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Stabilization of the Gear–Grimshaw system on a periodic domain

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This paper is devoted to the study of a nonlinear coupled system of two Korteweg–de Vries equations in a periodic domain under the effect of an internal damping term. The system was introduced by Gear and Grimshaw to model the interactions of two-dimensional, long, internal gravity waves propagation in a stratified fluid. Designing a time-varying feedback law and using a Lyapunov approach, we establish the exponential stability of the solutions in Sobolev spaces of any positive integral order.

Keywords: KdV equation; conservation laws; stabilization; decay rate.

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1. Introduction

The goal of this paper is to investigate the decay properties of the initial-value problem

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x + k(u - [u]) = 0, \\ b_1 v' + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x \\ \qquad + b_2 a_1 (uv)_x + k(v - [v]) = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{1.1}$$

with periodic boundary conditions. In (1.1), $r, a_1, a_2, a_3, b_1, b_2, k$ are given real constants with $b_1, b_2, k > 0$, $u(t,x), v(t,x)$ are real-valued functions of the time and space variables $t \geq 0$ and $0 \leq x \leq 1$, the subscript x and the prime indicate the partial differentiation with respect to x and t, respectively, and $[f]$ denotes the mean value of f defined by

$$[f] := \int_0^1 f(x)dx.$$

When $k = 0$, system was proposed by Gear and Grimshaw [8] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been object of intensive research in recent years. In what concerns the stabilization problems, most of the works have been focused on a bounded interval with a localized internal damping (see, for instance, [14] and the references therein). In particular, we also refer to [1] for an extensive discussion on the physical relevance of the system and to [3–7] for the results used in this paper.

We can (formally) check that the energy

$$E = \frac{1}{2} \int_0^1 b_2 u^2 + b_1 v^2 dx$$

associated with the model satisfies the inequality

$$E' = -k \int_0^1 b_2 (u - [u])^2 + (v - [v])^2 dx \leq 0$$

in $(0, \infty)$, so that the energy is nonincreasing. Therefore, the following basic questions arise: are the solutions asymptotically stable for t sufficiently large? And if yes, is it possible to find a rate of decay? The aim of this paper is to answer these questions.

More precisely, we will prove that for any fixed integer $s \geq 3$, the solutions are exponentially stable in the Sobolev spaces

$$H_p^s(0,1) := \{u \in H^s(0,1) : \partial_x^n u(0) = \partial_x^n u(1), n = 0, \dots, s\}$$

with periodic boundary conditions. This extends an earlier theorem of Dávila in [5] for $s \leq 2$.

The proposed choice of the feedbacks is perhaps the simplest one to make the system dissipative. A similar feedback law has been applied before for the scalar KdV equation in [9] and [10]. The corresponding proof has used the fine structure of all the conservation laws.

Dávila and Chavez [7] established that, under some assumptions on the coefficients, the system (1.1) also has an infinite set of conservation laws. However, the generalization of the method used in [10] would require a deep study of the structure of these laws, and would lead to very lengthy and complex computations. Instead of doing this, we apply, following a remark of Bona concerning the scalar KdV equation, a Lyapunov function approach which uses only the first four conservation laws.

Before stating the stabilization result mentioned above, we first need to ensure the well-posedness of the system. This was addressed by Dávila in [3] (see also [4]) under the following conditions on the coefficients:

$$\begin{gathered} a_3^2 b_2 < 1 \quad \text{and} \quad r = 0, \\ b_2 a_1 a_3 - b_1 a_3 + b_1 a_2 - a_2 = 0, \\ b_1 a_1 - a_1 - b_1 a_2 a_3 + a_3 = 0, \\ b_1 a_2^2 + b_2 a_1^2 - b_1 a_1 - a_2 = 0. \end{gathered} \tag{1.2}$$

Indeed, under conditions (1.2), Dávila and Chaves [7] derived some conservation laws for the solutions of (1.1). Combined with an approach introduced in [2, 17], improving [18], these conservation laws allow them to establish the global well-posedness in $H_p^s(0,1)$, for any $s \geq 0$. Moreover, the authors also give a simpler derivation of the conservation laws discovered by Gear and Grimshaw, and Bona *et al.* [1]. We also observe that these conservation properties were obtained employing the techniques developed in [13] for the scalar KdV equation; see also [12].

The well-posedness result reads as follows.

Theorem 1.1. *Assume that condition* (1.2) *holds. If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \geq 3$, *then the system* (1.1) *has a unique solution satisfying*

$$u, v \in C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)).$$

Moreover, the map $(\phi,\psi) \mapsto (u,v)$ *is continuous from* $(H_p^s(0,1))^2$ *into*

$$(C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)))^2.$$

For $k = 0$, the analogous theorem on the whole real line $-\infty < x < \infty$ was proved in [1], for all $s \geq 1$.

With the global well-posedness result in hand, we can focus on the stabilization problem. For simplicity of notation we consider only the case

$$b_1 = b_2 = 1. \tag{1.3}$$

Then the conditions (1.2) take the simplified form

$$r = 0, \quad a_1^2 + a_2^2 = a_1 + a_2, \quad |a_3| < 1, \quad \text{and} \quad (a_1 - 1)a_3 = (a_2 - 1)a_3 = 0. \tag{1.4}$$

Hence either $a_3 = 0$ and $a_1^2 + a_2^2 = a_1 + a_2$, or $0 < |a_3| < 1$ and $a_1 = a_2 = 1$.

We prove the following theorem.

Theorem 1.2. *Assume* (1.3) *and* (1.4). *If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \geq 3$, *then the solution of* (1.1) *satisfies the estimate*

$$\|u(t) - [u(t)]\|_{H_p^s(0,1)} + \|v(t) - [v(t)]\|_{H_p^s(0,1)} = o(e^{-k't}), \quad t \to \infty$$

for each $k' < k$.

In order to obtain the result, we prove a number of identities and estimates for the solutions of (1.1). In view of Theorem 1.1 it suffices to establish these estimates for *smooth solutions*, i.e. for solutions corresponding to C^∞ initial data ϕ, ψ with periodic boundary conditions. For such solutions all formal manipulations in the sequel will be justified.

For the scalar KdV equation stabilization results are also available by using localized feedback laws. For example, following Russell and Zhang [15–16], Laurent, Rosier and Zhang [11] study a model on a periodic domain from a control point of view with a forcing term f supported in a given open set of the domain. It is shown that the system is globally exactly controllable and globally exponentially stable. The stabilization is established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the corresponding linear system. We also refer to [11] for a quite complete review on the subject. We plan to investigate the system (1.1) with localized feedbacks in the near future by a similar approach, based on Bourgain spaces.

One of the referees suggested us to change the feedback laws to $k(u - v)$ and $-k(u - v)$, respectively, as in some two-component fluid systems. These feedbacks are quite interesting to study; this also will require a different approach.

The paper is organized as follows. In Sec. 2 we introduce the basic notations and we prove some technical lemmas. Sections 3–6 are devoted to the proof of the exponential decay in H_p^s, for $s = 0, 1, 2$ and $s \geq 3$, respectively.

2. Some Technical Lemmas

In the sequel all integrals are taken over the interval $(0, 1)$ so we omit the integration limits.

As explained in Sec. 1, all integrations by parts will be done for smooth periodic functions. Therefore, we will regularly use the simplified formulas

$$\int f_x g dx = -\int f g_x dx \quad \text{and} \quad \int f^n f_x dx = 0 \quad (n = 0, 1, \ldots)$$

without further explanation, and we will also use the simplified notation

$$f_n := \frac{d^n f}{dx^n}, \quad n = 1, 2, \ldots$$

As an example of the application of these rules we show that the mean values of the solutions are conserved.

Lemma 2.1. *The mean values* $[u]$ *and* $[v]$ *of the solutions of* (1.1) *do not depend on* t.

Proof. We have

$$\begin{aligned}
[u]' &= -\int uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x + k(u - [u])dx \\
&= -\int \left(\frac{u^2}{2} + u_{xx} + a_3 v_{xx} + a_1 \frac{v^2}{2} + a_2 uv\right)_x + k(u - [u])dx \\
&= -k \int (u - [u])dx \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
[v]' &= -\int vv_x + v_{xxx} + a_3 u_{xxx} + a_2 uu_x + a_1(uv)_x + k(v - [v])dx \\
&= -\int \left(\frac{v^2}{2} + v_{xx} + a_3 u_{xx} + a_2 \frac{u^2}{2} + a_1 uv\right)_x + k(v - [v])dx \\
&= -k \int (v - [v])dx \\
&= 0
\end{aligned}$$

by a straightforward computation. $\square$

Motivated by this result we set $M = [\varphi]$, $N = [\psi]$ and we rewrite (1.1) by changing u, v, φ and ψ to $u - [u] = u - M$, $v - [v] = v - N$, $\varphi - [\varphi] = \varphi - M$ and $\psi - [\psi] = \psi - N$, respectively. Under our assumptions $r = 0$ and $b_1 = b_2 = 1$ we obtain the equivalent system

$$\begin{cases}
u' + (u + M)u_x + u_{xxx} + a_3 v_{xxx} + a_1(v + N)v_x \\
\qquad + a_2((u + M)(v + N))_x + ku = 0, \\
v' + (v + N)v_x + v_{xxx} + a_3 u_{xxx} + a_2(u + M)u_x \\
\qquad + a_1((u + M)(v + N))_x + kv = 0, \\
u(0, x) = \phi(x), \\
v(0, x) = \psi(x)
\end{cases} \tag{2.1}$$

with periodic boundary conditions, corresponding to initial data ϕ, ψ with zero mean values. Theorem 1.2 will thus follow from the following proposition.

Proposition 2.2. *Under the assumptions of Theorem* 1.2 *the smooth solutions of* (2.1) *satisfy the identity*

$$\int u(t)^2 + v(t)^2 dx = e^{-2kt} \int \phi^2 + \psi^2 dx, \quad t \geq 0, \tag{2.2}$$

and the estimates

$$e^{2k't} \int (\partial_x^n u(t))^2 + (\partial_x^n v(t))^2 dx \to 0 \quad \text{as } t \to \infty$$

for all positive integers n and for all $k' < k$.

Remark. For $n = 1$ the proposition and its proof remain valid under the weaker assumption that $|a_3| < 1$. We can also add the term rv_x to the equation by changing g to $g - rv^2$ in Lemma 4.1.

Proposition 2.2 is proved by using the Lyapunov method. More precisely, we shall use the following lemma.

Lemma 2.3. *Let $f : (0, \infty) \to \mathbb{R}$ be a non-negative function, and write $h_1 \approx h_2$ if $h_1 - h_2 = o(f)$ as $t \to \infty$.*

If there exists a function $g : (0, \infty) \to \mathbb{R}$ such that $g \approx 0$, $f + g$ is continuously differentiable, and $(f + g)' \approx -2kf$ for some positive number k, then

$$e^{2k't} f(t) \to 0 \quad \text{as } t \to \infty$$

for each $k' < k$.

Proof. Fix $k'' > 0$ such that $k' < k'' < k$, and then fix $\varepsilon > 0$ such that

$$\frac{1 - \varepsilon}{1 + \varepsilon} = \frac{k''}{k}.$$

Finally, choose a sufficiently large $t' > 0$ such that

$$(1 - \varepsilon)f(t) \leq (f + g)(t) \leq (1 + \varepsilon)f(t)$$

and

$$2k(1 - \varepsilon)f(t) \leq -(f + g)'(t) \leq 2k(1 + \varepsilon)f(t)$$

for all $t \geq t'$. Then for $t \geq t'$ we have

$$-(f + g)'(t) \geq 2k(1 - \varepsilon)f(t) \geq 2k\frac{1 - \varepsilon}{1 + \varepsilon}(f + g)(t) = 2k''(f + g)(t),$$

whence

$$\frac{d}{dt}(e^{2k''t}(f + g)(t)) \leq 0.$$

It follows that

$$e^{2k''t}(f + g)(t) \leq e^{2k''t'}(f + g)(t')$$

for all $t \geq t'$, and hence

$$0 \leq e^{2k't} f(t) \leq \frac{e^{2k''t'}(f+g)(t')}{1-\varepsilon} e^{-2(k''-k')t}$$

for all $t \geq t'$. We conclude by observing that $e^{-2(k''-k')t} \to 0$ as $t \to \infty$. $\square$

For the proof of the next result, we shall use the Hölder and Poincaré-Wirtinger inequalities in the following form. The second estimate will be used only for functions with mean value zero: $[u] = 0$.

Lemma 2.4. *If $p, q \in [0, \infty)$, then*

$$\|u\|_p \leq \|u\|_q \quad \text{for all } u \in L^q(0,1) \text{ and } 1 \leq p \leq q \leq \infty; \tag{2.3}$$

$$\|u - [u]\|_p \leq \|u_x\|_q \quad \text{for all } u \in H^1(0,1) \text{ and } 1 \leq p, \ \ q \leq \infty. \tag{2.4}$$

We shall frequently use Lemma 2.3 together with the following result:

Lemma 2.5. *Let $n \geq 1$ and let α_m, β_m, $m = 0, \ldots, n$, be non-negative integers satisfying the two conditions*

$$2(\alpha_n + \beta_n) + \alpha_{n-1} + \beta_{n-1} \leq 4$$

and

$$d := \sum_{m=0}^{n} (\alpha_m + \beta_m) \geq 2.$$

Then

$$\left| \int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \, dx \right| \leq \left(\int u_n^2 + v_n^2 dx \right) \left(\int u_{n-1}^2 + v_{n-1}^2 dx \right)^{\frac{d-2}{2}}.$$

If, moreover, $d \geq 3$ and

$$\int u_{n-1}^2 + v_{n-1}^2 dx \to 0,$$

then it follows that

$$\int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \, dx = o\left(\int u_n^2 + v_n^2 dx \right)$$

as $t \to \infty$.

Proof. Setting

$$z_m := \sqrt{u_m^2 + v_m^2} \quad \text{and} \quad \gamma_m := \alpha_m + \beta_m, \ \ m = 0, \ldots, n$$

we have

$$\left| \int \prod_{m=0}^{n} u_m^{\alpha_m} v_m^{\beta_m} \, dx \right| \leq \int \prod_{m=0}^{n} z_m^{\gamma_m} \, dx.$$

We are going to majorize the right side by using the Hölder and Poincaré–Wirtinger inequalities (2.3)–(2.4). We distinguish five cases according to the value of $\gamma_n+\gamma_{n-1}$: since $2\gamma_n+\gamma_{n-1}\le 4$ by our assumption, $\gamma_n+\gamma_{n-1}\le 4$.

If $\gamma_n+\gamma_{n-1}=0$, then we have

$$\left|\int\prod_{m=0}^{n} z_m^{\gamma_m}\,dx\right| \le \prod_{m=0}^{n-2}\|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2\|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n+\gamma_{n-1}=1$, then

$$\left|\int\prod_{m=0}^{n} z_m^{\gamma_m}\,dx\right| \le \|z_n\|_1\prod_{m=0}^{n-2}\|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2\|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n+\gamma_{n-1}=2$, then

$$\left|\int\prod_{m=0}^{n} z_m^{\gamma_m}\,dx\right| \le \|z_n\|_2^2\prod_{m=0}^{n-2}\|z_m\|_\infty^{\gamma_m} \le \|z_n\|_2^2\|z_{n-1}\|_2^{d-2}.$$

If $\gamma_n+\gamma_{n-1}=3$, then we have necessarily $\gamma_n=1$ and $\gamma_{n-1}=2$, so that

$$\begin{aligned}\left|\int\prod_{m=0}^{n} z_m^{\gamma_m}\,dx\right| &\le \|z_n\|_2\|z_{n-1}\|_\infty\|z_{n-1}\|_2\prod_{m=0}^{n-2}\|z_m\|_\infty^{\gamma_m}\\ &\le \|z_n\|_2^2\|z_{n-1}\|_2^{d-2}.\end{aligned}$$

Finally, if $\gamma_n+\gamma_{n-1}=4$, then we have necessarily $\gamma_n=0$ and $\gamma_{n-1}=4$, so that

$$\begin{aligned}\left|\int\prod_{m=0}^{n} z_m^{\gamma_m}\,dx\right| &\le \|z_{n-1}\|_\infty^2\|z_{n-1}\|_2^2\prod_{m=0}^{n-2}\|z_m\|_\infty^{\gamma_m}\\ &\le \|z_n\|_2^2\|z_{n-1}\|_2^{d-2}.\end{aligned}$$

□

3. Proof of Proposition 2.2 for $n = 0$

Our proof is based on the following identity.

Lemma 3.1. *The solutions of* (2.1) *satisfy the following identity for all* $n = 0, 1, \ldots$:

$$\begin{aligned}\left(\int u_n^2+v_n^2\,dx\right)' &= -2k\int u_n^2+v_n^2\,dx\\ &\quad-2\int u_n(u_1u)_n+v_n(v_1v)_n\,dx\\ &\quad-2a_1\int u_n(vv_1)_n+v_n(uv)_{n+1}\,dx\\ &\quad-2a_2\int v_n(uu_1)_n+u_n(uv)_{n+1}\,dx.\end{aligned}\tag{3.1}$$

Proof. We have

$$\begin{aligned}\left(\int u_n^2+v_n^2dx\right)' &= \int 2u_nu_n'+2v_nv_n'dx\\ &= \int -2u_n((u+M)u_1+u_3+a_3v_3+a_1(v+N)v_1\\ &\quad +a_2((u+M)(v+N))_1+ku)_ndx\\ &\quad +\int -2v_n((v+N)v_1+v_3+a_3u_3+a_2(u+M)u_1\\ &\quad +a_1((u+M)(v+N))_1+kv)_ndx.\end{aligned}$$

This yields the stated identity because

$$\begin{aligned}&\int -2u_nu_{n+3}-2v_nv_{n+3}dx=\int 2u_{n+1}u_{n+2}+2v_{n+1}v_{n+2}dx\\ &\qquad =\int (u_{n+1}^2)_1+(v_{n+1}^2)_1dx=0,\\ &a_3\int -2u_nv_{n+3}-2v_nu_{n+3}dx=a_3\int -2u_nv_{n+3}+2v_{n+3}u_ndx=0,\\ &-2M\int u_nu_{n+1}+a_2u_nv_{n+1}+a_2v_nu_{n+1}+a_1v_nv_{n+1}dx\\ &\qquad =-M\int (u_n^2+2a_2u_nv_n+a_1v_n^2)_1dx=0,\\ &-2N\int a_1u_nv_{n+1}+a_2u_nu_{n+1}+v_nv_{n+1}+a_1v_nu_{n+1}dx\\ &\qquad =-N\int (2a_1u_nv_n+a_2u_n^2+v_n^2)_1dx=0\end{aligned}$$

and $(MN)_1=0$. □

Proof of the proposition for $n = 0$. In this case the last three integrals of the identity (3.1) vanish because

$$\int uu_1u+vv_1vdx=\frac{1}{3}\int (u^3+v^3)_1dx=0,$$

$$\int uvv_1+v(uv)_1dx=\int (uvv)_1dx=0$$

and

$$\int vuu_1+u(uv)_1dx=\int (vuu)_1dx=0. \qquad \square$$

Proceeding by induction on n, let $n\geq 1$ and assume that the estimates

$$\int u_m^2+v_m^2dx=o(e^{-2k't})\quad \text{as } t\to\infty \tag{3.2}$$

hold for all integers $m = 0, \dots, n-1$ and for all $k' < k$. For $n = 1$ this follows from the stronger identity (2.2).

4. Proof of Proposition 2.2 for $n = 1$

For the proof of the case $n = 1$ we shall use an identity suggested by a conservation law discovered by Bona *et al.* [1].

Lemma 4.1. *Setting*

$$f := \int u_1^2 + v_1^2 + 2a_3 u_1 v_1 \, dx$$

and

$$g := -\frac{1}{3} \int (u^3 + v^3) + 3(a_1 u v^2 + a_2 u^2 v) dx,$$

we have the following identity:

$$(f + g)' = -2kf - 3kg. \tag{4.1}$$

Proof. The equality (4.1) will follow by combining the following four identities:

$$\begin{aligned}\left(\int u_1^2 + v_1^2 dx\right)' &= -2k \int u_1^2 + v_1^2 dx \\ &\quad - \int u_1^3 + v_1^3 dx \\ &\quad - 3a_1 \int u_1 v_1^2 dx \\ &\quad - 3a_2 \int u_1^2 v_1 dx;\end{aligned} \tag{4.2}$$

$$\begin{aligned}\left(\int u_1 v_1 dx\right)' &= -2k \int u_1 v_1 dx + \int u u_1 v_2 + v v_1 u_2 dx \\ &\quad - \frac{a_1}{2} \int 2 v_2 u_1 u + 3 v_1 u_1^2 + v_1^3 dx \\ &\quad - \frac{a_2}{2} \int 2 u_2 v_1 v + 3 u_1 v_1^2 + u_1^3 dx;\end{aligned} \tag{4.3}$$

$$\begin{aligned}\left(\int u^3 + v^3 dx\right)' &= -3k \int u^3 + v^3 dx - 3 \int u_1^3 + v_1^3 dx \\ &\quad - a_1 \int 3u^2 v v_1 + 2v^3 u_1 dx \\ &\quad - a_2 \int 3v^2 u u_1 + 2u^3 v_1 dx. \\ &\quad + 6a_3 \int u u_1 v_2 + v v_1 u_2 dx;\end{aligned} \tag{4.4}$$

$$\begin{aligned}\left(\int a_1uv^2+a_2u^2vdx\right)' &= -3k\int a_1uv^2+a_2u^2vdx\\ &\quad +a_1\int \frac{2}{3}v^3u_1+u^2vv_1-3v_1^2u_1dx\\ &\quad +a_2\int \frac{2}{3}u^3v_1+v^2uu_1-3u_1^2v_1dx\\ &\quad -a_1a_3\int 2v_2u_1u+3v_1u_1^2+v_1^3dx\\ &\quad -a_2a_3\int 2u_2v_1v+3u_1v_1^2+u_1^3dx. \end{aligned}\tag{4.5}$$

Proof of (4.2). We transform the identity (3.1) for $n=1$ as follows. We have

$$\begin{aligned}\int u_1(u_1u)_1+v_1(v_1v)_1dx &= \int u_2u_1u+u_1^3+v_2v_1v+v_1^3dx\\ &= \int u_1^3+v_1^3+\frac{1}{2}(u_1^2)_1u+\frac{1}{2}(v_1^2)_1vdx\\ &= \frac{1}{2}\int u_1^3+v_1^3dx,\\ \int u_1(vv_1)_1+v_1(uv)_2dx &= \int u_1v_1^2+u_1vv_2-v_2(uv)_1dx\\ &= \int u_1v_1^2-v_2uv_1dx\\ &= \int u_1v_1^2-\frac{1}{2}u(v_1^2)_1dx\\ &= \frac{3}{2}\int u_1v_1^2dx,\end{aligned}$$

and by symmetry

$$\int v_1(uu_1)_1+u_1(uv)_2dx=\frac{3}{2}\int u_1^2v_1dx.$$

Using them (3.1) implies (4.2).

Proof of (4.3). We have

$$\begin{aligned}\left(\int u_1v_1dx\right)' &= \int u_1'v_1+u_1v_1'dx\\ &= \int -(uu_1+u_3+a_3v_3+a_1vv_1+a_2(uv)_1+ku)_1v_1dx\\ &\quad +\int -u_1(vv_1+v_3+a_3u_3+a_2uu_1+a_1(uv)_1+kv)_1dx\end{aligned}$$

$$
\begin{aligned}
&= -2k \int u_1 v_1 dx + \int (uu_1 + u_3)v_2 + (vv_1 + v_3)u_2 dx \\
&\quad - a_1 \int (vv_1)_1 v_1 + u_1 (uv)_2 dx - a_2 \int (uv)_2 v_1 + u_1 (uu_1)_1 dx \\
&\quad - a_3 \int v_4 v_1 + u_4 u_1 dx \\
&= -2k \int u_1 v_1 dx + \int uu_1 v_2 + vv_1 u_2 dx \\
&\quad + a_1 \int vv_1 v_2 + u_2 (uv)_1 dx + a_2 \int (uv)_1 v_2 + u_2 uu_1 dx
\end{aligned}
$$

because

$$
\int u_3 v_2 + v_3 u_2 dx = \int u_3 v_2 - v_2 u_3 dx = 0
$$

and

$$
\int v_4 v_1 + u_4 u_1 dx = -\int v_3 v_2 + u_3 u_2 dx = -\frac{1}{2} \int (v_2^2 + u_2^2)_1 dx = 0.
$$

Since

$$
\begin{aligned}
\int vv_1 v_2 + u_2 (uv)_1 dx &= \int \frac{1}{2} v (v_1^2)_1 + \frac{1}{2} (u_1^2)_1 v + u_2 u v_1 dx \\
&= \int -\frac{1}{2} v_1^3 - \frac{1}{2} u_1^2 v_1 - u_1^2 v_1 - u_1 u v_2 dx \\
&= -\frac{1}{2} \int 2 v_2 u_1 u + 3 v_1 u_1^2 + v_1^3 dx,
\end{aligned}
$$

and by symmetry

$$
\int uu_1 u_2 + v_2 (uv)_1 dx = -\frac{1}{2} \int 2 u_2 v_1 v + 3 u_1 v_1^2 + u_1^3 dx,
$$

Eq. (4.3) follows from the previous identity.

Proof of (4.4). We have

$$
\begin{aligned}
\left(\int u^3 dx \right)' &= \int 3u^2 u' dx \\
&= \int -3u^2 (uu_1 + u_3 + a_3 v_3 + a_1 vv_1 + a_2 (uv)_1 + ku) dx \\
&= \int -\frac{3}{4} (u^4)_1 + 3u (u_1^2)_1 - 3k u^3 dx \\
&\quad - 3a_3 \int u^2 v_3 dx - 3a_1 \int u^2 v v_1 dx \\
&\quad - 3a_2 \int u^3 v_1 + \frac{1}{3} (u^3)_1 v dx
\end{aligned}
$$

$$
\begin{aligned}
&= -3\int u_1^3 + ku^3 dx - 3a_1 \int u^2 v v_1 dx - 2a_2 \int u^3 v_1 dx \\
&\quad + 6a_3 \int u u_1 v_2 dx.
\end{aligned}
$$

We have an analogous identity for $\int v^3 dx$ by symmetry; adding them we get (4.4).

Proof of (4.5). We have

$$
\begin{aligned}
\left(\int u^2 v dx\right)' &= \int u'(2uv) + u^2 v' dx \\
&= \int -2uv(uu_1 + u_3 + a_3 v_3 + a_1 v v_1 + a_2 (uv)_1 + ku) dx \\
&\quad + \int -u^2 (v v_1 + v_3 + a_3 u_3 + a_2 u u_1 + a_1 (uv)_1 + kv) dx \\
&= \int -2u^2 u_1 v + 2u_2 (uv)_1 - u^2 v v_1 + 2v_2 u u_1 dx - 3k \int u^2 v dx \\
&\quad - a_1 \int 2uvvv_1 + u^2 (uv)_1 dx - a_2 \int 2uv(uv)_1 + u^3 u_1 dx \\
&\quad - a_3 \int 2uvv_3 + u^2 u_3 dx.
\end{aligned}
$$

Here

$$
\begin{aligned}
\int -2u^2 u_1 v dx &= -\frac{2}{3}\int (u^3)_1 v dx = \frac{2}{3}\int u^3 v_1, \\
\int -u^2 v v_1 dx &= -\frac{1}{2}\int u^2 (v^2)_1 dx = \frac{1}{2}\int (u^2)_1 v^2 dx \\
&= \int v^2 u u_1 dx, \\
\int 2u_2 (uv)_1 + 2v_2 u u_1 dx &= \int (2u_2 u_1 v + 2u_2 u v_1) - (2v_1 u_1^2 + 2v_1 u u_2) dx \\
&= \int (u_1^2)_1 v - 2v_1 u_1^2 dx \\
&= -3\int u_1^2 v_1 dx, \\
\int 2uvvv_1 + u^2 (uv)_1 dx &= \int \frac{2}{3} u (v^3)_1 + u^3 v_1 + \frac{1}{3}(u^3)_1 v dx \\
&= \frac{2}{3}\int u^3 v_1 - v^3 u_1 dx, \\
\int 2uv(uv)_1 + u^3 u_1 dx &= \int \left((uv)^2 + \frac{1}{4}u^4\right)_1 dx = 0
\end{aligned}
$$

and

$$\begin{aligned}\int 2uvv_3 + u^2u_3dx &= \int -2(u_1v + uv_1)v_2 - 2uu_1u_2dx\\ &= \int 2(u_2v + u_1v_1)v_1 - u(v_1^2)_1 - u(u_1^2)_1dx\\ &= \int 2(u_2v + u_1v_1)v_1 + u_1v_1^2 + u_1^3dx\\ &= \int 2u_2v_1v + 3u_1v_1^2 + u_1^3dx,\end{aligned}$$

so that

$$\begin{aligned}\left(\int u^2vdx\right)' = \int \frac{2}{3}u^3v_1 + v^2uu_1 - 3u_1^2v_1dx - 3k\int u^2vdx\\ - \frac{2}{3}a_1\int u^3v_1 - v^3u_1dx - a_3\int 2u_2v_1v + 3u_1v_1^2 + u_1^3dx.\end{aligned}$$

By symmetry, we also have

$$\begin{aligned}\left(\int v^2udx\right)' = \int \frac{2}{3}v^3u_1 + u^2vv_1 - 3v_1^2u_1dx - 3k\int v^2udx\\ - \frac{2}{3}a_2\int v^3u_1 - u^3v_1dx - a_3\int 2v_2u_1u + 3v_1u_1^2 + v_1^3dx.\end{aligned}$$

Combining the last two identities (4.5) follows (some terms annihilate each other). □

Proof of the proposition for $n = 1$. It suffices to show that the functions f and g of Lemma 4.1 satisfy the conditions of Lemma 2.3. Since $|a_3| < 1$, we have $f \geq 0$. The other conditions follow from the already proven case $n = 0$ and from the second part of Lemma 2.5. We conclude by applying the lemma and then by observing that

$$\int u_1^2 + v_1^2dx \leq \frac{1}{1 - |a_3|}\int u_1^2 + v_1^2 + 2a_3u_1v_1dx.$$ □

5. Proof of Proposition 2.2 for $n = 2$

Lemma 5.1. *Setting*

$$f := \int u_2^2 + v_2^2 + 2a_3u_2v_2dx,$$

$$g := -\frac{5}{3}\int (u_1^2u + v_1^2v) + a_1(2u_1v_1v + v_1^2u) + a_2(2u_1v_1u + u_1^2v)dx$$

and

$$h := \frac{2}{3}a_3\int (1 - a_1)(2u_3v_2u + u_2v_2u_1) + (1 - a_2)(2v_3u_2v + u_2v_2v_1)dx,$$

we have

$$(f+g)' \approx -2kf + h. \tag{5.1}$$

Proof. The relationship (5.1) will follow by combining the following relations:

$$\begin{aligned}\left(\int u_2^2 + v_2^2 dx\right)' &= -2k \int u_2^2 + v_2^2 dx - 5 \int u_2^2 u_1 + v_2^2 v_1 dx\\ &\quad - 5a_1 \int 2u_2v_2v_1 + v_2^2 u_1 dx\\ &\quad - 5a_2 \int 2u_2v_2u_1 + u_2^2 v_1 dx;\end{aligned} \tag{5.2}$$

$$\begin{aligned}\left(\int u_2v_2 dx\right)' &= -2k \int u_2v_2 dx\\ &\quad - \int u_3v_2u + v_3u_2v + 3u_2v_2(u_1+v_1)dx\\ &\quad - a_1 \int \frac{5}{2}(u_2^2+v_2^2)v_1 + 2u_2v_2u_1 - u_3v_2u dx\\ &\quad - a_2 \int \frac{5}{2}(u_2^2+v_2^2)u_1 + 2u_2v_2v_1 - v_3u_2v dx;\end{aligned} \tag{5.3}$$

$$\begin{aligned}\left(\int u_1^2 u + v_1^2 v dx\right)' &\approx -3 \int u_2^2 u_1 + v_2^2 v_1 dx\\ &\quad - 2a_3 \int u_3v_2u + v_3u_2v + 2u_2v_2(u_1+v_1)dx;\end{aligned} \tag{5.4}$$

$$\begin{aligned}\left(\int 2u_1v_1v + v_1^2 u dx\right)' &\approx -3 \int 2u_2v_2v_1 + v_2^2 u_1 dx\\ &\quad + a_3 \int -3(u_2^2+v_2^2)v_1 + 2u_3v_2u - 2u_2v_2u_1 dx;\end{aligned} \tag{5.5}$$

$$\begin{aligned}\left(\int 2u_1v_1u + u_1^2 v dx\right)' &\approx -3 \int 2u_2v_2u_1 + u_2^2 v_1 dx\\ &\quad + a_3 \int -3(u_2^2+v_2^2)u_1 + 2v_3u_2v - 2u_2v_2v_1 dx.\end{aligned} \tag{5.6}$$

Proof of (5.2). We transform the last three integrals of the identity (3.1) in the following way:

$$\begin{aligned}-2\int u_2(u_1u)_2 + v_2(v_1v)_2 dx &= -2\int 3u_2^2u_1 + u_2u_3u + 3v_2^2v_1 + v_2v_3v dx\\ &= -2\int 3u_2^2u_1 + \frac{1}{2}(u_2^2)_1 u + 3v_2^2v_1 + \frac{1}{2}(v_2^2)_1 v dx\\ &= -5\int u_2^2u_1 + v_2^2v_1 dx,\end{aligned}$$

$$\begin{aligned}
-2a_1\int u_2(vv_1)_2+v_2(uv)_3dx &= -2a_1\int 3u_2v_1v_2+u_2vv_3-v_3(uv)_2dx\\
&= -2a_1\int 3u_2v_1v_2-2v_3u_1v_1-v_3uv_2dx\\
&= -2a_1\int 3u_2v_1v_2+2v_2(u_1v_1)_1-\frac{1}{2}u(v_2^2)_1dx\\
&= -2a_1\int 5u_2v_1v_2+\frac{5}{2}u_1v_2^2dx\\
&= -5a_1\int 2u_2v_2v_1+v_2^2u_1dx,
\end{aligned}$$

and by symmetry

$$-2a_2\int v_2(uu_1)_2+u_2(uv)_3dx=-5a_2\int 2u_2v_2u_1+u_2^2v_1dx.$$

Combining these identities with (3.1) we obtain (5.2).

Proof of (5.3). We have

$$\begin{aligned}
\left(\int u_2v_2dx\right)' &= \int u_2'v_2+u_2v_2'dx\\
&= -\int (u_1u+u_3+ku+a_3v_3+a_1v_1v+a_2(uv)_1)_2v_2dx\\
&\quad -\int u_2(v_1v+v_3+kv+a_3u_3+a_2u_1u+a_1(uv)_1)_2dx\\
&= -2k\int u_2v_2dx-a_3\int v_5v_2+u_2u_5dx-\int u_5v_2+u_2v_5dx\\
&\quad -\int (uu_1)_2v_2+u_2(vv_1)_2dx\\
&\quad -a_1\int (vv_1)_2v_2+u_2(uv)_3dx\\
&\quad -a_2\int (uv)_3v_2+u_2(uu_1)_2dx.
\end{aligned}$$

Here

$$\begin{aligned}
\int v_5v_2+u_2u_5dx &= -\int v_4v_3+u_3u_4dx\\
&= -\frac{1}{2}\int (v_3^2+u_3^2)_1dx=0,
\end{aligned}$$

$$\int u_5v_2+u_2v_5dx=\int u_5v_2-u_5v_2dx=0,$$

$$\int (uu_1)_2v_2+u_2(vv_1)_2dx=\int 3u_1u_2v_2+uv_2u_3+vu_2v_3+3v_1v_2u_2dx,$$

$$\begin{aligned}\int (vv_1)_2 v_2 + u_2(uv)_3 dx &= \int 3v_2^2 v_1 + v_3 v_2 v + u_3 u_2 v + 3u_2^2 v_1 \\ &\quad + 3u_2 v_2 u_1 + v_3 u_2 u dx \\ &= \int 3v_2^2 v_1 + \frac{1}{2}(v_2^2)_1 v + \frac{1}{2}(u_2^2)_1 v + 3u_2^2 v_1 \\ &\quad + 3u_2 v_2 u_1 + v_3 u_2 u dx \\ &= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 3u_2 v_2 u_1 + v_3 u_2 u dx \\ &= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 3u_2 v_2 u_1 - v_2 u_3 u - v_2 u_2 u_1 dx \\ &= \int \frac{5}{2}(u_2^2 + v_2^2) v_1 + 2u_2 v_2 u_1 - u_3 v_2 u dx.\end{aligned}$$

By symmetry, we also have

$$\int (uu_1)_2 u_2 + v_2(uv)_3 dx = \int \frac{5}{2}(u_2^2 + v_2^2) u_1 + 2u_2 v_2 v_1 - v_3 u_2 v dx.$$

This proves (5.3).

Henceforth in all computations we integrate by parts and we apply Lemma 2.5 several times.

Proof of (5.4). We have

$$\begin{aligned}\left(\int u_1^2 u dx\right)' &= \int 2u_1 u_1' u + u_1^2 u' dx \\ &= \int -u'(2u_2 u + u_1^2) dx \\ &= \int (2u_2 u + u_1^2)(u_1 u + u_3 + ku + a_1 v_1 v + a_2(uv)_1 + a_3 v_3) dx \\ &= k \int 2u_2 u^2 + u_1^2 u dx + \int u_1 u(2u_2 u + u_1^2) dx + \int u_3(2u_2 u + u_1^2) dx \\ &\quad + a_1 \int v_1 v(2u_2 u + u_1^2) dx + a_2 \int (uv)_1 (2u_2 u + u_1^2) dx \\ &\quad + a_3 \int v_3(2u_2 u + u_1^2) dx.\end{aligned}$$

Here all integrals are equivalent to zero by Lemma 2.5, except those containing u_3 or v_3. Since

$$\begin{aligned}\int u_3(2u_2 u + u_1^2) dx &= \int (u_2^2)_1 u + u_3 u_1^2 dx \\ &= -\int u_2^2 u_1 + 2u_2^2 u_1 dx = -3 \int u_2^2 u_1 dx\end{aligned}$$

and

$$\begin{aligned}\int v_3(2u_2u+u_1^2)dx &= 2\int v_3u_2u - v_2u_2u_1dx\\ &= 2\int -v_2u_3u - v_2u_2u_1 - v_2u_2u_1dx\\ &= -2\int u_3v_2u + 2u_2v_2u_1dx,\end{aligned}$$

we conclude that

$$\left(\int u_1^2udx\right)' \approx -3\int u_2^2u_1dx - 2a_3\int u_3v_2u + 2u_2v_2u_1dx.$$

Adding this to the analogous relationship for $\int v_1^2vdx$ we get (5.4).

Proof of (5.5) and (5.6). We have

$$\begin{aligned}\left(\int u_1v_1vdx\right)' &= \int u_1'v_1v + u_1v_1'v + u_1v_1v'dx\\ &= \int -u'(v_2v+v_1^2) - v'u_2vdx\\ &= \int (v_2v+v_1^2)(u_1u+u_3+ku+a_1v_1v+a_2(uv)_1+a_3v_3)dx\\ &\quad + \int u_2v(v_1v+v_3+kv+a_2u_1u+a_1(uv)_1+a_3u_3)dx\\ &\approx \int v_2vu_3 + v_1^2u_3 + u_2vv_3dx + a_3\int (v_2v+v_1^2)v_3 + u_2vu_3dx\\ &= \int (u_2v_2)_1v - u_2(v_1^2)_1dx + a_3\int (v_2v+v_1^2)v_3 + u_2vu_3dx\\ &= -3\int u_2v_2v_1dx + a_3\int (v_2v+v_1^2)v_3 + u_2vu_3dx.\end{aligned}$$

Since

$$\begin{aligned}\int (v_2v+v_1^2)v_3 + u_2vu_3dx &= \int \frac{1}{2}(v_2^2)_1v - 2v_2^2v_1 + \frac{1}{2}v(u_2^2)_1dx\\ &= \int -\frac{1}{2}v_2^2v_1 - 2v_2^2v_1 - \frac{1}{2}u_2^2v_1dx\\ &= \int -\frac{5}{2}v_2^2v_1 - \frac{1}{2}u_2^2v_1dx,\end{aligned}$$

it follows that

$$\left(\int 2u_1v_1vdx\right)' \approx -6\int u_2v_2v_1dx - a_3\int (5v_2^2+u_2^2)v_1dx,$$

and then by symmetry

$$\left(\int 2u_1v_1u\,dx\right)' \approx -6\int u_2v_2u_1dx - a_3\int(5u_2^2+v_2^2)u_1dx.$$

Next we have

$$\begin{aligned}
\left(\int u_1^2v\,dx\right)' &= \int 2u_1u_1'v + u_1^2v'dx\\
&= \int -(2u_2v+2u_1v_1)u' + u_1^2v'dx\\
&= \int (2u_2v+2u_1v_1)(u_1u+u_3+ku+a_1v_1v+a_2(uv)_1+a_3v_3)dx\\
&\quad + \int -u_1^2(v_1v+v_3+kv+a_2u_1u+a_1(uv)_1+a_3u_3)dx\\
&\approx \int 2u_3u_2v+2u_1v_1u_3-u_1^2v_3dx + a_3\int(2u_2v+2u_1v_1)v_3-u_1^2u_3dx\\
&= \int -u_2^2v_1 - 2u_2(u_1v_1)_1 + 2u_1u_2v_2dx\\
&\quad + a_3\int(2u_2v+2u_1v_1)v_3-u_1^2u_3dx\\
&= -3\int u_2^2v_1dx + a_3\int(2u_2v+2u_1v_1)v_3-u_1^2u_3dx.
\end{aligned}$$

Since

$$\begin{aligned}
\int(2u_2v+2u_1v_1)v_3-u_1^2u_3dx &= \int -2v_2(u_3v+2u_2v_1+u_1v_2)+2u_2^2u_1dx\\
&= \int -2u_3v_2v - 4u_2v_2v_1 - 2v_2^2u_1 + 2u_2^2u_1dx\\
&= 2\int v_3u_2v - u_2v_2v_1 + (u_2^2-v_2^2)u_1dx,
\end{aligned}$$

it follows that

$$\left(\int u_1^2v\,dx\right)' = -3\int u_2^2v_1dx + 2a_3\int v_3u_2v - u_2v_2v_1 + (u_2^2-v_2^2)u_1dx,$$

and then by symmetry

$$\left(\int v_1^2u\,dx\right)' = -3\int v_2^2u_1dx + 2a_3\int u_3v_2u - u_2v_2u_1 + (v_2^2-u_2^2)v_1dx.$$

Combining the four relations we get (5.5) and (5.6). $\square$

Proof of the proposition for $n = 2$. We consider the functions f, g, h of Lemma 5.1. If $a_3 = 0$ or if $a_1 = a_2 = 1$, then $h = 0$. If $|a_3| < 1$, then

$$\int u_n^2 + v_n^2dx \le \frac{1}{1-|a_3|}\int u_n^2+v_n^2+2a_3u_nv_ndx.$$

Since by Lemma 2.5 and the induction hypothesis f and g satisfy the assumptions of Lemma 2.3, we may conclude as in case $n = 1$ above. $\square$

6. Proof of the Proposition for $n \geq 3$

We proceed by induction on n, so we assume that the proposition holds for smaller values of n.

By Lemma 3.1 we have

$$\begin{aligned} \left(\int u_n^2 + v_n^2 dx\right)' &= -2k \int u_n^2 + v_n^2 dx \\ &\quad - 2\int u_n(u_1u)_n + v_n(v_1v)_n dx \\ &\quad - 2a_1 \int u_n(vv_1)_n + v_n(uv)_{n+1} dx \\ &\quad - 2a_2 \int v_n(uu_1)_n + u_n(uv)_{n+1} dx. \end{aligned} \tag{6.1}$$

If we differentiate the products in the last three integrals by using Leibniz's rule and the binomial formula, we obtain a sum of three-term products. Using the inequality $n \geq 3$, it follows from Lemma 2.5 that all terms are equivalent to zero, except those containing the factor u_{n+1} or v_{n+1}.

Indeed, the orders of differentiation of the three factors are n, j and $n+1-j$ with $1 \leq j \leq n$. Since the sum $2n+1$ of the differentiations satisfies the inequality $2n+1 < 2n + (n-1)$, we have

$$2(\alpha_n + \beta_n) + (\alpha_{n_1} + \beta_{n-1}) \leq 4,$$

and Lemma 2.5 applies.

Using again that $1 \leq n-2$, it follows that

$$\begin{aligned} \int u_n(u_1u)_n + v_n(v_1v)_n dx &\approx \int u_n u_{n+1} u + v_n v_{n+1} v dx \\ &= \frac{1}{2}\int (u_n^2)_1 u + (v_n^2)_1 v dx \\ &= -\frac{1}{2}\int u_n^2 u_1 + v_n^2 v_1 dx \\ &\approx 0, \\ \int u_n(vv_1)_n + v_n(uv)_{n+1} dx &\approx \int u_n v v_{n+1} + v_n u_{n+1} v + v_n u v_{n+1} dx \\ &= \int u_n v v_{n+1} - u_n(v_n v)_1 + \frac{1}{2} u (v_n^2)_1 dx \\ &= \int -u_n v_n v_1 - \frac{1}{2} u_1 v_n^2 dx \\ &\approx 0, \end{aligned}$$

and by symmetry

$$\int v_n(uu_1)_n + u_n(uv)_{n+1}dx \approx 0.$$

Using these relations we infer from (6.1) that

$$\left(\int u_n^2 + v_n^2 dx\right)' \approx -2k \int u_n^2 + v_n^2 dx,$$

and we conclude as usual.

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