The main purpose of this paper is to show the global stabilization and exact controllability properties of a fourth order nonlinear Schrödinger system on a periodic domain $\mathbb{T}$ with internal control supported on an arbitrary sub-domain of $\mathbb{T}$. More precisely, by certain properties of propagation of compactness and regularity in Bourgain spaces, for the solutions of the associated linear system, we show that the system is globally exponentially stabilizable. This property together with the local exact controllability shows that fourth order nonlinear Schrödinger is globally exactly controllable.

Keywords Bourgain spaces · Exact controllability · Fourth order nonlinear Schrödinger · Propagation of compactness · Propagation of regularity · Stabilization

Mathematics Subject Classification Primary 35Q55; Secondary 93B05 · 93D15 · 35A21

1 Introduction

1.1 Presentation of the Model

Fourth-order cubic nonlinear Schrödinger (4NLS) equation or biharmonic cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, \tag{1.1}$$

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has been introduced by Karpman [19] and Karpman and Shagalov [20] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1.1) arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references (see [3,9,19,31,32] and references therein).

The past twenty years such 4NLS has been deeply studied from different mathematical viewpoint. For example, Fibich et al. [14] worked various properties of the equation in the sub-critical regime, with part of their analysis relying on very interesting numerical developments. The well-posedness and existence of solutions for different domains have been shown (see, for instance, [8,22,30–32,34–36]) by means of the Fourier restriction method, energy method, forcing boundary operators, Laplace transform, harmonic analysis, Fokas method, etc.

It is interesting to point out that there are many works related to the Eq. (1.1) not only dealing with well-posedness theory. For example, recently Natali and Pastor [29], considered the fourth-order dispersive cubic nonlinear Schrödinger equation on the line with mixed dispersion. They proved the orbital stability, in the $H^2(\mathbb{R})$—energy space by constructing a suitable Lyapunov function. Considering the equation (1.1) on the circle, Oh and Tzvetkov [35], showed that the mean-zero Gaussian measures on Sobolev spaces $H^s(\mathbb{T})$, for $s > \frac{3}{4}$, are quasi-invariant under the flow. For instance, there has been a significant progress over the recent years and the reader can have a great view in [6,7] for the nonlinear Schrödinger equation.

1.2 Setting of the Problem

In this article our purpose is to study properties of stabilization and, consequently, controllability for the periodic one-dimensional fourth order cubic nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{1.2}$$

In order to determine if the system (1.2) is controllable in large time for a control supported in any small open subset of $\mathbb{T}$, we will study the Eq. (1.2) from a control point of view with a forcing term $f = f(x,t)$ added on the equation as a control input

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u + f, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{1.3}$$

where f is assumed to be supported in a given open subset ω of $\mathbb{T}$. Therefore, the following classical issues related with the control theory are considered in this work:

Exact control problem Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input f so that the Eq. (1.3) admits a solution u which satisfies $u(\cdot,0) = u_0$ and $u(\cdot,T) = u_1$?

Stabilization problem Can one find a feedback control law f so that the system (1.3) is asymptotically stable as $t \to \infty$?

1.3 Previous Results

When we consider (1.1) on a periodic domain $\mathbb{T}$ is not of our knowledge any result about control theory. However, there are interesting results in a bounded domain of $\mathbb{R}$ or $\mathbb{R}^n$, which we will summarize on the paragraphs below for the following fourth order linear Schrödinger equation

$$i\partial_t u + \partial_x^4 u = 0. \tag{1.4}$$

The first result about the exact controllability of the linearized fourth order Schrödinger equation (1.4) on a bounded domain Ω of the $\mathbb{R}^n$ is due to Zheng and Zhongcheng in [40]. In this work, by means of an L^2—Neumann boundary control, the authors proved that the solution is exactly controllable in $H^s(\Omega)$, $s = -2$, for an arbitrarily small time. They used Hilbert Uniqueness Method (HUM) (see, for instance, [13,28]) combined with the multiplier techniques to get the main result of the article. More recently, in [39], Zheng proved another interesting problem related with the control theory. To do this, he showed a global Carleman estimate for the fourth order Schrödinger equation posed on a finite domain. The Carleman estimate is used to prove the Lipschitz stability for an inverse problem consisting in retrieving a stationary potential in the fourth order Schrödinger equation from boundary measurements.

Still on control theory Wen et al. in two works [36,37] studied well-posedness and control theory related with the Eq. (1.4) on a bounded domain of $\mathbb{R}^n$, for $n \geq 2$. In [36], they proved the Neumann boundary controllability with collocated observation. With this result in hands, the exponential stability of the closed-loop system under proportional output feedback control holds. Recently, the authors, in [37], gave positive answers when considered the equation with hinged boundary by either moment or Dirichlet boundary control and collocated observation, respectively.

Lastly, to get a general outline of the control theory already done for the system (1.4), two interesting problems were studied recently by Aksas and Rebiai [1] and Peng [16]: Uniform stabilization and stochastic control problem, in a smooth bounded domain Ω of $\mathbb{R}^n$ and on the interval $I = (0, 1)$ of $\mathbb{R}$, respectively. The first work, by introducing suitable dissipative boundary conditions, the authors proved that the solution decays exponentially in $L^2(\Omega)$ when the damping term is effective on a neighborhood of a part of the boundary. The results are established by using multiplier techniques and compactness/uniqueness arguments. Regarding the second work, above mentioned, the author showed a Carleman estimate for forward and backward stochastic fourth order Schrödinger equations which provided to prove the observability inequality, unique continuation property and, consequently, the exact controllability for the forward and backward stochastic system associated with (1.4).

1.4 Notations and Main Results

Before to present our main results, let us introduce the Bourgain spaces associated to the biharmonic Schrödinger equation. For given $b, s \in \mathbb{R}$ and a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, in $\mathcal{S}(\mathbb{T} \times \mathbb{R})$, defines the quantity

$$\|u\|_{X_{b,s}} := \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \langle \tau - p(k) \rangle^{2b} \left| \hat{u}(k, \tau) \right|^2 d\tau \right)^{\frac{1}{2}},$$

where $\hat{u}(k, \tau)$ denotes the Fourier transform of u with respect to the space variable x and the time variable t, $\langle \cdot \rangle = \sqrt{1 + | \cdot |^2}$ and $p(k) = -k^4 - k^2$. We denote by D^r the operator defined on $\mathcal{D}'(\mathbb{T})$ by

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \hat{u}(k) & \text{if } k \neq 0, \\ \hat{u}(0) & \text{if } k = 0. \end{cases} \tag{1.5}$$

The Bourgain space $X_{b,s}$ associated to the fourth order linear dispersive Schrödinger equation on $\mathbb{T}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X_{b,s}}$. Note that for any $u \in X_{b,s}$,

$$\|u\|_{X_{b,s}} = \|W(-t)\, u\|_{H^b(\mathbb{R}, H^s(\mathbb{T}))},$$

where $\{W(\cdot)\}$ is the free group associated to the linearized biharmonic Schödinger equation. For a given interval I, let $X_{b,s}(I)$ be the restriction space of $X_{b,s}$ to the interval I with the norm

$$\|u\|_{X_{b,s}(I)} = \inf \left\{ \|\tilde{u}\|_{X_{b,s}} \mid \tilde{u} = u \text{ on } \mathbb{T} \times I \right\}.$$

By simplicity, we denote $X_{b,s}(I)$ by $X_{b,s}^T$ when $I = (0, T)$.

To clarify, the first issue to be proved in this article is the following one. *Given $T > 0$ and $u_0, u_1 \in L^2(\mathbb{T})$, is there a control input $g \in C([0, T]; L^2(\mathbb{T}))$ in order to make the solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u + g, & (x, t) \in \mathbb{T} \times \mathbb{R}, \\ u(x, 0) = u_0(x), & x \in \mathbb{T} \end{cases} \tag{1.6}$$

satisfying $u(\cdot, T) = u_1$?

The strategy to answer the global controllability question is first to prove a local exact controllability result and to combine it with a global stabilization of the solutions to get the global controllability of the system (1.6). Thus, in this spirit, we first need to prove the control property of 4NLS (1.6) near to 0, that will be proved using a perturbation argument introduced by Zuazua in [41]. More precisely, we will show the following local controllability result:

Theorem 1.1 *Let ω be any nonempty open subset of $\mathbb{T}$ and $T > 0$. There exists $\epsilon > 0$ such that for any $u_0 \in L^2(\mathbb{T})$ with*

$$\|u_0\|_{L^2(\mathbb{T})} < \epsilon,$$

one can find a control input $g \in C([0,T]; L^2(\mathbb{T}))$, with $supp(g) \subset \omega \times (0,T)$ such that the unique solution $u \in X^T_{b,0}$, for $1/2 < b < 21/16$, of the system (1.6) satisfies $u(x,T) = 0$.

Let us now introduce our main results of the article. Consider $a(x) \in L^\infty(\mathbb{T})$ real valued, the stabilization system that we will consider is the following

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u + ia^2 u = \lambda |u|^2 u & \text{on } \mathbb{T}\times (0,T), \\ u(x,0) = u_0(x) & \text{on } \mathbb{T}, \end{cases} \tag{1.7}$$

where $\lambda \in \mathbb{R}$ and $u_0 \in L^2(\mathbb{T})$, in L^2–level.

Note that, easily, we can check that the solution of (1.7) satisfies the mass decay

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{T})} = \|u(\cdot,0)\|^2_{L^2(\mathbb{T})} - \int_0^t \|au(\tau)\|^2_{L^2(\mathbb{T})}\, d\tau, \quad \forall t \geq 0. \tag{1.8}$$

Observe that for $a(x) = 0$ we have, by (1.8), the mass of the system is indeed conserved. However, assuming that $a(x)^2 > \eta > 0$ on some nonempty open set ω of $\mathbb{T}$, identity (1.8) states that we have an possibility of a exponential decay of the solutions related of (1.7). In fact, following the ideas of Dehman and Lebeau [10], see also [12], by using techniques of semi-classical and microlocal analysis, the result that we are able to prove, for large data, can be read as follows:

Theorem 1.2 *Let $a(x) := a \in L^\infty(\mathbb{T})$ taking real values such that $a^2(x) > \eta$ on a nonempty open set $\omega \subset \mathbb{T}$, for some constant $\eta > 0$. Then, for every $R_0 > 0$ there exist $C := C(R_0) > 0$ and $\gamma > 0$ such that inequality*

$$\|u(\cdot,t)\|_{L^2(\mathbb{T})} \leq Ce^{-\gamma t}\|u_0\|_{L^2(\mathbb{T})}, \forall t > 0, \tag{1.9}$$

holds for every solution u of (1.7) with initial data $u_0 \in L^2(\mathbb{T})$ satisfying $\|u_0\|_{L^2(\mathbb{T})} \leq R_0$.

Finally, the global controllability result which one can be established is the following:

Theorem 1.3 *Let $1/2 < b < 21/16$ and ω be any nonempty open subset of $\mathbb{T}$. For any $R_0 > 0$, there exists $T := T(R_0) > 0$ such that for any u_0 and u_1 in $L^2(\mathbb{T})$ with*

$$\|u_0\|_{L^2(\mathbb{T})} \leq R_0 \quad \text{and} \quad \|u_1\|_{L^2(\mathbb{T})} \leq R_0$$

one can find a control input $g \in C([0,T]; L^2(\mathbb{T}))$, with $supp(g) \subset \omega \times (0,T)$, such that the unique solution $u \in X^T_{b,0}$ of the system (1.6) satisfies $u(x,T) = u_1$.

Let us describe briefly the main arguments of the proof of these theorems. Precisely, the control result for large data (Theorem 1.3) will be a combination of a global stabilization result (Theorem 1.2) and the local control result (Theorem 1.1), as is usual in control theory, see e.g. [11,12,24–26].

With respect to the proof of Theorem 1.2, in general lines, the functional spaces used here are the Bourgain spaces which are especially suited for solving dispersive equations. Thus, the step one is to prove the following *Strichartz estimate* for the operator of fourth order Schrödinger equation:

$$\|u\|_{L^4([0,T]\times\mathbb{T})} \leq C\,\|u\|_{X^T_{\frac{5}{16},0}},$$

this allows to prove the following *multilinear estimates* in $X^T_{b,s}$:

$$\|u_1u_2\overline{u}_3\|_{X^T_{b-1,s}} \leq C\,\|u_1\|_{X^T_{\frac{5}{16},0}}\|u_2\|_{X^T_{\frac{5}{16},0}}\|u_3\|_{X^T_{\frac{5}{16},s}},$$

$$\left\||u|^2u-|v|^2v\right\|_{X^T_{b-1,s}} \leq C\left(\|u\|^2_{X^T_{\frac{5}{16},s}}+\|v\|_{X^T_{\frac{5}{16},s}}\right)^2\|u-v\|_{X^T_{\frac{5}{16},s}},$$

where $s\geq 0$, $b<\frac{21}{16}$ and $T\leq 1$ are given and $C:=C(s)>0$. On the step two, $H^s(\mathbb{T})$ propagation of regularity and compactness (see Sect. 5 below) from the state to the control are obtained using these properties for the linear control and a local linear behavior. Lastly, results of propagation together with a *unique continuation property (UCP)*, bellow presented, guarantees the proof of Theorem 1.2.

Proposition 1.4 *For every $T>0$ and ω any nonempty open set of $\mathbb{T}$, the only solution $u\in C^\infty([0,T]\times\mathbb{T})$ of the system*

$$\begin{cases} i\partial_t u+\partial_x^2u-\partial_x^4u=b(x,t)u & \text{on } \mathbb{T}\times(0,T),\\ u=0 & \text{on } \omega\times(0,T),\end{cases}$$

where $b(x,t)\in C^\infty([0,T]\times\mathbb{T})$, is the trivial one

$$u\,(x,t)=0 \text{ on } \mathbb{T}\times(0,T)\,.$$

To end our introduction, we present the outline of our paper. Section 2 is to establish estimates needed in our analysis, namely, *Strichartz estimates* and *trilinear estimates*. Existence of solution for 4NLS with source and damping terms will be presented in Sect. 3. In Sect. 4, we prove the local controllability result, Theorem 1.1. Next, Sect. 5, the *propagation of compactness and regularity in Bourgain space* are proved and, with this in hands, Sect. 6 is aimed to present the proof of unique continuation property, Proposition 1.4. Section 7, is devoted to prove Theorem 1.2. Finally, we present concluding remarks in Sect. 8.

2 Linear Estimates

In this section, we introduce some results which are essential to establish the exact controllability and stabilization of the nonlinear system (1.6) and (1.7), respectively.

2.1 Strichartz and Trilinear Estimates

The next estimate is a *Strichartz type estimate*.

Lemma 2.1 *The following estimate holds*

$$\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \leq C\,\|u\|_{X^T_{\frac{5}{16},0}}. \tag{2.1}$$

Proof We closely follow the argument for the L^4-Strichartz estimate for the usual (second and fourth order) Schrödinger equation presented in [33] and [34].

Given a dyadic $M \geq 1$, let u_M the restriction of u onto the modulation size $\langle \tau - p(k)\rangle \sim M$. Then, it suffices to show that there exists $\epsilon > 0$ such that

$$\|u_M u_{2^m M}\|_{L^2_x L^2_t} \leq 2^{-\epsilon m} M^{\frac{5}{16}} \|u_M\|_{L^2_{x,t}} (2^m M)^{5/16} \|u_{2^m M}\|_{L^2_{x,t}}, \tag{2.2}$$

for any $M \geq 1$ and $m \in \mathbb{N}\cup\{0\}$. Indeed, assuming (2.2), by Cauchy–Schwarz inequality, we have that

$$\begin{aligned}
\|u\|^2_{L^4(\mathbb{T}\times\mathbb{R})} &= \sum_M \sum_{m\geq 0} \|u_M u_{2^m M}\|_{L^2_{x,t}} \\
&\lesssim \sum_M \sum_{m\geq 0} 2^{-\epsilon m} M^{\frac{5}{16}} \|u_M\|_{L^2_{x,t}} (2^m M)^{5/16} \|u_{2^m M}\|_{L^2_{x,t}} \\
&\lesssim \sum_{m\geq 0} 2^{-\epsilon m} \left(\sum_M M^{\frac{5}{8}} \|u_M\|^2_{L^2_{x,t}}\right)^{1/2} \left(\sum_M (2^m M)^{5/8} \|u_{2^m M}\|^2_{L^2_{x,t}}\right)^{1/2} \\
&\lesssim \|u\|^2_{X^T_{\frac{5}{16},0}}.
\end{aligned}$$

This proves (2.1).

Now we prove (2.2). By Plancherel's identity and Hölder's inequality, the following inequality holds

$$\begin{aligned}
\|u_M u_{2^m M}\|_{L^2_x L^2_t} &= \left\| \sum_{k=k_1+k_2} \int_{\tau=\tau_1+\tau_2} \hat{u}_M(k_1,\tau_1)\hat{u}_{M2^m}(k_2,\tau_2)\,d\tau_1 \right\|_{l^2_k L^2_\tau} \\
&\lesssim \left(\sup_{k,\tau} A(k,\tau)\right)^{1/2} \|u_M\|_{L^2_{x,t}} \|u_{2^m M}\|_{L^2_{x,t}},
\end{aligned}$$

where the function $A(k,\tau)$ is given by

$$A(k,\tau)=\sum_{k=k_1+k_2}\int_{\tau=\tau_1+\tau_2}\mathbf{1}_{\tau_1-p(k_1)\sim M}\mathbf{1}_{\tau_2-p(k_2)\sim 2^m M}d\tau_1. \tag{2.3}$$

Integrating in τ_1 holds that

$$A(k,\tau)\leq M\sum_{k=k_1+k_2}\mathbf{1}_{\tau\sim -p(k_1)-p(k_2)+2^m M}.$$

Here, we have used that the Lebesgue measure of set $\{\tau_1\in\mathbb{R};\tau_1-p(k_1)\sim M\}$ is comparable with M and $\tau=\tau_1+\tau_2\sim p(k_1)+p(k_2)+M+2^m M$.

Now, a direct calculation gives

$$\begin{aligned}(k-k_1)^4+k_1^4&=2k_1^4-4k_1^3k+6k_1^2k^2-4k_1k^3+k^4\\&=2\left(k_1-\frac{1}{2}k\right)^4+3k^2\left(k_1-\frac{1}{2}k\right)^4+\frac{1}{8}k^4\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}(k-k_1)^2+k_1^2&=2k_1^2-2k_1k+k^2\\&=2\left(k_1-\frac{1}{2}k\right)^2+\frac{1}{2}k^2.\end{aligned} \tag{2.5}$$

It follows that

$$\begin{aligned}p(k-k_1)+p(k_1)&=2\left(k_1-\frac{1}{2}k\right)^4+\frac{1}{2}k^2+(3k^2+2)\left(k_1-\frac{1}{2}k\right)^4+\frac{1}{8}k^4\\&=2\left(\left(k_1-\frac{1}{2}k\right)^2+\left(\frac{3}{2}k^2+1\right)^2\right)^2+\frac{1}{2}k^2+\frac{1}{8}k^4-2\left(\frac{3}{2}k^2+1\right)^2,\end{aligned} \tag{2.6}$$

which implies

$$\tau-p(k_1)-p(k-k_1)=-2\left(\left(k_1-\frac{1}{2}k\right)^2+\left(\frac{3}{2}k^2+1\right)^2\right)^2+C(k,\tau), \tag{2.7}$$

where $C(k,\tau)=\frac{1}{2}k^2+\frac{1}{8}k^4-2\left(\frac{3}{2}k^2+1\right)^2$. Thus, for fixed k and τ, $\mathbf{1}_{\tau}-p(k_1)-p(k_2)\sim 2^m M(k_1)=1$ only when

$$\left(\left(k_1-\frac{1}{2}k\right)^2+\left(\frac{3}{2}k^2+1\right)^2\right)^2=C(k,\tau)+O(2^m M), \tag{2.8}$$

that is, k_1 belongs at most four intervals of length $2^{m/4}M^{1/4}$. It yields that,

$$\sum_{k=k_1+k_2} \mathbf{1}_{\tau\sim -p(k_1)-p(k_2)+2^m M} = 2m^{1/4}M^{1/4}.$$

So, from (2.3), we have

$$A^{1/2}(k,\tau) \lesssim 2^{m/8}M^{5/8} \leq 2^{-\frac{3m}{16}}(2^m M)^{5/16}M^{5/16}.$$

This finishes the proof of Lemma 2.1. □

Lemma 2.1 allow us to prove the following multilinear estimates in $X^T_{b,s}$.

Lemma 2.2 *Let $s \geq 0$, $b \geq \frac{5}{16}$ and $T \leq 1$ be given. There exists a constant $C := C(s) > 0$ such that the following trilinear estimates holds*

$$\|u_1 u_2 \overline{u}_3\|_{X^T_{-b,s}} \leq C\, \|u_1\|_{X^T_{\frac{5}{16},s}} \|u_2\|_{X^T_{\frac{5}{16},0}} \|u_3\|_{X^T_{\frac{5}{16},0}},$$
$$\left\| |u|^2 u - |v|^2 v \right\|_{X^T_{-b,s}} \leq C(\|u\|^2_{X^T_{\frac{5}{16},s}} + \|v\|_{X^T_{\frac{5}{16},s}})^2 \|u - v\|_{X^T_{\frac{5}{16},s}}.$$

Moreover, there exist constants C and $C_s := C(s) > 0$, independent on $T \leq 1$, such that for every $s \geq 1$, follows that

$$\left\| |u|^2 u \right\|_{X^T_{-b,s}} \leq C\, \|u\|^2_{X^T_{\frac{5}{16},0}} \|u\|_{X^T_{\frac{5}{16},s}} + C_s\, \|u\|_{X^T_{\frac{5}{16},s-1}} \|u\|_{X^T_{\frac{5}{16},1}} \|u\|_{X^T_{\frac{5}{16},0}}.$$

Proof Here we will use the ideas contained in [5]. Let $w = u_1 u_2 \overline{u}_3$, by duality we have that

$$\begin{aligned} \|w\|_{X^T_{-b,s}} &= \sup_{\|c\|_{l^2_k L^2_\tau}\leq 1} \sum_{k=-\infty}^{+\infty} \int_\tau \langle k\rangle^s \langle \tau - p(k)\rangle^{-b} \hat{w}(k,\tau) c(k,\tau) d\tau \\ &= \sup_{\|c\|_{l^2_k L^2_\tau}\leq 1} \sum_{k,k_2,k_3=-\infty}^{+\infty} \int_{\tau,\tau_2,\tau_3} \langle k\rangle^s \langle \tau - p(k)\rangle^{-b} \hat{u}_1(k_1,\tau_1) \\ &\quad \times \hat{u}_2(k_2,\tau_2) \hat{\overline{u}}_3(k_3,\tau_3) c(k,\tau) d\tau d\tau_2 d\tau_3 \end{aligned} \tag{2.9}$$

where $k = k_1 + k_2 - k_3$ and $\tau = \tau_1 + \tau_2 - \tau_3$.

Assume that $\max\{|k_1|,|k_2|,|k_3|\}=|k_1|$ and define $\hat{f}(k,\tau)=c(k,\tau)\langle\tau-p(k)\rangle^{-b}$ and $\hat{v}(k_1,\tau_1)=\langle k_1\rangle^s\hat{u}_1(k_1,\tau_1)$. Then, last expression of (2.9) is bounded by

$$
\begin{aligned}
3 \sup_{\|c\|_{l_k^2 L_\tau^2}\le 1} \sum_{k,k_2,k_3=-\infty}^{+\infty} \int_{\tau,\tau_2,\tau_3} &|\hat{f}(\xi,\tau)\hat{v}(\xi_1,\tau_1)\hat{u}_2(k_2,\tau_2)\hat{\bar{u}}_3(k_3,\tau_3)|d\tau d\tau_2 d\tau_3 \\
&\le 3\|f\ v\ u_2\ u_3\|_{L_\tau^1 l_k^1} \\
&\le 3 \sup_{\|c\|_{l_k^2 L_\tau^2}\le 1} \|f\|_{L_\tau^4 l_k^4}\|v\|_{L_\tau^4 l_k^4}\|u_2\|_{L_\tau^4 l_k^4}\|u_3\|_{L_\tau^4 l_k^4} \\
&\le 3 \sup_{\|c\|_{l_k^2 L_\tau^2}\le 1} \|f\|_{X_{\frac{5}{16},0}}\|v\|_{X_{\frac{5}{16},0}}\|u_2\|_{X_{\frac{5}{16},0}}\|u_3\|_{X_{\frac{5}{16},0}} \\
&\le 3 \sup_{\|c\|_{l_k^2 L_\tau^2}\le 1} \|c\|_{L_\tau^2 l_k^2}\|u_1\|_{X_{\frac{5}{16},s}}\|u_2\|_{X_{\frac{5}{16},0}}\|u_3\|_{X_{\frac{5}{16},0}},
\end{aligned}
$$

where we have used Lemma 2.1 and the fact $b>\frac{5}{16}$. □

2.2 Auxiliary Lemmas

This subsection is devoted to present auxiliaries results related to the Bourgain space $X_{b,s}$ which are used several times in this work and play an important role on the main results of this article.

Lemma 2.3 *Let $-1\le b\le 1$, $s\in\mathbb{R}$ and $\varphi\in C^\infty(\mathbb{T})$. Then, for any $u\in X_{b,s}$, $\varphi(x)u\in X_{b,s-3|b|}$. Similarly, the multiplication by φ maps $X_{b,s}^T$ into $X_{b,s-3|b|}^T$.*

Proof We first consider the case of $b=0$ and $b=1$. The other cases of b will be derived later by interpolation and duality.

For $b=0$,

$$X_{0,s}=L^2\left(\mathbb{R};H^s(\mathbb{T})\right)$$

and the result is obvious. For $b=1$, we have $u\in X_{1,s}$ if and only if

$$u\in L^2(\mathbb{R};H^s(\mathbb{T})) \text{ and } i\partial_t u+\partial_x^2 u-\partial_x^4 u\in L^2(\mathbb{R};H^s(\mathbb{T})),$$

with the norm

$$\|u\|_{X_{1,s}}^2=\|u\|_{L^2(\mathbb{R};H^s(\mathbb{T}))}^2+\left\|i\partial_t u+\partial_x^2 u-\partial_x^4 u\right\|_{L^2(\mathbb{R};H^s(\mathbb{T}))}^2.$$

Thus,

$$
\begin{aligned}
\|\varphi(x)u\|_{X_{1,s-3}}^2 &= \|\varphi u\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2 + \left\|i\partial_t(\varphi u) + \partial_x^2(\varphi u) - \partial_x^4(\varphi u)\right\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2 \\
&\leq C\left(\|u\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2 + \left\|\varphi\left(i\partial_t u + \partial_x^2 u - \partial_x^4 u\right)\right\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2\right) \\
&\quad + \left\|\left[\varphi, \partial_x^2 - \partial_x^4\right]u\right\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2 \\
&\leq C\left(\|u\|_{L^2(\mathbb{R};H^{s-3}(\mathbb{T}))}^2 + \left\|i\partial_t u + \partial_x^2 u - \partial_x^4 u\right\|_{L^2(\mathbb{R};H^{s-4}(\mathbb{T}))}^2 + \|u\|_{L^2(\mathbb{R};H^{s}(\mathbb{T}))}^2\right) \\
&\leq C\,\|u\|_{X_{1,s}}^2 .
\end{aligned}
$$

Here, we have used the fact

$$
\left[\varphi, \partial_x^2 - \partial_x^4\right] = 4\left(\partial_x^3\varphi\right)\partial_x + 12\left(\partial_x^2\varphi\right)\partial_x^2 - 2\partial_x\varphi\partial_x + 4\partial_x\varphi\partial_x^3 + \left(\partial_x^4\varphi - \partial_x^2\varphi\right)
$$

is a differential operator of order 3.

To conclude, we prove that the $X_{b,s}$ spaces are in interpolation. Using Fourier transform, $X_{b,s}$ may be viewed as the weighted L^2 space $L^2\left(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s}\left\langle \tau + k^4 + k^2\right\rangle^{2b} \lambda \otimes \delta\right)$, where λ is the Lebesgue measure on $\mathbb{R}$ and δ is the discrete measure on $\mathbb{Z}$. Then, we use the complex interpolation theorem of Stein-Weiss for weighted L^p spaces (see [4, p. 114]): For $\theta \in (0, 1)$

$$
\left(X_{0,s}, X_{1,s'}\right)_{[\theta]} \approx L^2\left(\mathbb{R} \times \mathbb{Z}, \langle k\rangle^{2s(1-\theta)+2s'\theta}\left\langle \tau + k^4 + k^2\right\rangle^{2\theta} \mu \otimes \delta\right) \approx X_{\theta, s(1-\theta)+s'\theta}.
$$

Since the multiplication by φ maps $X_{0,s}$ into $X_{0,s}$ and $X_{1,s}$ into $X_{1,s-3}$, we conclude that for $b \in [0, 1]$, it maps $X_{b,s} = \left(X_{0,s}, X_{1,s}\right)_{[b]}$ into $\left(X_{0,s}, X_{1,s-3}\right)_{[b]} = X_{b,s-3b}$, which yields the $3b$ loss of regularity as announced.

Then, by duality, this also implies that for $b \in [0, 1]$, the multiplication by $\varphi(x)$ maps $X_{-b,-s+3b}$ into $X_{-b,-s}$. As the number s may take arbitrary values in $\mathbb{R}$, we also have the result for $b \in [-1, 0]$ with a loss of $-3b = 3|b|$. Finally, to get the same result for the restriction spaces $X_{b,s}^T$, consider

$$
\tilde{u} = \begin{cases} u, & \text{if } x \in \mathbb{T}, \\ 0, & \text{other cases,} \end{cases}
$$

thus

$$
\|\varphi u\|_{X_{b,s-3|b|}^T} \leq \|\varphi\tilde{u}\|_{X_{b,s-3|b|}} \leq C\,\|\tilde{u}\|_{X_{b,s}} .
$$

Taking the infimum on all the $\tilde{u}$, the result is achieved. □

Finally, to close this section, more four auxiliaries lemmas are enunciated. We follow [15], where the reader can also find the proofs, thus will be omitted it.

Lemma 2.4 *Let $\frac{1}{2} \leq b < 1$, $s \in \mathbb{R}$ and $T > 0$ be given. Then, there exists a constant $C > 0$ such that:*

(i) *For any* $\phi \in H^s(\mathbb{T})$,

$$\|W(t)\phi\|_{X^T_{b,s}} \leq C\|\phi\|_{H^s(\mathbb{T})};$$

(ii) *For any* $f \in X^T_{b-1,s}$,

$$\left\|\int_0^t W(t-\tau) f(\tau)\, d\tau\right\|_{X^T_{b,s}} \leq C\|f\|_{X^T_{b-1,s}}.$$

Lemma 2.5 *Let* $b \in [0,1]$ *and* $u \in X^T_{b,s}$, *then the function*

$$f : (0,T) \to \mathbb{R}$$
$$t \mapsto \|u\|_{X^t_{b,s}}$$

is continuous. Moreover, if $b > \frac{1}{2}$, *there exists* $C := C(b) > 0$ *such that*

$$\lim_{t\to 0^+} f(t) \leq C\|u(0)\|_{H^s(\mathbb{R})}. \tag{2.10}$$

Lemma 2.6 *Let* $b \in [0,1]$. *If* $\cup_{k=1}^n (a_k, b_k)$ *is a finite covering of* $(0,1)$, *then there exists a constant* $C > 0$ *depending only of the covering such that for every* $u \in X_{b,s}$, *we have*

$$\|u\|_{X_{b,s}[0,1]} \leq C\sum_{k=1}^n \|u\|_{X_{b,s}[a_k,b_k]}.$$

Lemma 2.7 *Let* $s \in \mathbb{R}$.

(i) *For any* $b \in \mathbb{R}$, *we have that*

$$\|\psi(t)e^{it(\partial_x^2-\partial_x^4)}\|_{X_{b,s}} \leq c\|\psi(t)\|_{H^b(\mathbb{R})}\|u_0\|_{H^s(\mathbb{T})}.$$

(ii) *Let* $\psi \in C_0^\infty(\mathbb{R})$ *and* $b,\ b'$ *satisfying* $0 < b' < \frac{1}{2} < b$ *and* $b + b' \leq 1$. *We have the following inequality*

$$\left\|\psi\left(\frac{t}{T}\right)\int_0^t e^{i(t-t')(\partial_x^2-\partial_x^4)}F(t')dt'\right\|_{X_{b,s}} \leq CT^{1-b-b'}\|F\|_{X_{-b',s}},$$

for $T \leq 1$.

3 Well-Posedness for 4NLS

In this section, we are interested in the existence of solution for 4NLS with source and damping terms. More precisely, the following result can be proved:

Theorem 3.1 *Let $T > 0$, $s \geq 0$, $\lambda \in \mathbb{R}$ and $\frac{1}{2} \leq b < \frac{11}{16}$. Let $a \in C_0^\infty(\mathbb{T})$ and $\varphi \in C_0^\infty(\mathbb{R})$ taking real values. For every $g \in L^2([-T,T]; H^s(\mathbb{T}))$ and $u_0 \in H^s(\mathbb{T})$, there exists a unique solution $u \in X_{b,s}^T$ of the Cauchy problem*

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u + i\varphi^2(t)a^2(x)u = \lambda |u|^2 u + g, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u_0(x) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{3.1}$$

Furthermore, the flow map

$$\begin{aligned} F : H^s(\mathbb{T}) \times L^2([-T,T]; H^s(\mathbb{T})) &\rightarrow X_{b,s}^T \\ (u_0, g) &\mapsto u \end{aligned}$$

is Lipschitz on every bounded subset. The same result is valid for $a \in L^\infty(\mathbb{T})$.

Proof Initially, we notice that $g \in X_{-b',s}^T$, for $b' \geq 0$. Let us restrict yourself to positive times. The solution on $[-T, 0]$ can be obtained similarly.

Define the integral operator by

$$\Lambda(u)(t) = e^{-it(\partial_x^2 - \partial_x^4)} u_0 - i \int_0^t e^{-i(t-t')(\partial_x^2 - \partial_x^4)} (\lambda |u|^2 u - i\varphi^2(t) a^2 u + g) dt'. \tag{3.2}$$

We are interested in applying the fixed point argument on the space $X_{b,s}^T$. To do this, let $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $t \in [-1,1]$. By Lemmas 2.7 and 2.2, for $0 < b' < \frac{1}{2} < b$ and $b + b' \leq 1$, we have that

$$\begin{aligned} \|\Lambda(u)\|_{X_{b,s}^T} &\leq C\|u_0\|_{H^s(\mathbb{T})} + CT^{1-b-b'} \|\lambda|u|^2 u - i\varphi^2(t) a + g\|_{X_{-b',s}^T} \\ &\leq C\|u_0\|_{H^s(\mathbb{T})} + CT^{1-b-b'} \left(\|\varphi^2 a^2 u\|_{X_{0,s}^T} + \||u|^2 u\|_{X_{b',s}^T} + \|g\|_{X_{-b',s}^T} \right) \\ &\leq C\|u_0\|_{H^s(\mathbb{T})} + CT^{1-b-b'} \|u\|_{X_{b,s}^T} \left(1 + \|u\|_{X_{b,0}^T}^2 \right) + CT^{1-b-b'} \|g\|_{X_{-b',s}^T}. \end{aligned} \tag{3.3}$$

In the same way, we get that

$$\|\Lambda(u) - \Lambda(v)\|_{X_{b,s}^T} \leq CT^{1-b-b'} (1 + \|u\|_{X_{b,s}^T}^2 + \|v\|_{X_{b,s}^T}^2) \|u - v\|_{X_{b,s}^T}. \tag{3.4}$$

These estimates imply that if T is chosen small enough Λ is a contraction on a suitable ball of $X_{b,s}^T$.

Now, we prove the uniqueness in the class $X_{b,s}^T$ for the integral equation (3.2). Set

$$w(t) = e^{-it(\partial_x^2 - \partial_x^4)} u_0 - i \int_0^t e^{-i(t-t')(\partial_x^2 - \partial_x^4)} (\lambda |u|^2 u - i\varphi^2(t) a + g) dt'. \tag{3.5}$$

By using (2.2) we have that $|u|^2u \in X^T_{-b',s}$, for any b' satisfying $-\frac{5}{16} < b' < \frac{1}{2}$. Moreover, we have that

$$\partial_t \left[\int_0^t e^{-i\tau(\partial_x^2 - \partial_x^4)}(-ia^2\varphi^2 u + \lambda|u|^2u + g)(\tau) \right] d\tau \\ = e^{-it(\partial_x^2 - \partial_x^4)}(-ia^2\varphi^2 u + \lambda|u|^2u + g)(\tau), \tag{3.6}$$

in the distributional sense. This implies that w solves the following equation

$$i\partial_t w + \partial_x^2 w - \partial_x^4 w + ia^2\varphi^2 u = \lambda|u|^2u + g. \tag{3.7}$$

It follows that $v(t) = e^{-it(\partial_x^2 - \partial_x^4)}(u - w)$ is solution of $\partial_t v = 0$ and $v(t) = 0$, respectively. Thus, $v = 0$ and u is solution of the integral equation.

Let us prove the propagation of regularity. Firstly, note that for $u_0 \in H^s(\mathbb{T})$, with $s > 0$, we have an existence of time T for the solution u of the integral equation in $X^T_{b,0}$ and another solution $\tilde{u}$ on time $\tilde{T}$ in $X^{\tilde{T}}_{b,s}$. By uniqueness in $X^T_{b,0}$ booth solutions u and $\tilde{u}$ are the same on $[0, \tilde{T}]$. Admitting $\tilde{T} < T$, we get that the blow up of $\|u(t)\|_{H^s(\mathbb{T})}$, as $t \to \tilde{T}$, while $\|u\|_{L^2(\mathbb{T})}$ remains bounded on this interval. By using local existence in $L^2(\mathbb{T})$ and Lemma 2.6 we see that $\|u\|_{X^{\tilde{T}}_{b,0}}$ is finite. Thus, estimate (3.3) on $[\tilde{T} - \epsilon, \tilde{T}]$, with ϵ small enough such that

$$C\epsilon^{1-b-b'}(1 + \|u\|^2_{X_{b,0[\tilde{T}-\epsilon,\tilde{T}]}}) < \frac{1}{2},$$

ensures

$$\|u\|_{X_{b,s[\tilde{T}-\epsilon,\tilde{T}]}} \leq C(\|u(T-\epsilon)\|_{H^s(\mathbb{T})} + \|g\|_{X_{-b',s}}).$$

Therefore, $u \in X^{\tilde{T}}_{b,s}$, contradicting the blow up of $\|u(t)\|_{H^s(\mathbb{T})}$ near $\tilde{T}$.

The second step is to use $L^2(\mathbb{T})$ energy estimates to obtain global existence in $L^2(\mathbb{T})$ and consequently, by using the above argument, in $H^s(\mathbb{T})$. Multiplying (3.1) by $\overline{u}$, taking imaginary part, integrating by parts and using Cauchy–Schwarz inequality, we get

$$\|u(t)\|^2_{L^2(\mathbb{T})} \leq \|u_0\|^2_{L^2(\mathbb{T})} + C\int_0^t \|u(\tau)\|^2_{L^2(\mathbb{T})} d\tau + C\|g\|_{L^2([-T,T];L^2(\mathbb{T}))}. \tag{3.8}$$

By using Gronwall inequality, we have that

$$\|u(t)\|^2_{L^2(\mathbb{T})} \leq C(\|u_0\|^2_{L^2(\mathbb{T})} + \|g\|_{L^2([-T,T];L^2(\mathbb{T}))})e^{C|t|}.$$

So, the $L^2(\mathbb{T})$-norm remains bounded and the solution u is global in time.

Lastly, we prove the continuity of flow. Let $\tilde{u}$ solution of (3.1) with $\tilde{u}_0$ and $\tilde{g}$, instead u_0 and g. A slight modification of (3.4) yields that

$$\|u-\tilde{u}\|_{X^T_{b,s}} \leq C\|u_0-\tilde{u}_0\|_{H^s(\mathbb{T})} + C\|g-\tilde{g}\|_{X_{-b',s}} + CT^{1-b-b'}(1+\|u\|^2_{X^T_{b,s}} + \|\tilde{u}\|^2_{X^T_{b,s}})\|u-v\|_{X^T_{b,s}}.$$

Then, for T small enough depending on the size of u_0, $\tilde{u}_0$, g and $\tilde{g}$, follows that

$$\|u-\tilde{u}\|_{X^T_{b,s}} \leq C\|u_0-\tilde{u}_0\|_{H^s(\mathbb{T})} + C\|g-\tilde{g}\|_{X^T_{-b',s}}.$$

Thus, the map data to solution is Lipschitz continuous on bounded sets for arbitrary T and, consequently, the proof is complete. □

The next two propositions are to give estimates that connect the solution u of the 4NLS (3.1) with the damping term and source term.

Proposition 3.2 *For every $T>0$, $\eta>0$ and $s\geq 0$, there exists $C(T,\eta,s)$ such that for every $u\in X^T_{b,s}$ solution of (3.1) with $\|u_0\|_{H^s(\mathbb{T})}+\|g\|_{L^2([0,T];H^s(\mathbb{T}))}\leq\eta$ the following estimate holds*

$$\|u\|_{X^T_{b,s}} \leq C(T,\eta,s)(\|u_0\|_{H^s(\mathbb{T})}+\|g\|_{L^2([0,T];H^s(\mathbb{T}))}).$$

Proof Initially, assume $T\leq 1$. Using (3.3) we have that

$$\|u\|_{X^T_{s,b}} \leq C\left(\|u_0\|_{H^s(\mathbb{T})}+\|g\|_{X^T_{-b',s}}\right) + C_S T^{1-b-b'}\|u\|_{X^T_{b,s}}\left(1+\|u\|^2_{X^T_{b,0}}\right).$$

Choose T such that $C_s T^{1-b-b'}\leq\frac{1}{2}$, the following inequality holds

$$\|u\|_{X^T_{s,b}} \leq C\left(\|u_0\|_{H^s(\mathbb{T})}+\|g\|_{X^T_{-b',s}}\right) + C_s T^{1-b-b'}\|u\|_{X^T_{b,s}}\|u\|^2_{X^T_{b,0}}. \quad (3.9)$$

By using estimate (3.9), for $s=0$, and choosing T_1 satisfying

$$T_1^{1-b-b'} < \frac{1}{2C_0\left(\|u_0\|_{H^s(\mathbb{T})}+\|g\|_{X^{T_1}_{-b',0}}\right)^2}, \quad (3.10)$$

we obtain

$$\|u\|_{X^{T_1}_{0,b}} \leq C\left(\|u_0\|_{L^2(\mathbb{T})}+\|g\|_{X^{T_1}_{-b',0}}\right) \leq C\left(\|u_0\|_{L^2(\mathbb{T})}+\|g\|_{L^2((T_1,T_1+\epsilon);L^2(\mathbb{T}))}\right). \quad (3.11)$$

On the other hand, estimate (3.8) implies

$$\|u(t)\|_{L^2(\mathbb{T})} \leq C\eta e^{C|t|} \leq C\eta e^{C} \leq C(\eta), \quad (3.12)$$

where we have used that $T \leq 1$. Then, thanks to (3.11) and (3.12), there exists a constant $\epsilon = \epsilon(\eta)$ such that

$$\|u(T_1^- + s)\|_{X^{\epsilon}_{b,0}} \leq C(\|u(T_1)\|_{L^2(\mathbb{T})} + \|g(t)\|_{L^2((T_1,T_1+\epsilon);L^2(\mathbb{T}))}), \tag{3.13}$$

and follows that the estimate (3.11) is valid for some large interval $[0, T]$, with $T \leq 1$, for any constant C depending of η.

Now, we back to the case $s > 0$. For T_s satisfying $C_s T_s^{1-b-b'} \leq \frac{1}{2}$, by using (2.9) and (3.9), we obtain

$$C_s T_s^{1-b-b'} \|u\|^2_{X^{T_s}_{b,0}} \leq C_s T_s^{1-b-b'} C(\eta)^2 \eta^2.$$

Thus, follows that for an appropriate $T \leq \epsilon(\eta, s)$ the last expression can be controlled by $1/2$, therefore, the following inequality is also true

$$\|u\|_{X^T_{s,b}} \leq C \left(\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2((0,T);H^s(\mathbb{T}))}\right).$$

Again, piecing solutions together, we get the same result for large $T \leq 1$ with C depending only of η and s. Finally, the assumption $T \leq 1$ is removed similarly with a final constant $C(s, \eta, T)$. □

Proposition 3.3 *For every $T > 0$ and $\eta > 0$, there exists a constant $C(T, \eta)$ such that for all $s \geq 1$, we can find $C(T, \eta, s)$ such that $u \in X^T_{b,s}$ solution of (3.1) with*

$$\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2([0,T];H^s(\mathbb{T}))} \leq \eta,$$

satisfies

$$\begin{aligned}\|u\|_{X^T_{b,s}} \leq{}& C(T, \eta)(\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2([0,T];H^s(\mathbb{T}))}) \\ &+ C(T, \eta, s) \left(\|u\|_{X^T_{b,s-1}} \|u\|_{X^T_{b,1}} \|u\|_{X^T_{b,0}} + \|u\|_{X^T_{b,s-1}}\right).\end{aligned}$$

Proof Initially, we assume $T \leq 1$. By using Lemma 2.7 we have a constant C, independent of s, such that

$$\begin{aligned}\|u\|_{X^T_{b,s}} \leq{}& C \left(\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2([0,T]:H^s(\mathbb{T}))}\right) \\ &+ CT^{1-b-b'} \left(\|a^2\varphi^2 u\|_{L^2([0,T];H^s(\mathbb{T}))} + \||u|^2 u\|_{X^T_{b',s-1}}\right),\end{aligned}$$

for $b,\ b'$ satisfying $0 \leq b' < \frac{1}{2} < b, b + b' \leq 1$.

[24, Lemma A.1] and Lemma 2.2 give us constants C and C_s which leads to the following inequality

$$\begin{aligned}\|u\|_{X^T_{b,s}} \leq C(\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2([0,T];H^s(\mathbb{T}))})\\ + T^{1-b-b'}(C\|u\|_{X^T_{b,s}} + C_s\|u\|_{X^T_{b,s-1}})\\ + T^{1-b-b'}(C\|u\|^2_{X^T_{b,0}}\|u\|_{X^T_{b,s}} + C_s\|u\|_{X^T_{b,s-1}}\|u\|_{X^T_{b,1}}\|u\|_{X^T_{b,0}}).\end{aligned} \tag{3.14}$$

From Proposition 3.2, we have that

$$\|u\|_{X^T_{b,0}} \leq C(\eta, T)\left(\|u_0\|_{L^2(\mathbb{T})} + \|g\|_{L^2([0,T];H^s(\mathbb{T}))}\right) \leq C(\eta, T)\eta. \tag{3.15}$$

Here, for $T \leq 1$ in the last inequality, we get $C(\eta) := C(\eta, 1)$.

By putting (3.15) into (3.14), for small enough T (depending only η), we get

$$\begin{aligned}\|u\|_{X^T_{b,s}} \leq C(\|u_0\|_{H^s(\mathbb{T})} + \|g\|_{L^2([0,T];H^s(\mathbb{T}))})\\ + C(s)(\|u\|_{X^T_{b,s-1}}\|u\|_{X^T_{b,1}}\|u\|_{X^T_{b,0}} + \|u\|_{X^T_{b,s-1}}).\end{aligned}$$

Then, the conclusion of lemma follows by a bootstrap argument. □

4 Local Controllability

This section is devoted to prove the local controllability near of the null trajectory of the 4NLS (1.6) by a perturbation argument near the one done by Zuazua in [41]. Then, we will use the fixed point theorem of Picard to deduce our result from the linear control.

First of all, we know (see, for instance, [39,40]) that any nonempty set ω satisfies an *observability inequality* in $L^2(\mathbb{T})$ for arbitrary small time $T > 0$. This means that:

For any $a(x) \in C^\infty(\mathbb{T})$ *and* $\varphi(t) \in C_0^\infty(0, T)$ *real valued such that* $a \equiv 1$ *on* ω *and* $\varphi \equiv 1$ *on* $[T/3, 2T/3]$, *there exists* $C > 0$ *such that*

$$\|\Psi_0\|^2_{L^2(\mathbb{T})} \leq C\int_0^T \left\|a(x)\varphi(t)e^{it(\partial_x^2-\partial_x^4)}\Psi_0\right\|^2_{L^2(\mathbb{T})} dt, \tag{4.1}$$

for every $\Psi_0 \in L^2(\mathbb{T})$.

Exact controllability property for a control system is equivalent to the observability of its adjoint system by using the Hilbert Uniqueness Method introduced by Lions [28]. Thus, observability inequality (4.1) implies the exact controllability in $L^2(\mathbb{T}) := L^2$ for the linear equation associated to (1.6).

To be precise, let us follow [12, Sect. 5] to construct an isomorphism of control

$$\begin{aligned}\mathcal{R} :L^2 &\to L^2\\ \Phi_0 &\to \mathcal{R}\Phi_0 = \Psi_0\end{aligned}$$

such that if Φ is solution of the adjoint system

$$\begin{cases} i\partial_t\Phi + \partial_x^2\Phi - \partial_x^4\Phi = 0, & (x,t) \in \mathbb{T}\times\mathbb{R}, \\ \Phi(x,0) = \Phi_0(x), & x \in \mathbb{T} \end{cases} \tag{4.2}$$

and Ψ solution of

$$\begin{cases} i\partial_t\Psi + \partial_x^2\Psi - \partial_x^4\Psi = a^2(x)\varphi^2(t)\Phi, & (x,t) \in \mathbb{T}\times\mathbb{R}, \\ \Psi(x,T) = \Psi_T(x) = 0, & x \in \mathbb{T}, \end{cases}$$

we get that $\Psi(x,0) = \Psi_0(x)$. First, notice that application $\mathcal{R}$ has the following property:

Lemma 4.1 *For every $s \geq 0$, $\mathcal{R}$ is an isomorphism of $H^s(\mathbb{T})$.*

Proof To get the result we need to prove that $\mathcal{R}$ maps $H^s(\mathbb{T})$ into itself and $\mathcal{R}\Phi_0 \in H^s(\mathbb{T})$ implies $\Phi_0 \in H^s(\mathbb{T})$, that is, $D^s\Phi_0 \in L^2$, where D^s is defined by (1.5). Is not difficult to see that $\mathcal{R}$ maps $H^s(\mathbb{T})$ into itself. Thus, we check the following:
Claim: $\mathcal{R}\Phi_0 \in H^s(\mathbb{T})$ implies $\Phi_0 \in H^s(\mathbb{T})$.

The claim is equivalent to show that $D^s\Phi_0 \in L^2$, D^s is defined by (1.5). Remember that,

$$\mathcal{R}\Phi_0 = i\int_0^T e^{-it(\partial_x^2-\partial_x^4)}\varphi^2a^2e^{it(\partial_x^2-\partial_x^4)}\Phi_0 dt.$$

Since $\mathcal{R}^{-1}$ is continuous from L^2 into itself we get, using [24, Lemma A.1], that

$$\begin{aligned} \left\|D^s\Phi_0\right\|_{L^2} &\leq C\left\|\mathcal{R}D^s\Phi_0\right\|_{L^2} \leq C\left\|\int_0^T e^{-it(\partial_x^2-\partial_x^4)}\varphi^2a^2e^{it(\partial_x^2-\partial_x^4)}D^s\Phi_0 dt\right\|_{L^2} \\ &\leq C\left\|D^s\int_0^T e^{-it(\partial_x^2-\partial_x^4)}\varphi^2a^2e^{it(\partial_x^2-\partial_x^4)}\Phi_0 dt\right\|_{L^2} \\ &\quad + C\left\|\int_0^T e^{-it(\partial_x^2-\partial_x^4)}[a^2, D^s]\varphi^2e^{it(\partial_x^2-\partial_x^4)}\Phi_0 dt\right\|_{L^2} \\ &\leq C\left\|\mathcal{R}\Phi_0\right\|_{H^s(\mathbb{T})} + C_sC\left\|\Phi_0\right\|_{H^{s-1}(\mathbb{T})}. \end{aligned}$$

Thus, the result for $s \in [0,1]$ is proved. The result for $s \geq 1$ can be guaranteed by iteration. Finally, previous computation, for $s \geq 1$, give us

$$\left\|\mathcal{R}^{-1}\Psi_0\right\|_{H^s(\mathbb{T})} \leq C(a,\psi,T)\left\|\Psi_0\right\|_{H^s(\mathbb{T})} + C(a,\psi,s,T)\left\|\Psi_0\right\|_{H^{s-1}(\mathbb{T})}. \tag{4.3}$$

Therefore, the claim is proved and, consequently, the lemma is verified. □

4.1 Proof of Theorem 1.1

Pick $a(x) \in C_0^\infty(\omega)$ and $\psi(t) \in C_0^\infty(0,T)$ different from zero, such that, observability inequality (4.1) holds. We look for the function g of the form $\varphi^2(t)a^2(x)\Phi$, where Φ is solution of (4.2) as in linear control problem.

We are interested in choosing an appropriate Φ_0 such that we can recover the controllability properties of the system (1.6). Consider the following two systems

$$\begin{cases} i\partial_t \Phi + \partial_x^2 \Phi - \partial_x^4 \Phi = 0, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ \Phi(x,0) = \Phi_0(x), & x \in \mathbb{T} \end{cases}$$

and

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda|u|^2 u + a^2\varphi^2\Phi, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,T) = 0, & x \in \mathbb{T}. \end{cases}$$

We also define the operator $\mathcal{L}$ as follows

$$\begin{aligned} \mathcal{L} : L^2(\mathbb{T}) &\to L^2(\mathbb{T}) \\ \Phi_0 &\mapsto \mathcal{L}\Phi_0 = u_0 = u(0). \end{aligned} \tag{4.4}$$

The goal is then to show that $\mathcal{L}$ is onto on a small neighborhood of the origin of $H^s(\mathbb{T})$, for $s \geq 0$. Split u as $u = v + \Psi$, with Ψ solution of

$$\begin{cases} i\partial_t \Psi + \partial_x^2 \Psi - \partial_x^4 \Psi = a^2(x)\varphi^2(t)\Phi, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ \Psi(x,T) = 0, & x \in \mathbb{T}. \end{cases} \tag{4.5}$$

It corresponds to the linear control, and thus, $\Psi(0) = \mathcal{R}\Phi_0$. Moreover, observe that v is solution of the system

$$\begin{cases} i\partial_t v + \partial_x^2 v - \partial_x^4 v = \lambda|u|^2 u, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ v(x,T) = 0, & x \in \mathbb{T}. \end{cases} \tag{4.6}$$

Therefore, u, v and Ψ belong to $X_{b,0}^T$ and $u(0) = v(0) + \Psi(0)$, which we can write $\mathcal{L}\Phi_0$ as follows

$$\mathcal{L}\Phi_0 = \mathcal{K}\Phi_0 + \mathcal{R}\Phi_0,$$

where $\mathcal{K}\Phi_0 = v(0)$. Observe that $\mathcal{L}\Phi_0 = u_0$, or equivalently, $\Phi_0 = -\mathcal{R}^{-1}\mathcal{K}\Phi_0 + \mathcal{R}^{-1}u_0$.

Define the operator

$$\begin{aligned} \mathcal{B} : &L^2 \to L^2 \\ &\Phi_0 \to \mathcal{B}\Phi_0 = \mathcal{R}^{-1}\mathcal{K}\Phi_0 + \mathcal{R}^{-1}u_0. \end{aligned}$$

We want to prove that $\mathcal{B}$ has a fixed point. To do it, let us firstly define the following set

$$F := B_{L^2(\mathbb{T})}(0,\eta) \cap \left(\bigcap_{i=1}^{[s]-1} B_{H^i(\mathbb{T})}(0,R_i)) \right) \cap B_{H^s(\mathbb{T})}(0,R_s),$$

for η small enough and for some large R_i. We may assume $T < 1$ and fix it, moreover, we will denote C and $C_s = C(s)$ any constant depending of a, φ, b, b', T and s, respectively.

By using Lemma 4.1, we have that $\mathcal{R}$ is an isomorphism of $H^s(\mathbb{T})$, thus

$$\|\mathcal{B}\Phi_0\|_{H^s(\mathbb{T})} \leq C_s \left(\|\mathcal{K}\Phi_0\|_{H^s(\mathbb{T})} + \|u_0\|_{H^s(\mathbb{T})} \right). \tag{4.7}$$

By the last inequality, we should estimate $\|\mathcal{K}\Phi_0\|_{H^s(\mathbb{T})} = \|v_0\|_{H^s(\mathbb{T})}$. Then, we will apply for equation (4.6) the same $X^T_{b,s}$ estimates which we used in Theorem 3.1, more precisely, Lemmas 2.7 and 2.2. Thus, we get

$$\begin{aligned} \|v(0)\|_{H^s(\mathbb{T})} &\leq C \|v\|_{X^T_{b,s}} \leq C T^{1-b-b'} \left\| |u|^2 u \right\|_{X^T_{-b',s}} \leq C \left\| |u|^2 u \right\|_{X^T_{-b',s}} \\ &\leq C_s \|u\|^2_{X^T_{b,0}} \|u\|_{X^T_{b,s}}. \end{aligned} \tag{4.8}$$

By the local linear behavior of u, that is, by using Proposition 3.2, we obtain

$$\|u\|_{X^T_{b,0}} \leq C \|\Phi_0\|_{L^2(\mathbb{T})},$$

for

$$\left\| \varphi^2 a^2 \Phi \right\|_{L^2([0,T];L^2(\mathbb{T}))} \leq C \|\Phi_0\|_{L^2(\mathbb{T})} < C\eta < 1.$$

Finally, applying (4.7) and (4.8), with $s = 0$, this ensures that

$$\|\mathcal{B}\Phi_0\|_{L^2(\mathbb{T})} \leq C \left(\|\Phi_0\|^3_{L^2(\mathbb{T})} + \|u_0\|_{L^2(\mathbb{T})} \right).$$

Then, by the last inequality, choosing η small enough and $\|u_0\|_{L^2(\mathbb{T})} \leq \frac{\eta}{2C}$, we have that

$$\|\mathcal{B}\Phi_0\|_{L^2(\mathbb{T})} \leq \eta$$

and, therefore, $\mathcal{B}$ reproduces the ball B_η in $L^2(\mathbb{T})$.

To prove the result on a small neighborhood of the origin $H^s(\mathbb{T})$, we will divide the proof in two steps.

Step 1 $s \in (0, 1]$

For $s \leq 1$, we came back to (4.8) with the following new estimates in $X^T_{b,s}$

$$\|v(0)\|_{H^s(\mathbb{T})} \leq C_s\eta^2 \|u\|^2_{X^T_{b,s}}$$

and

$$\|\mathcal{B}\Phi_0\|_{H^s(\mathbb{T})} \leq C_s \left(\eta^2 \|u\|_{X^T_{b,s}} + \|u_0\|_{H^s(\mathbb{T})}\right).$$

Thus, using Proposition 3.2 for

$$\left\|\varphi^2 a^2 \Phi\right\|_{L^2([0,T];L^2(\mathbb{T}))} \leq C \|\Phi_0\|_{L^2(\mathbb{T})} < C\eta < 1,$$

we have

$$\|u\|_{X^T_{b,s}} \leq C_s \|\Phi_0\|^2_{X^T_{b,s}} \tag{4.9}$$

and

$$\|\mathcal{B}\Phi_0\|_{H^s(\mathbb{T})} \leq C_s \left(\eta^2 \|\Phi_0\|_{H^s(\mathbb{T})} + \|u_0\|_{H^s(\mathbb{T})}\right).$$

Then, by these two inequalities, for $C_s\eta^2 < 1/2$, $\mathcal{B}$ reproduces any ball in $H^s(\mathbb{T})$ of the radius greater than $2C_s \|u_0\|_{H^s(\mathbb{T})}$. Therefore, we conclude that $\mathcal{B}$ reproduces the ball in F, if $\eta < \tilde{C}_s$, $\|u_0\|_{H^s(\mathbb{T})} \leq C(\eta)$ and $R \geq C(\|u_0\|_{H^s(\mathbb{T})})$. Furthermore, since these estimates are uniform in $s \in (0, 1]$, the bound on η is also uniform.

Step 2 $s > 1$

We will start choosing R_i by induction as follows: Chosen R_1 as the previous case so that $\mathcal{B}$ reproduces $B_{H^1(\mathbb{T})}(0, R_1)$. It is important, in this point, to make some assumptions of smallness on η which on will be independent of i and s. Firstly, using the estimate (4.3) we get

$$\|\mathcal{B}\Phi_0\|_{H^i(\mathbb{T})} \leq C \|\mathcal{K}\Phi_0\|_{H^i(\mathbb{T})} + C_i \|\mathcal{K}\Phi_0\|_{H^{i-1}(\mathbb{T})} + C_i \|u_0\|_{H^i(\mathbb{T})}.$$

Analogously, for $s \in (0, 1]$, we have that

$$\|\mathcal{K}\Phi_0\|_{H^{i-1}(\mathbb{T})} \leq C_{i-1}\eta^2 \|\Phi_0\|_{H^{i-1}(\mathbb{T})} \leq C_{i-1}\eta^2 R_{i-1}.$$

Using multilinear estimate, Lemma 2.2, the following holds

$$\begin{aligned}\|v(0)\|_{H^i(\mathbb{T})} &\leq C \|v\|_{X^T_{b,i}} \leq C \left\||u|^2 u\right\|_{X^T_{-b',i}} \\ &\leq C \|u\|^2_{X^T_{b,0}} \|u\|_{X^T_{b,i}} + C_i \|u\|_{X^T_{b,i-1}} \|u\|_{X^T_{b,1}} \|u\|_{X^T_{b,0}}.\end{aligned}$$

Right now, we would like to bound the term with maximum derivative. For this, we use Proposition 3.3 and [24, Corollary A.2] to obtain

$$\begin{aligned}\|u\|_{X^T_{b,1}(\mathbb{T})} &\leq C\left\|\varphi^2a^2\Phi\right\|_{L^2([0,T];H^i(\mathbb{T}))}+C_i\|u\|_{X^T_{b,i-1}}+C_i\|u\|_{X^T_{b,i-1}}\|u\|_{X^T_{b,1}}\|u\|_{X^T_{b,0}}\\ &\leq C\|\Phi_0\|_{H^i(\mathbb{T})}+C_i\|\Phi_0\|_{H^{i-1}(\mathbb{T})}+C_i\|u\|_{X^T_{b,i-1}}\\ &\quad+C_i\|u\|_{X^T_{b,i-1}}\|u\|_{X^T_{b,1}}\|u\|_{X^T_{b,0}}.\end{aligned}$$

By (4.9), we can also bound the lower derivative, which yields

$$\begin{aligned}\|v(0)\|_{H^i(\mathbb{T})} &\leq C\eta^2\|u\|^2_{X^T_{b,i}}+C_iR_{i-1}R_1\eta\\ &\leq C\eta^2\|\Phi_0\|^2_{H^i(\mathbb{T})}+C\eta^2(C_iR_{i-1}+C_iR_{i-1}R_1\eta)+C_iR_{i-1}R_1\eta.\end{aligned}$$

Finally, we ensures that

$$\|\mathcal{B}\Phi_0\|_{H^i(\mathbb{T})}\leq C\eta^2\|\Phi_0\|_{H^i(\mathbb{T})}+C(i,\eta,R_1,R_{i-1},\|u_0\|_{H^i(\mathbb{T})}).$$

Choosing $C\eta^2<1/2$ independent of s and $R_i=C(i,\eta,R_1,R_{i-1},\|u_0\|_{H^i(\mathbb{T})})$, then B reproduces $B_{H^i(\mathbb{T})}(0,R_i)$. The same argument is still valid for $s\geq 1$ and Step 2 is thus proved.

To finalize, $\mathcal{B}$ is contracting for $L^2(\mathbb{T})-$norm. Indeed, consider the following systems

$$\begin{cases} i\partial_t(u-\tilde{u})+\partial_x^2(u-\tilde{u})-\partial_x^4(u-\tilde{u})=\lambda(|u|^2u-|\tilde{u}|^2\tilde{u})+a^2\varphi^2(\Phi-\tilde{\Phi}), & (x,t)\in\mathbb{T}\times\mathbb{R},\\ (u-\tilde{u})(x,T)=0, & x\in\mathbb{T}\end{cases}\tag{4.10}$$

and

$$\begin{cases} i\partial_t(v-\tilde{v})+\partial_x^2(v-\tilde{v})-\partial_x^4(v-\tilde{v})=\lambda(|u|^2u-|\tilde{u}|^2\tilde{u}), & (x,t)\in\mathbb{T}\times\mathbb{R},\\ (v-\tilde{v})(x,T)=0, & x\in\mathbb{T}.\end{cases}$$

Lemma 2.2 ensures that

$$\begin{aligned}\left\|\mathcal{B}\Phi_0-\mathcal{B}\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})} &\leq\|(v-\tilde{v})(0)\|_{L^2(\mathbb{T})}\leq C\|(v-\tilde{v})\|_{X^T_{b,0}}\\ &\leq CT^{1-b-b'}\left\||u|^2u-|\tilde{u}|^2\tilde{u}\right\|_{X^T_{-b',0}}\\ &\leq C\left(\|u\|^2_{X^T_{b,0}}+\|\tilde{u}\|^2_{X^T_{b,0}}\right)\|u-\tilde{u}\|_{X^T_{b,0}}\\ &\leq C\eta^2\|u-\tilde{u}\|_{X^T_{b,0}}.\end{aligned}\tag{4.11}$$

To bound $\|u-\tilde{u}\|_{X^T_{b,0}}$ in the last inequality (4.11), we use the equation (4.10) to deduce

$$
\begin{aligned}
\|u-\tilde{u}\|_{X^T_{b,0}(\mathbb{T})} &\leq C\left\|\varphi^2 a^2(\Phi-\tilde{\Phi})\right\|_{L^2([0,T];L^2(\mathbb{T}))}+CT^{1-b-b'}\left\||u|^2 u-|\tilde{u}|^2\tilde{u}\right\|_{X^T_{-b',0}}\\
&\leq C\left\|\Phi_0-\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})}+C\left(\|u\|^2_{X^T_{b,0}}+\|\tilde{u}\|^2_{X^T_{b,0}}\right)\|u-\tilde{u}\|_{X^T_{b,0}}\\
&\leq C\left\|\Phi_0-\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})}+C\eta^2\|u-\tilde{u}\|_{X^T_{b,0}}.
\end{aligned}
$$

Taking η small enough (independent on s) it yields

$$
\|u-\tilde{u}\|_{X^T_{b,0}(\mathbb{T})} \leq C\left\|\Phi_0-\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})}. \tag{4.12}
$$

To finish, combining (4.12) into (4.11) follows that

$$
\left\|\mathcal{B}\Phi_0-\mathcal{B}\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})} \leq C\eta^2\left\|\Phi_0-\tilde{\Phi}_0\right\|_{L^2(\mathbb{T})}.
$$

Therefore, $\mathcal{B}$ is a contraction of a closed set F of $L^2(\mathbb{T})$, for η small enough (independent on s). In addition, $\mathcal{B}$ has a fixed point which, by construction, belongs to $H^s(\mathbb{T})$. This completes the proof of Theorem 1.1. □

5 Propagation of Compactness and Regularity in Bourgain Spaces

We present, in this section, some properties of propagation in Bourgain spaces for the linear differential operator $L = i\partial_t + \partial_x^2 - \partial_x^4$ associated with the fourth order Schrödinger equation. We will adapt the results due Dehman et al. [12, Propositions 13 and 15], in the case of $X_{b,s}$ spaces, of the Schrödinger operator. These results of propagation are the key to prove the global stabilization. The main ingredient is basically pseudo-differential analysis. Let us begin with a result of propagation of compactness which will ensure strong convergence in appropriate spaces for the study of the global stabilization.

Proposition 5.1 (Propagation of compactness) *Let $T > 0$ and $0 \leq b' \leq b \leq 1$ be given. Suppose that $u_n \in X^T_{b,0}$ and $f_n \in X^T_{-b,-3+3b}$ satisfying*

$$
i\partial_t u_n + \partial_x^2 u_n - \partial_x^4 u_n = f_n,
$$

for $n = 1, 2, 3, \ldots$. Assume that there exists a constant $C > 0$ such that

$$
\|u_n\|_{X^T_{b,0}} \leq C \tag{5.1}
$$

and

$$
\|u_n\|_{X^T_{-b,-3+3b}} + \|f_n\|_{X^T_{-b,-3+3b}} + \|u_n\|_{X^T_{-b',-1+3b'}} \to 0,\ as\ n \to +\infty. \tag{5.2}
$$

In addition, assume that for some nonempty open set $\omega \subset \mathbb{T}$

$$u_n \to 0 \text{ strongly in } L^2\left(0, T; L^2(\omega)\right).$$

Then

$$u_n \to 0 \text{ strongly in } L^2_{loc}\left([0, T]; L^2(\mathbb{T})\right).$$

Proof Pick $\varphi \in C^\infty(\mathbb{T})$ and $\psi \in C_0^\infty(0, T)$ real valued and set

$$B = \varphi(x) D^{-3} \text{ and } A = \psi(t) B,$$

where D^{-3} is defined by (1.5). Then

$$A^* = \psi(t) D^{-3}\varphi(x).$$

For $\epsilon > 0$, we denote $A_\epsilon = Ae^{\epsilon\partial_x^2} = \psi(t) B_\epsilon$ for the regularization of A. By a classical way, we can write

$$\begin{aligned}\alpha_{n,\epsilon} &= i\left(-\psi'(t) B_\epsilon u_n, u_n\right) + \left(A_\epsilon u_n, \left(\partial_x^2 - \partial_x^4\right) u_n\right)\\ &= \left(\left[A_\epsilon, \partial_x^2 - \partial_x^4\right] u_n, u_n\right) - i\left(\psi'(t) B_\epsilon u_n, u_n\right).\end{aligned}$$

On the other hand, we have

$$\alpha_{n,\epsilon} = \left(f_n, A_\epsilon^* u_n\right)_{L^2(\mathbb{T}\times(0,T))} - \left(A_\epsilon u_n, f_n\right)_{L^2(\mathbb{T}\times(0,T))}. \tag{5.3}$$

By using Hölder inequality and Lemma 2.3, we get that

$$\begin{aligned}\left|\left(f_n, A_\epsilon^* u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\le \|f_n\|_{X^T_{-b,-3+3b}} \left\|A_\epsilon^* u_n\right\|_{X^T_{b,3-3b}}\\ &\le \|f_n\|_{X^T_{-b,-3+3b}} \|u_n\|_{X^T_{b,0}}.\end{aligned}$$

Therefore, from (5.1) and (5.2), follows that

$$\lim_{n\to\infty} \sup_{0<\epsilon\le 1} \left|\left(f_n, A_\epsilon^* u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{5.4}$$

Similar computations yields that

$$\begin{aligned}\lim_{n\to\infty} \sup_{0<\epsilon\le 1} \left|\left(A_\epsilon u_n, f_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &= 0\\ \lim_{n\to\infty} \sup_{0<\epsilon\le 1} \left|\left(\psi'(t) B_\epsilon u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &= 0.\end{aligned} \tag{5.5}$$

Thus, thanks to (5.3)–(5.5), the following holds

$$\lim_{n\to\infty}\sup_{0<\epsilon\leq 1}\left|\alpha_{n,\epsilon}\right|=0$$

and, therefore,

$$\lim_{n\to\infty}\sup_{0<\epsilon\leq 1}\left|\left(\left[A_\epsilon,\partial_x^2-\partial_x^4\right]u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|=0,$$

particularly,

$$\lim_{n\to\infty}\left|\left(\left[A,\partial_x^2-\partial_x^4\right]u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|=0.$$

As $D_x^{-3}:=D^{-3}$ commutes with ∂_x^k, for $k=1,2,3$, we have that

$$\begin{aligned}\left[A,\partial_x^2-\partial_x^4\right]&=\left[\psi(t)B,\partial_x^2-\partial_x^4\right]\\&=\left[\psi(t)\varphi(x)D^{-3},\partial_x^2-\partial_x^4\right]\\&=4\psi(t)\left(\partial_x^3\varphi\right)\partial_xD^{-3}+12\psi(t)\left(\partial_x^2\varphi\right)\partial_x^2D^{-3}+4\psi(t)(\partial_x\varphi)\partial_x^3D^{-3}\\&\quad-2\psi(t)(\partial_x\varphi)\partial_xD^{-3}-\psi(t)\left(\partial_x^2\varphi-\partial_x^4\varphi\right)D^{-3}=:\sum_{i=1}^{5}I_i\end{aligned}\tag{5.6}$$

Note that, using (5.1) and (5.2), we control I_5 by

$$\left(\psi(t)\left(\partial_x^2-\partial_x^4\right)D^{-3}u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}\leq\|u_n\|_{X^T_{-b,-3+3b}}\|u_n\|_{X^T_{b,0}}.$$

Indeed,

$$\begin{aligned}\left(\psi(t)\left(\partial_x^2-\partial_x^4\right)D^{-3}u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}&\leq C\left\|\psi(t)\left(\partial_x^2-\partial_x^4\right)\varphi\partial_xD^{-3}u_n\right\|_{X^T_{-b,0}}\|u_n\|_{X^T_{b,0}}\\&\leq C\left\|(L\varphi)D^{-3}u_n\right\|_{X^T_{-b,0}}\|u_n\|_{X^T_{b,0}}\\&\leq C\left\|D^{-3}u_n\right\|_{X^T_{-b,3b}}\|u_n\|_{X^T_{b,0}}\\&\leq C\|u_n\|_{X^T_{-b,-3+3b}}\|u_n\|_{X^T_{b,0}}.\end{aligned}$$

Arguing as made in (5.4), we infer that

$$\left(\psi(t)\left(\partial_x^2-\partial_x^4\right)D^{-3}u_n,u_n\right)\to 0,\text{ as }n\to+\infty.$$

Note that for the terms I_i, $i = 1, 2$ and 4 in (5.6), the loss of regularity is too large if we use the estimate with the same b. Using the index b' instead of b, we have

$$\begin{aligned}
(I_1 u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} &= 4\left(\psi(t)\left(\partial_x^3\varphi\right)\partial_x D^{-3}u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))} \\
&\le C\left\|\psi(t)\left(\partial_x^3\varphi\right)\partial_x D^{-3}u_n\right\|_{X^T_{b',2-3b'}}\|u_n\|_{X^T_{-b',-2+3b'}} \\
&\le C\|u_n\|_{X^T_{b',0}}\|u_n\|_{X^T_{-b',-2+3b'}}, \\
(I_2 u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} &= 12\left(\psi(t)\left(\partial_x^2\varphi\right)\partial_x^2 D^{-3}u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))} \\
&\le C\left\|\psi(t)\left(\partial_x^2\varphi\right)\partial_x^2 D^{-3}u_n\right\|_{X^T_{b',1-3b'}}\|u_n\|_{X^T_{-b',-1+3b'}} \\
&\le C\|u_n\|_{X^T_{b',0}}\|u_n\|_{X^T_{-b',-1+3b'}}
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
(I_4 u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} &= -2\left(\psi(t)(\partial_x\varphi)\partial_x D^{-3}u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))} \\
&\le C\left\|\psi(t)(\partial_x\varphi)\partial_x D^{-3}u_n\right\|_{X^T_{b',2-3b'}}\|u_n\|_{X^T_{-b',-2+3b'}} \\
&\le C\|u_n\|_{X^T_{b',0}}\|u_n\|_{X^T_{-b',-2+3b'}}.
\end{aligned} \tag{5.8}$$

Observe that

$$X^T_{-b',-1+3b'} \hookrightarrow X^T_{-b',-2+3b'}, \tag{5.9}$$

where $\hookrightarrow$ denotes a compact embedding. Thus, from (5.1), (5.2) and (5.9), we have that (5.7)–(5.8) tends to 0 as $n \to +\infty$.

To conclude the proof we need to analyze the third term of (5.6), that is, I_3. Remark that $-\partial_x^3 D^{-3}$ is the orthogonal projection on the subspace of functions with $\hat{u}(0) = 0$. Furthermore,

$$X^T_{b,0} \hookrightarrow X^T_{0,0} \hookrightarrow X^T_{-b',0}, \text{ for } 0 \le b' \le b \le 1,$$

thus, using the Rellich Theorem, we see that

$$\hat{u}_n(0,t) \longrightarrow \hat{u}(0,t) = 0 \text{ in } X^T_{0,0} \equiv L^2(0,T) \text{ strongly,}$$

and hence

$$\left(\psi(t)(\partial_x\varphi)\hat{u}_n(0,t), u_n\right)_{L^2(\mathbb{T}\times(0,T))} \longrightarrow 0.$$

We have proved that, for any $\varphi \in C^{\infty}(\mathbb{T})$ and $\psi \in C_0^{\infty}((0,T))$,

$$\left(\psi(t)(\partial_x\varphi)\partial_x^3 D^{-3}u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))} \longrightarrow 0.$$

Observe that $\phi \in C^{\infty}(\mathbb{T})$ can be written in the form $\partial_x\varphi$ for some function $\varphi \in C^{\infty}(\mathbb{T})$ if and only if $\int_{\mathbb{T}}\phi(x)\,dx = 0$. So, for any $\chi \in C_0^{\infty}(\omega)$ and any $x_0 \in \mathbb{T}$, $\phi(x) = \chi(x) - \chi(x-x_0)$ can be written as $\phi = \partial_x\varphi$ for some $\varphi \in C^{\infty}(\mathbb{T})$. Since u_n is strongly convergent to 0 in $L^2\left(0,T;L^2(\omega)\right)$,

$$\lim_{n\to\infty}(\psi(t)\chi u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} = 0.$$

Then, for any $x_0 \in \mathbb{T}$,

$$\lim_{n\to\infty}(\varphi(t)\chi(\cdot - x_0)u_n, u_n)_{L^2(\mathbb{T}\times(0,T))} = 0.$$

Finally, we closed the proof constructing a partition of unity on $\mathbb{T}$ with some functions of the form $\chi_i\left(\cdot - x_0^i\right)$, with $\chi_i \in C_0^{\infty}(\omega)$ and $x_0^i \in \mathbb{T}$. □

To close this section, we prove the gain of regularity of the linear fourth order Schrödinger equation.

Proposition 5.2 (Propagation of regularity) *Let $T>0$, $0 \le b < 1$ and $f \in X_{-b,r}^T$ be given. Let $u \in X_{b,r}^T$ be a solution of*

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u = f.$$

If there exists a nonempty $\omega \subset \mathbb{T}$ such that $u \in L^2_{loc}\left([0,T];H^{r+\rho}(\omega)\right)$ for some ρ with

$$0 < \rho \le \min\left\{\frac{3}{2}(1-b), \frac{1}{2}\right\},$$

then

$$u \in L^2_{loc}\left([0,T];H^{r+\rho}(\mathbb{T})\right).$$

Proof We first regularize $u_n = \exp\left(\frac{1}{n}\partial_x^2\right)u := \Theta_n u$ and $f_n := \Theta_n f$, with

$$\|u_n\|_{X_{b,r}^T} \le C \text{ and } \|f_n\|_{X_{-b,r}^T} \le C,$$

for some constant $C > 0$ and $n = 1, 2, \ldots$.

Let $s = r + \rho$, $\varphi \in C^{\infty}(\mathbb{T})$ and $\psi \in C_0^{\infty}(0,T)$ taking real values. Set $Bu = D^{2s-3}\varphi(x)$ and $A = \psi(t)B$, where D^{-3} is defined by (1.5). If $L = i\partial_t + \partial_x^2 - \partial_x^4$, we write

$$\begin{aligned}&\left(Lu_n, A^*u_n\right)_{L^2(\mathbb{T}\times(0,T))}-(Au_n, Lu_n)_{L^2(\mathbb{T}\times(0,T))}\\&\quad=\left(\left[A, \partial_x^2-\partial_x^4\right]u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}-i\left(\psi'(t)\,Bu_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}\end{aligned}$$

and deduce that

$$\begin{aligned}\left|(Au_n, Lu_n)_{L^2(\mathbb{T}\times(0,T))}\right|&=\left|(Au_n, f_n)_{L^2(\mathbb{T}\times(0,T))}\right|\\&\leq\|Au_n\|_{X^T_{b,-r}}\|f_n\|_{X^T_{-b,r}}\\&\leq C\|u_n\|_{X^T_{b,r+2\rho-3+3b}}\|f_n\|_{X^T_{-b,r}},\end{aligned}$$

since $r+2\rho-3+3b\leq r$. The same estimates for the other terms imply that

$$\left|\left(\left[A, \partial_x^2-\partial_x^4\right]u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|\leq C.$$

Note that

$$\begin{aligned}\left[A, \partial_x^2-\partial_x^4\right]&=4\psi(t)\,D^{2s-3}\left(\partial_x^3\varphi\right)\partial_x+12\psi(t)\,D^{2s-3}\left(\partial_x^2\varphi\right)\partial_x^2+4\psi(t)\,D^{2s-3}(\partial_x\varphi)\,\partial_x^3\\&\quad-2\psi(t)\,D^{2s-3}(\partial_x\varphi)\,\partial_x-\psi(t)\,D^{2s-3}\left(\partial_x^2\varphi-\partial_x^4\varphi\right)\\&=:\sum_{i=1}^{5}I_i.\end{aligned}\tag{5.10}$$

Also, observe that

$$2s-3+2=2r+2\rho-1\leq 2r\tag{5.11}$$

and

$$2s-3+1=2r+2\rho-2\leq 2r.\tag{5.12}$$

Now, we will bound the terms of (5.10). As (5.11) and (5.12) are verified, taking

$$\rho\leq\min\left\{\frac{3}{2}(1-b), \frac{1}{2}\right\},$$

we have that

$$\begin{aligned}\left|(I_1u_n,u_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\le C\left\|\psi D^{2s-3}\left(\partial_x^3\varphi\right)\partial_x u_n\right\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))}\\ &\le C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))}\le C,\\ \left|(I_2u_n,u_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\le C\left\|\psi D^{2s-3}\left(\partial_x\varphi\right)\partial_x^2 u_n\right\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))}\\ &\le C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))}\le C\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\left|(I_5u_n,u_n)_{L^2(\mathbb{T}\times(0,T))}\right| &= \left|\left(\psi(t)D^{2s-3}\left(\partial_x^2\varphi-\partial_x^4\varphi\right)u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|\\ &\le\left\|\psi(t)D^{2s-3}\left(\partial_x^2\varphi-\partial_x^4\varphi\right)u_n\right\|_{L^2(0,T;H^{-r}(\mathbb{T}))}\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))}\\ &\le C\|u_n\|^2_{L^2(0,T;H^r(\mathbb{T}))}\le C,\end{aligned}$$

for any $n\ge 1$. Similarly of I_1 estimate we can get

$$|I_4|\le C.$$

Finally, we will control I_3. For any $\chi\in C_0^\infty(\omega)$, we have that

$$\begin{aligned}&\left(\psi(t)D^{2s-3}\chi^2\partial_x^3u_n,u_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &=\left(\psi(t)D^{s-3}\chi\partial_x^3u_n,\chi D^su_n\right)+\left(\psi(t)\left[D^{s-3},\chi\right]\chi\partial_x^3u_n,D^su_n\right)\\ &=\left(\psi(t)D^{s-3}\chi\partial_x^3u_n,D^s\chi u_n\right)+\left(\psi(t)D^{s-3}\chi\partial_x^3u_n,\left[\chi,D^s\right]u_n\right)\\ &\quad+\left(\psi(t)\left[D^{s-3},\chi\right]\chi\partial_x^3u_n,D^su_n\right):=\tilde I_1+\tilde I_2+\tilde I_3.\end{aligned} \tag{5.14}$$

In this moment, we need control the right hand side of (5.14). First, note that we infer from the assumptions that

$$\chi u\in L^2_{loc}\left(0,T;H^s(\mathbb{T})\right)$$

and

$$\chi\partial_x^3u\in L^2_{loc}\left(0,T;H^{s-3}(\mathbb{T})\right).$$

Then, as $s=r+\rho\le r+1$, we have

$$\chi u_n=\Theta_n\chi u+[\chi,\Theta_n]u\in L^2_{loc}\left(0,T;H^s(\mathbb{T})\right),$$

due [24, Lemma A.3]. Applying the same argument to $\chi \partial_x^3 u_n$, follows that

$$\left|\tilde{I}_1\right| \leq C. \tag{5.15}$$

Moreover, from [24, Lemma A.1] and the fact $u \in L^2(0,T;H^r(\mathbb{T}))$, $\tilde{I}_2$ can be bounded in the following way

$$\begin{aligned}
\left|\tilde{I}_2\right| &= \left|\left(\psi(t)\, D^{s-3}\chi\,\partial_x^3 u_n, \left[\chi, D^s\right] u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&= \left|\left(\psi(t)\, D^{r+\rho-3}\chi\,\partial_x^3 u_n, \left[\chi, D^s\right] u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&= \left|\left(\psi(t)\, D^{\rho} D^{r-3}\chi\,\partial_x^3 u_n, \left[\chi, D^s\right] u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&\leq \left\|\psi(t)\, D^{r-3}\chi\,\partial_x^3 u_n\right\|_{L^2(\mathbb{T}\times(0,T))} \left\|D^{\rho}\left[\chi, D^s\right] u_n\right\|_{L^2(\mathbb{T}\times(0,T))} \\
&\leq C\,\|u_n\|_{L^2(0,T;H^r(\mathbb{T}))}\,\|u_n\|_{L^2(0,T;H^{\rho+s-1}(\mathbb{T}))} \leq C.
\end{aligned} \tag{5.16}$$

Lastly, by similar computations, we ensure that

$$\left|\tilde{I}_3\right| \leq C. \tag{5.17}$$

Consequently,

$$\left|\left(\psi(t)\, D^{2s-3}\chi^2\partial_x^3 u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leq C,$$

for any $n \geq 1$. Then, writing $\partial_x\varphi = \chi^2(x) - \chi^2(x - x_0)$, from (5.14), (5.15), (5.16) and (5.17) yields,

$$\left|\left(\psi(t)\, D^{2s-3}\chi^2(\cdot - x_0)\,\partial_x^3 u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leq C,$$

for all $n \geq 1$. To conclude the proof, is necessary to use a partition of unity as in the proof of Proposition 5.2, to obtain

$$\left|\left(\psi(t)\, D^{2s-3}\partial_x^3 u, u\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leq C,$$

that is

$$\int_0^T \psi(t)\left(\sum_{k\neq 0}|k|^{2s}\left|\hat{u}(k,t)\right|^2 dt\right) \leq C,$$

or equivalently,

$$\|u\|^2_{L^2_{loc}([0,T];H^s(\mathbb{T}))} \leq C.$$

Thus, the proof is complete. □

6 Unique Continuation Property

We present, in this section, the unique continuation property (UCP) for 4NLS. However, before to enunciate the UCP, let us prove an auxiliary lemma which is a consequence of Proposition 5.2.

Lemma 6.1 *Let $u \in X^T_{b,0}$ be a solution of*

$$i\partial_t u + \partial_x^2 u - \partial_x^4 u + \lambda |u|^2 u = 0 \ \ on\ \mathbb{T}\times (0,T). \tag{6.1}$$

Here $b > \frac{1}{2}$ and we assume that $u \in C^\infty(\omega \times (0,T))$, where $\omega \subset \mathbb{T}$ nonempty set. Then,

$$u \in C^\infty(\mathbb{T}\times (0,T)).$$

Proof Note that $\lambda |u|^2 u \in X^T_{-b,0}$, by Lemma 2.2. Thus, from Proposition 5.2, we get

$$u \in L^2_{loc}([0,T]; H^{1+\frac{3}{2}(1-b)}(\mathbb{T})).$$

Choose t_0 such that $u(t_0) \in H^{1+\frac{3}{2}(1-b)}(\mathbb{T})$. We can then solve (6.1) in $X^T_{b,1+\frac{3}{2}(1-b)}$ with the initial data $u(t_0)$. By uniqueness of solution in $X^T_{b,0}$, we conclude that $u \in X^T_{b,1+\frac{3}{2}(1-b)}$. An iterated application of Proposition 5.2 give us

$$u \in L^2\left(0,T; H^r(\mathbb{T})\right), \quad \forall r \in \mathbb{R},$$

and, hence $u \in C^\infty(\mathbb{T}\times (0,T))$. □

The UCP is presented as follows:

Proposition 6.2 (Unique continuation property) *For every $T > 0$ and ω any nonempty open set of $\mathbb{T}$, the only solution $u \in C^\infty([0,T]\times\mathbb{T})$ of the system*

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = b(x,t)u & on\ \mathbb{T}\times(0,T), \\ u = 0 & on\ \omega\times(0,T), \end{cases}$$

where $b(x,t) \in C^\infty([0,T]\times\mathbb{T})$, is the trivial one

$$u(x,t) = 0 \ on\ \mathbb{T}\times(0,T).$$

Proof Proposition 6.2 is a direct consequence of the Carleman estimate for the operator $P = i\partial_t + \partial_x^2 - \partial_x^4$, proved by Zheng [39, Theorem 1.1] (see also [17, Corollary 6.1]), together with Lemma 6.1. □

Corollary 6.3 *Let ω be any nonempty open set of $\mathbb{T}$ and $u \in X^T_{\frac{1}{2},0}$ solution of*

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda \left|u\right|^2 u & on\ \mathbb{T}\times(0,T), \\ u = 0 & on\ \omega\times(0,T), \end{cases}$$

then $u(x,t) = 0$ on $\mathbb{T}\times(0,T)$.

Proof By using Lemma 6.1, we infer that $u \in C^\infty(\mathbb{T}\times(0,T))$. An application of Proposition 6.2 give us $u = 0$, as desired. □

Remark 1 Proposition 6.2 assures us that for $u \in X^T_{b,0}$ solution of

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = 0 & \text{on } \mathbb{T}\times(0,T), \\ u = 0 & \text{on } \omega\times(0,T), \end{cases}$$

we also have $u(x,t) = 0$ on $\mathbb{T}\times(0,T)$.

7 Stabilization: Global Result

This section is to establish the main result of this article. The propagation results and unique continuation property will play a key role for this study. Thus, we are concerned with stability properties of the following system

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u + ia^2 u = \lambda \left|u\right|^2 u & \text{on } \mathbb{T}\times(0,T), \\ u(x,0) = u_0(x) & \text{on } \mathbb{T}, \end{cases} \tag{7.1}$$

where $\lambda \in \mathbb{R}$ and $u_0 \in L^2(\mathbb{T})$, in L^2–level.

7.1 Proof of Theorem 1.2

Theorem 1.2 is a consequence of the following *observability inequality*:
Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\gamma > 0$ such that for any $u_0 \in L^2(\mathbb{T})$ satisfying $\|u_0\|_{L^2(\mathbb{T})} \le R_0$, the corresponding solution u of (7.1) satisfies

$$\|u_0\|^2_{L^2(\mathbb{T})} \le \gamma \int_0^T \|au\|^2_{L^2(\mathbb{T})}\, dt. \tag{7.2}$$

In fact, if (7.2) holds, the energy estimate give us

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{T})} = \|u(\cdot,0)\|^2_{L^2(\mathbb{T})} - \int_0^t \|au\|^2_{L^2(\mathbb{T})}(\tau)d\tau, \quad \forall t \ge 0.$$

The last equality ensures that

$$\|u(\cdot, T)\|^2_{L^2(\mathbb{T})} \leq (1-\gamma^{-1}) \|u_0\|^2_{L^2(\mathbb{T})}.$$

Thus,

$$\|u(\cdot, mT)\|^2_{L^2(\mathbb{T})} \leq (1-\gamma^{-1})^m \|u_0\|^2_{L^2(\mathbb{T})},$$

which yields

$$\|u(\cdot, t)\|_{L^2(\mathbb{T})} \leq Ce^{-\gamma t} \|u_0\|_{L^2(\mathbb{T})}, \forall t > 0.$$

Finally, we obtain a constant γ independent of R_0 by noticing that for $t > c\left(\|u_0\|_{L^2(\mathbb{T})}\right)$, the L^2 norm of $u(\cdot, t)$ is smaller than 1, so that we can take the γ corresponding to $R_0 = 1$, proving the result. □

7.2 Proof of the Observability Inequality

If (7.2) does not occurs, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} = u_n$ solution of (7.1) satisfying

$$\|u_n(0)\|_{L^2(\mathbb{T})} \leq R_0$$

and

$$\int_0^T \|au_n\|^2_{L^2(\mathbb{T})} dt < \frac{1}{n} \left\|u_{0,n}\right\|^2_{L^2(\mathbb{T})}, \tag{7.3}$$

where $u_{0,n} = u_n(0)$. Since $\gamma_n := \left\|u_{0,n}\right\|_{L^2(\mathbb{T})} \leq R_0$, one can choose a subsequence of $\gamma_n = \{\gamma_n\}_{n\in\mathbb{N}}$, still denote by γ_n, such that,

$$\lim_{n\to\infty} \gamma_n = \gamma.$$

Thus, we will analyze two cases for γ: $\gamma > 0$ or $\gamma = 0$. In both cases we will get a contradiction.

Case one $\lim_{n\to\infty} \gamma_n = \gamma > 0$:

Observe that u_n is bounded in $L^\infty\left(0, T; L^2(\mathbb{T})\right)$ and, therefore, in $X^T_{b,0}$, for $b > \frac{1}{2}$. Then, as $X^T_{b,0}$ is a separable Hilbert space, we can extract a subsequence such that

$$u_n \rightharpoonup u \quad \text{in} \quad X^T_{b,0},$$

for some $u \in X^T_{b,0}$. By compact embedding, as we have $b < 1$ and $-b < 0$, we can (also) extract a subsequence such that we have strong convergence in $X^T_{-b,-1+b}$. Now, we prove that the weak limit u is a solution of (7.1). Since $|u_n|^2 u_n$ is bounded in $X^T_{-b',0}$, for $b' > \frac{5}{16}$, there is a subsequence u_n, still denote by u_n, such that

$$|u_n|^2 u_n \rightharpoonup f \quad \text{in} \quad X^T_{-b',0}, \quad \text{for} \quad b' > \frac{5}{16}$$

and

$$|u_n|^2 u_n \to f \quad \text{in} \quad X^T_{-1+b,-b}, \quad \text{for} \quad \frac{5}{16} < b < 1.$$

Moreover, from (7.3) follows that

$$\int_0^T \|au_n\|^2_{L^2(\mathbb{T})} \, dt \longrightarrow \int_0^T \|au\|^2_{L^2(\mathbb{T})} \, dt = 0,$$

which implies that $u(x,t) = 0$ on $\omega \times (0,T)$. Therefore, letting $n \to \infty$, we obtain from (7.1) that

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = f & \text{on } \mathbb{T} \times (0,T), \\ u(x,t) = 0 & \text{on } \omega \times (0,T). \end{cases}$$

We affirm that

$$f = -ia^2 u + \lambda |u|^2 u.$$

In fact, let $w_n = u_n - u$ and $f_n = -ia^2 u_n + \lambda |u_n|^2 u_n - f$. Remark that from (7.3),

$$\int_0^T \|aw_n\|^2_{L^2(\mathbb{T})} \, dt \longrightarrow 0.$$

Thus,

$$f_n \to 0 \quad \text{in} \quad X^T_{-1+b,-b}.$$

It also implies

$$u_n \to 0 \text{ in } L^2\left(0,T; L^2(\omega)\right)$$

and

$$w_n \to 0 \text{ in } L^2\left(0,T; L^2(\omega)\right).$$

Applying Proposition 5.1, we get

$$w_n \longrightarrow 0 \text{ in } L^2_{loc}\left([0,T]; L^2(\mathbb{T})\right).$$

Then, we can pick one $t_0 \in [0, T]$ such that $w_n(t_0)$ tends to 0 strongly in $L^2(\mathbb{T})$.

Let v the solution of

$$\begin{cases} i\partial_t v + \partial_x^2 v - \partial_x^4 v + ia^2 v = \lambda |v|^2 v \quad \text{on } \mathbb{T}\times(0,T), \\ v(t_0) = u(t_0). \end{cases} \tag{7.4}$$

We claim that $u = v$. Indeed, by Theorem 3.1 we have that the map data-to-solution of (7.1) is locally Lipschitz continuous. Since $u_n(t_0) \to v(t_0)$ in $L^2(\mathbb{T})$ and $ia^2u_n \to ia^2u$ in $L^2([0,T]; L^2(\mathbb{T}))$, we get $u_n \to v$ in $X^T_{0,b}$, thus $u = v$ and u is a solution of (7.4). Unique continuation property, Corollary 6.3, implies $u = 0$. It follows that $\|u_n(0)\|_{L^2(\mathbb{T})} \to 0$, which leads a contradiction of our hypothesis $\gamma > 0$.

Case two $\lim_{n\to\infty} \gamma_n = \gamma = 0$:

Consider $v_n = \frac{u_n}{\gamma_n}$, $\forall n \geq 1$. Thus, v_n satisfies

$$\begin{aligned} & i\partial_t v_n + \partial_x^2 v_n - \partial_x^4 v_n + ia^2 v_n = \lambda\gamma_n^2 |v_n|^2 v_n, \\ & \int_0^T \|av_n\|^2_{L^2(\mathbb{T})}\, dt < \frac{1}{n} \end{aligned} \tag{7.5}$$

and

$$\|v_n(0)\|_{L^2(\mathbb{T})} = 1. \tag{7.6}$$

Observe that $v_n := \{v_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty\left(0,T; L^2(\mathbb{T})\right) \cap X^T_{b,0}$. Thus, we can extract a subsequence, still denoted by v_n, such that

$$v_n \rightharpoonup v \text{ in } X^T_{b,0}.$$

Furthermore, by Duhamel formula and multilinear estimates (2.2), we obtain

$$\|v_n\|_{X^T_{b,0}} \leq C\|v_n(0)\|_{L^2(\mathbb{T})} + CT^{1-b-b'}\left(\|v_n\|_{X^T_{b,0}} + \gamma_n^2 \|v_n\|^3_{X^T_{b,0}}\right),$$

for $0 < b' < \frac{1}{2} < b$ and $b + b' \leq 1$.

If we take $CT^{1-b-b'} < 1/2$, independent of v_n, we get

$$\|v_n\|_{X^T_{b,0}} \leq C + C\gamma_n^2 \|v_n\|^3_{X^T_{b,0}}.$$

Lemma 2.5 states that $\|v_n\|_{X^T_{b,0}}$ is continuous in T. Since it is bounded near $t = 0$ and $\gamma_n \to 0$, we obtain by a classical bootstrap argument (see, e.g, [2, Lemma 2.2]) that v_n is bounded on $X^T_{b,0}$. Using Lemma 2.6, we can conclude that it is bounded in $X^T_{b,0}$ even for large T. Thus,

$$\gamma_n^2 |v_n|^2 v_n \to 0 \quad \text{in} \quad X^T_{-b',0}$$

and so

$$\gamma_n^2 |v_n|^2 v_n \to 0 \quad \text{in} \quad X^T_{-b,-1+b}.$$

Then, we can extract a subsequence such that

$$v_n \rightharpoonup v \quad \text{in} \quad X^T_{b,0}$$

and

$$v_n \to v \quad \text{in} \quad X^T_{-1+b,-b}.$$

Therefore, the weak limit v satisfies

$$\begin{cases} i\partial_t v + \partial_x^2 v - \partial_x^4 v + ia^2 v = 0 & \text{on } \mathbb{T}\times(0,T), \\ v(x,t) = 0 & \text{on } \omega\times(0,T), \end{cases}$$

which implies that $v(x,t)$ is the trivial solution, that is, $v(x,t) = 0$, thanks to Remark 1 of Proposition 6.2.

Argument of contradiction (7.5) yields that

$$ia^2 v_n \longrightarrow 0 \quad \text{in} \quad L^2(0,T;L^2(\mathbb{T})), \tag{7.7}$$

and so

$$ia^2 v_n \longrightarrow 0 \quad \text{in} \quad X^T_{-1+b,-b}.$$

An application of Proposition 5.1, as in the case one $\gamma > 0$, ensures that

$$v_n \longrightarrow 0 \text{ in } L^2_{loc}\left([0,T];L^2(\mathbb{T})\right). \tag{7.8}$$

From the energy estimate for $t_0 \in (0,T)$, we get

$$\|v_n(0)\|^2_{L^2(\mathbb{T})} = \|v_n(t_0)\|^2_{L^2(\mathbb{T})} + \int_0^{t_0} \|av_n\|^2_{L^2(\mathbb{T})}\,dt.$$

Passing the limit on the last equality, by using (7.7) and (7.8), we have that $\|v_n(0)\|_{L^2(\mathbb{T})} \to 0$, which contradicts (7.6). Therefore, the proof is complete. □

8 Concluding Remarks

Let us consider the following system

$$\begin{cases} i\partial_t u + \partial_x^2 u - \epsilon\partial_x^4 u = \lambda|u|^2 u, & (x,t)\in\mathcal{M}\times\mathbb{R}, \\ u(x,0) = u_0(x), & x\in\mathcal{M}. \end{cases} \tag{8.1}$$

When $\epsilon = 0$ and $\mathcal{M} = \mathbb{T}$, system (8.1) is the well-know nonlinear Schrödinger system (NLS) posed on a periodic domain, that can be considered with Dirichlet or Neumann boundary conditions.

The focus of our discussion in this paper are the global properties of the control for the fourth order nonlinear Schrödinger (4NLS) equation on periodic domain, that is, considering $\epsilon = 1$ and $\mathcal{M} = \mathbb{T}$ in (8.1). The main results of this work ensure that system (8.1) is globally exponentially stabilizable and globally exactly controllable in the space $H^s(\mathbb{T})$, for any $s \geq 0$.

Remarks The following remarks are now in order.

1. If $u_0 \in H^s(\mathbb{T})$, with $s \geq 0$, one can impose that $g \in C([0, T]; H^s(\mathbb{T}))$ on Theorem 1.1. Analogously, if $u_0, u_1 \in H^s(\mathbb{T})$, with $s \geq 0$, one can impose that $g \in C([0, T]; H^s(\mathbb{T}))$ on Theorem 1.3.
2. We should point out that time T used to guide the system from the initial state u_0 for the terminal state u_1, on Theorem 1.3, depend on the amplitude of u_0 and u_1, that is,

$$\|u_0\|_{L^2(\mathbb{T})} \leq R_0 \quad \text{and} \quad \|u_1\|_{L^2(\mathbb{T})} \leq R_0.$$

   In general, the larger amplitude R_0, the longer time T is needed to conduct the control. That is why we call such controllability as *large time* controllability.
3. The global results, as described in Theorems 1.2 and 1.3, are truly nonlinear and their proofs demand new tools in addition to the Bourgain spaces and Bourgain smoothing properties. The needed tools turns out to be propagation properties of compactness and regularity for the fourth order Schrödinger equation.
4. Theses results of propagation are in concordance with the results of controllability and stabilization in the literature, more precisely, they are inspired by those established first in [12] for the wave equation, after that [24] for the Schrödinger equation, [26] for the Benjamin–Ono equation, [25] for the KdV equation and [38] for the Kawahara equation. Thus, the results presented in this article give us a complete picture of the study of controllability and stabilization to more classical nonlinear dispersive equations, posed on a periodic domain $\mathbb{T}$, by using certain propagation properties given by Bourgain spaces, closing the last gap that was missing when discussing nonlinear dispersive equations of order between 2 and 5.

8.1 Controllability Results for NLS and 4NLS

The results showed in this work can be compared with those already known for the NLS. One of the first results to the system (8.1), considering $\epsilon = 0$ and $\mathcal{M}$ a compact Riemannian manifold of dimension 2 without boundary, is due Dehman et al. in [12].

In [12], the authors consider the stabilization and exact controllability problem for NLS, more precisely, to prove the control properties, the authors were able to prove the propagation of regularity in $\mathcal{M}$. However, these properties are shown considering ω be an open subset of $\mathcal{M}$ and the following two assumptions:

(A) ω geometrically controls $\mathcal{M}$; i.e. there exists $T_0 > 0$, such that every geodesic of $\mathcal{M}$ traveling with speed 1 and issued at $t = 0$, enters the set ω in a time $t < T_0$.

(B) For every $T > 0$, the only solution lying in the space $C[0, T], H^2(\mathcal{M}))$ of the system

$$\begin{cases} i\partial_t u + \Delta^2 u + b_1(x,t)u + b_2(x,t)\overline{u}, & (x,t) \in \mathcal{M} \times (0,T), \\ u = 0, & (x,t) \in \omega \times (0,T), \end{cases}$$

where $b_1(t, x)$ and $b_2(t, x) \in L^\infty(0, T, L^p(\mathcal{M}))$ for some $p > 0$ large enough, is the trivial one $u \equiv 0$.

The global controllability result can be read as follows:

Theorem 8.1 (Dehman et al. [12]) *Assume that the open set ω satisfies conditions (A) and (B). Then for every $R_0 > 0$, there exists $T > 0$ such that for every data u_0 and v_0 in $H^1(\mathcal{M})$, satisfying*

$$\|u_0\|_{H^1(\mathcal{M})} \leq R_0 \quad \textit{and} \ \|v_0\|_{H^1(\mathcal{M})} \leq R_0,$$

there exists a control $g \in C\left([0,T], H^1(\mathcal{M})\right)$, with support in $[0,T] \times \omega$, such that the system

$$\begin{cases} i\partial_t u + \Delta u - P'\left(|u|^2\right)u = \widetilde{g} \\ \qquad\qquad\qquad u(0) = u_0 \end{cases}$$

where $\widetilde{g} = g$ if $0 \leq t \leq T$, 0 if $t > T$, and P is a polynomial function with real coefficients, satisfies $u(\cdot, T) = v_0$.

Considering the NLS on a periodic domain $\mathbb{T}$, recently, Laurent [24] applied the method introduced by Dehman *et al.* to prove that the nonlinear Schrödinger equation is globally internal controllable when posed on periodic Dirichlet or Neumann boundary conditions. The main result is the following.

Theorem 8.2 (Laurent [24]) *Let $b \in (1/2, 5/8)$. For any nonempty open set $\omega \subset \mathbb{T}$ and $R_0 > 0$, there exist a $T > 0$ and a constant $C > 0$ such that for every u_0 and u_1 in $L^2(\mathbb{T})$ with*

$$\|u_0\|_{L^2} \leq R_0 \quad \textit{and} \quad \|u_1\|_{L^2} \leq R_0$$

there exists a control $g \in C([0,T]; L^2(\mathbb{T}))$ with $supp(g) \subset \omega \times (0,T)$, such that the unique solution $u \in X^T_{b,0}$ of the system

$$\begin{cases} i\partial_t u + \partial_x^2 u = \lambda|u|^2 u + g, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases}$$

satisfies $u(x,T) = u_1$.

The result described above is consequence of the following result.

Theorem 8.3 (Laurent [24]) *Assume that $a(x)^2 > n > 0$ on some nonempty open set. Then, for every $R_0 > 0$, there exist $C > 0$ and $\gamma > 0$ such that inequality*

$$\|u(t)\|_{L^2} \leq C\mathrm{e}^{-\gamma t} \|u_0\|_{L^2}, \quad t > 0,$$

holds for every solution u of system

$$\begin{cases} i\partial_t u + \partial_x^2 u + ia^2 u = \lambda |u|^2 u, & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases}$$

with initial datau_0 such that $\|u_0\|_{L^2} \leq R_0$

Now, let us provide a comparison between the results presented for the NLS and for 4NLS. Taking $\epsilon = 1$ and $\mathcal{M} = \mathbb{T}$ in (8.1), Theorems 1.2 and 1.3 give us good global properties of stabilization and controllability to 4NLS, however there are some important issues to be considered.

Concerning the independence of C, γ and the time of control T on the bound R_0, our work does not give any answer, therefore are an open problem. Nevertheless, if we want g in $H^s(\mathbb{T})$, the time of controllability only depends on the size of the data in $L^2(\mathbb{T})$, as in the case of NLS.

About the time of controllability, the approach used here does not give any information about the minimal time to drive the initial data u_0 to a final data u_1. This is in strong contrast with the linear case where exact controllability occurs in arbitrary small time and the conditions are only geometric for the open set ω, in concordance with the results for the NLS (see Theorem 8.2).

We should be note that the results presented in this work are more complex than those that have been shown in Theorems 8.2 and 8.3. Taking into account the results of [12] on a compact surface, in our work, we adapted the propositions related with propagation (Propositions 5.1 and 5.2) for the linear system associated to 4NLS, with the following main difference between the results proved in [24]: The central points to prove the propagation results of the Schrödinger operator ($\tilde{L} = i\partial_t + \partial_x^2$) and fourth order Schrödinger operator ($L = i\partial_t + \partial_x^2 - \partial_x^4$) are the analysis of the terms given by

$$\begin{aligned} \alpha_{n,\epsilon} &= i\left(-\psi'(t)\, B_\epsilon u_n, u_n\right) \\ &\quad + \left(A_\epsilon u_n, \left(\partial_x^2 - \partial_x^4\right) u_n\right) \\ &= \left(\left[A_\epsilon, \partial_x^2 - \partial_x^4\right] u_n, u_n\right) - i\left(\psi'(t)\, B_\epsilon u_n, u_n\right) \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} &\left(Lu_n, A^* u_n\right)_{L^2(\mathbb{T}\times(0,T))} - \left(Au_n, Lu_n\right)_{L^2(\mathbb{T}\times(0,T))} \\ &= \left(\left[A, \partial_x^2 - \partial_x^4\right] u_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))} - i\left(\psi'(t)\, Bu_n, u_n\right)_{L^2(\mathbb{T}\times(0,T))}. \end{aligned} \tag{8.3}$$

Observe that the operator L instead of $\tilde{L}$ in the equalities (8.2) and (8.3) demand more attention. In fact, the bracket defined by

$$\begin{aligned}\left[A, \partial_x^2 - \partial_x^4\right] = & 4\psi(t) D^{2s-3}\left(\partial_x^3\varphi\right)\partial_x + 12\psi(t) D^{2s-3}\left(\partial_x^2\varphi\right)\partial_x^2 \\ & + 4\psi(t) D^{2s-3}(\partial_x\varphi)\partial_x^3 \\ & - 2\psi(t) D^{2s-3}(\partial_x\varphi)\partial_x - \psi(t) D^{2s-3}\left(\partial_x^2\varphi - \partial_x^4\varphi\right),\end{aligned}$$

is more complex than the usual Schrödinger operator, we need to bound the terms of the bracket in appropriate norms. This analysis is crucial to get the propagation results and, consequently, to show the global controllability and stabilization problem mentioned at the beginning of the introduction.

It is important to realize that Proposition 5.1 applied for the fourth order Schrödinger operator allows a source term f_n bounded in a lower order Sobolev norm $L^2(0, T; H^{-3}(\mathbb{T}))$ while u_n is bounded in $L^2(0, T; L^2(\mathbb{T}))$ instead of f_n bounded in $L^2(0, T; H^{-1}(\mathbb{T}))$ while u_n is bounded in $L^2(0, T; L^2(\mathbb{T}))$ for the Schrödinger operator (see [24, Theorem 4.1]). This fact can be extremely useful in a nonlinear context, where the source term comes from the nonlinearity.

To finish our discussion, we would like to mention a fact related with the assumption (A), i.e., *Geometric Control Condition*. The exact controllability is known to be true when geometric control condition is realized for NLS, see for instance, Lebeau [27], but also for any open set ω of $\mathbb{T}^n$, see Jaffard [18] and Komornik and Loreti [21]. Additionally, the exact controllability holds also for general manifolds considering the assumption (A), see for instance, Laurent [23]. We conjecture that these results can also be extended for the fourth order nonlinear Schrödinger system. In addition, due the regularity of the solutions to the 4NLS we expect more manifolds, in higher dimensions, for which the controllability and stabilization are established (with or without geometric control condition). These results are being prepared and will be present in a forthcoming paper.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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