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open problems*

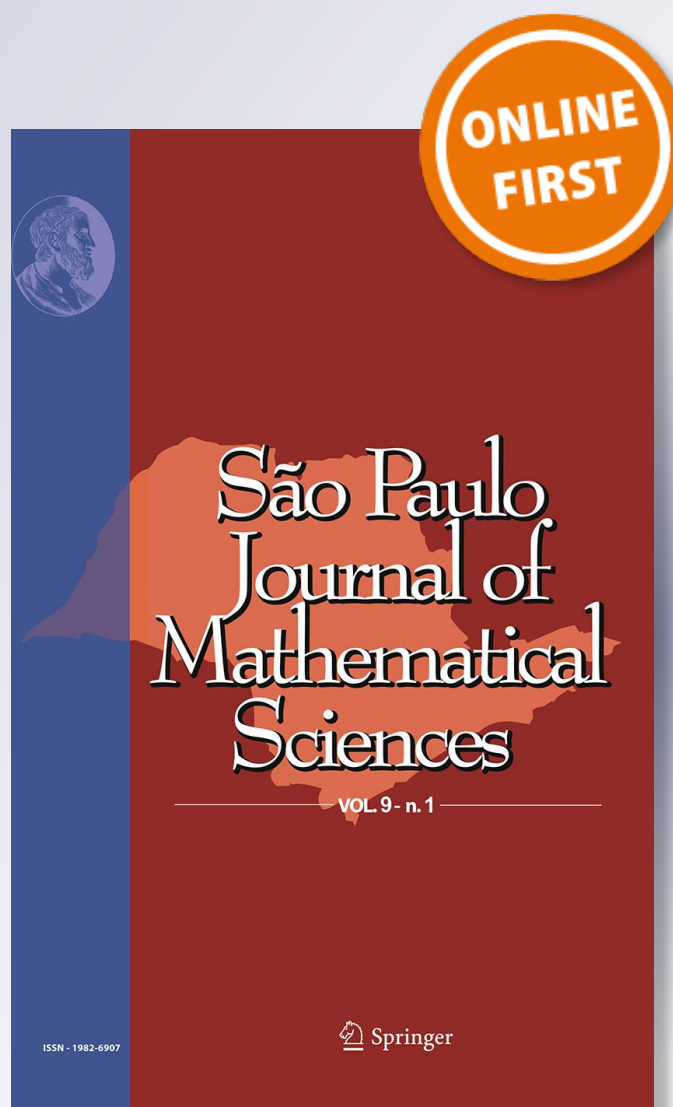
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Initial boundary value problem for Korteweg–de Vries equation: a review and open problems

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Abstract

In the last 40 years the study of initial boundary value problem for the Korteweg–de Vries equation has had the attention of researchers from various research fields. In this note we present a review of the main results about this topic and also introduce interesting open problems which still requires attention from the mathematical point of view.

Keywords KdV equation · Well-posedness · Non-homogeneous boundary value problem · Boundary integral operators · Initial boundary value problem

Mathematics Subject Classification 35Q53 · 35Q35 · 53C35

1 Introduction

In 1834 John Scott Russell, a Scottish naval engineer, was observing the Union Canal in Scotland when he unexpectedly witnessed a very special physical phenomenon which he called a wave of translation [52]. He saw a particular wave traveling through this channel without losing its shape or velocity, and was so captivated by this event that he focused his attention on these waves for several years, not only built water wave tanks at his home conducting practical and theoretical research into these types

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of waves, but also challenged the mathematical community to prove theoretically the existence of his solitary waves and to give an a priori demonstration a posteriori.

A number of researchers took up Russell’s challenge. Boussinesq was the first to explain the existence of Scott Russell’s solitary wave mathematically. He employed a variety of asymptotically equivalent equations to describe water waves in the small-amplitude, long-wave regime. In fact, several works presented to the Paris Academy of Sciences in 1871 and 1872, Boussinesq addressed the problem of the persistence of solitary waves of permanent form on a fluid interface [11–14]. It is important to mention that in 1876, the English physicist Lord Rayleigh obtained a different result [48].

After Boussinesq theory, the Dutch mathematicians D. J. Korteweg and his student G. de Vries [44] derived a nonlinear partial differential equation in 1895 that possesses a solution describing the phenomenon discovered by Russell,

$$\frac{\partial \eta}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{l}} \frac{\partial}{\partial x} \left(\frac{1}{2} \eta^2 + \frac{3}{2} \alpha \eta + \frac{1}{3} \beta \frac{\partial^2 \eta}{\partial x^2} \right), \tag{1.1}$$

in which η is the surface elevation above the equilibrium level, l is an arbitrary constant related to the motion of the liquid, g is the gravitational constant, and $\beta = \frac{l^3}{3} - \frac{Tl}{\rho g}$ with surface capillary tension T and density ρ . Equation (1.1) is called the Korteweg–de Vries equation in the literature, often abbreviated as the KdV equation, although it had appeared explicitly in Boussinesq’s massive 1877 Memoir [14], as equation (283bis) in a footnote on page 360.¹

Eliminating the physical constants by using the following change of variables

$$t \rightarrow \frac{1}{2} \sqrt{\frac{g}{l\beta}} t, \quad x \rightarrow -\frac{x}{\beta}, \quad u \rightarrow -\left(\frac{1}{2} \eta + \frac{1}{3} \alpha \right)$$

one obtains the standard Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

which is now commonly accepted as a mathematical model for the unidirectional propagation of small-amplitude long waves in nonlinear dispersive systems.

This note is concerned with the main results already obtained for the initial-boundary value problem (IBVP) of the KdV equation posed on a finite interval $(0, L)$. The first paper which treated this problem was given by Bubnov [15] when he considered the IBVP of the KdV equation on the finite interval $(0, 1)$ with general boundary conditions. After that, many authors worked on improving the existing results and presenting new results in the last 30 years.

Our intention here is to present the main results on this field. Also, we give some further comments and, at the end, discuss open problems related to the IBVP of the KdV equation in a bounded domain.

¹ The interested readers are referred to [35,47] for history and origins of the Korteweg–de Vries equation.

2 A review of IBVP for KdV

Consider the IBVP of the KdV equation posed on a finite interval $(0, L)$

$$u_t + u_x + u_{xxx} + uu_x = 0, \quad u(x, 0) = \phi(x), \quad 0 < x < L, \quad t > 0 \quad (2.1)$$

with general non-homogeneous boundary conditions posed on the two ends of the interval $(0, L)$,

$$B_1u = h_1(t), \quad B_2u = h_2(t), \quad B_3u = h_3(t) \quad t > 0, \quad (2.2)$$

where

$$B_iu = \sum_{j=0}^2 \left(a_{ij} \partial_x^j u(0, t) + b_{ij} \partial_x^j u(L, t) \right), \quad i = 1, 2, 3,$$

and $a_{ij}, b_{ij}, j = 0, 1, 2, i = 1, 2, 3$, are real constants. The following natural question arises:

Under what assumptions on the coefficients a_{ij}, b_{ij} in (2.2), is the IBVP (2.1)–(2.2) well-posed in the classical Sobolev space $H^s(0, L)$?

As mentioned before, Bubnov [15] studied the following IBVP of the KdV equation on the finite interval $(0, 1)$:

$$\begin{cases} u_t + uu_x + u_{xxx} = f, & u(x, 0) = 0, \quad x \in (0, 1), \quad t \in (0, T), \\ \alpha_1 u_{xx}(0, t) + \alpha_2 u_x(0, t) + \alpha_3 u(0, t) = 0, \\ \beta_1 u_{xx}(1, t) + \beta_2 u_x(1, t) + \beta_3 u(1, t) = 0, \\ \chi_1 u_x(1, t) + \chi_2 u(1, t) = 0 \end{cases} \quad (2.3)$$

and obtained the following result.

Theorem A [15] *Assume that*

$$\begin{cases} \text{if } \alpha_1 \beta_1 \chi_1 \neq 0, & \text{then } F_1 > 0, \quad F_2 > 0, \\ \text{if } \beta_1 \neq 0, \chi_1 \neq 0, \alpha_1 = 0, & \text{then } \alpha_2 = 0, \quad F_2 > 0, \quad \alpha_3 \neq 0, \\ \text{if } \beta_1 = 0, \chi_1 \neq 0, \alpha_1 \neq 0, & \text{then } F_1 > 0, \quad F_3 \neq 0, \\ \text{if } \alpha_1 = \beta_1 = 0, \chi_1 \neq 0, & \text{then } F_3 \neq 0, \quad \alpha_2 = 0, \quad \alpha_3 \neq 0, \\ \text{if } \beta_1 = 0, \alpha_1 \neq 0, \chi_1 = 0, & \text{then } F_1 > 0, \quad F_3 \neq 0, \\ \text{if } \alpha_1 = \beta_1 = \chi_1 = 0, & \text{then } \alpha_2 = 0, \quad \alpha_3 \neq 0, \quad F_3 \neq 0, \end{cases} \quad (2.4)$$

where

$$F_1 = \frac{\alpha_3}{\alpha_1} - \frac{\alpha_2^2}{2\alpha_1^2}, \quad F_2 = \frac{\beta_2 \chi_2}{\beta_1 \chi_1} - \frac{\beta_3}{\beta_1} - \frac{\chi_2^2}{2\chi_1^2}, \quad F_3 = \beta_2 \chi_2 - \beta_1 \chi_1.$$

For any given

$$f \in H^1_{loc}(0, \infty; L^2(0, 1)) \text{ with } f(x, 0) = 0,$$

there exists a $T > 0$ such that (2.3) admits a unique solution

$$u \in L^2(0, T; H^3(0, 1)) \text{ with } u_t \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H^1(0, 1)).$$

The main tool used by Bubnov to prove his theorem is the following Kato type smoothing property for solution u of the linear system associated to the IBVP (2.3),

$$\begin{cases} u_t + u_{xxx} = f, & u(x, 0) = 0, & x \in (0, 1), & t \in (0, T), \\ \alpha_1 u_{xx}(0, t) + \alpha_2 u_x(0, t) + \alpha_3 u(0, t) = 0, \\ \beta_1 u_{xx}(1, t) + \beta_2 u_x(1, t) + \beta_3 u(1, t) = 0, \\ \chi_1 u_x(1, t) + \chi_2 u(1, t) = 0. \end{cases} \tag{2.5}$$

Under the assumptions (2.4):

$$f \in L^2(0, T; L^2(0, 1)) \implies u \in L^2(0, T; H^1(0, 1)) \cap L^\infty(0, T; L^2(0, 1))$$

and

$$\|u\|_{L^2(0,T;H^1(0,1))} + \|u\|_{L^\infty(0,T;L^2(0,1))} \leq C \|f\|_{L^2(0,T;L^2(0,1))}$$

where $C > 0$ is a constant independent of f .

In the past 30 years since the work of Bubnov, various boundary-value problems of the KdV equation have been studied. In particular, the following three classes of IBVPs of the KdV equation on the finite interval $(0, L)$,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x, 0) = \phi(x), & x \in (0, L), & t > 0, \\ u(0, t) = h_1(t), & u(L, t) = h_2(t), & u_x(L, t) = h_3(t), \end{cases} \tag{2.6}$$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x, 0) = \phi(x), & x \in (0, L), & t > 0, \\ u(0, t) = h_1(t), & u(L, t) = h_2(t), & u_{xx}(L, t) = h_3(t), \end{cases} \tag{2.7}$$

and

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & u(x, 0) = \phi(x), & x \in (0, L), & t > 0, \\ u_{xx}(0, t) = h_1(t), & u_x(L, t) = h_2(t), & u_{xx}(L, t) = h_3(t), \end{cases} \tag{2.8}$$

as well as the IBVPs of the KdV equation posed in a quarter plane have been intensively studied in the past 20 years (cf. [4,7,22,27,28,30,31,34,45,46,49] and the references therein) following the rapid advances of the study of the pure initial value problem of the KdV equation posed on the whole line \mathbb{R} or on the periodic domain \mathbb{T} (cf. [1,2,8–10,25,27–29,38–43] and the references therein).

The nonhomogeneous IBVP (2.6) was first studied by Faminskii [27,28] and was shown to be well-posed in the spaces $L^2(0, L)$ and $H^3(0, L)$.

Theorem B [27,28] *Let $T > 0$ be given. For any $\phi \in L^2(0, L)$ and $\mathbf{h} = (h_1, h_2, h_3)$ belonging of*

$$W^{\frac{1}{3},1}(0, T) \cap L^{6+\epsilon}(0, T) \cap H^{\frac{1}{6}}(0, T) \times W^{\frac{5}{6}+\epsilon,1}(0, T) \cap H^{\frac{1}{3}}(0, T) \times L^2(0, T),$$

the IBVP (2.6) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

Moreover, the solution map is continuous in the corresponding spaces. In addition, if $\phi \in H^3(0, L)$, $h'_1 \in W^{\frac{1}{3},1}(0, T) \cap L^{6+\epsilon}(0, T) \cap H^{\frac{1}{6}}(0, T)$, $h'_2 \in W^{\frac{5}{6}+\epsilon,1}(0, T) \cap H^{\frac{1}{3}}(0, T)$ and $h'_3 \in L^2(0, T)$ with

$$\phi(0) = h_1(0), \quad \phi(L) = h_2(0), \quad \phi'(L) = h_3(0),$$

then the solution $u \in C^1([0, T]; H^3(0, L)) \cap L^2(0, T; H^4(0, L))$.

Bona et al. [4] showed that the IBVP (2.6) is locally well-posed in the space $H^s(0, L)$ for any $s \geq 0$:

Theorem C [4] *Let $s \geq 0$, $r > 0$ and $T > 0$ be given. There exists a $T^* \in (0, T]$ such that for any s -compatible $\phi \in H^s(0, L)$ and*

$$\vec{h} = (h_1, h_2, h_3) \in H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T)$$

satisfying

$$\|\phi\|_{H^s(0,L)} + \|\vec{h}\|_{H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (2.6) admits a unique solution

$$u \in C([0, T^*]; H^s(0, L)) \cap L^2(0, T^*; H^{s+1}(0, L)).$$

Moreover, the corresponding solution map is analytic in the corresponding spaces.

Holmer [34] proved that IBVP (2.6) is locally well-posed in the space $H^s(0, L)$ for any $-\frac{3}{4} < s < \frac{1}{2}$, and Bona et al. [7] showed that the IBVP (2.6) is locally well-posed $H^s(0, L)$ for any $s > -1$.

As for the IBVP (2.7), its study began with the work of Colin and Ghidalia in late 1990s [22–24]. They obtained in [24] the following results.

- (i) *Given $h_j \in C^1([0, \infty))$, $j = 1, 2, 3$ and $\phi \in H^1(0, L)$ satisfying $h_1(0) = \phi(0)$, there exists a $T > 0$ such that the IBVP (2.7) admits a solution (in the sense of distribution)*

$$u \in L^\infty(0, T; H^1(0, L)) \cap C([0, T]; L^2(0, L)).$$

- (ii) The solution u of the IBVP (2.7) exists globally in $H^1(0, L)$ if the size of its initial value $\phi \in H^1(0, L)$ and its boundary values $h_j \in C^1([0, \infty))$, $j = 1, 2, 3$ are all small.

In addition, they showed that the associate linear IBVP

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & u(x, 0) = \phi(x) & x \in (0, L), \quad t \in \mathbb{R}^+ \\ u(0, t) = 0, & u_x(L, t) = 0, & u_{xx}(L, t) = 0 \end{cases} \quad (2.9)$$

possesses a strong smoothing property:

For any $\phi \in L^2(0, L)$, the linear IBVP (2.9) admits a unique solution

$$u \in C(\mathbb{R}^+; L^2(0, L)) \cap L^2_{loc}(\mathbb{R}^+; H^1(0, L)).$$

Aided by this smoothing property, Colin and Ghidaglia showed that the homogeneous IBVP (2.7) is locally well-posed in the space $L^2(0, L)$.

Theorem D [24] *Assuming $h_1 = h_2 = h_3 \equiv 0$, then for any given $\phi \in L^2(0, L)$, there exists a $T > 0$ such that the IBVP (2.7) admits a unique weak solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$.*

Returning the attention to the IBVP (2.7), Rivas et al. [49] showed that the solutions exist globally as long as their initial values and the associated boundary data are small, they proved the following result:

Theorem E [49] *Let $s \geq 0$ with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3 \dots$. There exist positive constants δ and T such that for any s -compatible*

$$\phi \in H^s(0, L) \text{ and } \vec{h} = (h_1, h_2, h_3) \text{ on the class}$$

$$B^s_{(t,t+T)} := H^{\frac{s+1}{3}}(t, t+T) \times H^{\frac{s}{3}}(t, t+T) \times H^{\frac{s-1}{3}}(t, t+T)$$

with $\|\phi\|_{H^s(0,L)} + \|\vec{h}\|_{B^s_{(t,t+T)}} \leq \delta$, and $\sup_{t \geq 0} \|\vec{h}\|_{B^s_{(t,t+T)}} < \infty$, the IBVP (2.9) admits a unique solution

$$u \in Y^s_{(t,t+T)} := C([t, t+T]; H^s(0, L)) \cap L^2(t, t+T; H^{s+1}(0, L))$$

such that for any $t \geq 0$, $\sup_{t \geq 0} \|\vec{v}\|_{Y^s_{(t,t+T)}} < \infty$.

More recently, Kramer et al. [46] showed that the IBVP (2.7) is locally well-posedness in the classical Sobolev space $H^s(0, L)$, for $s > -\frac{3}{4}$, which provides a positive answer to one of the open questions of Colin and Ghidalia [24].

Kramer and Zhang [45] studied the following non-homogeneous boundary value problem,

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, & u(x, 0) = \phi(x), & x \in (0, 1), \quad t \in (0, T), \\ \alpha_1 u_{xx}(0, t) + \alpha_2 u_x(0, t) + \alpha_3 u(0, t) = h_1(t), \\ \beta_1 u_{xx}(1, t) + \beta_2 u_x(1, t) + \beta_3 u(1, t) = h_2(t), \\ \chi_1 u_x(1, t) + \chi_2 u(1, t) = h_3(t). \end{cases} \quad (2.10)$$

They showed that the IBVP (2.10) is locally well-posed in the space $H^s(0, 1)$ for any $s \geq 0$ under the assumption (2.4).

Theorem F [45] *Let $s \geq 0$ and $T > 0$ be given and assume (2.4) holds. For any $r > 0$, there exists a $T^* \in (0, T]$ such that for any s -compatible $\phi \in H^s(0, 1)$, $h_j \in H^{\frac{s+1}{3}}(0, T)$, $j = 1, 2, 3$ with*

$$\|\phi\|_{H^s(0,1)} + \|h_1\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|_{H^{\frac{s+1}{3}}(0,T)} \leq r,$$

the IBVP (2.10) admits a unique solution

$$u \in C([0, T^*]; H^s(0, 1)) \cap L^2(0, T^*; H^{s+1}(0, 1)).$$

Moreover, the solution u depends continuously on its initial data ϕ and the boundary values h_j , $j = 1, 2, 3$ in the respective spaces.

Recently, Capistrano–Filho et al. [16] studied the IBVP (2.8). The authors proved the local well-posedness for this system. More precisely:

Theorem G [16] *Let $T > 0$ and $s \geq 0$. There exists a $T^* \in (0, T]$ such that for any $(\phi, \vec{h}) \in X_T$, where*

$$X_T := H^s(0, L) \times H^{\frac{s-1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T)$$

the IBVP (2.8) admits a unique solution

$$u \in C([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+1}(0, L))$$

In addition, the solution u possesses the hidden regularities

$$\partial_x^l u \in L^\infty(0, L; H^{\frac{s+1-l}{3}}(0, T^*)) \quad \text{for } l = 0, 1, 2.$$

and, moreover, the corresponding solution map is Lipschitz continuous.

Finally, in a recently work, Capistrano–Filho et al. [17] studied the well-posedness of IBVP (2.1)–(2.2). The authors proposed the following hypotheses on those coefficients a_{ij} , b_{ij} , $j, i = 0, 1, 2, 3$:

- (A1) $a_{12} = a_{11} = 0$, $a_{10} \neq 0$, $b_{12} = b_{11} = b_{10} = 0$;
- (A2) $a_{12} \neq 0$, $b_{12} = 0$;
- (B1) $b_{22} = b_{21} = 0$, $b_{20} \neq 0$, $a_{22} = a_{21} = a_{20} = 0$;
- (B2) $b_{22} \neq 0$, $a_{22} = 0$;
- (C) $b_{32} = 0$, $b_{31} \neq 0$, $a_{32} = a_{31} = 0$.

For $s \geq 0$, consider the set

$$H_0^s(0, L) := \{\phi(x) \in H^s(0, L) : \phi^{(k)}(0) = \phi^{(k)}(L) = 0\}$$

with $k = 0, 1, 2, \dots, [s]$ and

$$H_0^s(0, T) := \{h(t) \in H^s(0, T) : h^{(j)}(0) = 0\},$$

for $j = 0, 1, \dots, [s]$. In addition, letting

$$\begin{cases} \mathcal{H}_1^s(0, T) := H_0^{\frac{s+1}{3}}(0, T) \times H_0^{\frac{s+1}{3}}(0, T) \times H_0^{\frac{s}{3}}(0, T), \\ \mathcal{H}_2^s(0, T) := H_0^{\frac{s+1}{3}}(0, T) \times H_0^{\frac{s-1}{3}}(0, T) \times H_0^{\frac{s}{3}}(0, T), \\ \mathcal{H}_3^s(0, T) := H_0^{\frac{s-1}{3}}(0, T) \times H_0^{\frac{s+1}{3}}(0, T) \times H_0^{\frac{s}{3}}(0, T), \\ \mathcal{H}_4^s(0, T) := H_0^{\frac{s-1}{3}}(0, T) \times H_0^{\frac{s-1}{3}}(0, T) \times H_0^{\frac{s}{3}}(0, T) \end{cases}$$

and

$$\begin{cases} \mathcal{W}_1^s(0, T) := H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T), \\ \mathcal{W}_2^s(0, T) := H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T), \\ \mathcal{W}_3^s(0, T) := H^{\frac{s-1}{3}}(0, T) \times H^{\frac{s+1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T), \\ \mathcal{W}_4^s(0, T) := H^{\frac{s-1}{3}}(0, T) \times H^{\frac{s-1}{3}}(0, T) \times H^{\frac{s}{3}}(0, T), \end{cases}$$

they proved the following well-posedness results for the IBVP (2.1)–(2.2):

Theorem H [17] *Let $s \geq 0$ with $s \neq \frac{2j-1}{2}$, $j = 1, 2, 3, \dots$, and $T > 0$ be given. If one of the assumptions below is satisfied,*

- (i) (A1), (B1) and (C) hold,
- (ii) (A1), (B2) and (C) hold,
- (iii) (A2), (B1) and (C) hold,
- (iv) (A2), (B2) and (C) hold,

then, for any $r > 0$, there exists a $T^* \in (0, T]$ such that for any

$$(\phi, \vec{h}) \in H_0^s(0, L) \times \mathcal{H}_1^s(0, T)$$

satisfying $\|(\phi, \vec{h})\|_{L^2(0,L) \times \mathcal{H}_1^s(0,T)} \leq r$, the IBVP (2.1)–(2.2) admits a solution

$$u \in C([0, T^*]; H^s(0, L)) \cap L^2(0, T^*; H^{s+1}(0, L))$$

possessing the hidden regularity (the sharp Kato smoothing properties)

$$\partial_x^l u \in L^\infty(0, L; H^{\frac{s+1-l}{3}}(0, T^*)) \text{ for } l = 0, 1, 2.$$

Moreover, the corresponding solution map is analytically continuous.

3 Further comments

Before presenting the main ideas to prove Theorem G, let us introduce the following boundary operators \mathcal{B}_k , $k = 1, 2, 3, 4$ as $\mathcal{B}_k = \mathcal{B}_{k,0} + \mathcal{B}_{k,1}$ with

$$\begin{aligned} \mathcal{B}_{1,0}v &:= (v(0, t), v(L, t), v_x(L, t)), & \mathcal{B}_{2,0}v &:= (v(0, t), v_x(L, t), v_{xx}(L, t)), \\ \mathcal{B}_{3,0}v &:= (v_{xx}(0, t), v(L, t), v_x(L, t)), & \mathcal{B}_{4,0}v &:= (v_{xx}(0, t), v_x(L, t), v_{xx}(L, t)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_{1,1}v &:= (0, 0, 0), \\ \mathcal{B}_{2,1}v &:= (0, b_{30}v(L, t), a_{21}v_x(0, t) + b_{20}v(L, t)), \\ \mathcal{B}_{3,1}v &:= (a_{10}v(0, t) + a_{11}v_x(0, t), 0, a_{30}v(0, t)), \\ \mathcal{B}_{4,1}v &:= \left(\sum_{j=0}^1 a_{1j} \partial_x^j v(0, t) + b_{10}v(L, t), a_{30}v(0, t) \right. \\ &\quad \left. + b_{30}v(L, t), \sum_{j=0}^1 a_{2j} \partial_x^j v(0, t) + b_{20}v(L, t) \right). \end{aligned}$$

Thus, the assumptions imposed on the boundary conditions in Theorem G can be reformulated as follows:

- (i) $((A1), (B1), (C)) \Leftrightarrow \mathcal{B}_1v = \vec{h}$,
- (ii) $((A1), (C), (B2)) \Leftrightarrow \mathcal{B}_2v = \vec{h}$,
- (iii) $((A2), (B1), (C)) \Leftrightarrow \mathcal{B}_3v = \vec{h}$,
- (iv) $((A2), (C), (B2)) \Leftrightarrow \mathcal{B}_4v = \vec{h}$.

In [17], to prove Theorem G, the authors first studied the linear IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = f, & x \in (0, L), \quad t > 0 \\ u(x, 0) = \phi(x), \\ \mathcal{B}_{k,0}u = \vec{h}, \end{cases} \tag{3.1}$$

for $k = 1, 2, 3, 4$, to establish all the linear estimates needed for dealing with the nonlinear IBVP (2.1)–(2.2). Here $\delta_k = 0$ for $k = 1, 2, 3$ and $\delta_4 = 1$.

After that, they considered the nonlinear map Γ defined by the following IBVP:

$$\begin{cases} u_t + u_{xxx} + \delta_k u = -v_x - vv_x + \delta_k v, & x \in (0, L), \quad t > 0 \\ u(x, 0) = \phi(x), \\ \mathcal{B}_{k,0}u = \vec{h} - \mathcal{B}_{k,1}v, \end{cases} \tag{3.2}$$

showing thus that Γ is a contraction in an appropriate space whose fixed point will be the desired solution of the nonlinear IBVP (2.1)–(2.2) by using the sharp Kato smoothing property of the solution of the IBVP (3.1).

The main point here is to demonstrate the smoothing properties for solutions of the IBVP (3.1). In order to overcome this difficulty, Capistrano–Filho et al. [17] needed to study the following IBVP

$$\begin{cases} u_t + u_{xxx} + \delta_k u = 0, & x \in (0, L), \quad t > 0, \\ u(x, 0) = 0, \\ \mathcal{B}_{k,0} u = \vec{h}. \end{cases} \tag{3.3}$$

The corresponding solution map $\vec{h} \rightarrow u$ will be called the *boundary integral operator* denoted by $\mathcal{W}_{bdr}^{(k)}$. An explicit representation formula is given for this boundary integral operator that plays an important role in showing the solution of the IBVP (3.3) possesses the smoothing properties. The needed smoothing properties for solutions of the IBVP (3.1) will then follow from the smoothing properties for solutions of the IBVP (3.3) and the well-known sharp Kato smoothing properties for solutions of the Cauchy problem

$$u_t + u_{xxx} + \delta_k u = 0, \quad u(x, 0) = \psi(x), \quad x, t \in \mathbb{R}.$$

Finally, the following comments are now given in order:

Remark 1 The temporal regularity conditions imposed on the boundary values \vec{h} on Theorem G are optimal (cf. [3,5,6]).

Remark 2 As a comparison, note that the assumptions of Theorem A are equivalent to one of the following boundary conditions imposed on the equation in (2.3):

a)

$$u(0, t) = 0, \quad u(1, t) = 0, \quad u_x(1, t) = 0;$$

b)

$$u_{xx}(0, t) + au_x(0, t) + bu(0, t) = 0, \quad u_x(1, t) = 0, \quad u(1, t) = 0$$

with

$$a > b^2/2; \tag{3.4}$$

c)

$$u(0, t) = 0, \quad u_{xx}(1, t) + au_x(1, t) + bu(1, t) = 0, \quad u_x(1, t) + cu(1, t) = 0,$$

with

$$ac > b - c^2/2; \tag{3.5}$$

d)

$$\begin{aligned} u_{xx}(0, t) + a_1 u_x(0, t) + a_2 u(0, t) &= 0, \\ u_{xx}(1, t) + b_1 u_x(1, t) + b_2 u(1, t) &= 0, \end{aligned}$$

and

$$u_x(1, t) + cu(1, t) = 0,$$

with

$$a_2 > a_1^2/2, \quad b_1c > b_2 - c^2/2. \tag{3.6}$$

It follows from Theorem G that conditions (3.4), (3.5) and (3.6) for Theorem A can be removed.

4 Open problems

While the results reported in this paper gave a significant improvement in the theory of initial boundary value problems of the KdV equation on a finite interval, there are still many questions to be addressed for the following IBVP:

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & 0 < x < L, \quad t > 0, \\ u(x, 0) = \phi(x), \\ \mathcal{B}_k u = \vec{h}. \end{cases} \tag{4.1}$$

Here we list a few of them which are most interesting to us.

- *Is the IBVP (4.1) globally well-posed in the space $H^s(0, L)$ for some $s \geq 0$ or equivalently, does any solution of the IBVP (4.1) blow up in the some space $H^s(0, L)$ in finite time?*

It is not clear if the IBVP (4.1) is globally well-posed or not even in the case of $\vec{h} \equiv 0$. It follows Theorem G (see [17]) that a solution u of the IBVP (4.1) blows up in the space $H^s(0, L)$ for some $s \geq 0$ at a finite time $T > 0$ if and only if

$$\lim_{t \rightarrow T^-} \|u(\cdot, t)\|_{L^2(0,L)} = +\infty.$$

Consequently, it suffices to establish a global a priori $L^2(0, L)$ estimate

$$\sup_{0 \leq t < \infty} \|u(\cdot, t)\|_{L^2(0,L)} < +\infty, \tag{4.2}$$

for solutions of the IBVP (4.1) in order to obtain the global well-posedness of the IBVP (4.1) in the space $H^s(0, L)$ for any $s \geq 0$. However, estimate (4.2) is known to be held only in one case

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = f, & 0 < x < L, \quad t > 0 \\ u(x, 0) = \phi(x) \\ u(0, t) = h_1(t), \quad u(L, t) = h_2(t), \quad u_x(L, t) = h_3(t). \end{cases}$$

- Is the IBVP well-posed in the space $H^s(0, L)$ for some $s \leq -1$?

Theorem G ensures that the IBVP (4.1) is locally well-posed in the space $H^s(0, L)$ for any $s \geq 0$. Theorem G can also be extended to the case of $-1 < s \leq 0$ using the same approach developed in [7]. For the pure initial value problems (IVP) of the KdV equation posed on the whole line \mathbb{R} or on torus \mathbb{T} ,

$$u_t + uu_x + u_{xxx} = 0, \quad u(x, 0) = \phi(x), \quad x, t \in \mathbb{R} \tag{4.3}$$

and

$$u_t + uu_x + u_{xxx} = 0, \quad u(x, 0) = \phi(x), \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \tag{4.4}$$

it is well-known that the IVP (4.3) is well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -\frac{3}{4}$ and is (conditionally) ill-posed in the space $H^s(\mathbb{R})$ for any $s < -\frac{3}{4}$ in the sense the corresponding solution map cannot be uniformly continuous. As for the IVP (4.4), it is well-posed in the space $H^s(\mathbb{T})$ for any $s \geq -1$. The solution map corresponding to the IVP (4.4) is real analytic when $s > -\frac{1}{2}$, but only continuous (not even locally uniformly continuous) when $-1 \leq s < -\frac{1}{2}$. Whether the IVP (4.3) is well-posed in the space $H^s(\mathbb{R})$ for any $s < -\frac{3}{4}$ or the IVP (4.4) is well-posed in the space $H^s(\mathbb{T})$ for any $s < -1$ is still an open question. On the other hand, by contrast, the IVP of the KdV–Burgers equation

$$u_t + uu_x + u_{xxx} - u_{xx} = 0, \quad u(x, 0) = \phi(x), \quad x \in \mathbb{R}, \quad t > 0$$

is known to be well-posed in the space $H^s(\mathbb{R})$ for any $s \geq -1$, but is known to be ill-posed for any $s < -1$. We conjecture that the IBVP (4.1) is ill-posed in the space $H^s(0, L)$ for any $s < -1$.

Finally, still concerning with well-posedness problem, while the approach developed recently in [17] studies the nonhomogeneous boundary value problems of the KdV equation on $(0, L)$ with quite general boundary conditions, there are still some boundary value problems of the KdV equation that the approach do not work, for example

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, & x \in (0, L) \\ u(x, 0) = \phi(x), \\ u(0, t) = u(L, t), \quad u_x(0, t) = u_x(L, t), \quad u_{xx}(0, t) = u_{xx}(L, t) \end{cases} \tag{4.5}$$

and

$$\begin{cases} u_t + uu_x + u_{xxx} = 0, & x \in (0, L), \\ u(x, 0) = \phi(x), \\ u(0, t) = 0, \quad u(L, t) = 0, \quad u_x(0, t) = u_x(L, t). \end{cases} \tag{4.6}$$

A common feature for these two boundary value problems is that the L^2 -norm of their solutions are conserved:

$$\int_0^L u^2(x, t) dx = \int_0^L \phi^2(x) dx \quad \text{for any } t \in \mathbb{R}.$$

The IBVP (4.5) is equivalent to the IVP (4.4) which was shown by Kato [36,37] to be well-posed in the space $H^s(\mathbb{T})$ when $s > \frac{3}{2}$ as early as in the late 1970s. Its well-posedness in the space $H^s(\mathbb{T})$ when $s \leq \frac{3}{2}$, however, was established 24 years later in the celebrated work of Bourgain [8,9]. As for the IBVP (4.6), its associated linear problem

$$\begin{cases} u_t + u_{xxx} = 0, & x \in (0, L), \\ u(x, 0) = \phi(x), & u(0, t) = 0, \\ u(L, t) = 0, & u_x(0, t) = u_x(L, t) \end{cases}$$

has been shown by Cerpa (see, for instance, [19]) to be well-posed in the space $H^s(0, L)$ forward and backward in time. However, the following problem is still unknown:

- Is the nonlinear IBVP (4.6) well-posed in the space $H^s(0, L)$ for some s ?

4.1 Control theory

Control theory for KdV equation has been extensively studied in the past 2 decades and the interested reader is referred to [19] for an overall view of the subject. As it is possible to see in the paper above, several authors have addressed the study of control theory of the IBVP (see, e.g. [16,21,50]), who worked on the following four problems related to the IBVP (4.1):

$$\begin{aligned} \mathcal{B}_{1,0}v &:= \begin{cases} u(0, t) = h_{1,1}(t), & t \geq 0, \\ u(L, t) = h_{2,1}(t), & t \geq 0, \\ u_x(L, t) = h_{3,1}(t), & t \geq 0, \end{cases} & \mathcal{B}_{2,0}v &:= \begin{cases} u(0, t) = h_{1,2}(t), & t \geq 0, \\ u_x(L, t) = h_{2,2}(t), & t \geq 0, \\ u_{xx}(L, t) = h_{3,2}(t), & t \geq 0, \end{cases} \\ \mathcal{B}_{3,0}u &:= \begin{cases} u_{xx}(0, t) = h_{1,3}(t) & t \geq 0, \\ u(L, t) = h_{2,3}(t), & t \geq 0, \\ u_x(L, t) = h_{3,3}(t), & t \geq 0 \end{cases} & \mathcal{B}_{4,0}u &:= \begin{cases} u_{xx}(0, t) = h_{1,4}(t), & t \geq 0, \\ u_x(L, t) = h_{2,4}(t), & t \geq 0, \\ u_{xx}(L, t) = h_{3,4}(t), & t \geq 0. \end{cases} \end{aligned}$$

The first class of problem (4.1)– $\mathcal{B}_{1,0}v$ was studied by Rosier [50] considering only the control input $h_{3,1}$ (i.e. $h_{1,1} = h_{2,1} = 0$). It was shown in [50] that the exact controllability of the linearized system holds in $L^2(0, L)$ if and only if, L does not belong to the following countable set of critical lengths

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} : k, l \in \mathbb{N}^* \right\}.$$

The analysis developed in [50] shows that when the linearized system is controllable, the same is true for the nonlinear case. Note that the converse is false, as it was proved in [18,20,26], that is, the (nonlinear) KdV equation is controllable even when L is a critical length but the linearized system is non controllable.

The existence of a discrete set of critical lengths for which the exact controllability of the linearized equation fails was also noticed by Glass and Guerrero [33] when $h_{2,1}$ is taken as control input (i.e. $h_{1,1} = h_{3,1} = 0$). Finally, it is worth mentioning the

result by Rosier [51] and Glass and Guerrero [32] for which $h_{1,1}$ is taken as control input (i.e. $h_{2,1} = h_{3,1} = 0$). They proved that system (4.1) with boundary conditions $\mathcal{B}_{1,0}v$ is then null controllable, but not exactly controllable, because of the strong smoothing effect.

Recently, Cerpa et al. [21] proved similar results to those obtained by Rosier [50] for the system (4.1) with boundary conditions $\mathcal{B}_{2,0}v$. More precisely, the authors consider the system with one, two or three controls. In addition, using the well-posedness properties proved by Kramer et al. [46], they also proved that the controls $h_{i,2}$, $i = 1, 2, 3$ belong to sharp spaces and the locally exact controllability of the linear system associated to (4.1) holds if, and only if, L does not belong to the following countable set of critical lengths

$$\mathcal{F} := \left\{ L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} \right. \\ \left. \text{satisfying } \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \right\}. \tag{4.7}$$

Moreover, they showed that the nonlinear system (4.1) with boundary conditions $\mathcal{B}_{2,0}v$ is locally exactly controllable *via* the contraction mapping principle.

Finally, more recently, Caicedo et al. [16] proved the controllability results for the system (2.8), that is, system (4.1) with boundary conditions $\mathcal{B}_{4,0}v$. Naturally, they used the same approaches that have worked effectively for system (4.1) with boundary condition $\mathcal{B}_{1,0}v$ and $\mathcal{B}_{2,0}v$. In particular, if only $h_{2,4}(t)$ is used, they showed that the system (4.1) with boundary conditions $\mathcal{B}_{4,0}v$ is *locally exactly controllable* as long as

$$L \notin \mathcal{R} := \mathcal{N} \cup \{k\pi : k \in \mathbb{N}^*\}. \tag{4.8}$$

Thus, with respect of the control issue, a natural and interesting open problem arises here:

- *Is the IBVP (4.1), with general boundary condition, controllable?*

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