

NEUMANN BOUNDARY CONTROLLABILITY OF THE KORTEWEG–DE VRIES EQUATION ON A BOUNDED DOMAIN*

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Abstract. In this paper we study boundary controllability of the Korteweg–de Vries equation posed on a finite domain with the Neumann boundary conditions. We consider the cases where one, two, or all three of those boundary data are employed as boundary control inputs. To get the main result, the system is first linearized and the corresponding linear system is proved to be exactly boundary controllable if using one, two, or three boundary control inputs. In the case where only one control input is allowed to be used, the linearized system is shown to be exactly controllable if and only if the length of the spatial domain does not belong to a set of critical values. Then the nonlinear system is shown to be locally exactly controllable around a constant steady state if the associated linear system is exactly controllable.

Key words. Korteweg–de Vries equation, exact boundary controllability, Neumann boundary conditions, Dirichlet boundary conditions, critical set

AMS subject classifications. Primary, 35Q53; Secondary, 37K10, 93B05, 93D15

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1. Introduction. In this paper we study a class of distributed parameter control systems described by the Korteweg–de Vries (KdV) equation posed on a bounded interval $(0, L)$ with the Neumann boundary conditions:

$$(1.1) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = 0, \quad u_x(L, t) = h(t), \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

where the boundary value function $h = h(t)$ will be considered as a control input. We are mainly concerned with its exact control problem:

Given $T > 0$ and $u_0, u_T \in L^2(0, L)$, can one find an appropriate control input h such that the corresponding solution u of (1.1) satisfies

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x)?$$

The study of control and stabilization of the KdV equation begun with the works of Russell [19], Zhang [26], Russell and Zhang [20, 21] in which they studied internal control of the KdV equation posed on a finite domain $(0, L)$ with periodic boundary conditions. Aided by then newly discovered Bourgain smoothing properties [2, 3] they showed that the internal control system is locally exactly controllable and exponen-

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tially stabilizable.¹ Since then, control and stabilization of the KdV equation have been intensively studied (see [4, 5, 6, 8, 9, 10, 16, 17, 18, 25] and references therein). In particular, Rosier [16] studied boundary control of the KdV equation posed on the finite domain $(0, L)$ with the Dirichlet boundary conditions:

$$(1.2) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0, u(L, t) = 0, u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

where the boundary value function $g(t)$ is considered as a control input. Rosier considered first the associated linear system

$$(1.3) \quad \begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = 0, u(L, t) = 0, u_x(L, t) = g(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

and discovered the so-called *critical length* phenomena; whether the system (1.3) is exactly controllable depends on the length L of the spatial domain $(0, L)$.

THEOREM A (Rosier [16]). *The linear system (1.3) is exactly controllable in the space $L^2(0, L)$ if and only if the length L of the spatial domain $(0, L)$ does not belong to the set*

$$(1.4) \quad \mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} : k, l \in \mathbb{N}^* \right\}.$$

The controllability result of the linear system was then extended to the nonlinear system when $L \notin \mathcal{N}$.

THEOREM B (Rosier [16]). *Let $T > 0$ be given and assume $L \notin \mathcal{N}$. There exists $\delta > 0$ such for any $u_0, u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

one can find a control input $g \in L^2(0, T)$ such that the nonlinear system (1.2) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

In the case $L \in \mathcal{N}$, Rosier proved in [16] that the associated linear system (1.3) is not controllable; there exists a finite-dimensional subspace of $L^2(0, L)$, denoted by $\mathcal{M} = \mathcal{M}(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi \in \mathcal{M}$, $g \in L^2(0, T)$, and $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ satisfying (1.3) and $u(\cdot, 0) = 0$, one has $u(\cdot, T) \neq \psi$. A spatial domain $(0, L)$ is called *critical* for the system (1.3) if its domain length $L \in \mathcal{N}$.

When the spacial domain $(0, L)$ is critical, one usually would not expect the corresponding nonlinear system (1.2) to be exactly controllable as the linear system (1.3) is not. It thus came as a surprise when Coron and Crépeau showed in [8] that

¹The system has been shown recently to be globally exponentially stabilizable and large time exactly controllable by Laurent, Rosier, and Zhang [15].

the nonlinear system (1.2) is still locally exactly controllable even though its spatial domain is critical with its length $L = 2k\pi$ and $k \in \mathbb{N}^*$ satisfying

$$\exists(m, n) \in \mathbb{N}^* \times \mathbb{N}^* \text{ with } m^2 + mn + n^2 = 3k^2 \text{ and } m \neq n.$$

For those values of L , the unreachable space \mathcal{M} of the associated linear system is a one-dimensional linear space generated by the function $1 - \cos(x)$. As for the other types of critical domains, the nonlinear system (1.2) was shown later by Cerpa [4] and Cerpa and Crépeau in [6] to be locally, large time exactly controllable.

THEOREM C (Cerpa [4] and Cerpa and Crépeau [6]). *Let $L \in \mathcal{N}$ be given. There exists a $T_L > 0$ such that for any $T > T_L$ there exists $\delta > 0$ such for any $u_0, u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

there exists $g \in L^2(0, T)$ such that the system (1.2) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

In this paper, we will investigate the boundary control system (1.1) of the KdV equation posed on the finite domain $(0, L)$ with the Neumann boundary conditions to see if it possesses similar control properties as that possessed by the boundary control system (1.2). First we will study the following linearized system associated with (1.1),

$$(1.5) \quad \begin{cases} u_t + (1 + \beta)u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = 0, \quad u_x(L, t) = h(t), \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

where β is a given real constant. For any $\beta \neq -1$, we define

$$(1.6) \quad \mathcal{R}_\beta := \left\{ \frac{2\pi}{\sqrt{3(1 + \beta)}} \sqrt{k^2 + kl + l^2} : k, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{\beta + 1}} : k \in \mathbb{N}^* \right\}.$$

The following theorem is one of the main results in this paper.

THEOREM 1.1.

- (i) *If $\beta \neq -1$, the linear system (1.5) is exactly controllable in the space $L^2(0, L)$ if and only if the length L of the spatial domain $(0, L)$ does not belong to the set \mathcal{R}_β .*
- (ii) *If $\beta = -1$, then the system (1.5) is not exact controllable in the space $L^2(0, L)$ for any $L > 0$.*

The next theorem addressing controllability of the nonlinear system (1.1) is another main result of the paper.

THEOREM 1.2. *Let $T > 0$, $\beta \neq -1$, and $L \notin \mathcal{R}_\beta$ be given. There exists a $\delta > 0$ such that for any $u_0, u_T \in L^2(0, L)$ with*

$$\|u_0 - \beta\|_{L^2(0,L)} + \|u_T - \beta\|_{L^2(0,L)} \leq \delta,$$

one can find a control input $h \in L^2(0, T)$ such that the system (1.1) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

The following remarks are now in order.

Remark 1.3. In the case $\beta = 0$, \mathcal{N} is a proper subset of \mathcal{R}_0 . The linear system (1.5) has more critical length domains than that of the linear system (1.3). In the case $\beta = -1$, every $L > 0$ is critical for the system (1.5). By contrast, if we remove the term u_x from the equation in (1.3), every $L > 0$ is not critical for the system (1.3).

Theorem 1.1 will be proved using the same approach that Rosier used to establish Theorem A. However, one will encounter some difficulties that demand special attention. The adjoint system of the linear system (1.5) is given by

$$(1.7) \quad \begin{cases} \psi_t + (1 + \beta)\psi_x + \psi_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ (1 + \beta)\psi(0, t) + \psi_{xx}(0, t) = 0 & \text{in } (0, T), \\ (1 + \beta)\psi(L, t) + \psi_{xx}(L, t) = 0 & \text{in } (0, T), \\ \psi_x(0, t) = 0 & \text{in } (0, T), \\ \psi(x, T) = \psi_T(x) & \text{in } (0, L). \end{cases}$$

It is well known that the exact controllability of system (1.5) is equivalent to the following observability inequality for the adjoint system (1.7):

$$(1.8) \quad \|\psi_T\|_{L^2(0,L)} \leq C\|\psi_x(L, \cdot)\|_{L^2(0,T)}.$$

However, the usual multiplier method and compactness arguments as those used in dealing with the system (1.7) only lead to

$$(1.9) \quad \|\psi_T\|_{L^2(0,L)}^2 \leq C_1\|\psi_x(L, \cdot)\|_{L^2(0,T)}^2 + C_2\|\psi(L, \cdot)\|_{L^2(0,T)}^2.$$

One has to find a way to remove the extra term present in (1.9). For this, a technical lemma presented below, which reveals some hidden regularities (or sharp trace regularities) for solutions of the adjoint system (1.7), is needed.

LEMMA 1.4 (hidden regularities). *For any $\psi_T \in L^2(0, L)$, the solution*

$$\psi \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

of IBVP (1.7) possesses the following sharp trace properties

$$(1.10) \quad \sup_{x \in (0, L)} \|\partial_x^r \psi(x, \cdot)\|_{H^{\frac{1-r}{3}}(0, T)} \leq C_r \|\psi_0\|_{L^2(0, L)}$$

for $r = 0, 1, 2$.

The sharp Kato smoothing properties of solutions of the Cauchy problem of the KdV equation posed on the whole line \mathbb{R} due to Kenig, Ponce, and Vega [12] will play an important role in the proof of Lemma 1.4.

Following the work of Rosier [16], the boundary control system of the KdV equation posed on the finite interval $(0, L)$ with various control inputs has been intensively studied (cf. [9, 10, 11] and see [5, 18] for more complete reviews):

$$(1.11) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u(0, t) = g_1(t), \quad u(L, t) = g_2(t), \quad u_x(L, t) = g_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L). \end{cases}$$

The system (1.11) has been found to have an interesting property: it behaves like a parabolic system if control only applied on the left end of the spatial domain $(0, L)$ ($g_2 = g_3 \equiv 0$): the system is only null controllable; but if control is allowed to apply on the right end of the spatial domain $(0, L)$, the system behaves like a hyperbolic system which is exactly controllable. Moreover, the critical length phenomenon occurs only in the case that just one control is applied to the right end of the spatial domain $(0, L)$.

In this paper we will also show that the boundary control systems of the KdV equation posed on $(0, L)$ with Neumann boundary conditions,

$$(1.12) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

possess similar properties.

THEOREM 1.5. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0, L)$ with*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \delta,$$

one can find h_1, h_2 , and h_3 satisfying one of the conditions below:

- (i) $h_1 \in H^{-\frac{1}{3}}(0, T)$ and $h_2 \in L^2(0, T)$, $h_3 = 0$;
- (ii) $h_2 \in L^2(0, T)$, $h_3 \in H^{-\frac{1}{3}}(0, T)$, and $h_1 = 0$;
- (iii) $h_1, h_3 \in H^{-\frac{1}{3}}(0, T)$, $h_2 = 0$,

such that the system (1.12) admits unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

If all three boundary control inputs are used, then the system (1.12) has much stronger controllability; it is locally exactly controllable around any smooth solution of the KdV equation in the space $H^s(0, L)$ for any $s \geq 0$ and is large time globally exactly controllable in the space $H^s(0, L)$ for any $s \geq 0$.

THEOREM 1.6. *Let $T > 0$, $s \geq 0$, and $L > 0$ be given. Assume that $y \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ satisfies*

$$y_t + y_x + yy_x + y_{xxx} = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}.$$

Then, there exists $\delta > 0$ such that for any $u_0, u_T \in H^s(0, L)$ with

$$\|u_0 - y(\cdot, 0)\|_{H^s(0,L)} + \|u_T - y(\cdot, T)\|_{H^s(0,L)} \leq \delta,$$

one can find

$$h_1 \in H^{\frac{s-1}{3}}(0, T), \quad h_2 \in H^{\frac{s}{3}}(0, T), \quad h_3 \in H^{\frac{s-1}{3}}(0, T)$$

such that system (1.12) admits a unique solution

$$u \in C([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+1}(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

THEOREM 1.7. *Let $L > 0$ and $\gamma > 0$ be given. There exists a $T > 0$ such that for any $u_0, u_T \in L^2(0, L)$ satisfying*

$$\|u_0\|_{L^2(0,L)} + \|u_T\|_{L^2(0,L)} \leq \gamma,$$

one can find

$$h_1 \in H^{-\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T), \quad h_3 \in H^{-\frac{1}{3}}(0, T)$$

such that system (1.12) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

Finally, if we consider the system with control only acting on the left end of the spatial domain $(0, L)$,

$$(1.13) \quad \begin{cases} u_t + u_x + u_{xxx} + uu_x = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = 0, \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

we have the following null controllability result.

THEOREM 1.8 (null controllability). *Let $T > 0$ be fixed. For $\bar{u}_0 \in L^2(0, L)$, let*

$$\bar{u} \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

be the solution of the following system

$$(1.14) \quad \begin{cases} \bar{u}_t + \bar{u}_x + \bar{u}_{xxx} + \bar{u}\bar{u}_x = 0 & \text{in } (0, L) \times (0, T), \\ \bar{u}_{xx}(0, t) = 0, \quad \bar{u}_x(L, t) = 0, \quad \bar{u}_{xx}(L, t) = 0 & \text{in } (0, T), \\ \bar{u}(x, 0) = \bar{u}_0(x) & \text{in } (0, L). \end{cases}$$

Then, there exists $\delta > 0$ such that for any $u_0 \in L^2(0, L)$ satisfying

$$\|u_0 - \bar{u}_0\|_{L^2(0,L)} < \delta,$$

there exists $h_1(t) \in L^2(0, T)$ such that the solution $u(x, t)$ of the system (1.13) belongs to the space

$$C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

and satisfies

$$u(x, T) = \bar{u}(x, T) \quad \text{in } (0, L).$$

The paper is organized as follows.

In section 2, we study the nonhomogeneous boundary value problem of the KdV equation on the finite interval $(0, L)$,

$$(1.15) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

for its well-posedness in the space $L^2(0, L)$. We will show that the system (1.15) is locally well-posed in the space $L^2(0, L)$: for any $u_0 \in L^2(0, L)$,

$$h_1 \in H^{-\frac{1}{3}}(\mathbb{R}^+), \quad h_2 \in L^2(\mathbb{R}^+), \quad \text{and} \quad h_3 \in H^{-\frac{1}{3}}(\mathbb{R}^+),$$

there exists a $T > 0$ such that (1.15) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

Various linear estimates including hidden regularities will be presented for solutions of the linear system associated with (1.15), which will play important roles in studying controllability of the system.

In section 3, the boundary control system (1.1) will be investigated for its controllability. We investigate first the linearized system (1.5) and its corresponding adjoint system (1.7) for their controllability and observability. In particular, the hidden regularities for solutions of the adjoint system (1.7) will be presented and then be used to prove Theorems 1.1 and 1.2.

The sketch of proofs of Theorems 1.5 and 1.8 will be presented in section 4 together with the proofs of Theorems 1.6 and 1.7.

We end our introduction with a few more comments. Having shown the nonlinear system (1.1) is locally exactly controllable (Theorem 1.2) if the length of the spatial domain is not critical, one naturally would like to show the system (1.1) is still locally exactly controllable when the length of its spatial domain is critical as in the case of system (1.2) (see Theorem C). We believe that the same approach developed in [4, 7, 8] to prove Theorem C for the system (1.2) can be adapted to obtain similar results for system (1.1). However, the Neumann boundary conditions present some extra difficulties. In particular, the adjoint linear system associated with (1.1) is different from the adjoint linear system associated with (1.2). The unobservable solutions of the adjoint system associated with (1.1) are not the solutions of the forward linear system associated with (1.1). When using the power series expansion method proposed in [7], there are more terms appearing that demand special handling. We plan to continue to study system (1.1) with the critical length for its controllability and present our results in a forthcoming paper.

2. Well-posedness. In this section, we will show the IBVP

$$(2.1) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

is locally well-posed in the space $L^2(0, L)$ with $u_0(x) \in L^2(0, L)$ and

$$h_1, h_3 \in H^{-\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T).$$

We begin by considering the following linear nonhomogeneous boundary value problem,

$$(2.2) \quad \begin{cases} w_t + w_{xxx} = 0, \quad w(x, 0) = 0, & x \in (0, L), t > 0, \\ w_{xx}(0, t) = h_1(t), \quad w_x(L, t) = h_2(t), \quad w_{xx}(L, t) = h_3(t), & t > 0. \end{cases}$$

First, we derive an explicit solution formula for its solution. Applying the Laplace transform with respect to t , (2.2) is converted to

$$(2.3) \quad \begin{cases} s\hat{w} + \hat{w}_{xxx} = 0, \\ \hat{w}_{xx}(0, s) = \hat{h}_1(s), \quad \hat{w}_x(L, s) = \hat{h}_2(s), \quad \hat{w}_{xx}(L, s) = \hat{h}_3(s), \end{cases}$$

where

$$\hat{w}(x, s) = \int_0^{+\infty} e^{-st} w(x, t) dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st} h_j(t) dt, \quad j = 1, 2, 3.$$

The solution $\hat{w}(x, s)$ can be written in the form

$$\hat{w}(x, s) = \sum_{j=1}^3 c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1, 2, 3$, are the solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and $c_j(s)$, $j = 1, 2, 3$, solve the linear system

$$(2.4) \quad \underbrace{\begin{pmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ \lambda_1^2 e^{\lambda_1 L} & \lambda_2^2 e^{\lambda_2 L} & \lambda_3^2 e^{\lambda_3 L} \end{pmatrix}}_A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix}}_{\vec{h}}.$$

By Cramer's rule,

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \quad j = 1, 2, 3,$$

with Δ the determinant of A and Δ_j the determinant of the matrix A with the j th column replaced by \vec{h} . The solution $w(x, t)$ for (2.2) can be written in the form

$$(2.5) \quad w(x, t) = \sum_{m=1}^3 w_m(x, t),$$

where $w_m(x, t)$ solves (2.2) with $h_j \equiv 0$ when $j \neq m$, $j, m = 1, 2, 3$. Using the inverse Laplace transform yields

$$w(x, t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} e^{st} \hat{w}(x, s) ds = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{\Delta_j(s)}{\Delta(s)} e^{\lambda_j(s)x} ds$$

for $r > 0$. Combining this with (2.5) we can write the values of w_m as follows for $m = 1, 2, 3$,

$$w_m(x, t) = \sum_{j=1}^3 \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{\Delta_{j,m}(s)}{\Delta(s)} e^{\lambda_j(s)x} \hat{h}_m(s) ds \equiv [W_{m,j}(t) h_m](x).$$

In the last two formulas, the right-hand sides are continuous with respect to r for $r \geq 0$. As the left-hand sides do not depend on r , we can take $r = 0$ in these formulas. Moreover,

$$w_{j,m}(x, t) = w_{j,m}^+(x, t) + w_{j,m}^-(x, t),$$

where

$$w_{j,m}^+(x, t) = \frac{1}{2\pi i} \int_0^{+\infty} e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} \hat{h}_m(s) e^{\lambda_j(s)x} ds$$

and

$$w_{j,m}^-(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^0 e^{st} \frac{\Delta_{j,m}(s)}{\Delta(s)} \hat{h}_m(s) e^{\lambda_j(s)x} ds$$

for $j, m = 1, 2, 3$. Here $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m, k, m = 1, 2, 3$. Make the substitution $s = i\rho^3$ with $\rho \geq 0$ in the characteristic equation

$$s + \lambda^3 = 0.$$

The three roots are given in terms of s by

$$(2.6) \quad \lambda_1(s) = is, \quad \lambda_2(s) = -is \left(\frac{1 + i\sqrt{3}}{2} \right), \quad \lambda_3(s) = -is \left(\frac{1 - i\sqrt{3}}{2} \right),$$

thus $w_{j,m}^+$ has the following form

$$w_{j,m}^+(x, t) = \frac{1}{2\pi i} \int_0^{+\infty} e^{i\rho^3 t} \frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)} \hat{h}_m^+(\rho) e^{\lambda_j^+(\rho)x} 3i\rho^2 d\rho$$

and

$$w_{j,m}^-(x, t) = \overline{w_{j,m}^+(x, t)},$$

where $\hat{h}_m^+(\rho) = \hat{h}_m(i\rho^3)$, $\Delta^+(\rho) = \Delta(i\rho^3)$, $\Delta_{j,m}^+(\rho) = \Delta_{j,m}(i\rho^3)$, and $\lambda_j^+(\rho) = \lambda_j(i\rho^3) = \lambda_j(s)$.

Then the solution of the IBVP (2.2) has the following representation.

LEMMA 2.1. *Given $\vec{h} = (h_1, h_2, h_3)$, the solution w of the IBVP (2.2) can be written in the form*

$$w(x, t) = [W_{bdr}\vec{h}](x, t) := \sum_{j,m=1}^3 [W_{j,m}h_m](x, t).$$

Let $\vec{h} := (h_1, h_2, h_3) \in \mathcal{H}_T$ with

$$\mathcal{H}_T = H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T)$$

and

$$\mathcal{Z}_T := C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

The following lemma holds, for solution of the system (2.2).

PROPOSITION 2.2. *Let $T > 0$ be given. For any $\vec{h} \in \mathcal{H}_T$ the system (2.2) admits a unique solution $w \in \mathcal{Z}_T$. Moreover, there exists a constant $C > 0$ such that*

$$\|w\|_{\mathcal{Z}_T} + \sum_{j=0}^2 \|\partial_x^j w\|_{L^\infty(0, L; H^{\frac{1-j}{3}}(0, T))} \leq C \|\vec{h}\|_{\mathcal{H}_T}.$$

Proof. As we stated above, the solution w can be written as

$$w(x, t) = w_1(x, t) + w_2(x, t) + w_3(x, t).$$

We just prove Proposition 2.2 for w_1 . The proof for w_2 and w_3 are similar. Some

straightforward calculations show that the asymptotic behavior of the ratios $\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}$ as $\rho \rightarrow +\infty$ are

$$\begin{aligned} \frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2} e^{-\frac{\sqrt{3}}{2}\rho L}, & \frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2} e^{-\frac{\sqrt{3}}{2}\rho L}, & \frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2} e^{-\frac{\sqrt{3}}{2}\rho L}, \\ \frac{\Delta_{1,2}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-1}, & \frac{\Delta_{2,2}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-1} & \frac{\Delta_{3,2}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-1}, \\ \frac{\Delta_{1,3}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2}, & \frac{\Delta_{2,3}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2} e^{-\frac{\sqrt{3}}{2}\rho L} & \frac{\Delta_{3,3}^+(\rho)}{\Delta^+(\rho)} &\sim \rho^{-2}. \end{aligned}$$

Since

$$w_1(x, t) = \frac{3}{\pi} \sum_{j=1}^3 \int_0^{+\infty} e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \hat{h}_1^+(\rho) \rho^2 d\rho,$$

we have

$$\begin{aligned} \sup_{t \in (0, T)} \|w_1(\cdot, t)\|_{L^2(0, L)}^2 &\leq C \int_0^{+\infty} \mu^{-2/3} |\hat{h}_1(i\mu)|^2 d\mu \\ &\leq C \|h_1\|_{H^{-\frac{1}{3}}(\mathbb{R}^+)}^2 \\ &\leq C \|\vec{h}\|_{\mathcal{H}_T}^2. \end{aligned}$$

Furthermore, for $l = 0, 1, 2$, let us to consider $\theta(\mu)$ the real solution of $\mu = \rho^3$, $\rho \geq 0$, thus

$$\begin{aligned} \partial_x^l w_1(x, t) &= \frac{3}{\pi} \sum_{j=1}^3 \int_0^{+\infty} (\lambda_j^+(\rho))^l e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \hat{h}_1^+(\rho) \rho^2 d\rho \\ &= \frac{3}{\pi} \sum_{j=1}^3 \int_0^{+\infty} (\lambda_j^+(\theta(\mu)))^l e^{i\rho^3 t} e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} \hat{h}_1(i\mu) d\mu. \end{aligned}$$

Applying the Plancherel theorem, in time t , yields that, for all $x \in (0, L)$

$$\begin{aligned} \|\partial_x^l w_1(x, \cdot)\|_{H^{\frac{1-l}{3}}(0, T)}^2 &\leq C \sum_{j=1}^3 \int_0^{+\infty} \mu^{\frac{2(1-l)}{3}} \left| (\lambda_j^+(\theta(\mu))^{\rho+1}) e^{\lambda_j^+(\theta(\mu))x} \frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))} \hat{h}_1(i\mu) \right|^2 d\mu \\ &\leq C \int_0^{+\infty} \mu^{-\frac{2}{3}} |\hat{h}_1(i\mu)|^2 d\mu \\ &\leq C \|h_1\|_{H^{-\frac{1}{3}}(0, T)}^2 \\ &\leq C \|\vec{h}\|_{\mathcal{H}_T}^2 \end{aligned}$$

for $l = 0, 1, 2$. Consequently

$$\sup_{x \in (0, L)} \|\partial_x^l w_1(x, \cdot)\|_{H^{\frac{(1-l)}{3}}(0, T)} \leq C \|\vec{h}\|_{\mathcal{H}_T}, \quad l = 0, 1, 2$$

which ends the proof of Proposition 2.2 for w_1 . □

Next we turn to consider the following IBVP:

$$(2.7) \quad \begin{cases} v_t + v_{xxx} = f, & x \in (0, L), t > 0, \\ v_{xx}(0, t) = 0, v_x(L, t) = 0, v_{xx}(L, t) = 0, & t > 0, \\ v(x, 0) = \phi(x), & x \in (0, L). \end{cases}$$

By semigroup theory, for any $\phi \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, it possesses a unique mild solution $u \in C([0, T]; L^2(0, L))$ which can be written as

$$u(x, t) = W_0(t)\phi + \int_0^t W_0(t - \tau)f(\tau)d\tau.$$

Here $\{W_0(t)\}_{t \geq 0}$ is the C_0 -semigroup in the space $L^2(0, L)$ generated by linear operator

$$Ag = -g'''$$

with domain

$$\mathcal{D}(A) = \{g \in H^3(0, L) : g''(0) = g'(L) = g''(L) = 0\}.$$

In order to show that the solution u of (2.7) also possesses the Kato smoothing property

$$u \in L^2(0, T; H^1(0, L))$$

and the hidden regularity (the sharp Kato smoothing property)

$$\partial_x^k u \in L_x^\infty(0, L; H^{\frac{1-k}{3}}(0, T)), \quad k = 0, 1, 2,$$

we rewrite u in terms of boundary integral operator $W_{bdr}(t)$ and the solution of the following initial value problem of the linear KdV equation posed on the whole line \mathbb{R} ,

$$(2.8) \quad \begin{cases} v_t + v_{xxx} = g(x, t), & x \in \mathbb{R}, t \in \mathbb{R}^+, \\ v(x, 0) = \psi(x). \end{cases}$$

Recall that the solution $v(x, t)$ can be written as

$$v(x, t) = W_{\mathbb{R}}(t)\psi + \int_0^t W_{\mathbb{R}}(t - \tau)g(\tau)d\tau,$$

where $\{W_{\mathbb{R}}(t)\}_{t \in \mathbb{R}}$ is the C_0 group in the space $L^2(0, L)$ generated by the operator $Kg = -g'''$ with domain $\mathcal{D}(K) = H^3(\mathbb{R})$. The following results are well known for solutions of (2.8) (see, e.g., [12]).

PROPOSITION 2.3. *For any $\psi \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}; L^2(\mathbb{R}))$, (2.8) admits a unique mild solution $v \in C(\mathbb{R}; L^2(\mathbb{R}))$ satisfying*

$$\|v(\cdot, t)\|_{L^2(\mathbb{R})} \leq C (\|\psi\|_{L^2(\mathbb{R})} + \|g\|_{L^1(\mathbb{R}; L^2(\mathbb{R}))}) \quad \text{for any } t \in \mathbb{R}.$$

Moreover, the solution v possesses the sharp Kato smoothing properties

$$\partial_x^k v \in L_x^\infty(\mathbb{R}; H^{(k-1)/3}(\mathbb{R}))$$

and

$$\|\partial_x^k v\|_{L_x^\infty(\mathbb{R}; H^{(k-1)/3}(\mathbb{R}))} \leq C_k (\|\psi\|_{L^2(\mathbb{R})} + \|g\|_{L^1(\mathbb{R}; L^2(\mathbb{R}))})$$

for $k = 0, 1, 2$.

For $\phi \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, let

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in (0, L), \\ 0 & \text{if } x \notin (0, L) \end{cases}$$

and

$$\tilde{f}(x, t) = \begin{cases} f(x, t) & \text{if } x \in (0, L) \times (0, T), \\ 0 & \text{if } x \notin (0, L) \times (0, T). \end{cases}$$

We have $\tilde{\phi} \in L^2(\mathbb{R})$, $\tilde{f} \in L^1(\mathbb{R}; L^2(\mathbb{R}))$, and

$$\|\tilde{\phi}\|_{L^2(\mathbb{R})} = \|\phi\|_{L^2(0,L)}, \quad \|\tilde{f}\|_{L^1(\mathbb{R};L^2(\mathbb{R}))} = \|f\|_{L^1(0,T;L^2(0,L))}.$$

Let

$$v(x, t) = W_{\mathbb{R}}(t)\tilde{\phi} + \int_0^t W_{\mathbb{R}}(t - \tau)\tilde{f}(\tau)d\tau$$

and

$$q_1(t) = v_{xx}(0, t), \quad q_2(t) = v_x(L, t), \quad q_3(t) = v_{xx}(L, t), \quad \vec{q}(t) = (q_1(t), q_2(t), q_3(t)).$$

Then $v(x, t)$ solves (2.8) with ψ and g replaced by $\tilde{\phi}$ and \tilde{f} , respectively, and

$$u(x, t) = W_{\mathbb{R}}(t)\tilde{\phi} + \int_0^t W_{\mathbb{R}}(t - \tau)\tilde{f}(\tau)d\tau - W_{bd,r}(t)\vec{q}$$

solves the IBVP (2.7). The following proposition then follows from Propositions 2.2 and 2.3.

PROPOSITION 2.4. *Let $T > 0$ be given. For any $\phi \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$, the IBVP (2.7) admits a unique mild solution $v \in C(0, T; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ satisfying*

$$\|v\|_{L^\infty(0,T;L^2(0,L))} + \|v\|_{L^2(0,T;H^1(0,L))} \leq C (\|\phi\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}).$$

Moreover, the solution v possesses the sharp Kato smoothing properties

$$\partial_x^k v \in L_x^\infty(0, L; H^{(k-1)/3}(0, T))$$

and

$$\|\partial_x^k v\|_{L_x^\infty(0,L;H^{(k-1)/3}(0,T))} \leq C_k (\|\phi\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))})$$

for $k = 0, 1, 2$.

Combining Propositions 2.3 and 2.4 together leads to the result for the following IBVP:

$$(2.9) \quad \begin{cases} v_t + v_{xxx} = f & \text{in } (0, L) \times (0, T), \\ v_{xx}(0, t) = h_1(t), \quad v_x(L, t) = h_2(t), \quad v_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ v(x, 0) = v_0(x) & \text{in } (0, L). \end{cases}$$

PROPOSITION 2.5. *Let $T > 0$ be given, for any $v_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and*

$$\vec{h} := (h_1, h_2, h_3) \in \mathcal{H}_T = H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T),$$

the IBVP (2.9) admits a unique solution

$$v \in \mathcal{Z}_T := C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

with

$$\partial_x^k v \in L_x^\infty(0, L; H^{\frac{1-k}{3}}(0, T)) \quad \text{for } k = 0, 1, 2.$$

Moreover, there exists $C > 0$ such that

$$\|v\|_{\mathcal{Z}_T} + \sum_{k=0}^2 \|v\|_{L_x^\infty(0, L; H^{\frac{1-k}{3}}(0, T))} \leq C \left(\|v_0\|_{L^2(0, L)} + \|\vec{h}\|_{\mathcal{H}_T} + \|f\|_{L^1(0, T; L^2(0, L))} \right).$$

The next proposition states similar hidden (or sharp trace) regularities for the linear system

$$(2.10) \quad \begin{cases} y_t + y_x + y_{xxx} = f, & x \in (0, L), t > 0, \\ y(0, t) = g_1(t), \quad y_x(L, t) = g_2(t), \quad y_{xx}(L, t) = g_3(t), & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \end{cases}$$

associated with (1.2).

PROPOSITION 2.6. *Let $T > 0$ be given, for any $y_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and*

$$\vec{g} := (g_1, g_2, g_3) \in \mathcal{G}_T = H^{\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T),$$

the IBVP (2.10) admits a unique solution $y \in \mathcal{Z}_T$. Moreover, there exists $C > 0$ such that

$$\|y\|_{\mathcal{Z}_T} \leq C \left(\|y_0\|_{L^2(0, L)} + \|\vec{g}\|_{\mathcal{G}_T} + \|f\|_{L^1(0, T; L^2(0, L))} \right).$$

In addition, the solution y possesses the following sharp trace estimates

$$(2.11) \quad \sup_{x \in (0, L)} \|\partial_x^r y(x, \cdot)\|_{H^{\frac{1-r}{3}}(0, T)} \leq C_r \left(\|y_0\|_{L^2(0, L)} + \|\vec{g}\|_{\mathcal{G}_T} + \|f\|_{L^1(0, T; L^2(0, L))} \right),$$

for $r = 0, 1, 2$.

The proof of Proposition 2.6 can be found in [1, 14].

Remark 2.7. The systems

$$(2.12) \quad \begin{cases} v_t + v_x + v_{xxx} = f & x \in (0, L), t > 0, \\ v_{xx}(0, t) = h_1(t), \quad v_x(L, t) = h_2(t), \quad v_{xx}(L, t) = h_3(t) & t > 0, \\ v(x, 0) = v_0(x) & x \in (0, L) \end{cases}$$

and

$$(2.13) \quad \begin{cases} y_t + y_x + y_{xxx} = f, & x \in (0, L), t > 0, \\ y(0, t) = g_1(t), \quad y_x(L, t) = g_2(t), \quad y_{xx}(L, t) = g_3(t), & t > 0, \\ y(x, 0) = y_0(x), & x \in (0, L), \end{cases}$$

are equivalent in the following sense:

Given $\{u_0, f, h_1, h_2, h_3\}$ one can find $\{y_0, f, g_1, g_2, g_3\}$ such that the corresponding solution u of (2.12) is exactly the same as the corresponding y for the system (2.13) and vice versa.

In fact, for given $u_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and $\vec{h} \in \mathcal{H}_T$, system (2.12) admits a unique solution $u \in \mathcal{Z}_T$. Let $y_0 = u_0$ and set

$$g_1(t) = h_1(t), \quad g_3(t) = h_2(t), \quad g_2(t) = h_3(t).$$

Then, according to Proposition 2.6, we have $\vec{g} \in \mathcal{G}_t$. Due to the uniqueness of the IBVP (2.13), with the selection $\{y_0, f, g_1, g_2, g_3\}$, the corresponding solution $y \in \mathcal{Z}_T$ of (2.13) must be equal to u since u also solves (2.13) with the given auxiliary data $\{y_0, f, g_1, g_2, g_3\}$. On the other hand, for any given $y_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and $\vec{g} \in \mathcal{G}_T$, let $y \in \mathcal{Z}_T$ be the corresponding solution of the system (2.13). From (2.11), we have $y_{xx}(0, \cdot)$ and $y_{xx}(L, \cdot) \in H^{-\frac{1}{3}}(0, T)$. Thus, if we set $u_0 = y_0$ and

$$h_1(t) = u_{xx}(0, t), \quad h_2(t) = g_3(t), \quad h_3(t) = u_{xx}(L, T),$$

then $\vec{h} \in \mathcal{H}_T$ and the corresponding solution $u \in \mathcal{Z}_T$ of (2.12) must be equal to y which also solves (2.12) with the auxiliary data (u_0, f, \vec{h}) .

Finally, we are at the stage to prove the well-posedness of the following nonlinear system,

$$(2.14) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) := \phi(x) & \text{in } (0, L). \end{cases}$$

For given $T > 0$, define

$$X_T := L^2(0, L) \times H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T)$$

and

$$\mathcal{Z}_T := C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

THEOREM 2.8. *Let $T > 0$ and $r > 0$ be given. There exists a $T^* \in (0, T]$ such that for any $(\phi, \vec{h}) \in X_T$ with*

$$\|(\phi, \vec{h})\|_{X_T} \leq r,$$

the IBVP (2.14) admits a unique solution

$$u \in \mathcal{Z}_{T^*}.$$

In addition, the solution u possesses the hidden regularities

$$\partial_x^k u \in L_x^\infty(0, L; H^{\frac{1-k}{3}}(0, T^*)), \quad k = 0, 1, 2,$$

and, moreover, the corresponding solution map is Lipschitz continuous.

Proof. Since the proof is similar to that presented in [1, 13], we will omit it. \square

3. Boundary controllability. In this section, we study exact boundary controllability of the system

$$(3.1) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = 0, \quad u_x(L, t) = h(t), \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

around a constant steady state $u \equiv c$. As is easy to see by letting $u = v + c$, it is equivalent to studying the exact boundary controllability of the following system

$$(3.2) \quad \begin{cases} v_t + (c + 1)v_x + vv_x + v_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ v_{xx}(0, t) = 0, v_x(L, t) = h(t), v_{xx}(L, t) = 0 & \text{in } (0, T), \\ v(x, 0) = v_0(x) & \text{in } (0, L) \end{cases}$$

around the origin 0 instead. We have the following exact controllability results for the system (3.2).

THEOREM 3.1. *Let $T > 0$, $c + 1 \neq 0$, and*

$$(3.3) \quad L \notin \mathcal{R}_c := \left\{ \frac{2\pi}{\sqrt{3(c+1)}} \sqrt{k^2 + kl + l^2} : k, l \in \mathbb{N}^* \right\} \cup \left\{ \frac{k\pi}{\sqrt{c+1}} : k \in \mathbb{N}^* \right\}$$

be given. Then there exists a $\delta > 0$ such that for any $v_0, v_T \in L^2(0, L)$ with

$$\|v_0\|_{L^2(0,L)} + \|v_T\|_{L^2(0,L)} \leq \delta,$$

one can find $h \in L^2(0, T)$ such that the system (3.2) admits a unique solution

$$v \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$v(x, 0) = v_0(x), \quad v(x, T) = v_T(x) \quad \text{in } (0, L).$$

To prove the Theorem 3.1, we first consider the linear system associated with (3.2)

$$(3.4) \quad \begin{cases} v_t + (c + 1)v_x + v_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ v_{xx}(0, t) = 0, v_x(L, t) = h(t), v_{xx}(L, t) = 0 & \text{in } (0, T), \\ v(x, 0) = v_0(x) & \text{in } (0, L) \end{cases}$$

and its adjoint system

$$(3.5) \quad \begin{cases} \psi_t + (c + 1)\psi_x + \psi_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ (c + 1)\psi(0, t) + \psi_{xx}(0, t) = 0 & \text{in } (0, T), \\ (c + 1)\psi(L, t) + \psi_{xx}(L, t) = 0 & \text{in } (0, T), \\ \psi_x(0, t) = 0 & \text{in } (0, T), \\ \psi(x, T) = \psi_T(x) & \text{in } (0, L). \end{cases}$$

Note that by transformations $x' = L - x$ and $t' = T - t$, the system (3.5) is equivalent of the following forward system

$$(3.6) \quad \begin{cases} \varphi_t + (c + 1)\varphi_x + \varphi_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ (c + 1)\varphi(0, t) + \varphi_{xx}(0, t) = 0 & \text{in } (0, T), \\ (c + 1)\varphi(L, t) + \varphi_{xx}(L, t) = 0 & \text{in } (0, T), \\ \varphi_x(L, t) = 0 & \text{in } (0, T), \\ \varphi(x, 0) = \varphi_0(x) & \text{in } (0, L). \end{cases}$$

PROPOSITION 3.2. *For any $\varphi_0 \in L^2(0, L)$, system (3.6) admits a unique solution $\varphi \in \mathcal{Z}_T$ which, moreover, possesses the following hidden regularities*

$$(3.7) \quad \sup_{x \in (0, L)} \|\partial_x^r \varphi(x, \cdot)\|_{H^{\frac{1-r}{3}}(0, T)} \leq C_r \|\varphi_0\|_{L^2(0, L)}$$

for $r = 0, 1, 2$.

Remark 3.3. Equivalently, for any $\psi_T \in L^2(0, L)$, the system (3.5) admits a unique solution $\psi \in Z_T$ which, moreover, possesses the hidden regularities

$$(3.8) \quad \sup_{x \in (0, L)} \|\partial_x^r \psi(x, \cdot)\|_{H^{\frac{1-r}{3}}(0, T)} \leq C_r \|\psi_T\|_{L^2(0, L)}$$

for $r = 0, 1, 2$.

Proof of Proposition 3.2. Let us consider the set

$$\mathcal{X}_T := \{u \in Z_T : \partial_x^r u \in L_x^\infty(0, L; H^{\frac{1-r}{3}}(0, T)), r = 0, 1, 2\}$$

which is a Banach space equipped with the norm

$$\|u\|_{\mathcal{X}_T} := \|u\|_{Z_T} + \sum_{r=0}^2 \|\partial_x^r u\|_{L_x^\infty(0, L; H^{\frac{1-r}{3}}(0, T))}.$$

According to Proposition 2.5, for any $v \in \mathcal{X}_\beta$ where $0 < \beta \leq T$ and any $\varphi_0 \in L^2(0, L)$, the system

$$(3.9) \quad \begin{cases} w_t + w_{xxx} = -(c+1)v_x & \text{in } (0, L) \times (0, T), \\ w_{xx}(0, t) = -(c+1)v(0, t) & \text{in } (0, T), \\ w_{xx}(L, t) = -(c+1)v(L, t) & \text{in } (0, T), \\ w_x(L, t) = 0 & \text{in } (0, T), \\ w(x, 0) = \varphi_0(x) & \text{in } (0, L), \end{cases}$$

admits a unique solution $w \in \mathcal{X}_\beta$ and, moreover,

$$\begin{aligned} & \|w\|_{\mathcal{X}_\beta} \\ & \leq C \left(\|\psi_0\|_{L^2(0, L)} + \|v(0, \cdot)\|_{H^{-\frac{1}{3}}(0, T)} + \|v(L, \cdot)\|_{H^{-\frac{1}{3}}(0, T)} + \|v_x\|_{L^1(0, \beta; L^2(0, L))} \right), \end{aligned}$$

where the constant $C > 0$ depends only on T . As we have,

$$\begin{aligned} \|v_x\|_{L^1(0, \beta; L^2(0, L))} & \leq C\beta^{1/2} \|v\|_{\mathcal{X}_\beta}, \\ \|v(0, \cdot)\|_{H^{-\frac{1}{3}}(0, \beta)} & \leq \|v(0, \cdot)\|_{L^2(0, \beta)} \leq \beta^{2/3} \|v(0, \cdot)\|_{L^6(0, \beta)} \\ & \leq C\beta^{2/3} \|v(0, \cdot)\|_{H^{\frac{1}{3}}(0, \beta)} \leq C\beta^{2/3} \|v\|_{\mathcal{X}_\beta}, \end{aligned}$$

and

$$\begin{aligned} \|v(L, \cdot)\|_{H^{-\frac{1}{3}}(0, \beta)} & \leq \|v(L, \cdot)\|_{L^2(0, \beta)} \leq \beta^{2/3} \|v(L, \cdot)\|_{L^6(0, \beta)} \\ & \leq C\beta^{2/3} \|v(L, \cdot)\|_{H^{\frac{1}{3}}(0, \beta)} \leq C\beta^{2/3} \|v\|_{\mathcal{X}_\beta}, \end{aligned}$$

the system (3.9) defines a map as follows:

$$\begin{aligned} \Gamma : \mathcal{X}_\beta & \longrightarrow \mathcal{X}_\beta, \\ v & \mapsto \Gamma(v) = w \end{aligned}$$

for any $v \in \mathcal{X}_T$ and $\beta \in (0, \min\{1, T\}]$. Here $w \in \mathcal{X}_\beta$ is the corresponding solution of (3.9) and

$$\|\Gamma(v)\|_{\mathcal{X}_\beta} \leq C_1 \|\psi_0\|_{L^2(0, L)} + C_2 \beta^{1/2} \|v\|_{\mathcal{X}_\beta},$$

where C_1 and C_2 are two positive constants depending only on T . Choosing $r > 0$ and $\beta \in (0, \min\{1, T\}]$ such that

$$r = 2C_1 \|\psi_0\|_{L^2(0,L)}$$

and

$$2C_2\beta^{1/2} \leq \frac{1}{2},$$

then, for any

$$v \in \mathcal{B}_{\beta,r} = \{v \in \mathcal{X}_\beta : \|v\|_{\mathcal{X}_\beta} \leq r\},$$

we have

$$\|\Gamma(v)\|_{\mathcal{X}_\beta} \leq r.$$

Moreover, for any $v_1, v_2 \in \mathcal{B}_{\beta,r}$, we get

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{\mathcal{X}_\beta} \leq 2C_2\beta^{1/2} \|v_1 - v_2\|_{\mathcal{X}_\beta} \leq \frac{1}{2} \|v_1 - v_2\|_{\mathcal{X}_\beta}.$$

Therefore, the map Γ is a contraction mapping on $\mathcal{B}_{\beta,r}$. Its fixed point $w = \Gamma(v) \in \mathcal{X}_\beta$ is the desired solution for $t \in (0, \beta)$. As the chosen β is independent of φ_0 , the standard continuation extension argument yields that the solution w belongs to \mathcal{X}_T . The proof is complete. \square

The system (3.6) possesses an elementary estimate as described below.

PROPOSITION 3.4. *Any solution φ of the adjoint system (3.6) with initial data $\varphi_0 \in L^2(0, L)$ satisfies*

$$(3.10) \quad \|\varphi_0\|_{L^2(0,L)}^2 \leq \frac{1}{T} \|\varphi\|_{L^2((0,L) \times (0,T))}^2 + \|\varphi_x(0, \cdot)\|_{L^2(0,T)}^2 + (c+1) \|\varphi(0, \cdot)\|_{L^2(0,T)}^2.$$

Proof. Multiplying the equation (3.6) by $(T - t)\varphi$ and integrating by parts over $(0, L) \times (0, T)$, we get

$$\begin{aligned} & \frac{T}{2} \int_0^L \varphi_0^2 dx \\ &= \frac{1}{2} \int_0^L \int_0^T \varphi^2 dx dt + \int_0^T \left(\frac{T-t}{2} \right) \left(-(c+1)\varphi^2(L) + (c+1)\varphi^2(0) + \varphi_x^2(0) \right) dt, \end{aligned}$$

which yields (3.10) since $c + 1 > 0$. \square

Equivalently, the following estimate holds for solutions ψ of the system (3.5):

$$(3.11) \quad \|\psi_T\|_{L^2(0,L)}^2 \leq \frac{1}{T} \|\psi\|_{L^2((0,L) \times (0,T))}^2 + (c+1) \|\psi(L, \cdot)\|_{L^2(0,T)}^2 + \|\psi_x(L, \cdot)\|_{L^2(0,T)}^2.$$

Remark 3.5. As a comparison, it is worth pointing out that for the adjoint system of (2.10), which is given by

$$(3.12) \quad \begin{cases} \xi_t + \xi_x + \xi_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ \xi(0, t) = 0, \xi(L, t) = 0, \xi_x(0, t) = 0 & \text{in } (0, T), \\ \xi(x, T) = \xi_T(x) & \text{in } (0, L), \end{cases}$$

the following inequality holds:

$$(3.13) \quad \|\xi_T\|_{L^2(0,L)} \leq \frac{1}{T} \|\xi\|_{L^2((0,L) \times (0,T))} + \|\xi_x(L, \cdot)\|_{L^2(0,T)}.$$

The extra term $\|\psi(L, \cdot)\|_{L^2(0,T)}^2$ in (3.11) brings a technique difficulty in establishing the observability inequality of the adjoint system (3.5), which calls the use of the hidden regularities established in Proposition 3.2.

Now we turn to analyze the exact controllability of the linear system (3.4).

PROPOSITION 3.6. *Assume $c + 1 \neq 0$. Let $T > 0$ and $L \notin \mathcal{R}_c$ be given. There exists a bounded linear operator*

$$\Psi : L^2(0, L) \times L^2(0, L) \longrightarrow L^2(0, T)$$

such that for any $v_0, v_T \in L^2(0, L)$, if one chooses $h_2 = \Psi(v_0, v_T)$, then system (3.4) admits a solution $v \in \mathcal{Z}_T$ satisfying

$$v|_{t=0} = v_0, \quad v|_{t=T} = v_T.$$

Proof. It suffices to prove the following.

For any given $L \in (0, +\infty) \setminus \mathcal{R}_c$ and $T > 0$, there exists a positive constant C depending only on T and L such that

$$(3.14) \quad \|\psi_T\|_{L^2(0,L)} \leq C \|\psi_x(L, t)\|_{L^2(0,T)}$$

holds for any $\psi_T \in L^2(0, L)$, where ψ is the solution of (3.5) with the terminal data ψ_T .

We proceed by contradiction as in [16, Proposition 3.3]. If (3.14) does not hold, then there exists a sequence $\{\psi_T^n\}_{n \in \mathbb{N}} \in L^2(0, L)$ with

$$(3.15) \quad \|\psi_T^n\|_{L^2(0,L)} = 1 \quad \forall n \in \mathbb{N}$$

such that the corresponding solutions of (3.5) satisfy

$$(3.16) \quad 1 = \|\psi_T^n\|_{L^2(0,L)} > n \|\psi_x^n(L, t)\|_{L^2(0,T)},$$

thus $\|\psi_x^n(L, t)\|_{L^2(0,T)} \rightarrow 0$ as $n \rightarrow \infty$. Thanks to Proposition 3.2 we have $\{\psi^n\}_{n \in \mathbb{N}}$ is bounded in $L^2(0, T; H^1(0, L))$ and $\{\psi^n(L, t)\}_{n \in \mathbb{N}}$ is bounded in $H^{\frac{1}{3}}(0, T)$. In addition, according to Proposition 3.4, we have

$$(3.17) \quad \|\psi_T^n\|_{L^2(0,L)} \leq \frac{1}{T} \|\psi^n\|_{L^2((0,L) \times (0,T))}^2 + \|\psi_x^n(L, \cdot)\|_{L^2(0,T)}^2 + (c + 1) \|\psi^n(L, \cdot)\|_{L^2(0,T)}^2.$$

Since $\psi_t^n = -(c + 1)\psi_x^n - \psi_{xxx}^n$ is bounded in $L^2(0, T; H^{-2}(0, L))$ and the embedding

$$H^1(0, L) \hookrightarrow L^2(0, L) \hookrightarrow H^{-2}(0, L),$$

the sequence $\{\psi^n\}_{n \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(0, L))$ (see [22]). Furthermore, the second term on the right in (3.17) converges to zero in $L^2(0, T)$, and by the compact embedding

$$H^{\frac{1}{3}}(0, T) \hookrightarrow L^2(0, T),$$

the sequence $\{\psi^n(L, t)\}_{n \in \mathbb{N}}$ has a convergent subsequence in $L^2(0, T)$. Therefore by (3.17), $\{\psi_T^n\}_{n \in \mathbb{N}}$ is an $L^2(0, L)$ -Cauchy sequence, thus, at least for a subsequence, we have

$$(3.18) \quad \psi_T^n \longrightarrow \psi_T \text{ in } L^2(0, L),$$

by Theorem 3.2 holds that

$$(3.19) \quad \psi_x^n(L, t) \longrightarrow \psi_x(L, t) \text{ in } L^2(0, T).$$

From (3.15), (3.18), and (3.19), we have ψ is a solution of

$$(3.20) \quad \begin{cases} \psi_t + (c + 1)\psi_x + \psi_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ (c + 1)\psi(0, t) + \psi_{xx}(0, t) = 0 & \text{in } (0, T), \\ (c + 1)\psi(L, t) + \psi_{xx}(L, t) = 0 & \text{in } (0, T), \\ \psi_x(0, t) = 0 & \text{in } (0, T), \end{cases}$$

satisfying the additional boundary condition

$$(3.21) \quad \psi_x(L, t) = 0$$

and

$$(3.22) \quad \|\psi_T\|_{L^2(0, L)} = 1.$$

Notice that (3.22) implies that the solutions of (3.20)–(3.21) cannot be identically zero. Therefore, by the following Lemma 3.7, one can conclude that $\psi \equiv 0$, therefore, $\psi_T(x) \equiv 0$, which contradicts (3.22). \square

LEMMA 3.7. *For any $T > 0$, let N_T denote the space of the initial states $\psi_T \in L^2(0, L)$ such that the mild solution ψ of (3.20) satisfies (3.21). Then, for $L \in (0, +\infty) \setminus \mathcal{R}_c$, $N_T = \{0\} \forall T > 0$.*

Proof. The proof uses the same arguments as those given in [16]. Therefore, if $N_T \neq \{0\}$, the map $\psi_T \in \mathbb{C}N_T \longrightarrow A(\psi_T) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denotes the complexification of N_T) has (at least) one eigenvalue, hence, there exists $\lambda \in \mathbb{C}$ and $\psi_0 \in H^3(0, L) \setminus \{0\}$ such that

$$(3.23) \quad \begin{cases} \lambda\psi_0 = -(c + 1)\psi_0' - \psi_0''', \\ (c + 1)\psi_0(0) + \psi_0''(0) = 0, \quad (c + 1)\psi_0(L) + \psi_0''(L) = 0, \quad \psi_0'(0) = 0, \quad \psi_0'(L) = 0. \end{cases}$$

To conclude the proof of Lemma 3.7, we prove that this does not hold if $L \notin \mathcal{R}_c$. To simplify the notation, henceforth we denote $\psi_0 := \psi$. \square

LEMMA 3.8. *Let $L > 0$. Consider the assertion*

$$(F) \quad \exists \lambda \in \mathbb{C}, \exists \psi \in H^3(0, L) \setminus \{0\} \text{ such that } \begin{cases} \lambda\psi = -(c + 1)\psi' - \psi''', \\ (c + 1)\psi(0) + \psi''(0) = 0, \\ (c + 1)\psi(L) + \psi''(L) = 0, \\ \psi'(0) = 0, \quad \psi'(L) = 0. \end{cases}$$

Then, (F) holds if and only if $L \in \mathcal{R}_c$.

Proof. We will use the argument developed in [16, Lemma 3.5]. Assume that ψ satisfies \mathcal{F} . Let us introduce the notation $\hat{\psi}(\xi) = \int_0^L \psi(x)e^{-ix\xi} dx$. Then, multiplying (3.23) by $e^{-ix\xi}$, integrating by parts in $(0, L)$, and using the boundary condition we obtain

$$(3.24) \quad (\lambda + (c + 1)(i\xi) + (i\xi)^3)\hat{\psi}(\xi) = (i\xi)^2\psi(0) - (i\xi)^2\psi(L)e^{-iL\xi}.$$

Setting $\lambda = -ip$, we have

$$(3.25) \quad \hat{\psi}(\xi) = -i\xi^2 \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - (c + 1)\xi + p}$$

with

$$\alpha = \psi(0), \quad \beta = \psi(L).$$

Using the Paley–Wiener theorem (see [23, Section 4, p. 161]) and the usual characterization of $H^2(\mathbb{R})$ by means of the Fourier transform we see that \mathcal{F} is equivalent to the existence of $p \in \mathbb{C}$ and

$$(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\},$$

such that

$$f(\xi) := \xi^2 \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - (c+1)\xi + p}$$

satisfies

- (a) f is an entire function in \mathbb{C} ;
- (b) $\int_{\mathbb{R}} |f(\xi)|^2 (1 + |\xi|^2)^2 d\xi < \infty$;
- (c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c_1 (1 + |\xi|)^k e^{L|\text{Im} \xi|}$ for some positive constants c_1 and k .

Recall that f is an entire function if and only if the roots ξ_0, ξ_1, ξ_2 of $Q(\xi) := \xi^3 - (c+1)\xi + p$ are roots of

$$(3.26) \quad s(\xi) := \xi^2(\alpha - \beta e^{-iL\xi}).$$

In addition, all the roots of $\alpha - \beta e^{-iL\xi}$ are simple, otherwise $\alpha = \beta = 0$ which implies that $\psi(0) = \psi(L) = 0$. Using the system (3.23) we conclude that $\psi \equiv 0$. Besides, as $c + 1 \neq 0$, the three roots of $Q(\xi)$ must be simple too. Let us first assume that $Q(\xi)$ and $\alpha - \beta e^{-iL\xi}$ share the same roots, we can write the three roots of $Q(\xi)$ as

$$(3.27) \quad \xi_1 := \xi_0 + k \frac{2\pi}{L} \quad \text{and} \quad \xi_2 := \xi_1 + l \frac{2\pi}{L}$$

with k and l some positive integers, we have

$$(3.28) \quad Q(\xi) = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2),$$

that is

$$(3.29) \quad \begin{cases} \xi_0 + \xi_1 + \xi_2 = 0, \\ \xi_0 \xi_1 + \xi_0 \xi_2 + \xi_1 \xi_2 = -(c+1), \\ \xi_0 \xi_1 \xi_2 = -p. \end{cases}$$

Thus we have

$$(3.30) \quad \begin{cases} L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3(1+c)}}, \\ \xi_0 = -\frac{1}{3}(2k+l) \frac{2\pi}{L}, \\ p = -\xi_0 \left(\xi_0 + k \frac{2\pi}{L} \right) \left(\xi_0 + (k+l) \frac{2\pi}{L} \right). \end{cases}$$

Next we assume that $\xi = 0$ is a root of $Q(\xi)$, but not of $\alpha - \beta e^{-iL\xi}$. Then three roots of $Q(\xi)$ can be written as $0, \xi_1, \xi_1 + k\frac{2\pi}{L}$ with k being a positive integer. We have

$$(3.31) \quad \begin{cases} \xi_1 + \xi_2 = 0, \\ \xi_1 \xi_2 = -(c + 1), \\ 0 = -p, \end{cases}$$

and, consequently, it follows that

$$(3.32) \quad \begin{cases} L = \frac{k\pi}{\sqrt{(1+c)}}, \\ \xi_1 = -k\frac{\pi}{L}, \\ p = 0. \end{cases}$$

Hence, \mathcal{F} holds if and only if $L \in \mathcal{R}_c$. This completes the proof of Lemma 3.8 and, consequently, the proof of Lemma 3.7. \square

Finally we consider the case of $c + 1 = 0$. Then it is easy to see that $\xi = 0$ must be a root of $Q(\xi)$; otherwise, $L = \infty$. Hence

$$f(\xi) := \frac{\alpha - \beta e^{-iL\xi}}{\xi}.$$

We must have

$$\alpha = \beta \text{ or } \psi(0) = \psi(L).$$

(3.23) becomes

$$(3.33) \quad \begin{cases} \psi_0''' = 0, \\ \psi_0''(0) = 0, \quad \psi_0''(L) = 0, \quad \psi_0'(0) = 0, \quad \psi_0'(L) = 0, \quad \psi_0(0) = \psi_0(L) \end{cases}$$

which implies that $\psi_0(x) \equiv C$.

We are now ready to present the proof of Theorem 3.1.

Proof of Theorem 3.1. Rewrite the system (3.2) in its integral form

$$(3.34) \quad v(t) = W_0(t)v_0 + W_{bdr}(t)h - \int_0^t W_0(t - \tau)(vv_x)(\tau, x)d\tau.$$

For any $u \in \mathcal{Z}_T$, let us define

$$\nu(T, u) := \int_0^T W_0(T - \tau)(uu_x)d\tau.$$

By using Proposition 3.6, for any $v_0, v_T \in L^2(0, L)$, if we choose

$$h = \Psi(v_0, v_T + \nu(T, u)),$$

then

$$u(t) = W_0(t)v_0 + W_{bdr}\Psi(v_0, v_T + \nu(T, u)) - \int_0^t W_0(t - \tau)(uu_x)(\tau, x)d\tau$$

satisfies

$$u(x, 0) = v_0(x), \quad u(x, T) = v_T(x) + \nu(T, u) - \nu(T, u) = v_T(x).$$

This leads us to consider the map

$$\Gamma(u) = W_0(t)v_0 + W_{bdr}\Psi(v_0, v_T + \nu(T, u)) - \int_0^t W_0(t - \tau)(uu_x)(\tau, x)d\tau.$$

If we can show that the map Γ is a contraction in an appropriate metric space, then its fixed point u is a solution of (3.2) with $h = \Psi(v_0, v_T + \nu(T, u))$ that satisfies

$$u(x, 0) = v_0(x), \quad u(x, T) = v_T(x).$$

Let

$$B_r = \{z \in \mathcal{Z}_T : \|z\|_{\mathcal{Z}_T} \leq r\}.$$

By Proposition 2.5, there exists a constant $C_1 > 0$ such that for any $u \in \mathcal{Z}_T$,

$$\|\Gamma(u)\|_{\mathcal{Z}_T} \leq C_1 \left(\|v_0\|_{L^2(0,L)} + \|\Psi(v_0, v_T + \nu(T, u))\|_{L^2(0,L)} + \int_0^T \|uu_x\|_{L^2(0,L)}(t)dt \right).$$

Furthermore, as

$$\|\Psi(v_0, v_T + \nu(T, u))\|_{L^2(0,L)} \leq C_2 (\|v_0\|_{L^2(0,L)} + \|v_T\|_{L^2(0,L)} + \|\nu(T, u)\|_{L^2(0,L)})$$

and

$$\|\nu(T, u)\|_{L^2(0,L)} \leq \int_0^T \|uu_x\|_{L^2(0,L)}(t)dt \leq C_3 \|u\|_{\mathcal{Z}_T}^2,$$

we infer that

$$\|\Gamma(u)\|_{\mathcal{Z}_T} \leq C_3 (\|v_0\|_{L^2(0,L)} + \|v_T\|_{L^2(0,L)}) + C_4 \|u\|_{\mathcal{Z}_T}^2$$

for any $u \in \mathcal{Z}_T$, where C_3 and C_4 are constants depending only T . Thus, if we select r and δ satisfying

$$r = 2C_3\delta$$

and

$$4C_3C_4\delta < \frac{1}{2},$$

then the operator Γ maps B_r into itself for any $v \in B_r$. In addition, for any $\tilde{u}, u \in B_r$, the similar arguments yield that

$$\|\Gamma(u) - \Gamma(\tilde{u})\|_{\mathcal{Z}_T} \leq \gamma \|u - \tilde{u}\|_{\mathcal{Z}_T}$$

with $\gamma = 8C_3C_4\delta < 1$. Therefore the map Γ is a contraction. Its fixed point is a desired solution. The proof of Theorem 3.1 is completed and, consequently, Theorem 1.2 follows. \square

4. Multicontrols and null controllability. In this section we will first consider the following linear systems associated with (1.12):

$$(4.1) \quad \begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

$$(4.2) \quad \begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = 0, \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L), \end{cases}$$

and

$$(4.3) \quad \begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = 0, \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L). \end{cases}$$

PROPOSITION 4.1. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Theta : L^2(0, L) \times L^2(0, L) \longrightarrow H^{-\frac{1}{3}}(0, T) \times L^2(0, T)$$

such that for any $u_0, u_T \in L^2(0, L)$, if one chooses

$$(h_1, h_2) = \Theta(u_0, u_T),$$

then system (4.1) admits a solution $u \in \mathcal{Z}_T$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

PROPOSITION 4.2. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Pi : L^2(0, L) \times L^2(0, L) \longrightarrow L^2(0, T) \times H^{-\frac{1}{3}}(0, T)$$

such that for any $u_0, u_T \in L^2(0, L)$, if one chooses

$$(h_2, h_3) = \Pi(u_0, u_T),$$

then system (4.2) admits a solution $u \in \mathcal{Z}_T$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

PROPOSITION 4.3. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Lambda : L^2(0, L) \times L^2(0, L) \longrightarrow H^{-\frac{1}{3}}(0, T) \times H^{-\frac{1}{3}}(0, T)$$

such that for any $u_0, u_T \in L^2(0, L)$, if one chooses

$$(h_1, h_3) = \Lambda(u_0, u_T),$$

then system (4.3) admits a solution $u \in \mathcal{Z}_T$ satisfying

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x).$$

Propositions 4.1–4.3 follow as a consequence of the following observability inequalities for the solution of the backward system (1.7):

$$(4.4) \quad \|\psi_T\|_{L^2(0,L)} \leq C \left(\|\Delta_t^{\frac{1}{3}}\psi(0,t)\|_{L^2(0,T)} + \|\psi_x(L,t)\|_{L^2(0,T)} \right),$$

$$(4.5) \quad \|\psi_T\|_{L^2(0,L)} \leq C \left(\|\psi_x(L,t)\|_{L^2(0,T)} + \|\Delta_t^{\frac{1}{3}}\psi(L,t)\|_{L^2(0,T)} \right),$$

and

$$(4.6) \quad \|\psi_T\|_{L^2(0,L)} \leq C \left(\|\Delta_t^{\frac{1}{3}}\psi(0,t)\|_{L^2(0,T)} + \|\Delta_t^{\frac{1}{3}}\psi(L,t)\|_{L^2(0,T)} \right),$$

where $\Delta_t := (I - \partial_t^2)^{\frac{1}{2}}$. The proofs of (4.4)–(4.6) are similar to that of (3.14). Furthermore, Theorem 1.5 can be proved using the same arguments as that in the proof of Theorem 3.1, their proof is thus omitted. Concerning the null controllability, that is, the proof of Theorem 1.8, notes that for the linear system we can get the result using the Carleman estimate provided by [11, Proposition 3] together with the following remark.

Remark 4.4. The following systems

$$(4.7) \quad \begin{cases} u_t + u_x + u_{xxx} = f & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = 0, \quad u_{xx}(L, t) = 0 & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

and

$$(4.8) \quad \begin{cases} y_t + y_x + y_{xxx} = f & \text{in } (0, L) \times (0, T), \\ y(0, t) = k_1(t), \quad y_x(L, t) = 0, \quad y_{xx}(L, t) = 0 & \text{in } (0, T), \\ y(x, 0) = y_0(x) & \text{in } (0, L) \end{cases}$$

are equivalent in the following sense: for given $\{u_0, f, h_1\}$ one can find $\{y_0, f, k_1\}$ such that the corresponding solution u of (4.7) is exactly the same as the corresponding solution y for the system (4.8) and vice versa.

Indeed, for given $u_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and $h_1(t) \in H^{-\frac{1}{3}}(0, T)$, system (4.7) admits a unique solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$. Let $y_0 = u_0$ and set $k_1(t) = h_1(t)$. Then, according to Proposition 2.6, we have $k_1(t) \in H^{\frac{1}{3}}(0, T)$. Due to the uniqueness of IBVP (4.8), with the selection $\{y_0, f, k_1\}$, the corresponding solution $y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ of (4.8) must be equal to u , since u also solves (4.8) with the given auxiliary data $\{y_0, f, k_1\}$. On the other hand, for any given $y_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$, and $k_1(t) \in H^{\frac{1}{3}}(0, T)$, let $y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ be the corresponding solution of the system (4.8). From Proposition 2.6, we have $y_{xx}(0, \cdot) \in H^{-\frac{1}{3}}(0, T)$. Thus, if $u_0 = y_0$ and $h_1(t) = k_1(t)$, then $h_1(t) \in H^{-\frac{1}{3}}(0, T)$ and the corresponding solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ of (4.7) must be equal to y , which also solves (4.7) with the auxiliary data $\{u_0, f, h_1\}$.

Proof of Theorem 1.8. Consider u and \bar{u} fulfilling system (1.13) and (1.14), respectively. Then $q = u - \bar{u}$ satisfies

$$(4.9) \quad \begin{cases} q_t + q_x + \left(\frac{q^2}{2} + \bar{u}q\right)_x + q_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ q_{xx}(0, t) = h_1(t), \quad q_x(L, t) = 0, \quad q_{xx}(L, t) = 0 & \text{in } (0, T), \\ q(x, 0) = q_0(x) := u_0(x) - \bar{u}_0(x) & \text{in } (0, L). \end{cases}$$

The objective is to find h_1 such that the solution q of (4.9) satisfies

$$q(\cdot, T) = 0.$$

Given $\xi \in \mathcal{Z}_T$ and $q_0 := u_0 - \bar{u}_0 \in L^2(0, L)$, we consider the following control problem

$$(4.10) \quad q_t + q_x + (\xi q)_x + q_{xxx} = 1_\omega v(t, x) \quad \text{in } (0, L) \times (0, T),$$

$$(4.11) \quad q_{xx}(0, t) = q_x(L, t) = q_{xx}(L, t) = 0 \quad \text{in } (0, T),$$

$$(4.12) \quad q(x, 0) = q_0(x) \quad \text{in } (0, L),$$

where v is solution of the following adjoint system

$$(4.13) \quad \begin{cases} v_t + \xi(t, x)v_x + v_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ v(0, t) + v_{xx}(0, t) = 0 & \text{in } (0, T), \\ v(L, t) + v_{xx}(L, t) = 0 & \text{in } (0, T), \\ v_x(L, t) = 0 & \text{in } (0, T), \\ v(x, 0) = v_0(x) & \text{in } (0, L). \end{cases}$$

We can prove the following estimate

$$(4.14) \quad \|q\|_{L^\infty(0, T; L^2(0, L))}^2 + 2\|q_x\|_{L^2(0, T; L^2(0, L))}^2 \leq \tilde{C}(T, L, \|\xi\|_{\mathcal{Z}_T})(\|q_0\|_{L^2(0, L)}^2 + \|v\|_{L^2((0, T) \times \omega)}^2).$$

We introduce the space

$$E := C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)) \cap H^1(0, T; H^{-2}(0, L)),$$

and in $L^2((0, T) \times (0, L))$ the following set

$$B := \{z \in E; \|z\|_E \leq 1\}.$$

B is compact in $L^2((0, T) \times (0, L))$, by Aubin–Lions’s lemma. We will limit ourselves to v fulfilling the condition

$$(4.15) \quad \|v\|_{L^2((0, T) \times \omega)}^2 \leq C_* \|q_0\|_{L^2(0, L)}^2,$$

where $C_* := C_*(T, L, \|\bar{u}\|_{\mathcal{Z}_T} + \frac{1}{2})$. We associate with any $z \in B$, solutions of the linear system (4.7), the set

$$T(z) := \left\{ q \in B; \exists v \in L^2((0, T) \times \omega) \text{ such that } v \text{ satisfies (4.15) and } q \text{ solves (4.10)–(4.12) with } \xi = \bar{u} + \frac{z}{2} \text{ and } q(\cdot, T) = 0 \right\}.$$

By the result of the linear system (see [11, Theorem 1]) and (4.14), we see that if $\|q_0\|_{L^2(0, L)}$ and T are sufficiently small, then $T(z)$ is nonempty for all $z \in B$. We shall use the following version of the Kakutani fixed point theorem (see, e.g., [24, Theorem 9.B]).

THEOREM 4.5. *Let F be a locally convex space, let $B \subset F$, and let $T : B \rightarrow 2^B$. Assume that*

1. B is a nonempty, compact, convex set;
2. $T(z)$ is a nonempty, closed, convex set for all $z \in B$;

3. the set-valued map $T : B \rightarrow 2^B$ is upper semicontinuous, i.e., for every closed subset A of F , $T^{-1}(A) = \{z \in B; T(z) \cap A \neq \emptyset\}$ is closed. Then T has a fixed point, i.e., there exists $z \in B$ such that $z \in T(z)$.

Let us check that Theorem 4.5 can be applied to T and

$$F = L^2((0, T) \times (0, L)).$$

The convexity of B and $T(z)$ for all $z \in B$ is clear. Thus (1) is satisfied. For (2), it remains to check that $T(z)$ is closed in F for all $z \in B$. Pick any $z \in B$ and a sequence $\{q^k\}_{k \in \mathbb{N}}$ in $T(z)$ which converges in F towards some function $q \in B$. For each k , we can pick some control function $v^k \in L^2((0, T) \times \omega)$ fulfilling (4.15) such that (4.10)–(4.12) are satisfied with $\xi = \bar{u} + \frac{z}{2}$ and $q^k(\cdot, T) = 0$. Extracting subsequences if needed, we may assume that as $k \rightarrow \infty$

$$(4.16) \quad v^k \rightarrow v \text{ in } L^2((0, T) \times \omega) \text{ weakly,}$$

$$(4.17) \quad q^k \rightarrow q \text{ in } L^2(0, T; H^1(0, L)) \cap H^1(0, T; H^{-2}(0, L)) \text{ weakly.}$$

By (4.17), the boundedness of $\|q^k\|_{L^\infty(0, T, L^2(0, L))}$ and Aubin–Lions’ lemma, $\{q^k\}_{k \in \mathbb{N}}$ is relatively compact in $C^0([0, T], H^{-1}(0, L))$. Extracting a subsequence if needed, we may assume that

$$q^k \rightarrow q \text{ strongly in } C^0([0, T], H^{-1}(0, L)).$$

In particular, $q(x, 0) = q_0(x)$ and $q(x, T) = 0$. On the other hand, we infer from (4.17) that

$$\xi q^k \rightarrow \xi q \text{ in } L^2((0, T) \times (0, L)) \text{ weakly.}$$

Therefore, $(\xi q^k)_x \rightarrow (\xi q)_x$ in $\mathcal{D}'((0, T) \times (0, L))$. Finally, it is clear that

$$\|v\|_{L^2((0, T) \times \omega)}^2 \leq C_* \|q_0\|_{L^2(0, L)}^2$$

and that q satisfies (4.10) with $\xi = \bar{u} + \frac{z}{2}$ and $q(\cdot, T) = 0$. Thus $q \in T(z)$ and $T(z)$ is closed. Now, let us check (3). To prove that T is upper semicontinuous, consider any closed subset A of F and any sequence $\{z^k\}_{k \in \mathbb{N}}$ in B such that

$$(4.18) \quad z^k \in T^{-1}(A) \quad \forall k \geq 0$$

and

$$(4.19) \quad z^k \rightarrow z \text{ in } F$$

for some $z \in B$. We aim to prove that $z \in T^{-1}(A)$. By (4.18), we can pick a sequence $\{q^k\}_{k \in \mathbb{N}}$ in B with $q^k \in T(z^k) \cap A$ for all k , and a sequence $\{v^k\}_{k \in \mathbb{N}}$ in $L^2((0, T) \times \omega)$ such that

$$(4.20) \quad \begin{cases} q_t^k + q_x^k + ((\bar{u} + \frac{z^k}{2})q^k)_x + q_{xxx}^k = 1_\omega v^k(t, x) & \text{in } (0, L) \times (0, T), \\ q_{xx}^k(0, t) = q_x^k(L, t) = q_{xx}^k(L, t) = 0 & \text{in } (0, T), \\ q^k(x, 0) = q_0(x) & \text{in } (0, L), \end{cases}$$

$$(4.21) \quad q^k(x, T) = 0 \quad \text{in } (0, L),$$

and

$$(4.22) \quad \|v^k\|_{L^2((0, T) \times \omega)}^2 \leq C_* \|q_0\|_{L^2(0, L)}^2.$$

From (4.22) and the fact that $z^k, q^k \in B$, extracting subsequences if needed, we may assume that as $k \rightarrow \infty$,

$$\begin{aligned} v^k &\rightarrow v && \text{in } L^2((0, T) \times \omega) \text{ weakly,} \\ q^k &\rightarrow q && \text{in } L^2(0, T; H^1(0, L)) \cap H^1(0, T; H^{-2}(0, L)) \text{ weakly,} \\ q^k &\rightarrow q && \text{in } C^0([0, T], H^{-1}(0, L)) \text{ strongly,} \\ q^k &\rightarrow q && \text{in } F \text{ strongly,} \\ z^k &\rightarrow z && \text{in } F \text{ strongly,} \end{aligned}$$

where $v \in L^2((0, T) \times \omega)$ and $q \in B$. Again, $q(x, 0) = q_0(x)$ and $q(x, T) = 0$. We also see that (4.11) and (4.15) are satisfied. It remains to check that

$$(4.23) \quad q_t + q_x + \left(\left(\bar{u} + \frac{z}{2} \right) q \right)_x + q_{xxx} = 1_\omega v(t, x).$$

Observe that the only nontrivial convergence in (4.20) is that of the nonlinear term $(z^k q^k)_x$. Note first that

$$\|z^k q^k\|_{L^2(0, T, L^2(0, L))} \leq \|z^k\|_{L^\infty(0, T, L^2(0, L))} \|q^k\|_{L^2(0, T, L^\infty(0, L))} \leq C,$$

so that, extracting a subsequence, one can assume that $z^k q^k \rightarrow f$ weakly in $L^2((0, T) \times (0, L))$. To prove that $f = zq$, it is sufficient to observe that for any $\varphi \in \mathcal{D}(Q)$,

$$\int_0^T \int_0^L z^k q^k \varphi dx dt \rightarrow \int_0^T \int_0^L zq \varphi dx dt$$

for $z^k \rightarrow z$ and $q^k \varphi \rightarrow q\varphi$ in F . Thus

$$z^k q^k \rightarrow zq \quad \text{in } L^2((0, T) \times (0, L)) \text{ weakly.}$$

It follows that $(z^k q^k)_x \rightarrow (zq)_x$ in $\mathcal{D}'((0, T) \times (0, L))$. Therefore, (4.23) holds and $q \in T(z)$. On the other hand, $q \in A$, since $q^k \rightarrow q$ in F and A is closed. We conclude that $z \in T^{-1}(A)$, and hence $T^{-1}(A)$ is closed.

Thus, it follows from Theorem 4.5 that there exists $q \in B$ with $q \in T(q)$, i.e., we have found a control $h_1 \in L^2(0, T)$ such that the solution of (4.9) satisfies $q(\cdot, T) = 0$ in $(0, L)$. The proof of Theorem 1.8 is finished. \square

Finally we consider the boundary control system

$$(4.24) \quad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0 & \text{in } (0, L) \times (0, T), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t) & \text{in } (0, T), \\ u(x, 0) = u_0(x) & \text{in } (0, L) \end{cases}$$

with all three control inputs being used and present the proofs of Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Consider the initial value control of the KdV equation posed in the whole line \mathbb{R} :

$$(4.25) \quad w_t + w_x + ww_x + w_{xxx} = 0, \quad w(x, 0) = g(x), \quad x \in \mathbb{R}, \quad t \in (0, T),$$

where the initial value g is considered as a control input. By [25, Theorem 1.2] there exists a $\delta > 0$ such that, for $s \geq 0$, if $u_0, u_T \in H^s(0, L)$ satisfying

$$\|u_0(\cdot) - y(\cdot, 0)\|_{H^s(0, L)} + \|u_T(\cdot) - y(\cdot, T)\|_{H^s(0, L)} \leq \delta,$$

then one can choose $g \in H^s(\mathbb{R})$ so that (4.25) admits a solution

$$w \in C([0, T]; H^s(\mathbb{R}) \cap L^2(0, T; H^{s+1}(\mathbb{R})))$$

with

$$w(x, 0) = u_0(x), \quad w(x, T) = u_T(x) \quad \text{for } x \in (0, L).$$

Moreover, the solution w possesses the sharp Kato smoothing properties with

$$\begin{aligned} h_1(t) &:= w_{xx}(0, t) \in H^{\frac{s-1}{3}}(0, T), \\ h_2(t) &:= w_x(L, t) \in H^{\frac{s}{3}}(0, T), \\ h_3 &:= w_{xx}(L, t) \in H^{\frac{s-1}{3}}(0, T). \end{aligned}$$

Thus with such chosen control inputs $h_j, j = 1, 2, 3$,

$$u(x, t) := w(x, t) \quad \text{for } x \in (0, L), t \in (0, T),$$

solves system (4.24) and satisfies

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x) \quad \text{for } x \in (0, L).$$

We have, thus, completed the proof of Theorem 1.6. □

Proof of Theorem 1.7. Without loss of generality, we assume $u_T = 0$. Consider the feedback control system of the KdV equation posed on the interval $(-L, L)$:

$$(4.26) \quad \begin{cases} v_t + v_x + vv_x + v_{xxx} + b(x)v = 0, & v(x, 0) = \tilde{u}_0(x), & x \in (-L, L), t \in (0, T), \\ v(-L, t) = 0, & v(L, t) = 0, & v_x(L, t) = 0, & t \in (0, T), \end{cases}$$

where

$$b(x) = \begin{cases} 1, & x \in (-\frac{4}{3}L, -\frac{1}{2}L), \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{u}_0(x) = \begin{cases} u_0(x), & x \in (0, L), \\ 0, & \text{otherwise.} \end{cases}$$

It follows from [25] that, for given $u_0 \in L^2(0, L)$, we have

$$v \in C_b(\mathbb{R}^+, L^2(-L, L)) \cap L^2_{loc}(\mathbb{R}^+; H^1(-L, L))$$

and there exists a $\nu > 0$ such that

$$\|v(\cdot, t)\|_{L^2(-L, L)} \leq C\|u_0\|_{L^2(0, L)}e^{-\nu t} \quad \text{for any } t \geq 0.$$

For given $\delta > 0$, choose t^* large enough such that

$$\|v(\cdot, t^*)\|_{L^2(-L, L)} \leq C\|u_0\|_{L^2(0, L)}e^{-\nu t^*} \leq \delta.$$

Then, again by [25, Theorem 1.2], one can find a control $h \in L^2(t^*, t^* + 1)$ such that (1.1) admits a solution $z \in C([t^*, t^* + 1]; L^2(0, L)) \cap L^2(t^*, t^* + 1; H^1(0, L))$ satisfying

$$z(x, t^*) = v(x, t^*), \quad z(x, t^* + 1) = 0, \quad x \in (0, L).$$

Let $T = t^* + 1$,

$$h_1(t) := \begin{cases} v_{xx}(0, t), & t \in (0, t^*), \\ 0, & t \in (t^*, T), \end{cases} \quad h_2(t) := \begin{cases} v_x(L, t), & t \in (0, t^*), \\ h(t), & t \in (t^*, T), \end{cases}$$

and

$$h_3(t) := \begin{cases} v_{xx}(L, t), & t \in (0, t^*), \\ 0, & t \in (t^*, T). \end{cases}$$

Note that as the solutions of (4.26) possess the sharp Kato smoothing properties we have

$$h_1 \in H^{-\frac{1}{3}}(0, T), \quad h_2 \in L^2(0, T), \quad h_3 \in H^{-\frac{1}{3}}(0, T).$$

Thus if we let

$$u(x, t) := \begin{cases} v(x, t), & x \in (0, L), \quad t \in (0, t^*), \\ z(x, t), & x \in (0, L), \quad t \in (t^*, T), \end{cases}$$

then $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ solves (4.24) and satisfies

$$u(x, 0) = u_0(x), \quad u(x, T) = 0 \quad \text{for } x \in (0, L).$$

Thus, the proof of Theorem 1.7 is achieved. \square

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